Exercise 16.4

Chapter 16 Vector Calculus Exercise 16.4 1E

Consider the following line integral:

$$\oint_C (x-y)\,dx + (x+y)\,dy$$

Taking advantage of trigonometry, there is a standard parameterization of the curve which is the perimeter of a circle centered at the origin with radius 2.

$$x = 2\cos\theta$$
$$y = 2\sin\theta$$

With $0 \le \theta \le 2\pi$, the corresponding differentials are as follows:

$$dx = -2\sin\theta d\theta$$
$$dy = 2\cos\theta d\theta$$

Substitute into the integral.

$$\begin{aligned} I &= \int_C Pdx + Qdy \\ &= \int_0^{2\pi} \left(-2(2\cos\theta - 2\sin\theta)\sin\theta d\theta + 2(2\cos\theta + 2\sin\theta)\cos\theta\right) d\theta \\ &= \int_0^{2\pi} \left(-4\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta\right) d\theta \end{aligned}$$

Combine like terms, factor, and simplify the trigonometric identities to get a simple integral to evaluate.

$$\begin{aligned} I &= 4\int_0^{2\pi} 1 d\theta \\ &= 4|\theta|_0^{2\pi} \\ &= \boxed{8\pi} \end{aligned}$$

With Green's Theorem, this line integral can also be worked as a double integral.

Green's Theorem:

$$\begin{aligned} I &= \int_C Pdx + Qdy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \end{aligned}$$

Here, *D* is the region bounded counterclockwise by *C*.

The parameterization in part (a) was counterclockwise, so Green's Theorem should give the same result.

Compute the integrand.

$$\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= 1 - (-1) \\ &= 2 \end{aligned}$$

Use Green's Theorem to set up the integral.

$$I = \iint_D 2\,dA$$

Here, *D* is the circle of radius 2.

Since the integrand is constant and the formula for the area of a circle is $A = \pi r^2$, there is a shortcut easier than computing the integral.

$$\begin{aligned} I &= 2(\pi \cdot 2^2) \\ &= \boxed{8\pi} \end{aligned}$$

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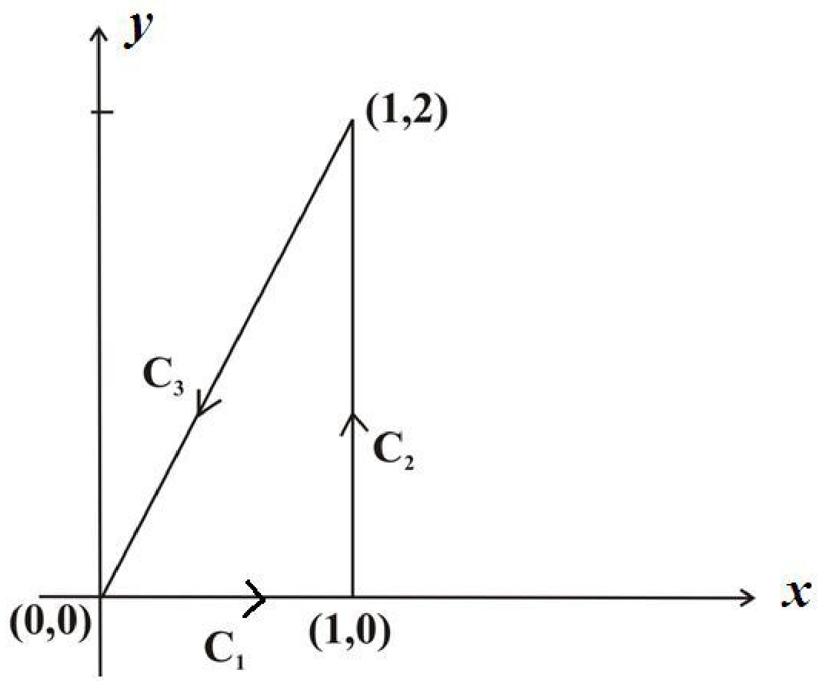
Consider:

$\oint_C xy\,dx + x^2y^3dy$, where C is a triangle with vertices (0, 0), (1, 0) and (1, 2).

Evaluate.

The following is the figure:



(a)

C consists of line segments with parametric equations:

$C_1: \quad y=0,\ x=t,\ 0\le t\le 1$

$C_2: \quad x=1,\ y=t,\ 0\le t\le 2$

$C_3: \quad x=1-t,\ y=2-2t,\ 0\le t\le 1$

Then $\int_C xy\,dx+x^2y^3dy=\int_{C_1} xy\,dx+x^2y^3dy+\int_{C_2} xy\,dx+x^2y^3dy$

$+\int_{C_3} xy\,dx+x^2y^3dy$

For C_1, $dy=0, dx=dt$

Now $\int_{C_1} xy\,dx+x^2y^3\,dy=\int_0^1(0)\,dt+(0)\,dt$

$=0$

For C_2, $dx=0, dy=dt$

$\int_{C_2} xy\,dx+x^2y^3dy=\int_0^2 t(0)\,dt+(1)t^3(1)\,dt$

$=\int_0^2 t^3\,dt$

$=\frac{1}{4}\left(t^4\right)_0^2$

$=\frac{16}{4}$

$=4$

For C_3, $dy=-2dt, dx=-dt$

And $\int_{C_3} xy\,dx+x^2y^3dy=\int_0^1(1-t)2(1-t)(-1)\,dt+(1-t)^2 8(1-t)^3(-2)\,dt$

$=\int_0^1\left[(-2)(1-t)^2-16(1-t)^5\right]dt$

$=\left[\frac{2}{3}(1-t)^3+\frac{16}{6}(1-t)^6\right]_0^1$ $\quad\left(\int x^n dx=\frac{x^{n+1}}{n+1}\right)$

$=\frac{2}{3}(0)+\frac{8}{3}(0)-\frac{2}{3}-\frac{8}{3}$

$=-\frac{10}{3}$

Therefore, $\int_C xy\,dx+x^2y^3dy=0+4-\frac{10}{3}$

$=\boxed{\frac{2}{3}}$

(b)

On comparing $\oint_C xy\,dx+x^2y^2dy$ with $\oint_C P\,dx-Q\,dy$ we have

$P=xy$ and $Q=x^2y^3$

Then $\frac{\partial P}{\partial y}=x$ and $\frac{\partial Q}{\partial x}=2xy^3$

By Green's theorem:

$\oint_C P\,dx-Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$

Here in this case $D=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 2x\}$

Then $\oint_C xy\,dx+x^2y^3\,dy=\int_0^1\int_0^{2x}\left(2xy^3-x\right)dy\,dx$

$=\int_0^1\int_0^{2x}x\left(2y^3-1\right)dy\,dx$

$=\int_0^1 x\left(\frac{1}{2}y^4-y\right)_{y=0}^{y=2x}dx$ $\quad\left(\int x^n dx=\frac{x^{n+1}}{n+1}\right)$

$=\int_0^1 x\left(8x^4-2x\right)dx$

The above is simplified to,

$=\int_0^1\left(8x^5-2x^2\right)dx$

$=\left(\frac{4}{3}x^6-\frac{2}{3}x^3\right)_0^1$ $\quad\left(\int x^n dx=\frac{x^{n+1}}{n+1}\right)$

$=\frac{4}{3}-\frac{2}{3}$

$=\boxed{\frac{2}{3}}$.

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(a) To evaluate the line integral, use the formulas

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt \quad \ldots\ldots (1)$$

And

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt \quad \ldots\ldots (2)$$

We must split the path C into three pieces. The first piece is the arc of the parabola $y = x^2$, from (0, 0) to (1, 1). We can parametrize this path by

$x = t$

$y = t^2$

With t ranging from 0 to 1. From this parametrization we get the derivatives

$x'(t) = 1$

$y'(t) = 2t$

Using equation (1) along this piece of the curve, we have

$$\oint_C x^2y^2dx = \int_0^1 (t)^2(t^2)^2(1)dt$$
$$= \int_0^1 (t^6)dt$$
$$= \left(\frac{t^7}{7}\right)\Bigg|_0^1$$
$$= \frac{1}{7}$$

And using equation (2) we have

$$\oint_C xydy = \int_0^1 (t)(t^2)(2t)dt$$
$$= \int_0^1 (2t^4)dt$$
$$= \left(\frac{2t^5}{5}\right)\Bigg|_0^1$$
$$= \frac{2}{5}$$

So the line integral $\oint_C x^2y^2dx + xydy$ along this piece of the path is $1/7 + 2/5 = 19/35$.

The next piece is the top of the curve, the line segment from (1, 1) to (0, 1). We can parametrize this as

$x = t$

$y = 1$

With t ranging from 1 to 0. Note that the limits must be in this order so we continue to traverse the curve counterclockwise. From this parametrization we get the derivatives

$x'(t) = 1$

$y'(t) = 0$

Using equation (1) along this line segment, we have

$$\oint_C x^2y^2dx = \int_1^0 (t)^2(1)^2(1)dt$$
$$= \int_1^0 t^2dt$$
$$= \frac{t^3}{3}\Bigg|_1^0$$
$$= 0 - \frac{1^3}{3}$$
$$= -\frac{1}{3}$$

And using equation (2) we have

$$\oint_C xydy = \int_1^0 (t)(1)(0)dt$$
$$= 0$$

So the line integral $\oint_C x^2y^2dx + xydy$ along this piece of the path is $-1/3 + 0 = -1/3$.

