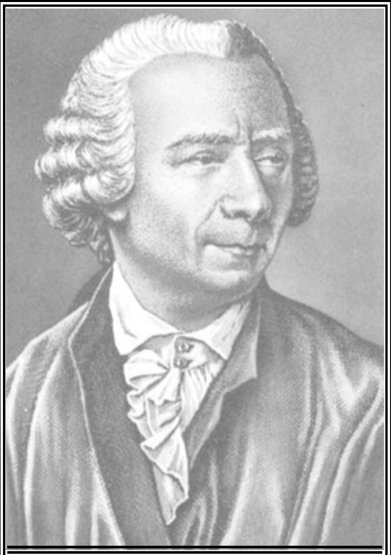
Chapter 2

Complex Numbers

Contents



Leonhard Euler

The fact that square root of a negative number does not exist in the real number system was recognized by the Greeks. But the credit goes to the Indian mathematician Mahavira (850 A.D.) *Who first stated this difficulty clearly "He mentions in his work* "Ganitasara Sangraha" *as in the nature of things a negative (quantity) is not a square (quantity), it has, therefore no square root".* Bhaskara, *another Indian mathematician, also writes in this work* 'Bijaganita' *written in* 1150 A.D.

Euler *was the first to introduce the symbol* i *for* $\sqrt{-1}$ *and* W.R. Hamilton *(about 1830 A.D.) regarded the complex number a + ib as an ordered pair of real numbers (a, b), thus giving it a purely mathematical definition and avoiding use of the so called "Imaginary numbers".*

Complex Numbers

2.1 Introduction

Number system consists of real numbers $(-5,7,\frac{1}{3},\sqrt{3}............etc.)$ and imaginary numbers $(\sqrt{-5},\sqrt{-9}$....etc.) If we combine these two numbers by some mathematical operations, the resulting number is known as Complex Number *i.e.*, "Complex Number is the combination of real and imaginary numbers".

(1) **Basic concepts of complex number**

(i) **General definition :** A number of the form $x+iy$, where $x,y\in R$ and $i=\sqrt{-1}$ is called a complex number so the quantity $\sqrt{-1}$ is denoted by 'i' called iota thus $i=\sqrt{-1}$.

A complex number is usually denoted by z and the set of complex number is denoted by c

i.e., $c=\{x+iy:x\in R,y\in R,i=\sqrt{-1}\}$

For example, $5+3i,-1+i,0+4i,4+0i$ etc. are complex numbers.

Note : ❑ Euler was the first mathematician to introduce the symbol i (iota) for the square root of – 1 with property $i^2=-1$. He also called this symbol as the imaginary unit.

- ❑ Iota (i) is neither 0, nor greater than 0, nor less than 0.
- ❑ The square root of a negative real number is called an imaginary unit.
- ❑ For any positive real number a, we have $\sqrt{-a}=\sqrt{-1\times a}=\sqrt{-1}\sqrt{a}=i\sqrt{a}$
- ❑ $i\sqrt{-a}=-\sqrt{a}$.
- ❑ The property $\sqrt{a}\sqrt{b}=\sqrt{ab}$ is valid only if at least one of a and b is non-negative. If a and b are both negative then $\sqrt{a}\sqrt{b}=-\sqrt{ab}$.
- ❑ If $a<0$ then $\sqrt{a}=\sqrt{|a|}\,i$.

(2) **Integral powers of iota (i) :** Since $i=\sqrt{-1}$ hence we have $i^2=-1$, $i^3=-i$ and $i^4=1$. To find the value of $i^n\ (n>4)$, first divide n by 4. Let q be the quotient and r be the remainder.

i.e., $n=4q+r$ where $0\le r\le 3$

$i^n=i^{4q+r}=(i^4)^q.(i)^r=(1)^q.(i)^r=i^r$

In general we have the following results $i^{4n}=1,\ i^{4n+1}=i,\ i^{4n+2}=-1,\ i^{4n+3}=-i,$ where n is any integer.

In other words, $i^n=(-1)^{n/2}$ if n is even integer and $i^n=(-1)^{n-1/2}i$ if n is odd integer.

