Chapter 1

Probability, Random Variables, and Random Process

CHAPTER HIGHLIGHTS

- *Probability*
- *Random Variable*
- *Standard Random Variable*
- *Random Process*
- *Probability Density Function*
- *Rayleigh Density Function*
- *Poisson Density*
- *Probability Distribution Function*
- *Central Limit Theorem*
- *Random Process X*
- *Wide Sense Stationary Random Process*
- *Properties of Autocorrelation*
- *Gaussian Process*
- *Additive White Gaussian Noise Process*
- *Fourier Transform of Important Signals*
- *Inverse Hilbert Transform*
- *Complex Envelope*
- *Phase Delay and Group Delay*

PROBABILITY

Deterministic Experiment Versus Random Experiment

An experiment whose outcome is completely predictable is called deterministic experiment.

An experiment whose outcome cannot be predicted with certainty is called random experiment. Tossing of a coin or tossing of a die are examples of random experiment.

Sample Space (Ω)

The entire set of all possible outcomes of a random experiment is called sample space.

For tossing of a coin experiment,

$$\Omega = \{T, H\}$$

For tossing of a die experiment,

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Event

Any subset of the sample space Ω is called an event.

If a random experiment has total 'n' outcomes, there exists a 2^n possible subset of n outcomes. Thus, 2^n possible events exists.

Definition of Probability

If A is an event in the sample space Ω of a random experiment, if the random experiment is performed 'n' times and event A occurred n_A times, the probability of event A is defined by

$$P(\mathrm{A}) = \underset{n\to\infty}{Lt}\ \frac{n_A}{n}.$$

Axioms of Probability

1. $P(\mathrm{A}) \geq 0$
2. $P(\Omega) = 1$ and $P(\varphi) = 0$
3. If events A and B are mutually exclusive, that is, there exist no common elements in A and B, then

$$P(A\cup B) = P(\mathrm{A}) + P(\mathrm{B}).$$

Conditional Probability *P(A/B)*

$P(A/B)$ is the probability of occurrence of event A after the occurrence of event B.

If $A \supset B$, $P(A/B) = 1$

If $A \subset B$ $P(A/B) \geq \mathrm{P(A)}$

If $A \cap B = \varnothing$ $P(A/B) = 0$

For example, if a fair die is tossed P(2/even number) is given by the probability of occurrence of two provided, an even number has occurred. The probability of 2 after occurrence of $\{2, 4, 6\}$ is 1/3.

Joint Probability *P*(*A*, *B*)

$P(A,B)$ is the probability of occurrence of both A and B.

If $$A \supset B$$

$$P(A, B) = P(B)$$

If $$A \cap B = \varnothing$$

$$P(A, B) = 0$$

$$P(A, B) = P(B, A).$$

Bayer's Rule

$$P(A, B) = P(A/B)\, P(B)$$

$$P(B, A) = P(B/A)\, P(A)$$

$$\therefore\ P(A/B)\, P(B) = P(B/A) P(A)$$

Repeated Trials

In experiment with sample space Ω is repeated n times, the sample space of the entire experiment is the Cartesian product of Ω with itself 'n' times.

i.e., $$\Omega_n = \Omega \times \Omega \times \ldots \times \Omega$$

If the tossing of the coin experiment is repeated three times, the new sample space would be

$$\{T, H\} \times \{T, H\} \times \{T, H\} = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

If the original experiment has 'k' outcomes, the 'n' times repeated experiment will have k^n outcomes.

Binomial Density Function

In a random experiment, if the probability of occurrence of an event 'A' is P, if the experiment is performed n *times*, then

$$P\{\text{'}A\text{' occurs } K \text{ times}\} = {}^{n}c_{k} P^{k} (1\text{-}p)^{n\text{-}k}$$

In a random experiment, if $A_1, A_2, \ldots A_n$ are mutually exclusive events, if $P(A_1) = P_1, P(A_2) = P_2, \ldots P(A_n) = P_n$ $[P_1 + P_2 + \ldots + P_n = 1]$. Let the experiment performed 'n' times.

$$P\{A_1 \text{ occurs } K_1 \text{ times}, A_2 \text{ occurs } K_2 \text{ times}, \ldots\ A_n \text{ occurs } K_n = n - k_1 - k_2 - k_n \text{ times}\} = \frac{n!}{k_1!k_2!\ldots k_n!} P_1^{k_1} P_2^{k_2} \ldots P_n^{k_n}$$

Probability

The concept of probability occurs naturally when we think the possible outcomes of an experiment. There are three kinds of event possible:

1. **Mutually Exclusive Events:** Two possible outcomes of an experiment are defined as being mutually exclusive if the occurrence of one outcome makes impossible the occurrence of the other event.

   If for two events A_1 and A_2 probabilities are $P(A_1)$ and $P(A_2)$, then the probability of occurrence of either A_1 or A_2 is given by

   $$P(A_1 \text{ or } A_2) = P(A_1) + P(A_2)$$

   Let us consider the following example.
   If we ask the probability of either 1 or 2? Then, we can say that probability of either even is $\frac{1}{6}$ and since one having occurred, the other one cannot take place. Therefore, it is a mutually exclusive event.

   Thus, the probability of 1 or 2 is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

2. **Joint probability of Related and Independent Events:**
   Suppose we have two experiments A and B with outcomes $A_1, A_2\ldots$ and $B_1, B_2\ldots$
   The probability of joint occurrence is $P(A_j, B_k)$, if we have a situation in which the outcome of the second experiment is conditional on the outcome of the first experiment. Then, the probability of the outcome B_k, given that A_j is known to have occurred is called conditional probability $(P(B_k/A_j)$.
   Let N_j be the number of times A_j occurs with or without B_k, N_k the number of times B_k occurs with or without A_j, and N_{jk} the number of times of joint occurrence. Then,

   $$P(B_k/A_j) = \frac{N_{jk}}{N_j} = \frac{N_{jk}/N}{N_j/N} = \frac{P(A_j, B_k)}{P(A_j)} \quad (1)$$

   or $$P(A_j/B_k) = \frac{P(A_j, B_k)}{P(B_k)} \quad (2)$$

   From Equation (1) and (2),

   $$P(A_j/B_k)\, P(B_k) = P(B_k/A_j)\, P(A_j) = P(A_j, B_k)$$

   or $$P(A_j/B_k) = \frac{P(A_j)}{P(B_k)} P(B_k/A_j) \quad (3)$$

   Equation (3) is Bayes' Theorem.
   Therefore, we can say that a single experiment whose outcome is characterized by two events is known as Bayes' theorem.

3. **Statistical Independence:**
   If A_j and B_k are the possible outcomes of two successive experiments or the joint outcome of a single experiment and if the probability of the occurrence of the outcome B_k does not depend on outcome A_j, then the outcomes A_j and B_k are independent.

   $$P(B_k/A_j) = P(B_k)$$

   $$P(A_j, B_k) = P(A_j)\, P(B_k)$$

   CDF (Cumulative Distribution Function): The cumulative distribution function associated with a random variable that is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(\lambda) \le x$, where x is any given number. The probability will depend on the number x and also on $X(\lambda)$.

CDF is denoted by $F(\text{x}) \equiv P[X(\lambda) \le x]$ and $F(\text{x})$ is always having value.

$$0 \le F(\text{x}) \le 1$$

Now, $F(-\infty)$ includes no possible events and $F(\infty)$ includes all possible events.

Let us consider the following example.

We throw two dice showing numbers 1 to 6, then there are 36 possible events.

Let i = number on 1st dice

and j = number on 2nd dice

Now, we are interested to know the sum of the numbers appearing on the dice. Then, the random variable function $N(\lambda ij)$ is defined as

$n = N(\lambda ij) = i + j$, then $N(\lambda ij) = N(\lambda ji)$ is started from

$$n = 2 \text{ to } n = 12$$

$$P(1) = 0,\ P(2) = P(12) = \frac{1}{36}$$

$$P(3) = P(11) = \frac{2}{36},\ P(4) = P(10) = \frac{3}{36}$$

$$P(5) = P(9) = \frac{4}{36},\ P(6) = P(8) = \frac{5}{36}$$

$$P(7) = \frac{6}{36}$$

Now, we are interested to know the CDF for $n = 4$, then $F(4) = P[N(\lambda ij) \le 4) = P(\text{n} \le 4)$

$$F(4) = P(1) + P(2) + P(3) + P(4)$$
$$= 0 + \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36} = \frac{1}{6}$$

Random Variable

Random variable is a function that maps the experimental outcomes to a real line.

The domain of a random variable is the experimental outcome and the range is the real number.

If x is the random variable,

$$X(\xi_{\text{i}}) \to R$$

For example, if $\Omega = \{TT, HT, HH, TH\}$

RV X = number of heads, then $X(TT) = 0$, $X(TH) = 1$ $X(HT) = 1$, $X(HH) = 2$.

X converted the experimental outcome into real line.

For a given experiment, any number of random variables can be defined based on the requirement.

Standard Random Variable

To standardize a random variable, we subtract the expected value μ from each value of the random variable can assume. It shifts all of the values so that the expected value μ is centred at the origin.

Random Process

To determine the probabilities of the various possible outcomes of an experiment, it is necessary to repeat the experiment many times.

To determine the statistic of noise, there are two methods: First, we can make repeated measurements of the noise voltage output of a single noise source.

Second, we can make simultaneous measurements of the output of a very large collection of statistically identical noise sources.