The third piece is the left side of the curve. This is a vertical line from (0, 1) to (0, 0). We can parametrize it as

$x = 0$

$y = t$

With t ranging from 1 to 0. Notice again that the limits must be in this order so we continue to traverse the curve counterclockwise. From this parametrization we get the derivatives

$x'(t) = 0$

$y'(t) = 1$

Using equation (1) along this line segment, we have

$$\oint_C x^2y^2dx = \int_1^0 (0)^2(t)^2(0)dt$$
$$= 0$$

And using equation (2) we have

$$\oint_C xydy = \int_1^0 (0)(t)(1)dt$$
$$= 0$$

So the line integral $\oint_C x^2y^2dx + xydy$ along this piece of the path is 0.

The total value of the line integral is the addition of all the pieces, or

$\frac{19}{35} - \frac{1}{3} + 0 = \boxed{\frac{22}{105}}$.

(b) Green's Theorem is

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

To apply Green's Theorem to this integral, find the limits of integration for the region enclosed by the curve. In the y, the region ranges from $y = x^2$ to $y = 1$. The x limits are 0 and 1.

The integrand comes from $P = x^2y^2$ and $Q = xy$. Take the necessary partial derivatives:

$$\frac{\partial Q}{\partial x} = y$$

$$\frac{\partial P}{\partial y} = 2x^2y$$

Therefore, applying Green's Theorem to the integral leads to:

$$\oint_C x^2y^2dx + xydy = \int_0^1\int_{x^2}^1 (y - 2x^2y)dydx$$

Hold x constant and integrate in terms of y:

$$\begin{aligned}\oint_C x^2y^2dx + xydy &= \int_0^1\int_{x^2}^1 (y - 2x^2y)dydx \\ &= \int_0^1 \left(\frac{y^2}{2} - x^2y^2 \right)\Bigg|_{x^2}^1 dx \\ &= \int_0^1 \left(\frac{1^2}{2} - x^2(1)^2 - \left(\frac{(x^2)^2}{2} - x^2(x^2)^2 \right) \right) dx \\ &= \int_0^1 \left(\frac{1}{2} - x^2 - \frac{x^4}{2} + x^6 \right) dx\end{aligned}$$

Integrate in terms of x:

$$\begin{aligned}\oint_C x^2y^2dx + xydy &= \int_0^1 \left(\frac{1}{2} - x^2 - \frac{x^4}{2} + x^6 \right) dx \\ &= \left(\frac{x}{2} - \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{7} \right)\Bigg|_0^1 \\ &= \left(\frac{1}{2} - \frac{1^3}{3} - \frac{1^5}{10} + \frac{1^7}{7} \right) - 0 \\ &= \boxed{\frac{22}{105}}\end{aligned}$$

As expected, this is the same value reached in part (a).

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Consider the integral, $\int_C xy^2dx + 2x^2ydy$. Here, C is the triangle with vertices.

$(0,0),(2,2,),$ and $(2,4)$.

Use Green's theorem to evaluate the line integral.

Recollect Green's theorem.

Green's Theorem:

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

Here, $P = xy^2$, and $Q = 2x^2y$

Find the partial derivatives.

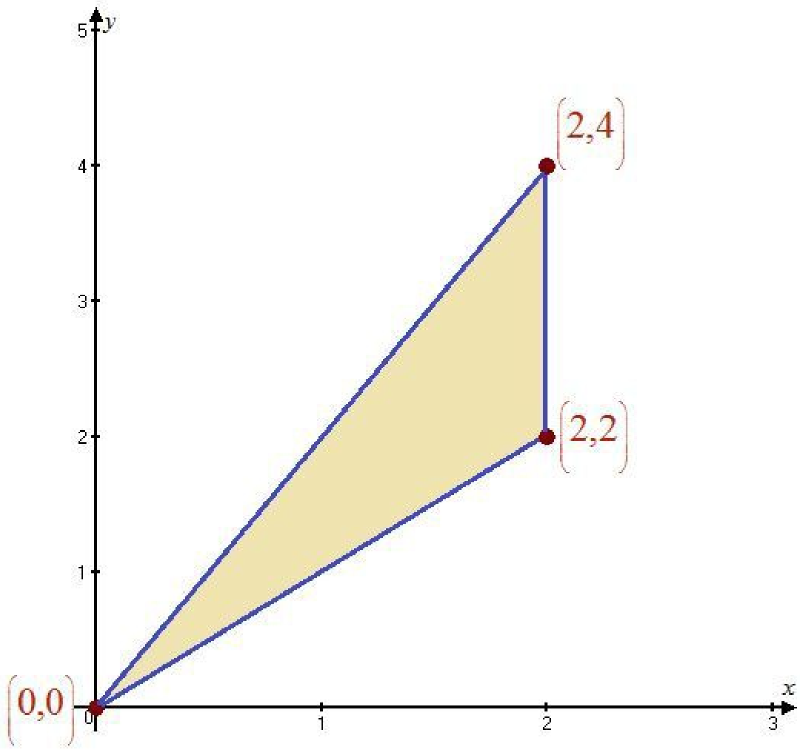
Differentiate Q with respect to x.

$$Q(x,y) = 2x^2y$$
$$\frac{\partial Q}{\partial x} = 4xy$$

Differentiate Q with respect to y.

$$P(x,y) = xy^2$$
$$\frac{\partial P}{\partial y} = 2xy$$

Sketch the region of the curve with vertices, $(0,0),(2,2,),$ and $(2,4)$.



Find the limits of the region.

From the region, observe the following:

Change in $x-$axis is 0 to 2.

Change in $y-$axis is $y = x$ to $y = 2x$.

By the Green's theorem,

$$\int_C xy^2 dx + 2x^2 y dy = \int_0^2 \int_x^{2x} (4xy - 2xy) dy dx$$

$$= \int_0^2 \int_2^4 2xy dy dx$$

$$= \int_0^2 2x \left[\frac{y^2}{2} \right]_x^{2x} dx$$

$$= \int_0^2 2x \left[\frac{(2x)^2}{2} - \frac{x^2}{2} \right] dx$$

On continuation, obtain as follows:

$$= \int_0^2 2x \left[\frac{4x^2}{2} - \frac{x^2}{2} \right] dx$$

$$= \int_0^2 2x \left[\frac{3x^2}{2} \right] dx$$

$$= 3\int_0^2 x^3 dx$$

$$= 3\left[\frac{x^4}{4} \right]_0^2$$

$$= 3\left[\frac{2^4}{4} - \frac{0^4}{4} \right]$$

$$= 3\left[\frac{16}{4} \right]$$

$$= 3 \cdot 4$$

$$= 12$$

Hence, the value of the line integral is $\boxed{12}$.

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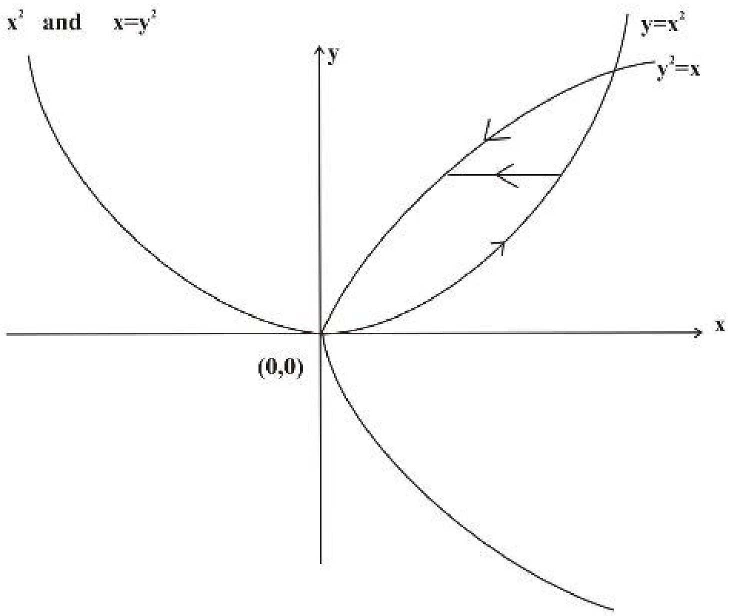
$$\int_C \left(y+e^{\sqrt{x}}\right)dx+\left(2x+\cos y^2\right)dy$$

Where C is the region bounded by parabolas $y=x^2$ and $x=y^2$



On comparing with $\int P\,dx+Q\,dy$ we have

$$P=y+e^{\sqrt{x}}$$
$$Q=2x+\cos y^2$$

Then $\dfrac{\partial P}{\partial y}=1$

And $\dfrac{\partial Q}{\partial x}=2$

By Green's theorem

$$\int_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

Here $D=\left\{(x,y):\ 0\le y\le 1,\ y^2\le x\le\sqrt{y}\right\}$

Then $\int_C \left(y+e^{\sqrt{x}}\right)dx+\left(2x+\cos y^2\right)dy$

$$=\int_0^1\int_{y^2}^{\sqrt{y}}(2-1)\,dx\,dy$$
$$=\int_0^1\int_{y^2}^{\sqrt{y}}dx\,dy$$
$$=\int_0^1 (1)\Big|_{y^2}^{\sqrt{y}}\,dy$$
$$=\int_0^1\left(\sqrt{y}-y^2\right)dy$$
$$=\left(\frac{2}{3}y^{3/2}-\frac{1}{3}y^3\right)_0^1$$
$$=\frac{2}{3}-\frac{1}{3}$$
$$=\boxed{\frac{1}{3}}$$

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Use Green's Theorem:

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

In this case, $P(x, y) = y^4$ and $Q(x, y) = 2xy^3$. Find the appropriate partial derivatives and the limits of integration of the region, and plug into Green's Theorem to find the solution to the line integral.