The value of the negative integral powers of i are found as given below :

$$i^{-1}=\frac{1}{i}=\frac{i^3}{i^4}=i^3=-i, i^{-2}=\frac{1}{i^2}=\frac{1}{-1}=-1,\ i^{-3}=\frac{1}{i^3}=\frac{i}{i^4}=\frac{i}{1}=i,\ i^{-4}=\frac{1}{i^4}=\frac{1}{1}=1$$

Important Tips

- ☞ *The sum of four consecutive powers of i is always zero i.e.,* $i^n+i^{n+1}+i^{n+2}+i^{n+3}=0,\ n\in I.$
- ☞ $i^n=1, i, -1, -i,$ *where n is any integer.*
- ☞ $(1+i)^2=2i,\ (1-i)^2=-2i$
- ☞ $\frac{1+i}{1-i}=i,\ \frac{1-i}{1+i}=-i,\ \frac{2i}{i-1}=1-i$

Example: 1 If $i^2=-1,$ then the value of $\sum_{n=1}^{200} i^n$ is **[MP PET 1996]**

(a) 50 (b) – 50 (c) 0 (d) 100

Solution: (c) $\sum_{n=1}^{200} i^n=i+i^2+i^3+.....+i^{200}=\frac{i(1-i^{200})}{1-i}$ (since G.P.) $=\frac{i(1-1)}{1-i}=0.$

Example: 2 If $i=\sqrt{-1}$ and n is a positive integer, than $i^n+i^{n+1}+i^{n+2}+i^{n+3}=$ **[Rajasthan PET 2001; Karnataka CET 1994]**

(a) 1 (b) i (c) i^n (d) 0

Solution: (d) $i^n+i^{n+1}+i^{n+2}+i^{n+3}=i^n(1+i+i^2+i^3)=i^n(1+i-1-i)=o.$

Trick: Since the sum of four consecutive powers of i is always zero.

$\Rightarrow i^n+i^{n+1}+i^{n+2}+i^{n+3}=0,\ \ n\in I.$

Example: 3 If $\left(\frac{1+i}{1-i}\right)^x=1$ then **[AIEEE 2003; Rajasthan PET 2003]**

(a) $x=4n$, where n is any positive integer (b) $x=2n$, where n is any positive integer

(c) $x=4n+1$, where n is any positive integer (d) $x=2n+1$, where n is any positive integer

Solution: (a) $\frac{1+i}{1-i}=\frac{1+i}{1-i}.\frac{1+i}{1+i}\Rightarrow\left(\frac{1+i}{1-i}\right)^x=1\Rightarrow i^x=1\Rightarrow x=4n,\ n\in I^+.$

Example: 4 $1+i^2+i^4+i^6+.....+i^{2n}$ is **[EAMCET 1980]**

(a) Positive (b) Negative (c) Zero (d) Can not be determined

Solution: (d) $1+i^2+i^4+i^6+.....+i^{2n}=\sum_{k=0}^{n} i^{2k}=1$ or – 1 (which is depend upon the value of n).

Example: 5 If $x=3+i$, then $x^3-3x^2-8x+15=$ **[UPSEAT 2003]**

(a) 6 (b) 10 (c) – 18 (d) – 15

Solution: (d) Given that; $x-3=i\Rightarrow(x-3)^2=i^2\Rightarrow x^2-6x+10=0$

Now, $x^3-3x^2-8x+15=x(x^2-6x+10)+3(x^2-6x+10)-15=0+0-15=-15.$

Example: 6 The complex number $\frac{2^n}{(1-i)^{2n}}+\frac{(1+i)^{2n}}{2^n},(n\in Z)$ is equal to

(a) 0 (b) 2 (c) $[1+(-1)^n].i^n$ (d) None of these

Solution: (d) $(1+i)^{2n}=((1+i)^2)^n=(1+i^2+2i)^n=(1-1+2i)^n=2^n i^n$

$(1-i)^{2n}=((1-i)^2)^n=(1+i^2-2i)^n=(1-1-2i)^n=(-2)^n i^n$

$$\therefore \frac{2^n}{(1-i)2^n}+\frac{(1+i)^{2n}}{2^n}=\frac{2^n}{(-2)^n i^n}+\frac{2^n i^n}{2^n}=\frac{1}{(-1)^n i^n}+i^n=\frac{1+(-1)^n i^{2n}}{(-1)^n i^n}=\frac{1+(-1)^n (i^2)^n}{(-1)^n i^n}$$

$$=\frac{1+(-1)^n(-1)^n}{(-1)^n i^n}=\frac{1+(-1)^{2n}}{(-1)^n i^n}=\frac{1+1}{(-1)^n i^n}=\frac{2}{(-1)^n i^n}.$$

2.2 Real and Imaginary Parts of a Complex Number

If x and y are two real numbers, then a number of the form $z = x+iy$ is called a complex number. Here 'x' is called the real part of z and 'y' is known as the imaginary part of z. The real part of z is denoted by $Re(z)$ and the imaginary part by $\text{Im}(z)$.