Therefore, the average determined by the second method is called *ensemble average*. The average determined by the first method is called *time average.*

The ensemble average or statistical average both are same. Therefore, it can be expressed as

$$E\left[n_1^2(t)\right] = \overline{n_1^2(t)}$$

When time average is expressed as $\langle n^2(\text{t})\rangle$:

1. **Stationary Process:** When the statistical characteristics of the sample function do not change with time, the random process is known as stationary process.
2. **Ergodic Process:** A process $X(\text{t})$ is said to be ergodic, if time average is same as ensemble average, that is, $\langle x(\text{t})\rangle$

   $$= E[X(\text{t})] = m$$

   and it is called ergodic autocorrelation, if $\langle x(\text{t})\, x(t + \tau)\rangle = E[X(\text{t})\, X(t + \tau] = R_{\text{xx}}(\tau)$.
3. **White Noise Process:** A random process $X(\text{t})$ is said to be white noise process, when its PSD $G(\text{f}) = \eta/2$ is constant over entire frequency spectrum and mean is assumed to be zero.

   $$\text{Autocorrelation } R(\tau) = \frac{\eta}{2}\ \delta(\tau)$$

   Band-limited white noise: Band-limited white noise has similar flat spectrum as white noise has but only over the pass band of frequencies only.
4. **Bandpass random process:** If a random process $x(\text{t})$ can be expressed in terms of in-phase and in-quadrature components, then it is called bandpass random process.

   $$x(\text{t}) = x_{\text{c}}(\text{t}) \cos\ (2\pi f_c t)\ + x_{\text{s}}(\text{t}) \sin\ (2\pi f_c t)$$

Probability Density Function

If x is a random variable, the probability density function $f_{\text{x}}(\text{x})$ is defined by the probability that the random variable X takes a particular value of X.

$$f_{\text{x}}(\text{x}) = p(X = x)$$

Properties of $f_x(x)$:<NL>

1. $f_x(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_x(x)dx = 1$

   (total area under probability density function is one)
3. $p(a < x < b) = \int_a^b f_x(x)\,dx$

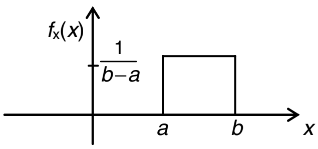
   Examples of probability density function.

Uniform Density Function

Random variable X is said to have uniform density, if

$$f_x(x) = \frac{1}{b-a} \text{ for } x \in (a,b)$$

$$= 0 \text{ for } x \notin (a, b)$$



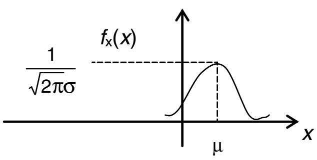
Total area under $f_x(x) = (b-a)\dfrac{1}{(b-a)}$

Gaussian Density Function

If X is a random variable with Gaussian density function,

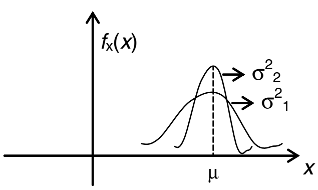
$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ is the mean of RV X and σ^2 is the variance of X.



Properties of Gaussian density function

1. Gaussian density function is symmetric with respect to mean μ.
2. Gaussian density is maximum at $x = \mu$. The maximum value of probability is $\dfrac{1}{\sqrt{2\pi\sigma}}$.
3. Gaussian density curve becomes sharper if the variance is less. If the variance is zero, the random variable becomes a constant or deterministic.



$\sigma^2_1 > \sigma^2_2$

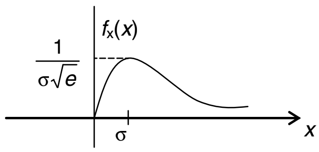
4. $\displaystyle\int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = \int_{\mu}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} = \frac{1}{2}$
5. At $x = \mu \pm 3.16\sigma$, the Gaussian density decays 99% of its maximum value.
6. Gaussian density is also referred as normal density. A Gaussian random variable with mean μ and variance σ^2 is denoted by $N(\mu, \sigma^2)$.
7. If $Y = x_1 + x_2 + \ldots + x_n$, the probability density function of Y approaches Gaussian density, if $n \to \infty$ and $x_1, x_2, \ldots x_n$ may have any density. This theorem is called central limit theorem.

Rayleigh Density Function

If x is a random variable with Rayleigh density function:

$$f_x(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$$

where σ^2 is the variance of x.



Properties of Rayleigh density

1. The envelope of a noise or fading channel is modelled as a Rayleigh density function.
2. The Rayleigh density function is maximum at $x = \sigma$.

   The maximum value of density is $\dfrac{1}{\sigma\sqrt{e}}$.

Poisson density

X is said to be Poisson random variable with mean λ, if

$$P\{x = k\} = \frac{e^{-\lambda}\lambda^k}{K!}$$

where $K = 0, 1, 2, \ldots. \infty$

Probability Distribution Function

Probability distribution function of a random variable X is defined as

$$F_x(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} f_x(x)\,dx$$

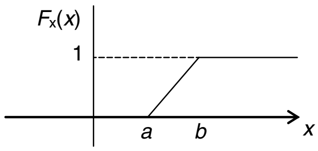
$F_x(x)$ is the total probability that random variable takes up to x. Properties of $F_x(x)$ are as follows:

1. $F_x(x)$ is non-decreasing function
2. $F_x(-\infty) = 0$, $F_x(\infty) = 1$

3. $f_x(x) = \dfrac{dF_x(x)}{dx}$

4. $\int_a^b f_x(x)dx = F_x(b) - F_x(a)$

5. if x is uniform density RV



6. If x is a Gaussian density RV

$$F_x(\mu) = 0.5$$

For any 'a'

$$F_x(\mu - a) = P$$

$$F_x(\mu + a) = 1 - P$$

Function of a Random Variable

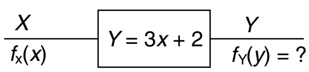
If random variable Y is a function of X

$Y = g(x)$ and probability density of X is $f_x(x)$, then $f_Y(y)$ is given by

$$f_Y(y) = \sum_{i=1}^{N} \frac{f_x(x_i)}{(dy/dx)_{x=x_i}}$$

where N is the number of roots of the equation $y = g(x)$.

For example, if x is uniform density, $f_x(x) = \dfrac{1}{3}\, x \in (2, 5)$
$= 0$ else



To find out $f_Y(y)$

$$y = 3x + 2$$

$$\frac{dy}{dx} = 3$$

$x = \dfrac{y-2}{3}$: only one root for every y

$$f_Y(y) = \frac{f_x(x)}{dy/dx}$$

$$= \frac{\frac{1}{3}}{3} = \frac{1}{9}$$

$$f_Y(y) = \frac{1}{9} \quad y \in (8, 17)$$

$$= 0 \text{ else.}$$

Joint Probability Density Function

If X and Y are two random variables, and the joint probability density function $f_{xy}(x, y)$ is the probability with which X takes the value of x and Y takes the value of y.

Properties of $f_{xy}(x, y)$

1. $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x, y)\, dy$

$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x, y)\, dx$

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x, y)\, dx\, dy = 1$

3. If RV's X and Y are independent, that is, the occurrence X does not depend on Y and vice versa, then

$$f_{XY}(x, y) = f_x(x)\, f_y(y)$$

4. If $X_1, X_2, \ldots X_n$ are n random variables, the n^{th} order probability density function is defined by $f_{x1,\, x2,\, \ldots\, xn}(x_1, x_2, \ldots x_n)$ is the probability that $X_1, X_2, \ldots X_n$ takes the values of $x_1, x_2, \ldots x_n$, respectively.

$$f_{x1,x2,\ldots xn-1}(x_1, x_2, \ldots x_{n-1})$$

$$= \int_{-\infty}^{\infty} f_{x_1, x_2 \ldots x_{n-1}, x_n}(x_1, x_2, \ldots x_{n-1}, x_n)\, dx_n$$

5. From nth order probability density function, we can obtain the pdf of $(n - 1)^{th}$ order or less by integrating with respect to the required random variables.
Joint probability distribution function is defined by

$$F_{XY}(x, y) = P(X \le x, Y \le y)$$

Properties of $F_{XY}(x, y)$

(i) $F_{XY}(\infty, \infty) = 1$
$F_{XY}(x, \infty) = F_X(x)$
$F_{XY}(\infty, y) = F_Y(y)$

(ii) $F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x^1, y^1)\, dx^1 . dy^1$

(iii) If X and Y are two independent random variables,

$$F_{XY}(x, y) = F_X(x)\, F_Y(y)$$

(iv) $f_{XY}(x, y) = \dfrac{d^2 F_{XY}{}^{(x,y)}}{dx\, dy}$

Bayes' rule for joint probability density is given by

$$f_{XY}(x, y) = f_{X/Y}(x/y)\, f_Y(y)$$

where $f_{X/Y}(x/y)$ is conditional probability of x given specific 'y'.
We know

$$f_{XY}(x, y) = f_{Y,X}(y, x)$$

Thus,

$$f_{XY}(x/y)\, f_Y(y) = f_{Y/X}(y/x)\, f_X(x).$$

The Bayes' rule can be extended for any nth order joint density function.

$$f_{W,X,Y,Z}(w,x,y,z) = f_{W/X,Y,Z}(w/x,y,z)\, f_{X,Y,Z}(x,y,z)$$
$$= f_{W/X,Y,Z}(w/x,y,z).f_{X/Y,Z}(x/y,z)\, f_{Y,Z}(y,z)$$
$$= f_{W/X,Y,Z}(w/x,y,z).f_{X/Y,Z}(x/y,z) f_{Y/Z}(y/z) f_Z(z)$$

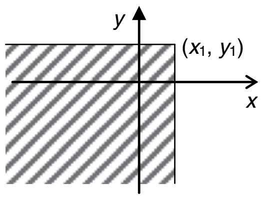
If two random variables X and Y are independent

$$f_{X/Y}(x/y) = f_X(x).$$

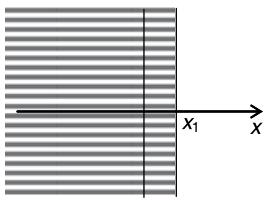
Thus, $f_{X,Y}(x,y) = f_{X/Y}(x,y) f_Y(y) = f_X(x) f_Y(y)$.

For two random variables X and Y, if $f_{XY}(x,y)$ is the probability of a point (x, y) is two-dimensional plane, then

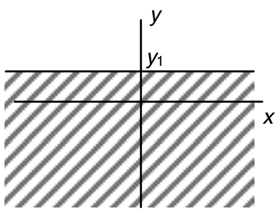
$F_{X,Y}(x_1,y_1)$ = total probability of shaded area in the following diagram.



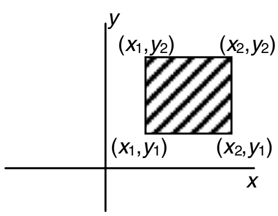
$F_X(x_1)$ is the total probability under the area



$F_Y(y_1)$ is total probability under area



The total probability in the given shaded area is given by



$$F_{XY}(x_2,y_2) + F_{XY}(x_1,y_1) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy\,.$$

Sum of Random Variables

If $x_1, x_2, \ldots x_n$ are independent random variable with pdf $f_{x1}(x), f_{x2}(x) \ldots f_{Xn}(x)$, if $Y = X_1 + X_2 + \ldots X_n$.

The pdf of Y is given by $f_Y(y) = f_{x1}(x) * f_{X2}(x) * \ldots * f_{Xn}(x)$, i.e., the pdf of Y is convolution of all pdfs.