Find the relevant partial derivatives:

$$Q(x, y) = 2xy^3$$

$$\frac{\partial Q}{\partial x} = 2y^3$$

$$P(x, y) = y^4$$

$$\frac{\partial P}{\partial y} = 4y^3$$

Find the limits of integration of the region. The region is an ellipse; the limits in y can be the positive and negative halves of the ellipse, which we get by solving the ellipse equation for y:

$$x^2 + 2y^2 = 2$$

$$\frac{x^2}{2} + y^2 = 1$$

$$y = \pm\sqrt{1 - \frac{x^2}{2}}$$

The top and bottom halves of the ellipse, and therefore the y limits, are $y = \pm\sqrt{1 - \frac{x^2}{2}}$.

The x limits will be the most extreme values of x. Since the standard form of this ellipse is $\frac{x^2}{2} + y^2 = 1$, the vertices in the x-direction, and therefore the limits in x, are $x = \pm\sqrt{2}$.

Therefore, applying Green's Theorem to the integral leads to:

$$\begin{aligned}
\int_C y^4 dx + 2xy^3 dy &= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{1-(x^2)/2}}^{\sqrt{1-(x^2)/2}} \left(2y^3 - 4y^3\right) dy dx \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{y^4}{2} - y^4 \right) \Bigg|_{-\sqrt{1-(x^2)/2}}^{\sqrt{1-(x^2)/2}} dx \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{(1-(x^2)/2)^2}{2} - (1-(x^2)/2)^2 - \left(\frac{(1-(x^2)/2)^2}{2} - (1-(x^2)/2)^2 \right) \right) dx \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} 0 dx \\
&= \boxed{0}
\end{aligned}$$

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Consider the line integral

$\int_C y^3 dx - x^3 dy$, where C is the circle $x^2 + y^2 = 4$

Evaluate the above integral by using the green's theorem.

Green's theorem states that let *C* be a positively oriented, and let *D* be the region bounded by *C*. if *P* and *Q* have continuous partial derivatives on an open region that contains *D*, then

$$\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

On comparing $\int_C y^3 dx - x^3 dy$ with $\int_C P\, dx + Q\, dy$ we have

$P = y^3$ and $Q = -x^3$

Then

$\frac{\partial P}{\partial y} = 3y^2$ and $\frac{\partial Q}{\partial x} = -3x^2$

By Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Here in polar co – ordinates

$x = r\cos\theta$ and $y = r\sin\theta$

And

$$x^2 + y^2 = r^2$$
$4 = r^2$ Since $x^2 + y^2 = 4$
$$r = 2$$

Therefore the region is

$$D = \{(r,\theta),\quad 0 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$$

Then the line integral becomes

$$\int_C y^3 dx - x^3 dy = \iint_D \left(-3x^2 - 3y^2\right) dA$$

$= -3\iint_D \left(x^2 + y^2\right) dA$ Since $x^2 + y^2 = r^2$

$$= -3\int_0^{2\pi}\int_0^2 r^2\, r\, dr\, d\theta$$

$= -3\int_0^{2\pi} d\theta \left[\int_0^2 r^3\, dr\right]$ Apply integration

$= -3(\theta)_0^{2\pi}\left(\frac{r^4}{4}\right)_0^2$ Use the formula $\int x^n dx = \frac{x^{n+1}}{n+1}$

$= -3(2\pi - 0)\left(\frac{16}{4} - 0\right)$ Apply limits

$$= -3(2\pi)(4)$$

$= \boxed{-24\pi}$ Simplify

Thus

$$\int_C y^3 dx - x^3 dy = \boxed{-24\pi}$$

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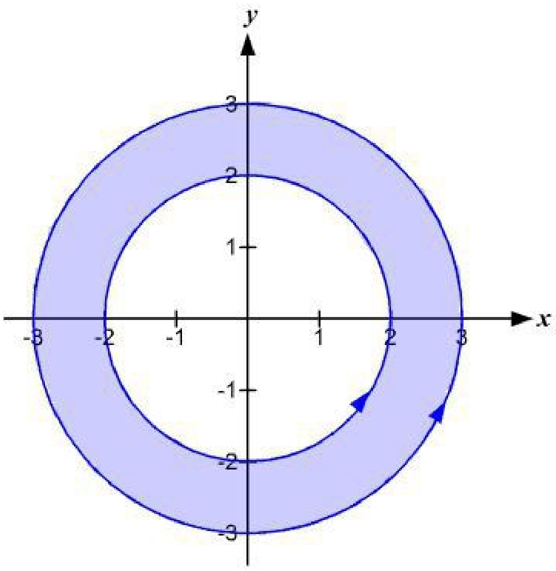
Let us first graph of the given region.



In polar coordinates, D is given by $2 \le r \le 3$ and $0 \le \theta \le 2\pi$. Also,
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - (-3y^2)$ or $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 + 3y^2$. On replacing x with $r\cos\theta$ and y with $r\sin\theta$, we get $3r^2$.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^{2\pi}\int_2^3 (3r^2) r \, dr d\theta$$
$$= \int_0^{2\pi} \left[\frac{3r^4}{4} \right]_2^3 d\theta$$
$$= \int_0^{2\pi} \left[\frac{195}{4} \right] d\theta$$

Evaluate the outer integral.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \left[\frac{195\theta}{4} \right]_0^{2\pi}$$
$$= \frac{195\pi}{2}$$

Thus, the integral evaluates to $\boxed{\frac{195\pi}{2}}$.

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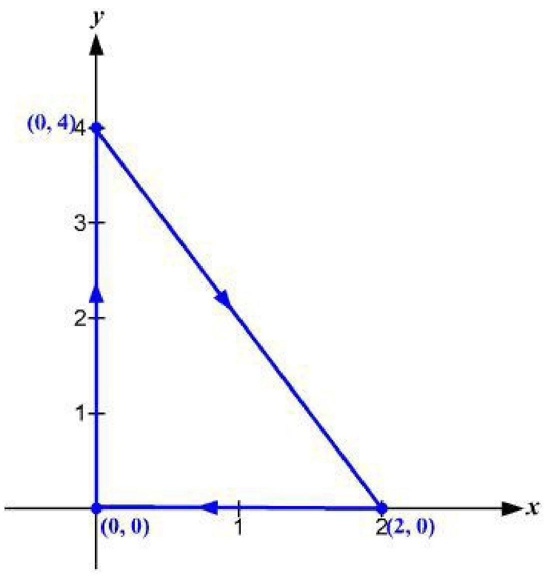
Consider the vector field function:

$\mathbf{F}(x, y) = \langle y\cos x - xy\sin x, xy + x\cos x \rangle$.

Thus, $P(x, y) = y\cos x - xy\sin x$ and $Q(x, y) = xy + x\cos x$.

The region D enclosed by C is the triangle and C has negative orientation.

Sketch the region.



Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Green's Theorem:

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Clearly, P and Q have continuous partial derivatives on an open region that contains D.

Compute $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$.

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\frac{\partial}{\partial x}(xy+x\cos x)-\frac{\partial}{\partial y}(y\cos x-xy\sin x)$$
$$=(y+\cos x-x\sin x)-(\cos x-x\sin x)$$
$$=y$$

Thus, $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y.$

The curve C moves from $(0,4)$ to $(2,0)$, thus the equation of the line passing through the points $(0,4)$ to $(2,0)$ is

$$y-4=\frac{4-0}{0-2}(x-0)$$
$$y-4=-2x$$
$$y=4-2x$$

So $0\le x\le 2, 0\le y\le 4-2x.$

Apply Green's Theorem:

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=-\int_0^2\int_0^{4-2x} y\,dy\,dx \qquad \text{since } C \text{ is negatively oriented}$$
$$=-\int_0^2\left[\frac{y^2}{2}\right]_0^{4-2x}dx \qquad \text{Since, } \int x^n dx=\frac{x^{n+1}}{n+1}+C$$
$$=-\int_0^2\left[\frac{(4-2x)^2}{2}\right]dx$$

Evaluate the outer integral.

$$\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=-\int_0^2\left[\frac{(4-2x)^2}{2}\right]dx$$
$$=-\frac{1}{2}\left[-\frac{(4-2x)^3}{3}\times\frac{1}{2}\right]_0^2$$
$$=-\frac{1}{2}\left(-\frac{(4-2(2))^3}{6}+\frac{(4-2(0))^3}{6}\right)$$
$$=-\frac{1}{2}\times\frac{64}{6}$$
$$=-\frac{16}{3}$$

Therefore, the value of the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is $\boxed{-\frac{16}{3}}$.

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Use Green's Theorem. For a positively oriented curve C,

$$\int_C Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

In this case, $P(x,y)$ is the $\mathbf{i}$-component of $\mathbf{F}$, meaning $P(x,y)=e^{-x}+y^2$, and $Q(x,y)$ is the $\mathbf{j}$-component of $\mathbf{F}$, meaning $Q(x,y)=e^{-y}+x^2$. Find the appropriate partial derivatives and the limits of integration of the region, and plug into Green's Theorem to find the solution to the line integral.

Find the relevant partial derivatives:

$Q(x,y) = e^{-y} + x^2$

$\frac{\partial Q}{\partial x} = 2x$

$P(x,y) = e^{-x} + y^2$

$\frac{\partial P}{\partial y} = 2y$

Find the limits of integration of the region. The y-values range from 0 to $\cos x$ and the x-values range from $-\pi/2$ to $\pi/2$, so these are the limits of the region in y and x. Notice that the curve is negatively oriented, however; we are traversing it clockwise instead of counterclockwise. Therefore we must add a negative to the double integral in Green's Theorem.