If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

Note :
- ❑ A complex number z is purely real if its imaginary part is zero *i.e.,* $\text{Im}(z) = 0$ and purely imaginary if its real part is zero *i.e.,* $\text{Re}(z) = 0$.
- ❑ i can be denoted by the ordered pair (0,1).
- ❑ The complex number (a, b) can also be split as $(a, 0) + (0, 1)(b, 0)$.

Important Tips

- ☞ *A complex number is an imaginary number if and only if its imaginary part is non-zero. Here real part may or may not be zero.*
- ☞ *All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or not be purely imaginary.*
- ☞ *A real number can be written as a + i.0, therefore every real number can be considered as a complex number whose imaginary part is zero. Thus the set of real number (R) is a proper subset of the complex number (C) i.e., R $\subset$ C.*
- ☞ *Complex number as an ordered pair : A complex number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (a,b). For a complex number to be uniquely specified, we need two real numbers in particular order.*

2.3 Algebraic Operations with Complex Numbers

Let two complex numbers $z_1 = a+ib$ and $z_2 = c+id$

Addition : $(a+ib)+(c+id)=(a+c)+i(b+d)$

Subtraction : $(a+ib)-(c+id)=(a-c)+i(b-d)$

Multiplication : $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

Division : $\dfrac{a+ib}{c+id}$ (when at least one of c and d is non-zero)

$$\frac{a+ib}{c+id}=\frac{(a+ib)}{(c+id)}\cdot\frac{(c-id)}{(c-id)} \quad \text{(Rationalization)}$$

$$\frac{a+ib}{c+id}=\frac{(ac+bd)}{c^2+d^2}+\frac{i(bc-ad)}{c^2+d^2}.$$

Properties of algebraic operations with complex numbers : Let z_1, z_2 and z_3 are any complex numbers then their algebraic operation satisfy following operations:

(i) Addition of complex numbers satisfies the commutative and associative properties

i.e., $z_1 + z_2 = z_2 + z_1$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(ii) Multiplication of complex number satisfies the commutative and associative properties.

i.e., $z_1 z_2 = z_2 z_1$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(iii) Multiplication of complex numbers is distributive over addition

i.e., $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$.

Note :

- $0 = 0 + 0i$ is the identity element for addition.
- $1 = 1 + 0i$ is the identity element for multiplication.
- The additive inverse of a complex number $z = a + ib$ is $-z$ (*i.e.* $-a - ib$).
- For every non-zero complex number *z*, the multiplicative inverse of *z* is $\frac{1}{z}$.

Example: 7 $\frac{1-2i}{2+i} + \frac{4-i}{3+2i} =$ **[Rajasthan PET 1987]**

(a) $\frac{24}{13} + \frac{10}{13}i$ (b) $\frac{24}{13} - \frac{10}{13}i$ (c) $\frac{10}{13} + \frac{24}{13}i$ (d) $\frac{10}{13} - \frac{24}{13}i$

Solution: (d) $\frac{1-2i}{2+i} + \frac{4-i}{3+2i} = = \frac{(1-2i)(3+2i)+(4-i)(2+i)}{(2+i)(3+2i)} = \frac{50-120i}{65} = \frac{10}{13} - \frac{24}{13}i.$

Example: 8 $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ **[Roorkee 1979; Rajasthan PET 1999]**

(a) $\frac{1}{2} + \frac{9}{2}i$ (b) $\frac{1}{2} - \frac{9}{2}i$ (c) $\frac{1}{4} - \frac{9}{4}i$ (d) $\frac{1}{4} + \frac{9}{4}i$

Solution: (d) $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right) = \left[\frac{1+2i}{1^2+2^2} + \frac{3-3i}{1^2+1^2}\right]\left[\frac{6-16+12i+8i}{2^2+4^2}\right] = \left(\frac{2+4i+15-15i}{10}\right)\left(\frac{-1+2i}{2}\right)$

$= \frac{(17-11i)(-1+2i)}{20} = \frac{5+45i}{20} = \frac{1}{4} + \frac{9}{4}i.$

Example: 9 The real value of θ for which the expression $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number, is

(a) $n\pi + \pi/2$ (b) $n\pi - \pi/2$ (c) $n\pi \pm \pi/2$ (d) None of these

Solution: (c) Given that $\frac{1+i\cos\theta}{1-2i\cos\theta} = \frac{(1+i\cos\theta)(1+2i\cos\theta)}{(1-2i\cos\theta)(1+2i\cos\theta)} = \left[\frac{(1-2\cos^2\theta)}{(1+4\cos^2\theta)}\right] + i\left[\frac{3\cos\theta}{1+4\cos^2\theta}\right]$

Since $\text{Im}(z) = 0$, then $3\cos\theta = 0 \Rightarrow \theta = n\pi \pm \pi/2$.