Central Limit Theorem

If $X_1, X_2, \ldots X_n$ are any random variables with finite mean and variance, the sum

$X_1 + X_2 + \ldots + X_n$ will approach Gaussian density function if $n \to \infty$

Expectation (Statistical Average)

If x is a random variable with pdf $f_x(x)$, the statistical average of x is given by

$$E(x) = \int_{-\infty}^{\infty} x.f_x(x)\,dx,$$

$E(x)$ is called mean of $X = \mu_x$.

Properties of Expectation

If 'c' is a constant

1. $E(c) = c$
2. $E(cx) = cE(x)$
3. $E(aX + bY) = aE(X) + bE(Y)$

*n*th Moment of X

The nth moment of X is given by

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_x(x)\,dx \quad \text{if } n = 2.$$

$E(X^2)$ is called mean square value.

*n*th Central Moment

$E((X - \mu_x)^n)$ is called nth central moment of RV X, that is, nth moment around the mean μ_x.

$$E((X - \mu_x)^n) = \int_{-\infty}^{\infty} (x - \mu_x)^n f_x(x)dx$$

if $\qquad n = 2$

$E((x - \mu_x)^2)$ is called variance of RV $X = \sigma_x^2$

$$\sigma_x^2 = E((x - \mu_x)^2)$$
$$= E(x^2 + \mu_x^2 - 2\mu_x)$$
$$= E(x^2) + E(\mu_x^2) - E(2\mu_x)$$
$$= E(x^2) + \mu_x^2 - 2\mu_x E(x)$$
$$= E(x^2) + \mu_x^2 - 2\,\mu_x^2$$
$$= E(x^2) - \mu_x^2$$

i.e., $\qquad \sigma_x^2 = E(x^2) - \mu_x^2$

If mean $\mu_x = 0$, variance = mean square value.

Expectation of a Function of RV

If $Y = g(X)$

$E(Y) = E(g(X))$ is given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx \quad .$$

Correlation between Two RV's *X* and *Y*

The correlation between two RV's X and Y is denoted by R_{XY} and it is defined by

$$R_{XY} = E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x.y\, f_{XY}(x, y)\, dx\, dy$$

R_{XY} indicates the dependence between two random variables X and Y.

If two random variables X and Y are independent

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

Thus,

$$R_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x.y\, f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \mu_X\, \mu_Y$$

If $R_{XY} = 0$, the two RV's X and Y are called orthogonal.

Covariance of RV's *X* and *Y*

Covariance between two RVs is defined by

$$C_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= E[XY - \mu_x Y - \mu_Y X + \mu_X\, \mu_Y]$$

$$= E(XY) - \mu_X\, \mu_Y$$

$$= R_{XY} - \mu_X\, \mu_Y$$

If μ_X or μ_Y or both $= 0$, then

$$R_{XY} = C_{XY}$$

If $C_{XY} = 0$, then random variables X and Y are called uncorrected.

If X and Y are independent, $R_{XY} = \mu_X\, \mu_Y$. Thus, $C_{XY} = 0$.

Independence is a superior condition to the uncorrelatedness of two random variables X and Y.

Correlation Coefficient

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y are standard deviations of RV's X and Y.

Properties of Correlation Coefficient

1. ρ_{XY} is in the range of [0, 1].
2. $\rho_{XY} = 0$ means X and Y are uncorrelated.
3. $\rho_{XY} = 1$ means X and Y are completely dependent, for example, $Y = 2x + 3$
   Y can be completely defined in terms of X, then $\rho_{XY} = 1$.
4. Correlation coefficient indicates the dependence of one RV on another.

Solved Examples

Example 1

The variance of random variable X is σ^2_x. Then, the variance of a random variable K^2X is (where K is a constant).

(A) σ^2_x (B) $K\sigma^2_x$

(C) $K^2\, \sigma^2_x$ (D) $K^4\, \sigma^4_x$

Solution

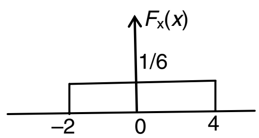
$$\text{Var}(X) = E(X^2) - \mu^2_X = \sigma^2_x$$

$$\text{Var}(K^2X) = E(K^4X^2) - (K^2\mu_x)^2$$

$$= (E(X^2) - \mu^2_X)\, K^4 = K^4\, \sigma^2_x$$

Example 2

For a random variable X, $f_x(x)$ is shown in the following figure.



The mean and variance of X, respectively, are

(A) 1, 3 (B) $2, \frac{8}{3}$ (C) $1, \frac{8}{3}$ (D) 2, 3

Solution

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \quad f_x(x)dx$$

$$\int_{-2}^{4} x \,.\, \frac{1}{6} dx$$

$$\frac{1}{6}\left[\frac{x^2}{2}\right]_{-2}^{4} = 1$$

$$\text{Variance} = E(x^2) - \mu_X^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \;\; fx(x)\; dx$$

$$= \int_{-2}^{4} x^2 \;\; \frac{1}{6} dx = \frac{1}{6}\left[\frac{x^3}{3}\right]_{-2}^{4} = 4$$

Variance = 4 – 1 = 3

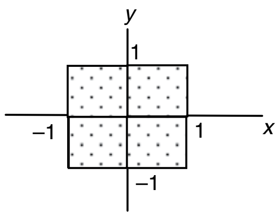
Example 3

Two independent random variables X and Y are uniformly distributed in the interval [–1,1]. The probability min[X, Y] is less than $\frac{1}{2}$ is

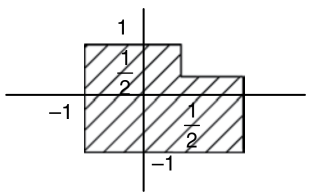
(A) $\frac{3}{4}$ (B) $\frac{15}{16}$ (C) $\frac{7}{8}$ (D) $\frac{1}{2}$

Solution

Considering X and Y are uniformly distributed, (X, Y) may take any point in the undermentioned square with equal probability.



$P\left[\min(X,Y) < \frac{1}{2}\right]$ = total probability under undermentioned area.



$$P\left[\min(X, Y) < \frac{1}{2}\right] = \frac{1-0.25}{4} = \frac{15}{16}$$

Example 4

During transmission over a communication channel, bit error occurs independently with probability 'P'. If a block of n bits are transmitted, the probability of at least one bit is in error equal to

(A) $(1 - P)^n$ (B) P^n
(C) $1 - (1 - P)^n$ (D) $1 - P^n$

Solution

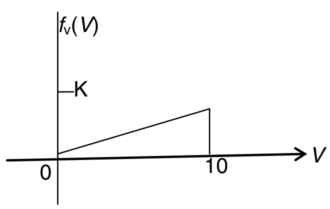
The probability of at least one error

$\lambda = 1$ – probability of no errors

$$= 1 - {}^{n}c_n(1-P)^n = 1 - (1 - P)^n$$

Example 5

The output of a communication channel is a random variable 'V' with probability density function shown in the following figure. The mean square value of 'V' is



(A) 50 (B) 100 (C) 200 (D) 10

Solution

Total area under $f_V(V) = 1$

$$\frac{1}{2}.10\,.K = 1$$

$$K = \frac{1}{5}$$

$$f_V(V) = aV$$

at $V = 10, f_v(V) = \frac{1}{5}$

$$\frac{1}{5} = 10a$$

$$\therefore\ a = \frac{1}{50}$$

$$E(V^2) = \int_{-\infty}^{\infty} V^2 f_v(V)dV$$

$$= \int_0^{10} V^2 . \frac{1}{50} V\,dV = \frac{1}{50}\int_0^{10} V^3\,dV$$

$$= \frac{1}{50}\left[\frac{V^4}{4}\right]_0^{10} = 50$$

Example 6

Consider two independent random variables X and Y with identical distributions. The variables X and Y take values 0, 1, and 2 with probabilities $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$, respectively.

The value of the condition of probability $P(X + Y = 3/X - Y = 1)$

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{6}$ (D) $\frac{1}{12}$

Solution

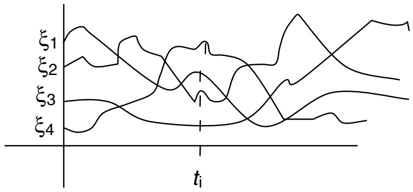
$P(X + Y = 3/X - Y = 1) = P(2, 1)/(1, 0), (2, 1))$ = probability of (2, 1) after the occurrence of either (1, 0), or (2, 1)

$$= \frac{\frac{1}{3}.\frac{1}{6}}{\frac{1}{3}.\frac{1}{6} + \frac{1}{2}.\frac{1}{3}} = \frac{1}{4}$$

Random Process $X(t, \xi)$

A random process is a random variable that is also a function of time. A random process can be seen as a collection of time function depending on the outcome of an experiment.

The random process at any particular time t_i is $X(t_i)$ is a random variable.



Each of the above four time functions corresponds to a particular experimental outcome ξ_i. At any time t_i, the value of a random process depends on experimental outcome.

Examples of random process are as follows:

1. $A_C \cos(2\pi f_c t + \varphi)$, where φ is an RV.
2. $r \cos(2\pi f_c t)$, where r is an RV.

In general random process, $X(t, \xi)$ have the density function $f_{x(t)}(x)$, which is a function of time.

Mean of Random Process

$$\mu_{x(ti)} = E[x(t_i)]$$

$$= \int_{-\infty}^{\infty} x\, f_x(x;t_i)\, dx$$

Wide-sense Stationary Random Process

A random process is called WSS process, if its first-order density function is independent of time and its second-order density function depends only on the difference between the time.

1. $f_x(x;t_i) = f_x(x;t_j)$ for all t_i and t_j
2. $f_x(x_1, x_2; t_i, t_j) = f_x(x_1, x_2; t_i + \tau, t_j + \tau)$

Autocorrelation of a WSS Process

$$R_X(t_i, t_j) = E[x(t_i)\, x(t_j)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 . x_2 . f_x(x_1, x_2; t_i, t_j)\, dx_1\, dx_2$$

Properties of WSS process If the process $x(t)$ is a WSS

1. mean = constant
2. Autocorrelation depends only on difference of time between two samples t_i and t_j
   i.e., $E[x(t_i)\, x(t_j)] = E(x(t_i + \lambda)\, x(t_j + \lambda)] = R_X(t_i - t_j) = R_X(t)$
   Any process satisfying the above two properties is a WSS process.