Applying Green's Theorem to the integral leads to:

$$\begin{aligned}-\int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(2x-2y)dydx &= -\int_{-\pi/2}^{\pi/2}\left(2xy-y^2\right)\Big|_0^{\cos x}dx \\ &= -\int_{-\pi/2}^{\pi/2}\left(2x\cos x-\cos^2 x-0\right)dx \\ &= -\int_{-\pi/2}^{\pi/2}\left(2x\cos x-\frac{(1+\cos 2x)}{2}\right)dx\end{aligned}$$

In the last step we have used the trigonometric identity $\cos^2 x = (1+\cos 2x)/2$.
Split the integral up across the addition sign. Integrate the first integral using integration by parts, given by the formula $\int u dv = uv - \int (v) du$. Let

$u = 2x$

$dv = \cos x dx$

$du = 2dx$

$v = \sin x$

This assignment for integration by parts gives us

$$\begin{aligned}-\int_{-\pi/2}^{\pi/2}(2x\cos x)dx &= -\left[(2x)(\sin x)\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2}\sin x(2)dx\right] \\ &= -\left[(2(\pi/2))(\sin(\pi/2)) - (2(-\pi/2))(\sin(-\pi/2)) - 2(-\cos x)\Big|_{-\pi/2}^{\pi/2}\right] \\ &= -\left[\pi(1)+\pi(-1)-2(-\cos(\pi/2)+\cos(-\pi/2))\right] \\ &= 0\end{aligned}$$

Evaluate the second integral:

$$\begin{aligned}-\int_{-\pi/2}^{\pi/2}\left(-\frac{(1+\cos 2x)}{2}\right)dx &= \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)\Big|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(-\frac{\pi}{2}+\frac{\sin(-\pi)}{2}\right)\right) \\ &= \frac{1}{2}(\pi+0) \\ &= \frac{\pi}{2}\end{aligned}$$

The result from Green's Theorem is the sum of these two integrals, or

$$0+\frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

Chapter 16 Vector Calculus Exercise 16.4 13E

Consider the vector field

$\mathbf{F}(x,y)=\langle y-\cos y, x\sin y\rangle$ ...... (1)

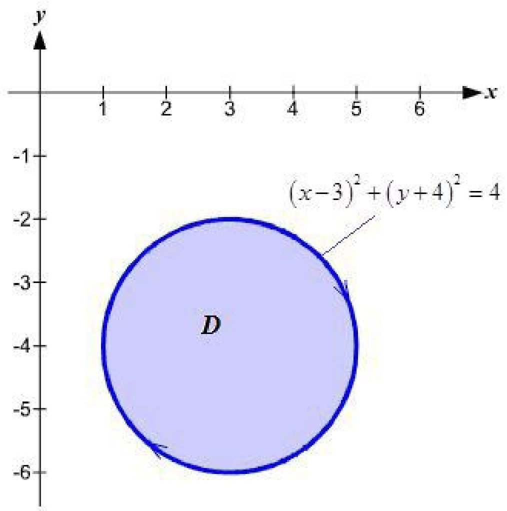
Use green's theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, here C is circle $(x-3)^2+(y+4)^2=4$ oriented clock wise.

Recall Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C Pdx+Qdy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA \;\; ...... (2)$$

Sketch the region D bounded by the circle $(x-3)^2+(y+4)^2=4$ oriented clock wise.



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Consider vector field

$\mathbf{F}(x,y)=(y-\cos y)\mathbf{i}+x\sin y\mathbf{j}$

Compare this vector field to $\mathbf{F}(x,y)=P\,\mathbf{i}+Q\,\mathbf{j}$.

So we have

$P=y-\cos y$

$Q=x\sin y$

">

Differentiate P partially with respect to y,

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}(y-\cos y)$$
$$=1+\sin y$$

Differentiate Q partially with respect to x,

$$\frac{\partial Q}{\partial x}=\frac{\partial}{\partial x}(x\sin y)$$
$$=\sin y$$

">

Substitute $\frac{\partial P}{\partial y}=1+\sin y$ and $\frac{\partial Q}{\partial x}=\sin y$ in $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=\sin y-(1+\sin y)$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-1$$

">

Use equation (1), to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \left((y - \cos y)\,dx + x\sin y\,dy\right) \\ &= \iint_D \left(\frac{\partial}{\partial x}(x\sin y) - \frac{\partial}{\partial y}(y - \cos y)\right) dA \\ &= \iint_D dA\end{aligned}$$

">

Evaluate the integral $\iint_D dA$,

First change the Cartesian coordinates into polar coordinates, by substituting

$x = 3 + 2\cos\theta$ and $y = -4 + 2\sin\theta$, then the bounded region D is

$D = \{(r,\theta) : 0 \le r \le 2 \text{ and } 0 \le \theta \le 2\pi\}$.

$$\begin{aligned}\iint_D dA &= \int_0^{2\pi}\int_0^2 (1)\, r\,dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta\end{aligned}$$

$$\begin{aligned}&= \int_0^{2\pi} [2]\,d\theta \\ &= 2\theta\Big|_0^{2\pi} \\ &= 2(2\pi - 0) \\ &= 4\pi\end{aligned}$$

The integral is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D dA \\ &= 4\pi\end{aligned}$$

Therefore the required integral is

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 4\pi}$$

">

Chapter 16 Vector Calculus Exercise 16.4 14E

Chapter 16 Vector Calculus Exercise 16.4 14E

Consider the following vector field

$\mathbf{F}(x,y) = \left\langle \sqrt{x^2+1}, \tan^{-1} x \right\rangle$ ...... (1)

Use green's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, here C is triangle from

$(0,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$.

Green's Theorem:

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Compare the vector field (1) with $\mathbf{F}(x,y) = P\,\mathbf{i} + Q\,\mathbf{j}$.

The functions are,

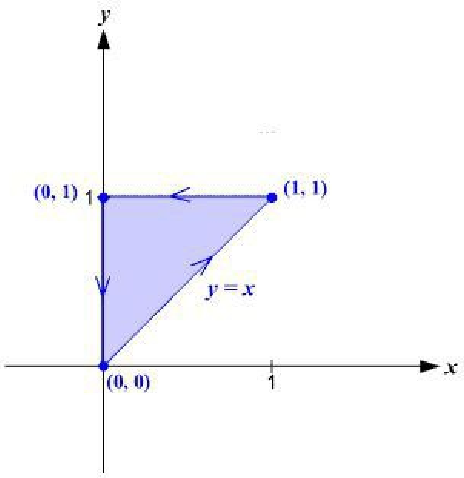
$$P = \sqrt{x^2+1}$$
$$Q = \tan^{-1} x$$

The line integral can be written as follows

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P dx + Q dy$$
$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_C \left(\left(\sqrt{x^2+1}\right)dx + \left(\tan^{-1} x\right)dy\right)$$
$$= \iint_D \left(\frac{\partial}{\partial x}\left(\tan^{-1} x\right) - \frac{\partial}{\partial y}\left(\sqrt{x^2+1}\right)\right) dA$$
$$= \iint_D \left(\frac{1}{x^2+1} - 0\right) dA$$
$$= \iint_D \frac{1}{x^2+1} dA \quad \ldots\ldots (2)$$

Sketch the triangular region is follows:



The region D consists of three lines and the orientation is clockwise. Thus, the value of the integral should be negative of the value from using Green's theorem.

The equation of the line segment from $(0,0)$ to $(1,1)$ is,

$$y-0=1(x-0)$$
$$y=x$$

And it will be the upper limit for y.

Here, x limits are from $x=0$ to $x=y$ and y limits are from $y=0$ to $y=1$.

From (2),

$$\iint_D\left(\frac{1}{1+x^2}\right)dA=\int_0^1\int_0^y\frac{1}{1+x^2}dxdy$$
$$=\int_0^1\tan^{-1}x\Big]_0^y dy$$
$$=\int_0^1\left(\tan^{-1}y-\tan^{-1}0\right)dx$$
$$=\int_0^1\tan^{-1}ydy$$

$$=\left[\left(\tan^{-1}y\right)y\right]_0^1-\int_0^1\frac{1}{1+y^2}ydy$$
$$=\left(\tan^{-1}1-0\right)-\frac{1}{2}\int_0^1\frac{2y}{1+y^2}$$
$$=\frac{\pi}{4}-\frac{1}{2}\int_0^1\frac{d\left(1+y^2\right)}{1+y^2}$$

Continue to the above,

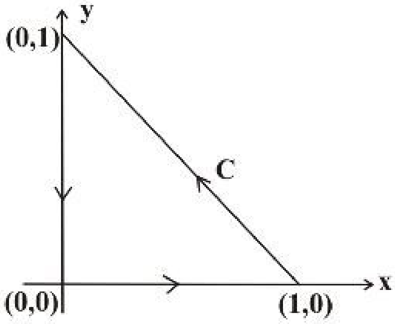
$$=\frac{\pi}{4}-\frac{1}{2}\left[\ln\left(1+y^2\right)\right]_0^1$$
$$=\frac{\pi}{4}-\frac{1}{2}\left[\ln(1+1)-\ln(1+0)\right]$$
$$=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

Thus, the value of integral $\int_C \mathbf{F}\cdot d\mathbf{r}=\boxed{\frac{\pi}{4}-\frac{1}{2}\ln 2}$.

Chapter 16 Vector Calculus Exercise 16.4 17E

$$\vec{F}(x,y)=x(x+y)\hat{i}+xy^2\hat{j}$$

The path followed by the particle is positively oriented and so we can use Green's theorem to find the work done in moving particle along C.