2.4 Equality of Two Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if their real parts and imaginary parts are separately equal.

i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

Thus , one complex equation is equivalent to two real equations.

Note : ❑ A complex number $z = x + iy = 0$ *iff* $x = 0, y = 0$.

❑ The complex number do not possess the property of order *i.e.,* $(a+ib) < (or) > (c+id)$ is not defined. For example, the statement $9+6i > 3+2i$ makes no sense.

Example: 10 Which of the following is correct

(a) $6+i > 8-i$ (b) $6+i > 4-i$ (c) $6+i > 4+2i$ (d) None of these

Solution: (d) Because, inequality is not applicable for a complex number.

Example: 11 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then **[MP PET 2000; IIT 1998]**

(a) $x = 3, y=1$ (b) $x=1, y=3$ (c) $x = 0, y=3$ (d) $x = 0, y=0$

Solution: (d) $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}$ Applying $C_2 \rightarrow C_2 + 3iC_3$

$\begin{vmatrix} 6i & 0 & 1 \\ 4 & 0 & -1 \\ 20 & 0 & i \end{vmatrix} = 0 = 0 + 0\,i$, Equating real and imaginary parts $x = 0$, $y = 0$

Example: 12 The real values of x and y for which the equation $(x^4 + 2xi) - (3x^2 + yi) = (3-5i) + (1+2yi)$ is satisfied, are **[Roorkee**

(a) $x = 2, y = 3$ (b) $x = -2, y = \frac{1}{3}$ (c) Both (a) and (b) (d) None of these

Solution: (c) Given equation $(x^4 + 2xi) - (3x^2 + yi) = (3-5i) + (1+2yi) \Rightarrow (x^4 - 3x^2) + i(2x - 3y) = 4 - 5i$

Equating real and imaginary parts, we get

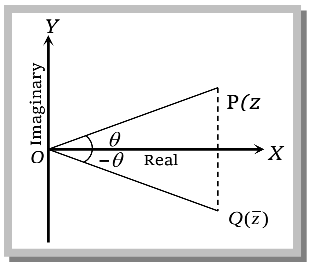
$x^4 - 3x^2 = 4$ .....(i) and $2x - 3y = -5$ .....(ii)

Form (i) and (ii), we get $x = \pm 2$ and $y = 3, \frac{1}{3}$.

Trick: Put $x = 2, y = 3$ and then $x = -2, y = \frac{1}{3}$, we see that they both satisfy the given equation.

2.5 Conjugate of a Complex Number

(1) **Conjugate complex number :** If there exists a complex number $z = a + ib, (a, b) \in R$, then its conjugate is defined as $\bar{z} = a - ib$.



Hence, we have $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) = \frac{z-\bar{z}}{2i}$. Geometrically, the conjugate of z is the reflection or point image of z in the real axis.

(2) **Properties of conjugate :** If z, z_1 and z_2 are existing complex numbers, then we have the following results:

(i) $\overline{(\bar{z})} = z$ (ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (iv) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, In general $\overline{z_1.z_2.z_3.....z_n} = \bar{z}_1.\bar{z}_2.\bar{z}_3.....\bar{z}_n$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}},\ z_2 \neq 0$ (vi) $(\bar{z})^n = \overline{(z^n)}$

(vii) $z + \bar{z} = 2\,\text{Re}(z) = 2\,\text{Re}(\bar{z}) =$ purely real (viii) $z - \bar{z} = 2i\,\text{Im}(z) =$ purely imaginary

(ix) $z\bar{z} =$ purely real (x) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,\text{Re}(z_1\bar{z}_2) = 2\,\text{Re}(\bar{z}_1 z_2)$

(xi) $z - \bar{z} = 0$ *i.e.,* $z = \bar{z} \Leftrightarrow z$ is purely real *i.e.,* $\text{Im}(z) = 0$

(xii) $z + \bar{z} = 0$ *i.e.,* $z = -\bar{z} \Leftrightarrow$ either $z = 0$ or z is purely imaginary *i.e.,* $\text{Re}(z) = 0$