Stationary Process

A random process $x(t)$ is stationary if the joint density of any set of random variables obtained by observing the random process $x(t)$ is invariant with respect to the location of origin $t = 0$

i.e.,

$$f_x(x_1, x_2, \ldots x_n; t_1, t_2, \ldots t_n) =$$
$$f_x(x_1, x_2, \ldots x_n; t_1 + \tau, t_2 + \tau, \ldots t_n + \tau) \text{ for any } n \text{ and } \tau.$$

For stationary process, the joint density only depends on relative positions of the time and does not depend on the shift or distance from origin $t = 0$.

WSS process satisfies stationary process condition for $n = 1$ and 2, but the stationary process satisfies abovementioned condition for all values of 'n'.

Thus, the condition for stationary process is stronger than WSS. This means all stationary processes are WSS but all WSS processes may not be stationary.

Properties of Autocorrelation If $X(t)$ is a stationary process and $R_X(\tau)$ is the autocorrelation of $X(t)$, then

1. $R_x(\tau) = R_x(-\tau)$, i.e., autocorrelation is an even function.
2. $R_x(0) \geq R_x(\tau)$
3. $R_x(0) = R(x(t)\, x(t + 0))$
   $= E(x^2(t))$ = mean square value
4. If $R_x(\tau) = R_x((\tau + T_0)$ for any T_0, then $R_X(\tau)$ is periodic.
5. The autocorrelation value $R_x(\tau)$ indicates the dependence of two samples of random process $x(t)$ taken at a distance τ.
6. If $R_x(\tau) = 0$ for $\tau > a$, the process is called a dependent process for a dependent process, $x(t_i)$ and $x(t_i + a + \delta)$ are uncorrelated for a positive value of δ, provided $E(x(t)) = 0$.

Power Spectral Density

For a stationary process $x(t)$, the power spectral density $S_x(f)$ is defined by the Fourier transform of its autocorrelation function.

$$R_x(\tau) \rightleftharpoons S_x(f)$$

where $S_x(f)$ indicates the amount of power contained in $x(t)$ with respect to frequency.

Properties of power spectral density

1. $S_x(f) \geq 0$
2. For a real-valued random process
   $S_x(f) = S_x(-f)$
3. $S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f \tau}\, d\tau$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f \tau}\, df$$

If we substitute $\tau = 0$ in the abovementioned equation

$$R_x(0) = \int_{-\infty}^{\infty} S_x(f) df$$

Total power in

$$x(t) = \int_{-\infty}^{\infty} S_x(f)df = R_x(0) = E(x^2(t))$$

If $\mu_x = 0$, total power $= \sigma^2_x$.

Gaussian Process

A process X(t) is Gaussian, if every sample of X(t) is a Gaussian density random variable and any linear combination of X(t) is a Gaussian random variable.

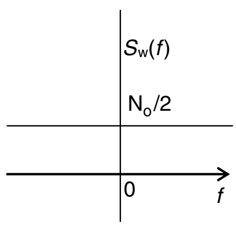
For any function g(t), and for any time T;

$$\int_0^T g(t)\,x(t)\,dt \text{ is a Gaussian random variable.}$$

Additive white Gaussian Noise Process (AWGN)

A random process W(t) is called AWGN if

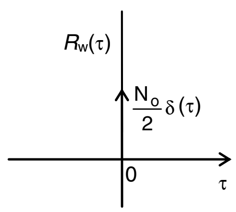
1. w(t) is a Gaussian process, that is, any sample of w(t) or any linear combination of w(t) should have Gaussian density.
2. $E(w(t)) = 0$
3. The power spectral density of w(t) should be uniform throughout the entire frequency.
   i.e.,



4. The autocorrelation of w(t) is given by

$$R_w(\tau) = \frac{N_o}{2}\delta(\tau)$$

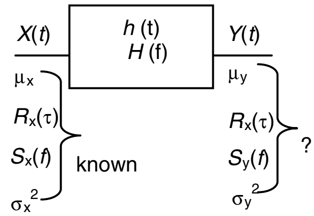
[Fourier transform S_w(f)]



If we take two samples of AWGN at two different times t_i and t_j, they are uncorrelated and independent.

Transmission of a Random Process through a Linear System

If h(t) is the impulse response of the system and H(f) is the transfer function of the system, the parameters of Y(t) are given by



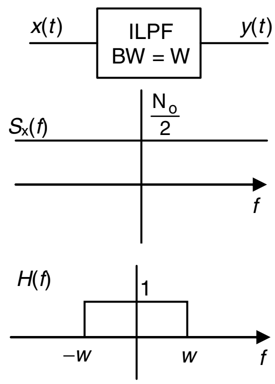
$$\mu_Y = \mu_X.H(0)$$

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

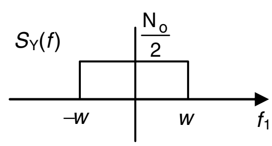
$$S_Y(f) = S_X(f)|H(f)|^2$$

$$\sigma_Y^2 = \int_{-\infty}^{\infty} S_x(f)|H(f)|^2\,df$$

If AWGN with spectral density $N_o/2$ is passed through ideal LPF with BW W Hz



$S_Y(f) = S_X(f).|H(f)|^2$



$$E(Y^2(t)) = \int_{-\infty}^{\infty} S_Y(f)\,df$$

$$= \text{area under } S_Y(f) = 2W.\frac{N_o}{2} = WN_o$$

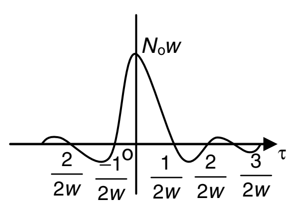
Being X(t) is AWGN $E(x(t)) = 0$

Thus, $E[Y(t)] = 0$

$$\therefore\ \sigma_Y^2 = E(Y^2(t)) = N_oW \text{ watts}$$

$$R_Y(\tau) \rightleftharpoons S_Y(f)$$

$$\therefore\ R_Y(\tau) = WN_o \text{ Sin } C\,(2W\tau)$$

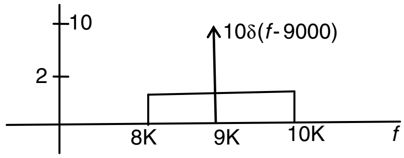


$$R_Y(\tau) = 0 \text{ at } \tau = \frac{n}{2w}$$

If we take two samples of output of LPF, which are separated by a distance of $\frac{n}{2w}$, then the two samples are uncorrelated and independent.

Example 7

The power spectral density of a real process $X(t)$ for positive frequencies is shown in the following figure. The value of $E[X^2(t)]$ is



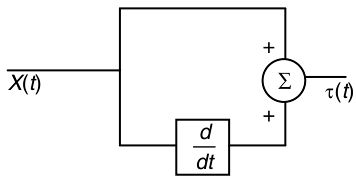
(A) 8,000 (B) 4,000 (C) 8,020 (D) 4,010

Solution

$E(x^2(t))$ = area under power spectral density
Area under $\delta(f) = 1$
$\therefore$ Area under power spectral density = $2(2{,}000 \times 2 + 10) = 8{,}020$

Example 8

$X(t)$ is a stationary random process with autocorrelation function. $R_x(\tau) = \exp(-\pi\tau^2)$. This process is passed through a system in the following figure.



The power spectral density to output process $y(t)$ is
(A) $(1 + 4\pi^2 f^2)\exp(-\pi f^2)$ (B) $(1 - 4\pi^2 f^2)\exp(-\pi f^2)$
(C) $(1 + 4\pi^2 f^2)\exp(-\pi f)$ (D) $(1 - 4\pi^2 f^2)\exp(-\pi f)$

Solution

If we convert the given relation into frequency domain

$$Y(f) = (1 + j2\pi f)\, X(f)$$

Given

$$R_X(\tau) = \exp(-\pi\tau^2)$$
$$R_X(\tau) \Leftrightarrow S_X(f)$$
$$S_X(f) = \exp(-\pi f^2)$$

We know

$$S_Y(f) = S_X(f)\,(H(f))^2$$
$$\therefore S_Y(f) = \exp(-\pi f^2)\,.\,(1 + 4\pi^2 f^2)$$

Revision of Signals and Systems

Fourier Transform

Fourier transform converts the time domain signal into frequency domain. For a signal $x(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

Similarly, inverse Fourier transform converts the frequency domain to time domain.

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

Properties of Fourier transform

If $x(t) \rightleftharpoons X(f)$

1. $\int_{-\infty}^{\infty} x(t)\, dt = X(0)$

   $\int_{-\infty}^{\infty} X(f)\, df = x(0)$
2. $x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{f}{a}\right)$
3. $x(t - t_0) \rightleftharpoons e^{-j2\pi f t_0} X(f)$
4. $e^{j2\pi f_c t} x(t) \rightleftharpoons X(f - f_c)$
5. $\frac{d\,x(t)}{dt} \rightleftharpoons j2\pi f X(f)$
6. $\int_{-\infty}^{t} x(t^1)\, dt^1 \rightleftharpoons \frac{X(f)}{j2\pi f}$
7. $X(t) \rightleftharpoons x(-f)$
8. $\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(f)|^2\, df$
9. $x_1(t) * x_2(t) \rightleftharpoons X_1(f) X_2(f)$

   where $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t - \tau)\, d\tau$
10. If $x(t)$ is real $x(t) \rightleftharpoons X(f) = |X(f)| e^{\beta(f)}$
    $X(-f) = X(f)$ $\therefore$ magnitude spectrum is an even function.
    $\beta(-f) = -\beta(f)$ $\therefore$ phase spectrum is an odd function.

Fourier Transform of important Signals

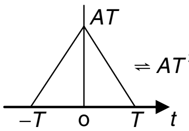
1. $\delta(t) \rightleftharpoons 1$

   $1 \rightleftharpoons \delta(f)$
2. $A\,\mathrm{rect}\left(\frac{t}{\tau}\right) \rightleftharpoons AT\,\mathrm{sinc}(fT)$
3. $e^{-at} u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$
4. $e^{at} u(-t) \rightleftharpoons \frac{1}{a - j2\pi f}$

5. $\text{sgn}(t) \rightleftharpoons \dfrac{1}{j\pi f}$

6. $u(t) \rightleftharpoons \dfrac{1}{2}\left[\delta(f) + \dfrac{1}{j\pi f}\right]$

7. $e^{j2\pi f_c t} \rightleftharpoons \delta(f - f_c)$

8. 