The work done is given by $\int_C \vec{F}.d\vec{r}$

Here $\vec{F}(x,y)=x(x+y)\hat{i}+xy^2\hat{j}$

Then $\oint_C \vec{F}.d\vec{r}=\oint_C x(x+y)dx+xy^2dy$

On comparing with $\oint_C P\,dx + Q\,dy$ we have

$$P = x^2 + xy \text{ and } Q = xy^2$$

Then $\dfrac{\partial P}{\partial y} = x$ and $\dfrac{\partial Q}{\partial x} = y^2$

By Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Here $D = \{(x, y);\ \ 0 \le x \le 1,\ \ 0 \le y \le 1 - x\}$

Then
$$\oint_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dA$$
$$= \int_0^1 \int_0^{1-x} (y^2 - x)\,dy\,dx$$
$$= \int_0^1 \left(\frac{y^3}{3} - xy\right)_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left(\frac{(1-x)^3}{3} - x(1-x)\right) dx$$
$$= \left[\frac{-(1-x)^4}{12} - \frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$
$$= \frac{-(0)}{12} - \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$$
$$= \frac{-1}{12}$$

Hence $\boxed{\text{work done is } -1/12}$

Chapter 16 Vector Calculus Exercise 16.4 18E

Consider the force field

$\vec{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$

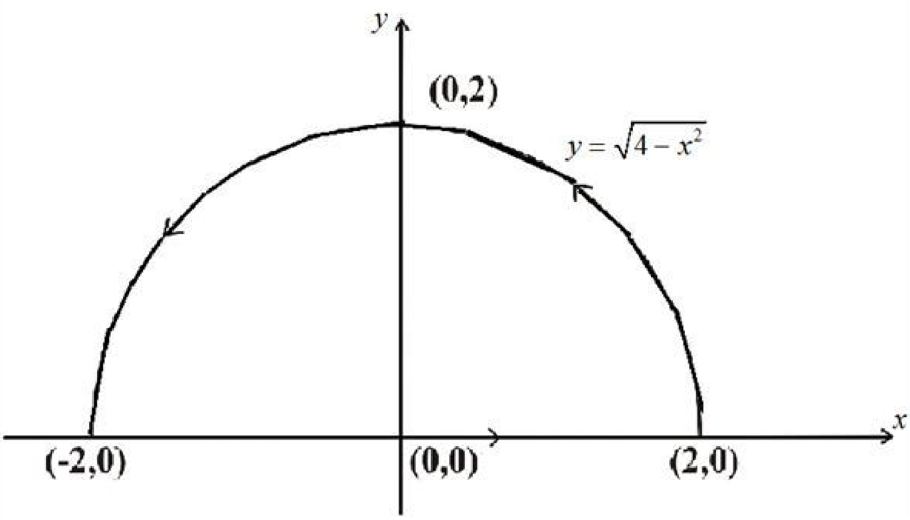
And semicircle $y = \sqrt{4 - x^2}$

Squaring on both sides

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

The path of the particle is positively oriented so use Green's theorem to find the work done in moving particle along C.



The work done is given by $\oint_C \vec{F}.d\vec{r}$

Here $\vec{F}(x,y) = < x,\ x^3 + 3xy^2 >$

Then $\oint_C \vec{F}.d\vec{r} = \oint_C x\,dx + \left(x^3 + 3xy^2\right)dy$

On comparing with $\oint_C P\,dx + Q\,dy$

$P = x$

Then taking partial derivative with respect to y

$\frac{\partial P}{\partial y} = 0$

And

$Q = x^3 + 3xy^2$

Take partial derivative with respect to x

$\frac{\partial Q}{\partial x} = 3x^2 + 3y^2$

Green's Theorem:

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P \text{ and } Q$ have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

Take Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

Then

$x^2 + y^2 = r^2$

$dA = dxdy = rdrd\theta$

Now

$x^2 + y^2 = 4$ And $x^2 + y^2 = r^2$

$r^2 = 4$

$r = 2$

And

$$\begin{array}{ll} 2\cos\theta = 2 & 2\cos\theta = -2 \\ \cos\theta = 1 & \cos\theta = -1 \\ \theta = 0 & \theta = \pi \end{array}$$

Here D in polar co – ordinates is

$D = \left\{(r,\theta):\ 0 \le r \le 2,\ 0 \le \theta \le \pi\right\}$

Then by using Green's Theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\oint_C x\,dx + \left(x^3 + 3xy^2\right) dy$$

$$= \iint_D \left\{\left(3x^2 + 3y^2\right) - 0\right\} dA$$

$$= \iint_D 3\left(x^2 + y^2\right) dA$$

$$= \int_0^{\pi}\int_0^{2} 3r^2 . r\,dr\,d\theta$$

$$= \int_0^{\pi} d\theta \int_0^{2} 3r^3\,dr$$

$$= (\theta)_0^{\pi}\left(\frac{3}{4}r^4\right)_0^2$$

$$= (\pi - 0)\left(\frac{3}{4}(16) - 0\right)$$

$$= 12\pi$$

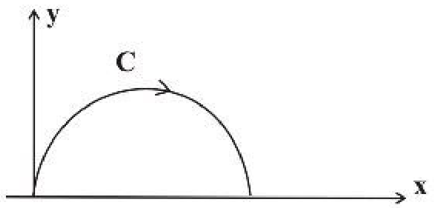
Hence Work done on this particle is $\boxed{12\pi}$

Chapter 16 Vector Calculus Exercise 16.4 19E

The area is given by

$$A = \oint_C x\,dy$$

For the cycloid, the curve C is negatively oriented



Then the area will be

$$A = -\oint_{C'} x\,dy$$

Where C' is same as C but having opposite orientation i.e. C' is positively oriented.

Since $x = t - \sin t,\ y = 1 - \cos t$

Then $A = -\int_0^{2\pi} (t - \sin t)(\sin t)\,dt$

$$= -\int_0^{2\pi} \left(t\sin t - \sin^2 t\right) dt$$

$$= -\left[\sin t - t\cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= -(-3\pi)$$

$$= 3\pi$$

That is area under an arch of cycloid is $\boxed{3\pi}$

Chapter 16 Vector Calculus Exercise 16.4 20E

Using the Green's Theorem the formula for the area of a region is:

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

Given,

$$x = 5\cos t - \cos(5t)$$
$$y = 5\sin t - \sin(5t)$$

Apply the formula and find the area,

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$
$$= \frac{1}{2}\int_C \left(5\cos t - \cos(5t)\right)\left(5\cos t - 5\cos(5t)\right)dt - \left(5\sin t - \sin(5t)\right)\left(-5\sin t + 5\sin(5t)\right)dt$$
$$= \frac{1}{2}\int_C \left(25\cos^2 t - 30\cos t\cos(5t) + 5\cos^2(5t)\right)dt - \left(-25\sin^2 t + 30\sin t\sin(5t) - 5\sin^2(5t)\right)dt$$
$$= \frac{1}{2}\int_C 25\cos^2 t - 30\cos t\cos(5t) + 5\cos^2(5t) + 25\sin^2 t - 30\sin t\sin(5t) + 5\sin^2(5t)\,dt$$

Simplify the integral and solve it

$$A = \frac{1}{2}\int_C 25\cos^2 t - 30\cos t\cos(5t) + 5\cos^2(5t) + 25\sin^2 t - 30\sin t\sin(5t) + 5\sin^2(5t)\,dt \quad \text{Simplify,}$$
$$= \frac{1}{2}\int_C \left(25\cos^2 t + 25\sin^2 t\right) + \left(5\cos^2(5t) + 5\sin^2(5t)\right) - 30\cos t\cos(5t) - 30\sin t\sin(5t) + dt \quad \text{Apply upper and lower limits,}$$
$$= \frac{1}{2}\int_C (25) + (5) - 30\cos t\cos(5t) - 30\sin t\sin(5t) + dt \quad \text{integrating trig functions over full period equal to zero,}$$
$$= \frac{1}{2}\int_0^{2\pi} 30\,dt$$
$$= \boxed{30\pi}$$

Complicated line integral can be rewritten into a double integral of simpler forms such that it can be solved in fewer and easier steps.

Chapter 16 Vector Calculus Exercise 16.4 21E

(A)

If C is the line segment connecting the point (x_1, y_1) to (x_2, y_2) then the parametric equations of C are:

$$x = (x_2 - x_1)t + x_1$$
$$y = (y_2 - y_1)t + y_1$$

i.e. $0 \le t \le 1$

Then $\int_C x\,dy - y\,dx$

$$= \int_0^1 \left\{(x_2 - x_1)t + x_1\right\}(y_2 - y_1) - \left\{(y_2 - y_1)t + y_1\right\}(x_2 - x_1)\,dt$$
$$= \int_0^1 (x_2 - x_1)(y_2 - y_1)t + x_1(y_2 - y_1) - (y_2 - y_1)(x_2 - x_1) - y_1(x_2 - x_1)\,dt$$
$$= \int_0^1 x_1(y_2 - y_1) - y_1(x_2 - x_1)\,dt$$
$$= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt$$
$$= (x_1 y_2 - x_2 y_1)(t)_0^1$$
$$= x_1 y_2 - x_2 y_1$$

(B)

As we know by Green's theorem if C is a closed path then $\int_C x\,dy - y\,dx = \iint_D 2\,dx\,dy = 2A(D)$

So to find the area of the polygon we need to integrate $\frac{1}{2}\int(x\,dy - y\,dx)$ around its boundary. But this integral can be split into n integrals along the line segments from (x_1', y_1') to $(x_1'+1, y_1'+1)$

i.e. $A = \frac{1}{2}\int_C x\,dy - y\,dx$

$$= \frac{1}{2}\int_{C_1}(x\,dy - y\,dx) + \frac{1}{2}\int_{C_2}(x\,dy - y\,dx) + \ldots\ldots$$

$$\ldots\ldots\ldots + \frac{1}{2}\int_{C_n}(x\,dy - y\,dx)$$

Where C_1 is the line segment from (x_1, y_1) to (x_2, y_2) and C_2 is the line segment from (x_2, y_2) to (x_3, y_3) and C_n is the line segment from (x_n, y_n) to (x_1, y_1)