$\rightleftharpoons AT^2 \,\text{sinc}^2(fT)$

9. $e^{-\pi t^2} \rightleftharpoons e^{-\pi f^2}$

10. $\dfrac{1}{t} \rightleftharpoons -j\pi\, \text{sgn(f)}$

Hilbert Transform

If x(t) is the time domain signal, $x(t \pm 90)$ is called the Hilbert transform of x(t).

$$\hat{x}(t) = x(t \pm 90)$$

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{(t-\tau)}\, d\tau$$

$$= x(t) * \frac{1}{\pi t}$$

If we convert into frequency domain

$$\hat{x}(f) = x(\text{f}).[-j\text{sgn (f)}]$$

$$= -j\text{sgn(f)}\, x(\text{f})$$

i.e., $$x(\text{f}) = -jx(\text{f}) \text{ for } f > 0$$

$$= jx(\text{f}) \text{ for } x < 0$$

Hilbert transform gives –90° phase shift for positive frequencies of x(t) and gives 90° phase shift for negative frequencies of x(t).

Inverse Hilbert Transform

$$x(t) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{(t-\tau)}\, d\tau$$

$$= \hat{x}(t) * -\frac{1}{\pi \text{t}}$$

Properties of Hilbert transform

1. x(t) and $\hat{x}(t)$ are orthogonal

 i.e., $\displaystyle\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$

2. x(t) and $\hat{x}(t)$ have same spectrum

 $$\left[\hat{x}(f) = -\text{jsgn(f)}\, x(f)\; |\hat{x}(f)| = |x(f)|\right]$$

3. $\hat{\hat{x}}(t) = -\text{x(t)}$

Pre-envelope

If x(t) is a real valued signal, the pre-envelope of x(t) is given by

$$x_+(\text{t}) = x(\text{t}) + \text{j}\,\hat{x}(t)$$

where $\hat{x}(t)$ is the Hilbert transform.

If we convert into frequency domain

$$X_+(\text{f}) = X(\text{f}) + j\,[-j\text{sgn(f)}\, X(\text{f})]$$

$$= X(\text{f}) + \text{sgn(f)}\, X(\text{f})$$

$$\therefore X_+(\text{f}) = 2X(\text{f}) \text{ for } f > 0$$

$$= 0 \text{ for } f < 0$$

i.e., pre-envelope eliminates the negative frequency components of x(t).

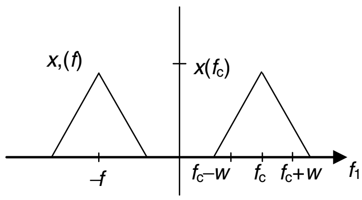
Complex Envelope

The complex envelope x(t) is defined as

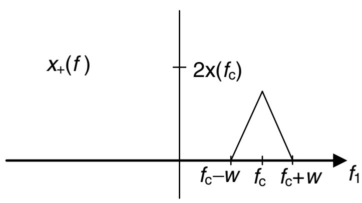
$$\tilde{x}(t) = x_+(t)e^{-j2\pi f_c t}$$

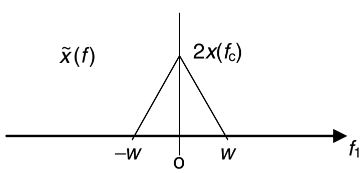
i.e., $$\tilde{x}\,(\text{t}) = x_+(f + f_c)$$

By using the concept of complex envelope, we can convert any bandpass signal into low pass signal.



Let





$$x_+(t) = \tilde{x}(t)\, e^{j2\pi f_c t}$$

$$x(t) + j\hat{x}(t) = \tilde{x}(t)\, e^{j2\pi f_c t}$$

$$\therefore x(t) = \text{real of}\left[\tilde{x}(t)e^{j2\pi f c^t}\right]$$

$$\hat{x}\,(t) = \text{imaginary of}\left[\tilde{x}(t)e^{j2\pi f c^t}\right]$$

Any low pass signal, $\hat{x}$ (t) can be expressed as

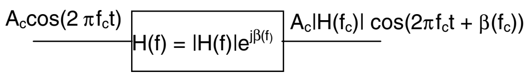
$$\hat{x}\,(\mathrm{t}) = x_I(\mathrm{t}) + x_Q(\mathrm{t})$$

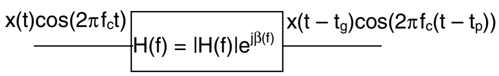
where x_I(t) and x_Q(t) are low pass in phase of quadrature components.

x(t) can be expressed as

$x(\mathrm{t}) = x_I\,(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$.

Phase Delay and Group Delay





where t_g is the delay of the envelope and x(t) is called group delay

$$T_g = \frac{-1}{2\pi}\frac{\delta\beta(f)}{\delta f}\bigg|_{f=fc}$$

where T_p is the delay of the carrier called phase delay.

$$T_p = \frac{-1}{2\pi f_c}.\beta(f_c)$$

For a distortionless communication channel, phase delay and group delay should be constant.

Example 9

The input to the channel is a bandpass signal. It is obtained by linearly modulating a sinusoidal carrier with single tone signal. The output of the channel due to this input is given by

$$y(t) = 10\cos(200t - 10^{-5})\cos(10^6\, t - 2)$$

The group delay and phase delay in seconds of the channel, respectively, are

(A) 50 ns, 2 μs (B) 50 ns, 20 μs
(C) 20 ns, 50 μs (D) 20 ns, 5 μs

Solution

If $m(t)\cos(2\pi f_c t)$ is the input to the channel, the output of the channel can be given by

$$y(t) = m(t - t_g)\cos(2\pi f_c(t - t_p))$$

where t_g is group delay and t_p is the phase delay.
In the given problem, y(t) can be written as

$$y(t) = 10\cos 200(t - 5\times 10^{-8})\,.\,\cos(10^6(t - 2\times 10^{-6})$$

Group delay $= 5 \times 10^{-8}$
Phase delay $= 2 \times 10^{-6}$

Example 10

A modulated signal is given by $s(t) = \cos((w_c + \Delta\omega)t)u(t)$. *where* $w_c >> \Delta\omega$. The complex envelope of s(t) is given by

(A) $\exp(j\Delta\omega t)\, u(t)$ (B) $\cos(\Delta\omega t)\, u(t)$
(C) $\sin(\Delta\omega t)\, u(t)$ (D) $\exp(j(w_c + \Delta\omega)t)\, u\,(t)$

Solution

$$s(t) = \cos((w_c + \Delta\omega)t)\, u(t)$$
$$= (\cos(w_c t)\cos(\Delta\omega t) - \sin(w_c t)\sin(\Delta\omega t))\, u(t)$$

If $s(t) = s_I(t)\cos 2\pi f_c t - s_Q(t)\sin(2\pi f_c t)$
Then, the complex envelope is given by

$$\tilde{S}\,(t) = s_I(t) + js_Q(t)$$

$$\tilde{S}\,(t) = (\cos(\Delta\omega t) + j\sin(\Delta\omega t))\, u(t)] = \exp\,(j\Delta\omega t)\, u\,(t)$$

Exercises

Practice Problems I

Direction for questions 1 to 22: Select the correct alternative from the given choices.

1. If a random variable X is uniform distributed with mean 3 and variance 2, the mean and variance of the random variable $4X + 5$ is
   (A) 17, 64 (B) 17, 32
   (C) 9, 64 (D) 9, 32

2. If a random variable X is uniformly distributed with mean zero and variance 10, the probability $P(X \geq 2)$ is
   (A) 0.42 (B) 0.52 (C) 0.32 (D) 0.22

3. If two random variables X and Y are jointly Gaussian with means 5 and 7, respectively, then probability density function of $2X + 3Y$ is
   (A) Gaussian with mean 12
   (B) non-Gaussian with mean 12
   (C) Gaussian with mean 31
   (D) non-Gaussian with mean 31

4. If a random variable X is uniformly distributed with mean zero and variance 12, the value of $f_x(4)$ and $f_x(8)$, respectively, are
   (A) $\frac{1}{10}, \frac{1}{10}$ (B) $\frac{1}{10}, 0$ (C) $\frac{1}{12}, \frac{1}{12}$ (D) $\frac{1}{12}, 0$

5. If two random variables U and V are uniformly distributed in the range (–2, 2), the probability $P(X + 1 > Y)$ is (assume both the random variables are independent)
(A) 0.72 (B) 0.64 (C) 0.42 (D) 0.34

6. The value $\int_{-\infty}^{\infty} e^{-\frac{(x-2)^2}{8}} dx$ is given by
(A) $\sqrt{2\pi}$ (B) $2\sqrt{2\pi}$ (C) 1 (D) $\frac{1}{\sqrt{2\pi}}$

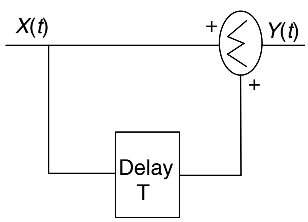
7. If X is a Gaussian random variables with mean zero and variance 10, the values of $E(X^2)$ and $E(X^3)$, respectively, are
(A) 10, 10 (B) 10, 100
(C) 10, 0 (D) 0, 0

8. If the autocorrelation of a random process X(t) is given by $R_X(\tau) = e^{-4\pi\tau^2}$, the power spectral density of X(t) is
(A) $\frac{1}{2}e^{\frac{-\pi f^2}{2}}$ (B) $\frac{1}{2}e^{\frac{-\pi f^2}{4}}$
(C) $e^{\frac{-\pi f^2}{4}}$ (D) $e^{-4\pi f^2}$

9. If a random process X(t) is a Gaussian random process, the probability density function of given random variable 'Y' is given by $Y = \int_0^T \sin(2t)\, X(t)\, dt$
(A) Gaussian for all T
(B) Gaussian for $T \le 2\pi$
(C) non-Gaussian for all T
(D) non-Gaussian for $T \le 2\pi$

10. A random process X(t) is defined by X(t) = 10 cos (1,000 $\pi t + \phi$), where ϕ is the random phase with uniform density function over ($-\pi$, π). Then, $R_X(\tau)$ is given by (Consider X(t) is a WSS process)
(A) 100 cos(2,000 πT) (B) 100 cos(1,000 πT)
(C) 50 cos(2,000 πT) (D) 50 cos (1,000 πT)

11. The relationship between two random process X(t) and Y(t) is given in the following figure.

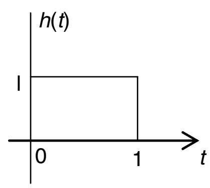


For low frequencies, the relationship between the power spectral densities S_x(f) and S_y(f) is given by
(A) $S_y(f) = 2S_x(f)$ (B) $S_y(f) = S_x(f)$
(C) $S_y(f) = 4S_x(f)$ (D) $S_y(f) = 0$

12. The autocorrelation of a random process is given by $R_X(\tau) = \text{sinc}(\tau)$. If the above process is passed through an ideal low pass filter with bandwidth 1 Hz, the autocorrelation of output of the low pass filter is
(A) sinc (T) (B) sinc^2 (T)
(C) sinc($2T$) (D) $\frac{1}{2}\sin c(2T)$

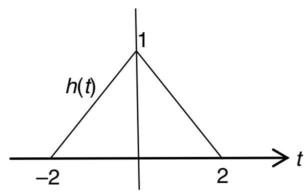
13. An AWGN process with power spectral density $\frac{N_0}{2}$ is passed through an ideal low pass filter with bandwidth 1 kHz. The output of the filter is sampled at t = 1.2 ms and 2.2 ms. The value of correlation between both the samples at t = 1.2 ms and 2.2 ms is
(A) $\frac{1}{2000}$ (B) $\frac{1}{1000}$ (C) $\frac{1}{500}$ (D) zero

14. A stationary Gaussian process X(t) with zero mean is applied to a linear filter whose impulse response is shown in the following figure. The mean of the output process Y(t) is



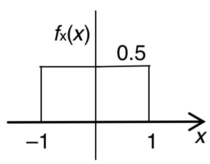
(A) 0
(B) infinity
(C) finite value
(D) given data is not adequate

15. A random process of mean 10 is passed through a system whose impulse response is given in the following figure.



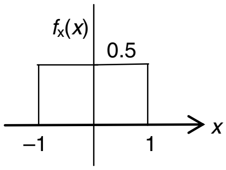
The mean of the output random process is
(A) 10 (B) 20 (C) 40 (D) 5

16. Probability density function of a random variable X is given in the following figure.



The probability $\int_{0.25}^{0.5} f_x(x/x \ge 0)\, dx$ is given by
(A) 0.75 (B) 0.5 (C) 0.40 (D) 0.25

17. Probability function of a random variable X is given in the following figure.



The probability $f_x(x/|x| < 0.2)$ is

(A) $5 f_x(x)$ (B) $10 f_x(x)$
(C) $2.5 f_x(x)$ (D) $2 f_x(x)$

18. A Gaussian distributed random variable 'X' is passed through a system $Y = \text{sgn}\ (X)$. The mean and variable of the random variable X are 0 and 10, respectively. $F_y(0.2)$ is

(A) 0.2 (B) 0.4 (C) 0.5 (D) 1.0

19. A random variable X is uniform distributed in the range $x \in (-2, 2)$. If X is passed through a square device with characteristic $y = x^2$, which of the following statements are not true.

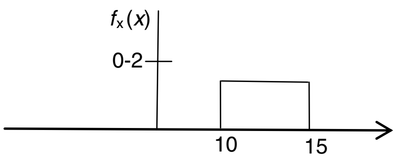
(A) $F_y(0) = 0$ (B) $F_y(4) = 1$
(C) $F_y(2) = 0.5$ (D) $F_y\ (8) = 1$

20. If random variables x and y are independent with Gaussian density functions, The probability density functions of x and y are as defined f_x (x): $N(1, 1)$ and f_y (y): $N(2, 1)$. Two new random variables u and v defined as under $u = x + y$ $v = x - y$.

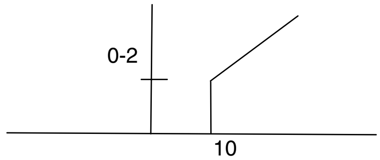
The correlation coefficient between u and v is

(A) 0.25 (B) 0.4 (C) 0.60 (D) 0

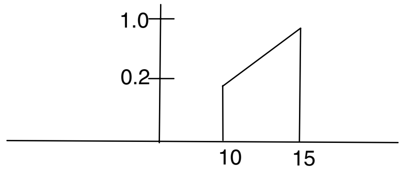
21. If the density function $f_x(x)$ of a random variable X is given in the following figure. Its distribution function $F_x(x)$ is



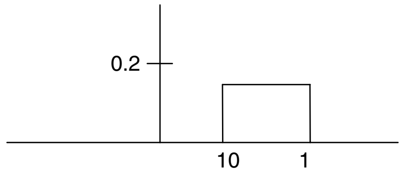
(A)



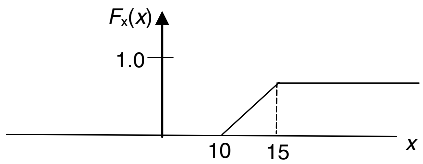
(B)



(C)



(D)



22. Let $w(t)$ is an additive white Gaussian noise with zero mean and power spectral density $\frac{N_0}{2}$, the mean and variance of new random variable $y = \frac{1}{T_b}\int_0^{T_b} w(t)\,dt$ is

(A) $0, \frac{N_0}{2}$ (B) $\frac{1}{T_b}, \frac{N_o}{2}$

(C) $0, \frac{N_0}{T_b}$ (D) $0, \frac{N_0}{2T_b}$

Practice Problems 2

Direction for questions 1 to 22: Select the correct alternative from the given choices.

1. If two random variables x and y are Gaussian and independent with mean zero and unity variance, Two new random variables defined as follows:

$U = x + y$

$V = x - y$

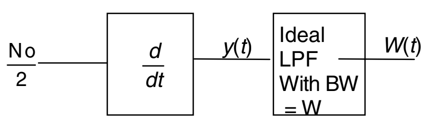
The correlation coefficient between u and v is

(A) 1 (B) 0 (C) 0.5 (D) 0.25

2. The autocorrelation functions of a random process $X(t)$ is given by $R_{XX}(\tau)$. If the process $X(t)$ is applied to a filter with impulse response $h(t)$, the autocorrelation function of the output process is given by

(A) $R_X(\tau) * h(t)$ (B) $R_X(\tau) * h(t) * h(-t)$
(C) $R_X(\tau) * h(t) * h(t)$ (D) $R_X(\tau) * h(-t)$

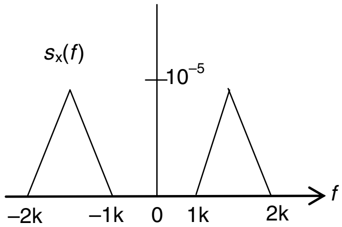
3. The output power in the undermentioned circuit is given by AWGN with S × (f) =



(A) $\frac{4}{3} N_0 \pi^2 w^2$ (B) $\frac{4}{3} N_0 \pi^2 w^3$

(C) $\frac{2}{3} N_0 \pi^2 w^2$ (D) $\frac{2}{3} N_0 \pi^2 w^3$

4. Power spectral density of a random process X(t) is mentioned as in the following figure.



The mean square value of the random process $E(X^2(t))$ is

(A) 10 mW (B) 100 mW
(C) 20 mW (D) 50 mW

5. $X_1, X_2, \ldots. X_n$ are independent random variables with finite mean and variance. If a new random variable X is defined as $X = X_1 + X_2 + \ldots\ldots X_n$. The density function of X is approximated for sufficiently large value of n is

(A) Uniform (B) Poisson
(C) Rayleigh (D) Gaussian

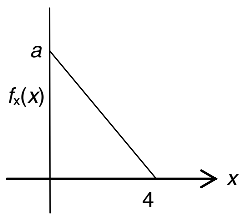
6. If two random variables X_1 and X_2 are independent with probability density functions $f_{x1}(x)$ and $f_{x2}(x)$, the probability density function of $X_1 + X_2$ is given by

(A) $f_{x1}(x) + f_{x2}(x)$
(B) $f_{x1}(x).f_{x2}(x)$
(C) $f_{x1}(x) * f_{x2}(x)$
(D) $f_{x1}(x) + f_{x2}(x) - f_{x1}(x).f_{x2}(x)$

7. X and Y are independent random variables, which takes the values 1, 2, and 3 with equal probability. The probability $P(x + Y > 4/X - Y < 0)$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{6}$ (C) $\frac{5}{6}$ (D) $\frac{1}{3}$

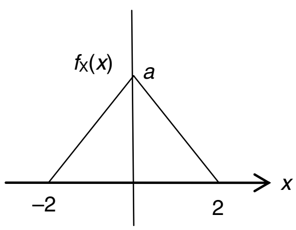
8. Probability density function of a random variable X is given in the following figure.



The mean of the random variable X is

(A) $\frac{8}{3}$ (B) 2 (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

9. Probability density function of random variable X is mentioned as in the following figure.



The value of $E(X^5)$ is

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) Zero

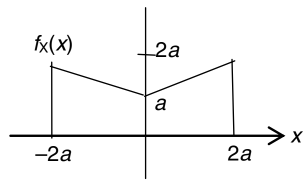
10. X is a Gaussian distributed random variable with mean 2 and variance 5. If $F_x(-4) = 0.06$ and $F_x(4) = 0.68$, the value of $P(X \geq 0)$ is

(A) 0.32 (B) 0.68 (C) 0.60 (D) 0.74

11. A random variable X is Gaussian distributed with $f_x(x) = Ke^{-\frac{(x-2)^2}{18}}$. The value of k and the value of x where the probability becomes approximately 0.01 k are

(A) $\frac{1}{\sqrt{6\pi}}$, -4.4 and 8.4
(B) $\frac{1}{3\sqrt{2\pi}}$, -4.4 and 8.4
(C) $\frac{1}{\sqrt{6\pi}}$, -7.6 and 11.6
(D) $\frac{1}{3\sqrt{2\pi}}$, -7.6 and 11.6

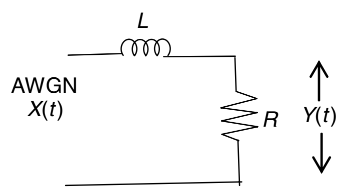
12. A random variable X is having the undermentioned probability density curve



The value of 'a' is given by

(A) 0.4 (B) 0.5 (C) 0.6 (D) 0.3

13. A WGN process with spectral density $\frac{N_0}{2}$ is passed through a low pass filter mentioned in the following figure. The variance of output of the filter is



(A) $\frac{N_0.R}{2L}$ (B) $\frac{N_0.R}{L}$ (C) $\frac{N_0.R}{4L}$ (D) $\frac{N_0R}{8L}$

14. If the autocorrelation function of a random process $X(t)$ is given by $R_X(\tau) = e^{-2|\tau|}$. Power spectral density of $X(t)$ is