By using part (A)

$$\int_{C_1} x\,dy - y\,dx = x_1y_2 - x_2y_1$$

$$\int_{C_2} x\,dy - y\,dx = x_2y_3 - x_3y_2$$

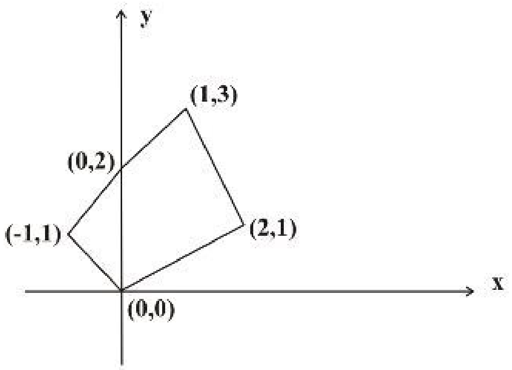
$\vdots$

$$\int_{C_n} x\,dy - y\,dx = x_ny_1 - x_1y_n$$

Hence $A = \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \ldots\ldots + (x_ny_1 - x_1y_n)\right]$

Hence proved

(c)



Using part (B)

$$A = \frac{1}{2}\left[\{0(1) - 2(0) +\}\{(2)(3) - (1)(1)\} + \{(1)(2) - 0(3)\} + \{(0)(1) - (1)(2)\} + \{(-1)(0) - (0)(1)\}\right]$$

$$= \frac{1}{2}[0 + 6 - 1 + 2 - 0 + 0 + 2 + 0 - 0]$$

$$= \frac{1}{2}(9)$$

$$= \boxed{\frac{9}{2}}$$

Chapter 16 Vector Calculus Exercise 16.4 22E

Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem we need to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ are

$\bar{x} = \frac{1}{2A}\int_C x^2\,dy$ and also $\bar{y} = \frac{1}{2A}\int_C y^2\,dx$, where A is the area of D.

The centroid is the same as the center of mass when the density ρ is constant. Referring to the formula the mass m equals ρA. So the density cancels in the center of mass formula, and it becomes this formula for the centroid:

$\bar{x} = \frac{1}{A}\int_C xdydx$ and $\bar{y} = \frac{1}{A}\int_C ydxdy$.

This can be interpreted as saying the coordinates of the centroid are the mean, or average values of x and y on D.

So we now have a double integral formula for $(\bar{x}, \bar{y})$ and a suggestion in the problem that the centroid can also be computed from line integrals on the boundary using Green's theorem. So we just plug in Green's theorem into these line integrals to convert them to double integrals and see what we get.

$$\bar{x} = \frac{1}{A}\int_C xdydx$$

$$\bar{x} = \frac{1}{A}\iint_D x\,dA$$

$$\bar{x} = \iint_D \frac{x}{A}dA \qquad \text{...... (1)}$$

And

$$\bar{y} = \frac{1}{A}\int_C ydxdy$$

$$\bar{y} = \frac{1}{A}\iint_D y\,dA$$

$$\bar{y} = \iint_D \frac{y}{A}dA \qquad \text{...... (2)}$$

Where A is the area of the region D

From Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA \qquad \text{...... (3)}$$

Comparing (1) and (3), for getting $\bar{x}$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{x}{A}$$

There are several possibilities

$$P(x,y) = 0 \quad P(x,y) = -\frac{xy}{2A} \quad P(x,y) = \frac{-xy}{A}$$

$$Q(x,y) = \frac{x^2}{2A} \quad Q(x,y) = \frac{x^2}{4A} \quad Q(x,y) = 0$$

On choosing $P = 0$ and $Q = \frac{x^2}{2A}$, we find that

$$\bar{x} = \oint_C (P\,dx + Q\,dy)$$

$$= \oint_C \left((0)\,dx + \left(\frac{x^2}{2A}\right)dy\right)$$

$$= \oint_C \left(\frac{x^2}{2A}\right)dy$$

$$= \frac{1}{2A}\oint_C x^2\,dy$$

Now comparing (2) and (3), for getting $\bar{y}$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y}{A}$$

There are several possibilities

$$P(x,y) = -\frac{y^2}{2A} \quad P(x,y) = 0 \quad P(x,y) = \frac{-y^2}{4A}$$
$$Q(x,y) = 0 \quad Q(x,y) = \frac{xy}{A} \quad Q(x,y) = \frac{xy}{2A}$$

On choosing $P = -\frac{y^2}{2A}$ and $Q = 0$,

We find that

$$\begin{aligned}\bar{y} &= \oint_C (P\,dx + Q\,dy) \\ &= \oint_C \left(\left(\frac{-y^2}{2A}\right)dx + (0)\,dy\right) \\ &= \oint_C \left(\frac{-y^2}{2A}\right)dx \\ &= -\frac{1}{2A}\oint_C y^2\,dx\end{aligned}$$

Hence the coordinates of the centroid are

$$\boxed{\bar{x} = \frac{1}{2A}\oint_C x^2\,dy \text{ and } \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx}.$$

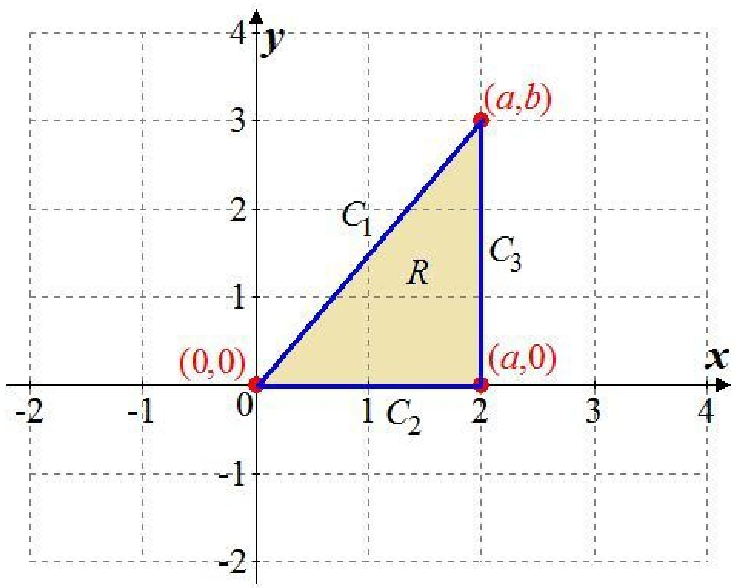
Chapter 16 Vector Calculus Exercise 16.4 24E

Suppose that C is the curve that encloses the triangle a triangle that has vertices $(0,0)$, $(a,0)$, and (a,b) where a and b are positive.

The objective is to find the coordinates $(\bar{x},\bar{y})$ of the centroid of the triangle.

Use the formulas $\bar{x} = \frac{1}{2A}\oint_C x^2 dy$ and $\bar{y} = -\frac{1}{2A}\oint_C y^2 dx$, where A is the area of the triangle.

First, sketch the triangle that has vertices $(0,0)$, $(a,0)$, and (a,b) as shown in the below figure:



Since the curve is broken into 3 curves C_1, C_2, and C_3, $\bar{x} = \frac{1}{2A}\left[\oint_{C_1} x^2 dy + \oint_{C_2} x^2 dy + \oint_{C_3} x^2 dy\right]$.

First parameterize the line segment C_1

$$\begin{aligned}\mathbf{r}(t) &= \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)\rangle \\ &= \langle 0 + t(a-0), 0 + t(0-0)\rangle \\ &= \langle at, 0\rangle\end{aligned}$$

So on C_1, $\mathbf{r}(t) = \langle at, 0\rangle$ where $0 \le t \le 1$.

Now parameterize segment C_2.

$$\mathbf{r}(t) = \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)\rangle$$
$$= \langle a + t(a - a), 0 + t(b - 0)\rangle$$
$$= \langle a, bt\rangle$$

So on C_2, $\mathbf{r}(t) = \langle a, bt\rangle$ where $0 \le t \le 1$.

Now parameterize segment C_3.

$$\mathbf{r}(t) = \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)\rangle$$
$$= \langle a + t(0 - a), b + t(0 - b)\rangle$$
$$= \langle a - at, b - bt\rangle$$

So on C_3, $\mathbf{r}(t) = \langle a - at, b - bt\rangle$ where $0 \le t \le 1$.

Now calculate the *x*-coordinate of the centroid. Recall that the area of a triangle is $\frac{1}{2}(\text{base})(\text{height})$. Since this triangle has a base of *a* and a height of *b*, the area of the triangle is $A = \frac{ab}{2}$.

Now calculate $\overline{x} = \frac{1}{2A}\left[\oint_{C_1} x^2 dy + \oint_{C_2} x^2 dy + \oint_{C_3} x^2 dy\right]$ using the parameterization of each curve.

$$\oint_{C_1} x^2 dy = \int_0^1 x^2(t)\, y'(t)\, dt$$
$$= \int_0^1 (at)^2 (0) dt$$
$$= 0$$

$$\oint_{C_2} x^2 dy = \int_0^1 x^2(t)\, y'(t)\, dt$$
$$= \int_0^1 (a)^2 (b) dt$$
$$= a^2 b[t]_{t=0}^{t=1}$$
$$= a^2 b[1 - 0]$$
$$= a^2 b$$

$$\oint_{C_3} x^2 dy = \int_0^1 x^2(t)\, y'(t)\, dt$$
$$= \int_0^1 (a - at)^2 (-b)\, dt$$
$$= \int_0^1 \left(-a^2 b + 2a^2 bt - a^2 bt^2\right) dt$$
$$= \left[-a^2 bt + 2a^2 b\frac{t^2}{2} - a^2 b\frac{t^3}{3}\right]_{t=0}^{t=1}$$
$$= \left[-a^2 b(1) + 2a^2 b\frac{(1)^2}{2} - a^2 b\frac{(1)^3}{3} - 0\right]$$
$$= \left[-a^2 b + a^2 b - \frac{a^2 b}{3}\right]$$
$$= -\frac{a^2 b}{3}$$

Now put all of these together

$$\overline{x} = \frac{1}{2A}\left[\oint_{C_1} x^2 dy + \oint_{C_2} x^2 dy + \oint_{C_3} x^2 dy\right]$$

$$= \frac{1}{2\left(\frac{ab}{2}\right)}\left[0 + a^2 b - \frac{a^2 b}{3}\right]$$

$$= \frac{2a}{3}$$

So the *x*-coordinate of the centroid is $\frac{2a}{3}$.

Now calculate $\overline{y} = -\frac{1}{2A}\left[\oint_{C_1} y^2 dx + \oint_{C_2} y^2 dx + \oint_{C_3} y^2 dx\right]$ by computing the individual line integrals first

$$\oint_{C_1} y^2 dx = \int_0^1 y^2(t)\, x'(t)\, dt$$

$$= \int_0^1 (0)^2 (a) dt$$

$$= 0$$

$$\oint_{C_2} y^2 dy = \int_0^1 y^2(t)\, x'(t)\, dt$$

$$= \int_0^1 (bt)^2 (0) dt$$

$$= 0$$

$$\oint_{C_3} y^2 dy = \int_0^1 y^2(t)\, x'(t)\, dt$$

$$= \int_0^1 (b - bt)^2 (-a) dt$$

$$= \int_0^1 \left(-ab^2 + 2ab^2 t - ab^2 t^2\right) dt$$

$$= \left[-ab^2 t + 2ab^2 \frac{t^2}{2} - ab^2 \frac{t^3}{3}\right]_{t=0}^{t=1}$$

$$= \left[-ab^2(1) + 2ab^2 \frac{(1)^2}{2} - ab^2 \frac{(1)^3}{3} - 0\right]$$

$$= -\frac{ab^2}{3}$$

Now put all of these together to get

$$\overline{y} = -\frac{1}{2A}\left[\oint_{C_1} y^2 dx + \oint_{C_2} y^2 dx + \oint_{C_3} y^2 dx\right]$$

$$= -\frac{1}{2\left(\frac{ab}{2}\right)}\left[0 + 0 - \frac{ab^2}{3}\right]$$

$$= \frac{b}{3}$$

Since $\overline{y} = \frac{b}{3}$, the coordinates of the centroid of the triangle are $\left(\frac{2a}{3}, \frac{b}{3}\right)$.

Chapter 16 Vector Calculus Exercise 16.4 25E

Consider a plane lamina with constant density $\rho(x,y)=\rho$ occupies a region in the $xy-\text{plane}$ bounded by a simple closed path $C.$

Show that its moments of inertia about the axes are

$$I_x=-\frac{\rho}{3}\oint_C y^3\,dx \qquad I_y=\frac{\rho}{3}\oint_C x^3\,dx$$

Now, recall the moment of inertia of the lamina about the $x-\text{axis}$ and $y-\text{axis}$ are

$$I_x=\iint_D y^2\rho(x,y)\,dA$$

And $I_y=\iint_D x^2\rho(x,y)\,dA$

Recall the Green's theorem

$$\oint_C P\,dx+Q\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA \quad \ldots\ldots (1)$$

Now, compare $-\frac{\rho}{3}\oint_C y^3\,dx$ with $\oint_C P\,dx+Q\,dy$

Thus,

$$P=-\frac{\rho}{3}y^3 \qquad Q=0$$

$$\frac{\partial P}{\partial y}=-\rho y^2 \qquad \frac{\partial Q}{\partial x}=0$$

These values substitute in (1), it becomes

$$\begin{aligned}-\frac{\rho}{3}\oint_C y^3\,dx&=\oint_C\left(-\frac{\rho}{3}y^3\right)dx+0dy && \text{Rewrite}\\ &=\iint_D\left(0-\left(-\rho y^2\right)\right)dA && \text{From Green's Theorem}\\ &=\iint_D \rho y^2 dA\\ &=I_x\end{aligned}$$

Therefore, $I_x=\boxed{-\frac{\rho}{3}\oint_C y^3\,dx}$.

Now, show that $I_y=\frac{\rho}{3}\oint_C x^3\,dx$

Compare $\frac{\rho}{3}\oint_C x^3\,dx$ with $\oint_C P\,dx+Q\,dy$

Thus,

$$P=0 \qquad Q=\frac{\rho}{3}x^3$$

$$\frac{\partial P}{\partial y}=0 \qquad \frac{\partial Q}{\partial x}=\rho x^2$$

These values substitute in (1), it becomes

$$\begin{aligned}\frac{\rho}{3}\oint_C x^3\,dx&=\oint_C(0)\,dx+\frac{\rho}{3}x^3dy && \text{Rewrite}\\ &=\iint_D\left(\rho x^2-0\right)dA && \text{From Green's Theorem}\\ &=\iint_D \rho x^2 dA\\ &=I_y\end{aligned}$$

Therefore, $I_y=\boxed{\frac{\rho}{3}\oint_C x^3\,dx}$.

Chapter 16 Vector Calculus Exercise 16.426E

As we know $I_x = -\frac{\rho}{3}\oint_C y^3\,dx$ and $I_y = \frac{\rho}{3}\oint_C x^3\,dy$

Now the parametric equations of circular disk C of radius a with constant density ρ are

$$x = a\cos t \quad y = a\sin t \quad 0 \le t \le 2\pi$$

Then

$$\begin{aligned}
I_x &= -\frac{\rho}{3}\oint_C y^3\,dx \\
&= -\frac{\rho}{3}\int_0^{2\pi}(a\sin t)^3(-a\sin t)dt \\
&= \frac{\rho}{3}\int_0^{2\pi} a^4\sin^4 t\,dt \\
&= \frac{\rho a^4}{3}\int_0^{2\pi}\sin^4 t\,dt \\
&= \frac{\rho a^4}{3}\left[\frac{-1}{4}\sin^3 t\cos t + \frac{3}{8}t - \frac{3}{16}\sin 2t\right]_0^{2\pi} \\
&= \frac{\rho a^4}{3}\left[0 + \frac{3}{8}(2\pi) - 0 + 0 - 0 + 0\right] \\
&= \frac{\rho\pi a^4}{4}
\end{aligned}$$

Chapter 16 Vector Calculus Exercise 16.4 27E

Consider a positive oriented circle C' with center as the origin and a as radius such that C' lies inside C. Let D be the region bounded by C and C'. Then,

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA.$$

We have $Q = \frac{y^2 - x^2}{(x^2+y^2)^2}$ and $P = \frac{2xy}{(x^2+y^2)^2}$. Find $\iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$.

$$\begin{aligned}
\iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA &= \iint_D\left(\frac{2x(x^2-3y^2)}{(x^2+y^2)^3} - \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}\right)dA \\
&= 0
\end{aligned}$$

Thus, we get $\int_C P\,dx + Q\,dy = \int_{-C'} P\,dx + Q\,dy$. This means that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-C'}\mathbf{F}\cdot d\mathbf{r}$.

Let $\mathrm{r}(t) = a\cos t\mathrm{i} + a\sin t\mathrm{j},\ 0 \le t \le 2\pi$. Then, $r'(t) = -a\sin t + a\cos t$.

Find $\mathrm{F}(\mathrm{r}(t))$.

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) &= \frac{2(a\cos\theta)(a\sin\theta)}{\left[(a\cos\theta)^2 + (a\sin\theta)^2\right]^2}\mathbf{i} + \frac{(a\sin\theta)^2 - (a\cos\theta)^2}{\left[(a\cos\theta)^2 + (a\sin\theta)^2\right]^2}\mathbf{j} \\
&= \frac{2a^2\cos\theta\sin\theta}{a^4}\mathbf{i} + \frac{a^2(\sin^2\theta - \cos^2\theta)}{a^4}\mathbf{j} \\
&= \frac{2\cos\theta\sin\theta}{a^2}\mathbf{i} + \frac{(\sin^2\theta - \cos^2\theta)}{a^2}\mathbf{j}
\end{aligned}$$

Now, evaluate $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)$.

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) &= \frac{2\cos t\sin t(-a\sin t)}{a^2} + \frac{(\sin^2 t - \cos^2 t)(a\cos t)}{a^2} \\
&= \frac{-2\cos t\sin^2 t + \cos t\sin^2 t - \cos^3 t}{a} \\
&= \frac{-\cos t\sin^2 t - \cos^3 t}{a}
\end{aligned}$$

Find $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-C} \mathbf{F} \cdot d\mathbf{r}$ given by $\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$.

Find $\int_0^{2\pi} \frac{-\cos t \sin^2 t - \cos^3 t}{a} dt$.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \frac{-\cos t \sin^2 t - \cos^3 t}{a} dt \\ &= \left[-\frac{\sin t}{a}\right]_0^{2\pi} \\ &= 0\end{aligned}$$

Thus, we get $\int_C \mathbf{F} \cdot d\mathbf{r}$ as $\boxed{0}$.

Chapter 16 Vector Calculus Exercise 16.4 28E

Use Green's Theorem. For a positively oriented curve C,

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In this case, $P(x,y)$ is the **i**-component of **F**, meaning $P(x,y) = x^2 + y$, and $Q(x,y)$ is the **j**-component of **F**, meaning $Q(x,y) = 3x - y^2$. Find the appropriate partial derivatives and plug into Green's Theorem to find the solution to the line integral.