(A) $\frac{1}{1+4\pi^2 f^2}$ (B) $\frac{1}{1+\pi^2 f^2}$
(C) $\frac{1}{4+\pi^2 f^2}$ (D) $\frac{2}{1+4\pi^2 f^2}$

15. A random process $X(t)$ with autocorrelation $R_X(\tau) = 4 + 3\cos(100\tau)$ is passed through a low pass filter with characteristic

$H(f) = \dfrac{3}{1 + j0.01f}$. The DC value at the output of the filter is

(A) 6 (B) 3 (C) 12 (D) 4

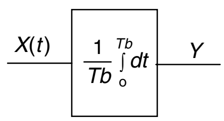
16. The autocorrelation function of a random process X(t) is given by $R_X(\tau) = 2 + 3\,e^{-5|\tau|}$. The total power of the random process is

(A) 2 W (B) 4 W (C) 5 W (D) 13 W

17. X(t) is AWGN noise with spectral density 1 μW/Hz. If X_I(t) and X_Q(t) are in phase and quadrature components of X(t), the spectral densities of in phase and quadrature components, respectively, are

(A) 1 μW/Hz, 1 μW/Hz
(B) 0.5 μW/Hz, 0.5 μW/Hz
(C) 2 μW/Hz, 2 μW/Hz
(D) 0.25 μW/Hz, 0.25 μW/Hz

18. X(t) is an AWGN with spectral density $\dfrac{N_0}{2}$. If X(t) is passed through a undermentioned system, the variance of output random variable Y is



(A) $\dfrac{N_0}{2}$ (B) $\dfrac{N_0}{2}.T_b$ (C) $N_0\,T_b$ (D) $\dfrac{N_0}{2T_b}$

19. An AWGN process X(t) is passed through R–L low pass filter with cut-off frequency $f_c = 1$ and unity DC gain. If $\dfrac{N_0}{2}$ is spectral density of X(t), the autocorrelation function of output process is given by

(A) $\dfrac{N_0}{2\pi}e^{-2\pi}|\tau|$ (B) $\dfrac{N_0}{2\pi}e^{-\pi}|\tau|$

(C) $\dfrac{N_0}{\pi}e^{-2\pi}|\tau|$ (D) $\dfrac{N_0}{\pi}e^{-\pi}|\tau|$

20. Select the correct answer from the following

(A) The power spectral density s_x(f) of a random process X(t) is always positive
(B) s_x(f) may be negative or positive
(C) s_x(f) is negative only at finite number of frequencies
(D) None of the above

21. Select the correct answer in respect to Gaussian process X(t)

(i) Any linear combination of X(t) have Gaussian density
(ii) Any sample of Gaussian process X(t) will have Gaussian density

(A) Only (i) (B) Only (ii)
(C) Both (i) and (ii) (C) Neither (i) nor (ii)

22. An additive white Gaussian process X(t) is passed through an ideal bandpass filter of centre frequency 100 MHz and bandwidth 2 MHz. Let X (t) have a power spectral density of $\dfrac{N_o}{2}$ and the output of BPF is P(t). The variance of P (t) is

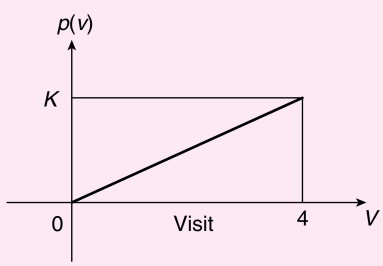
(A) $10^6\,N_0$ (B) $\dfrac{N_0}{2}$
(C) $2 \times 10^6\,N_0$ (D) N_0

Previous Years' Questions

1. Noise with uniform power spectral density of N_0W/Hz is passed through a filter H(w) = 2exp($-j\omega t_d$) followed by an ideal low pass filter of bandwidth B Hz. The output noise power in Watts is **[2005]**

(A) $2N_0B$ (B) $4N_0B$
(C) $8N_0B$ (D) $16N_0B$

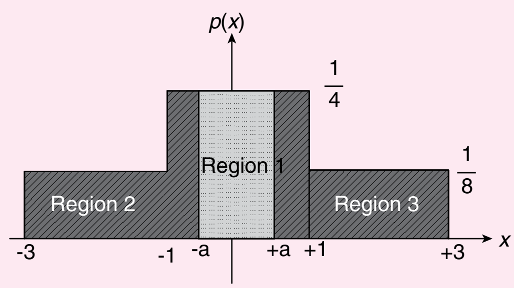
2. An output of a communication channel is a random variable ν with the probability density function as shown in the figure. The mean square value of ν is **[2005]**



(A) 4 (B) 6 (c) 8 (D) 9

Direction for questions 3 and 4:

A symmetric three-level mid-tread quantizer is to be designed assuming equiprobable occurrence of all quantization levels.



3. If the input probability density function is divided into three regions as shown in figure, the value of 'a' in the figure is **[2005]**

(A) $\dfrac{1}{3}$ (B) $\dfrac{2}{3}$ (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{4}$

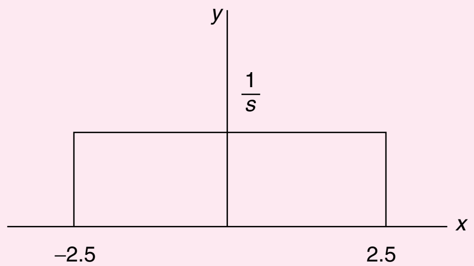
4. The quantization noise power for the quantization region between $-a$ and $+a$ in the figure is **[2005]**

(A) $\frac{4}{81}$ (B) $\frac{1}{9}$ (C) $\frac{5}{81}$ (D) $\frac{2}{81}$

5. A uniformly distributed random variable x with probability density function

$$f_x(x) = \frac{1}{10}(u(x+5) - u(x-5))$$

where $u(.)$ is the unit step function is passed through a transformation given in the figure. The probability density function of the transformed random variable y would be **[2006]**



(A) $f_y(y) = \frac{1}{5}(u(y+2.5) - u(y-2.5))$

(B) $f_y(y) = 0.5\delta(y) + 0.5\delta(y-1)$

(C) $f_y(y) = 0.25\delta(y+2.5) + 0.25\delta(y-2.5) + 0.5\delta(y)$

(D) $f_y(Y) = 0.25\delta(Y+2.5) + 0.25\delta(Y-2.5) + \frac{1}{10}[u(y+2.5) - u(y-2.5)]$

6. A zero-mean white Gaussian noise is passed through an ideal low pass filter of bandwidth 10 kHz. The output is then uniformly sampled with sampling period t_s= 0.03 ms. The samples so obtained would be **[2006]**

(A) correlated
(B) statistically independent
(C) uncorrelated
(D) orthogonal

Direction for questions 7 and 8:

The following two questions refer to wide-sense stationary stochastic processes

7. It is desired to generate a stochastic process (as voltage process) with power spectral density.

$$S(\omega) = \frac{16}{16+\omega^2}$$

By driving a linear time-Invariant system by zero mean white noise (as voltage process) with power spectral density being constant equal to 1. The system that can perform the desired task could be: **[2006]**

(A) first-order low pass R–L filter
(B) first-order high pass R–C filter
(C) tuned L–C filter
(D) series R–L–C filter

8. The parameters of the system obtained in 8 would be **[2006]**

(A) first-order R–L low pass filter would have R = 4 Ω, L = 1 H
(B) first-order R–C high pass filter would have R = 4 Ω, C = 0.25 F
(C) tuned L–C filter would have L = 4 H C = 4 F
(D) series R–L–C low pass filter would have R = 1 Ω, L = 4 H, and C = 4 F.

9. If $R(\tau)$ is the autocorrelation function of a real, wide-sense stationary random process, then which of the following is not true? **[2007]**

(A) $R(\tau) = R(-\tau)$
(B) $|R(\tau)| \le R(0)$
(C) $R(\tau) = -R(-\tau)$
(D) The mean square value of the process is $R(0)$.

10. If S(f) is the power spectral density of a real, wide-sense stationary random process, then which of the following is always true? **[2007]**

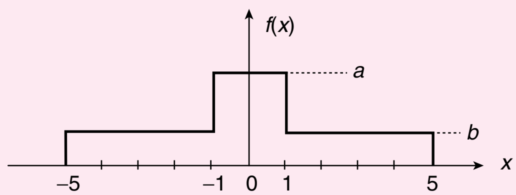
(A) $S(0) \ge S(f)$ (B) $S(f) \ge 0$
(C) $S(-f) = -S(f)$ (D) $\int_{-\infty}^{\infty} S(f)\,df = 0$

11. A Hilbert transformer is a **[2007]**

(A) non-linear system (B) non-causal system
(C) time varying system (D) low pass system

Direction for questions 12 and 13:

An input to a six-level quantizer has the probability density function f(x), as shown in the figure. Decision boundaries of the quantizer are chosen so as to maximize the entropy of the quantizer output. It is given that three consecutive decision boundaries are '–1' , '0', and '1'.



12. The values of a and b are as follows: **[2007]**

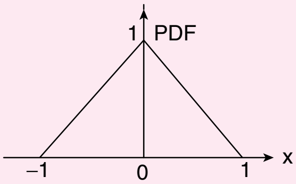
(A) $a = \frac{1}{6}$ and $b = \frac{1}{12}$ (B) $a = \frac{1}{5}$ and $b = \frac{3}{40}$

(C) $a = \frac{1}{4}$ and $b = \frac{1}{16}$ (D) $a = \frac{1}{3}$ and $b = \frac{1}{24}$

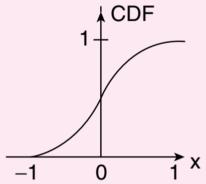
13. Assuming that the reconstruction levels of the quantizer are the mid-points of the decision boundaries, the ratio of signal power to quantization noise power is **[2007]**

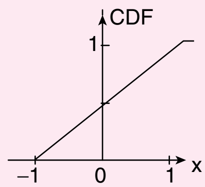
(A) $\frac{152}{9}$ (B) $\frac{64}{3}$
(C) $\frac{76}{3}$ (D) 28

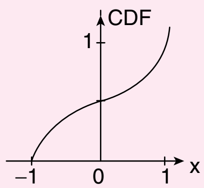
14. The probability density function (PDF) of a random variable X is as shown in the following figure.