Find the relevant partial derivatives:

$$Q(x,y) = 3x - y^2$$
$$\frac{\partial Q}{\partial x} = 3$$
$$P(x,y) = x^2 + y$$
$$\frac{\partial P}{\partial y} = 1$$

Plug into Green's Theorem:

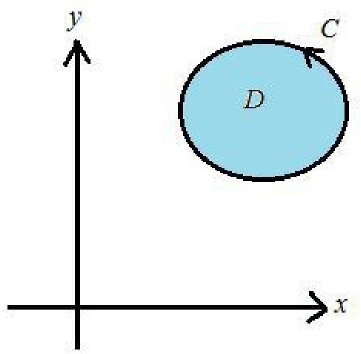
$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_D (3-1) dA \\ &= \iint_D (2) dA\end{aligned}$$

Chapter 16 Vector Calculus Exercise 16.4 29E

Consider the vector field $\mathbf{F}(x,y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.

Need to show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

Let D be region bounded by C which is shown below:



And clearly D is a simple closed path that does not pass through or enclose the origin.

Since $\mathbf{F}(x,y)=\frac{-y}{x^2+y^2}\mathbf{i}+\frac{x}{x^2+y^2}\mathbf{j}=P\mathbf{i}+Q\mathbf{j}.$

Find the partial derivatives:

$$\begin{aligned}\frac{\partial Q}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\\&=\frac{1}{x^2+y^2}-\frac{2x^2}{\left(x^2+y^2\right)^2}\\&=\frac{\left(x^2+y^2\right)-2x^2}{\left(x^2+y^2\right)^2}\\&=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial P}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\\&=-\frac{1}{x^2+y^2}+\frac{2y^2}{\left(x^2+y^2\right)^2}\\&=\frac{-x^2-y^2+2y^2}{\left(x^2+y^2\right)^2}\\&=\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

“>

Apply Green's Theorem, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r}&=\int_C Pdx+Qdy\\&=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA\\&=\iint_D\left(\frac{y^2-x^2}{\left(x^2+y^2\right)^2}-\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}\right)dA\\&=\iint_D\left(\frac{y^2-x^2+x^2-y^2}{\left(x^2+y^2\right)^2}\right)dA\\&=\iint_D\left(\frac{0}{\left(x^2+y^2\right)^2}\right)dA\\&=0\end{aligned}$$

Therefore, $\boxed{\int_C \mathbf{F}\cdot d\mathbf{r}=0}$.

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Chapter 16 Vector Calculus Exercise 16.4 30E

To prove Green's theorem, we first prove two equations:

$$\int_C Q\,dy=\iint_D \frac{\partial Q}{\partial x}dA$$

And $$\int_C P\,dx=-\iint_D \frac{\partial P}{\partial y}dA$$

Expressing D as a type II region

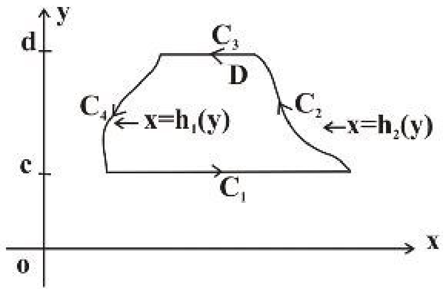
$$D=\{(x,y):\ c\le y\le d\quad h_1(y)\le x\le y\le h_2(y)\}$$

Where h_1 and h_2 are continuous functions

Then $$\iint_D \frac{\partial Q}{\partial x}dA=\int_c^d\int_{h_1(y)}^{h_2(y)}\frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

$$=\int_c^d\left[Q\left(h_2(y),y\right)-Q\left(h_1(y),\ y\right)\right]dy \quad \text{-------------} \ (1)$$

Let C be the union of the four curves C_1, C_2, C_3 and C_4



On C_2 we take y as the parameter and write parametric equations as

$$x = h_2(y) \quad y = y \quad c \le y \le d$$

Thus $\int_{C_2} Q(x,y)dy = \int_{C_2}^{c} Q(h_2(y),y)dy$

Now C_4 is in the direction opposite to C_2. Then we can write parametric equations for C_4 as

$$x = h_1(y) \quad y = y \quad c \le y \le d$$

Then $\int_{C_4} Q(x,y)dy = -\int_{-C_4} Q(x,y)dy$

$$= -\int_c^d Q(h_1(y),y)dy$$

On C_1 and C_3 y is constant so dy = 0 and

$$\int_{C_1} Q(x,y)dy = 0 = \int_{C_4} Q(x,y)dy$$

Hence $\int_C Q(x,y)dy = \int_{C_1} Q(x,y)dy + \int_{C_2} Q(x,y)dy + \int_{C_3} Q(x,y)dy$

$$+\int_{C_4} Q(x,y)dy$$

$$= \int_c^d Q(h_2(y),y)dy - \int_c^d Q(h_1(y),y)dy$$

$$= \int_c^d \left[Q(h_2(y),y) - Q(h_1(y),y)\right]dy \quad \text{---------------- (2)}$$

Comparing (1) and (2): -

$$\int_C Q(x,y)dy = \iint_D \frac{\partial Q}{\partial x} dA \quad \text{--------------- (3)}$$

Now on expressing 0 as a type 1 region

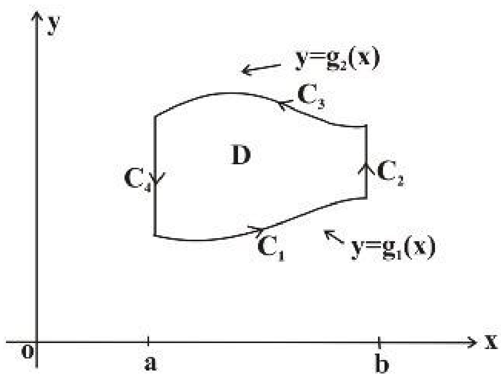
$$D = \{(x,y): \ a \le x \le b \quad g_1(x) \le y \le g_2(x)\}$$

Where g_1 and g_2 are continuous

Then $\iint_D \frac{\partial P}{\partial y} dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)dy\,dx$

$$= \int_a^b \left[P(x,g_2(x)) - P(x,g_1(x))\right]dx \quad \text{----------------- (4)}$$

We express C as the union of four curves C_1, C_2, C_3 and C_4



On C_1 we take x as parameter and writer parametric equations as

$$x = x, \quad y = g_1(x), \; a \le x \le b$$

Thus $\int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx$

Where C_3 is in the opposite direction of C_1 then we can write the parametric equations of C_3 as:

$$x = x, \quad y = g_2(x), \quad a \le x \le b$$

Therefore $\int_{C_3} P(x,y)\,dx = -\int_{-C_3} P(x,y)\,dx$

$$= -\int_a^b P(x,y)\,dx$$

$$= -\int_a^b P(x, g_2(x))\,dx$$

On C_2 and C_4 x is a constant, so $dx = 0$ and

$$\int_{C_2} P(x,y)\,dx = 0 = \int_{C_4} P(x,y)\,dx$$

Hence $\int_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx + \int_{C_3} P(x,y)\,dx$

$$+\int_{C_4} P(x,y)\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

$$= \int_a^b \left[P(x, g_1(x)) - P(x, g_2(x))\right]dx \quad \text{-----------} \; (5)$$

On comparing (4) and (5)

$$\int_C P(x,y)\,dx = -\iint \frac{\partial P}{\partial y}\,dA \qquad \text{----------------} \; (6)$$

Now on adding (5) and (6)

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

Which is Green's theorem

Chapter 16 Vector Calculus Exercise 16.4 31E

Start with the left side of the equation and keep equating until the right side is reached. The integral

$$\iint_R dxdy$$

is equal to the area over region R. We also have a formula for area involving line integrals, namely

$$A = \oint_C xdy$$

Now we convert to the uv-plane using some unknown transformation $x(u,v)$, $y(u,v)$. To find dy, we take the total differential. The formula for the total differential is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

So in this case, we can find the total differential dy from $y(u,v)$ as

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

The transformation will also convert the curve along which we are integrating, δR to some curve in the uv-plane—call it δS—that bounds some region S. We make all of these substitutions:

$$\iint_R dxdy = \oint_{\delta R} xdy$$

$$= \oint_{\delta S} x(u,v)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \quad \ldots\ldots (1)$$

$$= \oint_{\delta S}\left(x\frac{\partial y}{\partial u}du + x\frac{\partial y}{\partial v}dv\right)$$

Now we apply Green's Theorem, which is:

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

We are in u and v instead of x and y, but (1) is recognizable as the left side of Green's Theorem, with $P = x\frac{\partial y}{\partial u}$ and $Q = x\frac{\partial y}{\partial v}$. We take the relevant partial derivatives, this time in terms of u and v instead of x and y.

For both partial derivatives we will need to apply the Product Rule, which states that the derivative of a product equals the derivative of the first factor times the second factor plus the first factor times the derivative of the second factor.

$$Q = x\frac{\partial y}{\partial v}$$

$$\frac{\partial Q}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + x\frac{\partial^2 y}{\partial u \partial v}$$

$$P = x\frac{\partial y}{\partial u}$$

$$\frac{\partial P}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} + x\frac{\partial^2 y}{\partial v \partial u}$$

Plug the partial derivatives into Green's Theorem:

$$\oint_{\delta S} \left(x\frac{\partial y}{\partial u}du + x\frac{\partial y}{\partial v}dv \right) = \iint_S \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + x\frac{\partial^2 y}{\partial u \partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - x\frac{\partial^2 y}{\partial v \partial u} \right) dudv$$

Since mixed partials are equal under the conditions we are in, we can cancel them out to get:

$$\iint_S \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) dudv$$

But this is exactly

$$\iint_S \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

Otherwise known as $\iint_S \left| \frac{\partial (x,y)}{\partial (u,v)} \right| dudv$, which is exactly what we were trying to get to. We have now successfully proven that

$$\boxed{\iint_R dxdy = \iint_S \left| \frac{\partial (x,y)}{\partial (u,v)} \right| dudv}$$