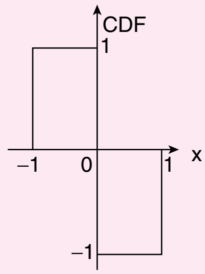


The corresponding cumulative distribution function (CDF) has the form **[2008]**

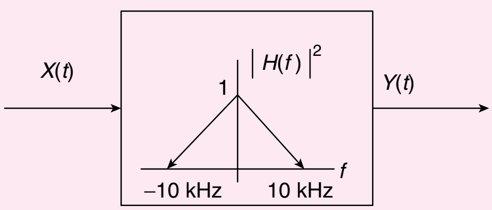
(A)


(B)


(C)


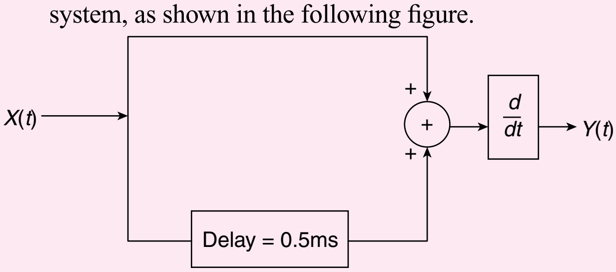
(D)


15. A white noise process $X(t)$ with two-sided power spectral density 1×10^{-10} W/Hz is input to a filter whose magnitude squared response is shown in the following figure. **[2009]**



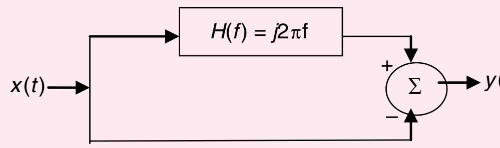
The power of the output process $Y(t)$ is given by
(A) 5×10^{-7} W (B) 1×10^{-6} W
(C) 2×10^{-6} W (D) 1×10^{-5} W

16. $X(t)$ is a stationary process with the power spectral density $S_x(f) > 0$ for all f. The process is passed through a system, as shown in the following figure.



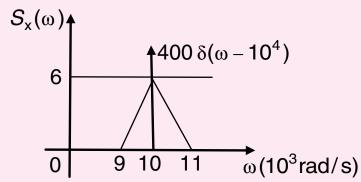
Let $S_y(f)$ be the power spectral density of $Y(t)$. Which one of the following statements is correct? **[2010]**
(A) $S_y(f) > 0$ for all f
(B) $S_y(f) = 0$ for $|f| > 1$ kHz
(C) $S(f) = 0$ for $f = nf_0$, $f_0 = 2$ kHz, n any integer
(D) $S(f) = 0$ for $f = (2n + 1)$, $f_0 = 1$ kHz, n any integer

17. $x(t)$ is a stationary random process with autocorrelation function $R_x(\tau) = \exp(-\pi\tau^2)$. This process is passed through the system, as shown in the following figure. The power spectral density of the output process $Y(\tau)$ is **[2011]**



(A) $(4\pi^2 f^2 + 1)\exp(-\pi f^2)$ (B) $(4\pi^2 f^2 - 1)\exp(-\pi f^2)$
(C) $(4\pi^2 f^2 + 1)\exp(-\pi f)$ (D) $(4\pi^2 f^2 - 1)\exp(-\pi f)$

18. The power spectral density of a real process $X(t)$ for positive frequencies is shown in the following figure. The values of $E[X^2(t)]$ and $|E[X(t)]|$, respectively, are **[2012]**



(A) $6{,}000/\pi$, 0 (B) $6{,}400/\pi$, 0
(C) $6{,}400/\pi$, $20/\left(\pi\sqrt{2}\right)$ (D) $6{,}000/\pi$, $20/\left(\pi\sqrt{2}\right)$

19. Consider two identically distributed zero mean random variables U and V. Let the cumulative distribution functions of U and $2V$ be $F(x)$ and $G(x)$, respectively. Then, for all values of x **[2013]**
(A) $F(x) - G(x) \le 0$ (B) $F(x) - G(x) \ge 0$
(C) $(F(x) - G(x))\, x \le 0$ (D) $(F(x) - G(x))\, x \ge 0$

20. Let X be a real-valued random variable with $E[X]$ and $E[X^2]$ denoting the mean values of X and X^2, respectively. The relation which always holds true is **[2014]**
(A) $(E[X])^2 > E[X^2]$ (B) $E[X^2] \ge (E[X])^2$
(C) $E[X^2] = (E[X])^2$ (D) $E[X^2] > (E[X])^2$

21. A fair coin is tossed repeatedly until a 'Head' appears for the first time. Let L be the number of tosses to get this first 'Head'. The entropy H (L) in bits is ________. **[2014]**

22. The capacity of a band-limited additive white Gaussian noise (AWGN) channel is given by $C = W \log_2\left(1 + \frac{P}{\sigma^2 W}\right)$ bits per second (bps), where W is the channel bandwidth, P is the average power received, and σ^2 is the one-sided power spectral density of the AWGN.

For a fixed $\frac{P}{\sigma^2} = 1000$, the channel capacity (in kbps) with infinite bandwidth ($W \to \infty$) is approximately **[2014]**

(A) 1.44 (B) 1.08 (C) 0.72 (D) 0.36

23. The input to a 1-bit quantizer is a random variable X with PDF $f_x(x) = 2e^{-2x}$ for $x \geq 0$ and $f_x(x) = 0$ for $x < 0$. For outputs to be of equal probability, the quantizer threshold should be ___________. **[2014]**

24. The power spectral density of a real stationary random process $X(t)$ is given by $S_x(f) = \begin{cases} \frac{1}{W}, & |f| \leq W \\ 0, & |f| > W \end{cases}$

The value of the expectation $E\left[\pi X(t) X\left(t - \frac{1}{4w}\right)\right]$ is ______ **[2014]**

25. Let $X(t)$ be a wide sense stationary (WSS) random process with power spectral density $S_X(f)$. If $Y(t)$ is the process defined as $Y(t) = X(2t - 1)$, the power spectral density $S_Y(f)$ is **[2014]**

(A) $S_Y(f) = \frac{1}{2} S_x\left(\frac{f}{2}\right) e^{-j\pi f}$

(B) $S_Y(f) = \frac{1}{2} S_x\left(\frac{f}{2}\right) e^{-j\pi f/2}$

(C) $S_Y(f) = \frac{1}{2} S_X\left(\frac{f}{2}\right)$

(D) $S_Y(f) = \frac{1}{2} S_X\left(\frac{f}{2}\right) e^{-j2\pi f}$

26. A real band-limited random process $X(t)$ has two-sided power spectral density

$$S_X(f) = \begin{cases} 10^{-6}(3000 - |f|)\,\text{Watts/Hz} & \text{for } |f| \leq 3\text{kHz} \\ 0 & \text{otherwise} \end{cases}$$

where f is the frequency expressed in Hz. The signal $x(t)$ modulates a carrier cos 16,000 πt and the resultant signal is passed through an ideal bandpass filter of unity gain with centre frequency of 8 kHz and bandwidth of 2 kHz. The output power (in Watts) is ______. **[2014]**

27. If calls arrive at a telephone exchange such that the time of arrival of any call is independent of the time of arrival of earlier or future calls, the probability distribution function of the total number of calls in a fixed time interval will be **[2014]**

(A) Poisson (B) Gaussian
(C) Exponential (D) Gamma

28. Consider a communication scheme where the binary valued signal X satisfies $P\{X = +1\} = 0.75$ and $P\{X = -1\} = 0.25$. The received signal $Y = X + Z$, where Z is a Gaussian random variable with zero mean and variance σ^2. The received signal Y is fed to the threshold detector. The output of the threshold detector $\hat{X}$ is:

$$\hat{X} = \begin{cases} +1, & Y > \tau \\ -1, & Y \leq \tau \end{cases}$$

To achieve a minimum probability of error $P\{\hat{X} \neq X\}$, the threshold τ should be **[2014]**

(A) strictly positive

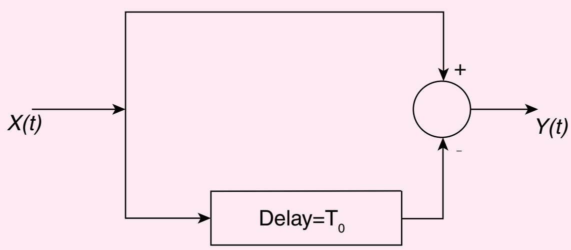
(B) zero

(C) strictly negative

(D) strictly positive, zero, or strictly negative depending on the nonzero value of σ^2.

29. Consider a random process $x(t) = 3V(t) - 8$, where $V(t)$ is a zero mean stationary random process with autocorrelation $R_v(\tau) = 4e^{-5|\tau|}$. The power in $X(t)$ is ________. **[2016]**

30. A wide sense stationary random process $X(t)$ passes through the LTI system shown in the figure. If the autocorrelation function of $X(t)$ is $Rx(t)$, then the autocorrelation function $R_y(t)$ of the output $Y(t)$ is equal to **[2016]**



(A) $2R_x(\tau) + R_x(\tau - T_0) + R_x(\tau + T_0)$
(B) $2R_x(\tau) - R_x(\tau - T_0) - R_x(\tau + T_0)$
(C) $2R_x(\tau) + 2R_x(\tau - 2T_0)$
(D) $2R_x(\tau) - 2R_x(\tau - 2T_0)$

Answer Keys

Exercises

Practice Problems 1

1. B	**2.** C	**3.** C	**4.** D	**5.** A	**6.** B	**7.** C	**8.** B	**9.** A	**10.** D
11. C	**12.** A	**13.** D	**14.** A	**15.** B	**16.** D	**17.** A	**18.** C	**19.** C	**20.** D
21. D	**22.** D								

Practice Problems 2

1. B	**2.** B	**3.** B	**4.** A	**5.** D	**6.** C	**7.** D	**8.** C	**9.** D	**10.** B
11. D	**12.** A	**13.** C	**14.** B	**15.** A	**16.** C	**17.** C	**18.** D	**19.** A	**20.** A
21. C	**22.** A								

Previous Years' Questions

1. B	**2.** C	**3.** B	**4.** A	**5.** B	**6.** B	**7.** A	**8.** A	**9.** C	**10.** B
11. B	**12.** A	**13.** D	**14.** C	**15.** B	**16.** D	**17.** A	**18.** B	**19.** C	**20.** B
21. 1.99 to 2.01		**22.** A	**23.** 0.34 to 0.36		**24.** 3.9 to 4.1		**25.** C	**26.** 2.5	**27.** A
28. C	**29.** 100	**30.** B							