(xiii) $z_1 = z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$ (xiv) $z = 0 \Leftrightarrow \bar{z} = 0$

(xv) $z\bar{z} = 0 \Leftrightarrow z = 0$ (xvi) If $w = f(z)$ then $\bar{w} = f(\bar{z})$ (xvii) $\overline{re^{i\theta}} = re^{-i\theta}$

Important Tips

- *Complex conjugate is obtained by just changing the sign of i.*
- *Conjugate of* $i = -i$
- *Conjugate of* $iz = -i\bar{z}$
- $(z_1 + z_2)$ *and* $(z_1 . z_2)$ *real* $\Leftrightarrow z_1 = \bar{z}_2$ *or* $z_2 = \bar{z}_1$
- $\overline{z_1\bar{z}_2} = \bar{z}_1 z_2$

(3) **Reciprocal of a complex number :** For an existing non-zero complex number $z = a + ib$, the reciprocal is given by $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$ *i.e.,* $z^{-1} = \frac{1}{a+ib} \Rightarrow \frac{a-ib}{a^2+b^2} = \frac{\text{Re}(z)}{|z|^2} + \frac{i[-\text{Im}(z)]}{|z|^2} = \frac{\bar{z}}{|z|^2}$.

Example: 13 If the conjugate of $(x+iy)(1-2i)$ be $1+i$, then **[MP PET 1996]**

(a) $x = \frac{1}{5}$ (b) $y = \frac{3}{5}$ (c) $x + iy = \frac{1-i}{1-2i}$ (d) $x - iy = \frac{1-i}{1+2i}$

Solution: (c) Given that $\overline{(x+iy)(1-2i)} = 1+i \Rightarrow x - iy = \frac{1+i}{1+2i} \Rightarrow x + iy = \frac{1-i}{1-2i}$

Example: 14 For the complex number z, one from $z + \bar{z}$ and $z\bar{z}$ is

(a) A real number (b) An imaginary number
(c) Both are real numbers (d) Both are imaginary numbers

Solution: (c) Here $z + \bar{z} = (x+iy) + (x-iy) = 2x$ (Real) and $z\bar{z} = (x+iy)(x-iy) = x^2 + y^2$ (Real).

Example: 15 The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for **[IIT 1988]**

(a) $x = n\pi$ (b) $x = \left(n + \frac{1}{2}\right)\pi$ (c) $x = 0$ (d) No value of x

Solution: (d) $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other if $\sin x = \cos x$ and $\cos 2x = \sin 2x$

or $\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$ (i) and $\tan 2x = 1 \Rightarrow 2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...$ or $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, ...$ (ii) There exists no value of x common in (i) and (ii). Therefore there is no value of x for which the given complex numbers are conjugate.

Example: 16 The conjugate of complex number $\frac{2-3i}{4-i}$ is **[MP PET 2004]**

(a) $\frac{3i}{4}$ (b) $\frac{11+10i}{17}$ (c) $\frac{11-10i}{17}$ (d) $\frac{2+3i}{4i}$

Solution: (b) $\frac{2-3i}{4-i}=\frac{(2-3i)(4+i)}{(4-i)(4+i)}=\frac{8+3-12i+2i}{16+1}=\frac{11-10i}{17}$ $\Rightarrow$ Conjugate $=\frac{11+10i}{17}$.

Example: 17 The real part of $(1-\cos\theta+2i\sin\theta)^{-1}$ is **[Karnataka CET 2001; IIT 1978, 86]**

(a) $\frac{1}{3+5\cos\theta}$ (b) $\frac{1}{5-3\cos\theta}$ (c) $\frac{1}{3-5\cos\theta}$ (d) $\frac{1}{5+3\cos\theta}$

Solution: (c) $\{(1-\cos\theta)+i.2\sin\theta\}^{-1}=\left\{2\sin^2\frac{\theta}{2}+i.4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^{-1}=\left(2\sin\frac{\theta}{2}\right)^{-1}\left\{\sin\frac{\theta}{2}+i.2\cos\frac{\theta}{2}\right\}^{-1}$

$$=\left(2\sin\frac{\theta}{2}\right)^{-1}.\frac{1}{\sin\frac{\theta}{2}+i.2\cos\frac{\theta}{2}}\times\frac{\sin\frac{\theta}{2}-i.2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}-i.2\cos\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}-i.2\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(\sin^2\frac{\theta}{2}+4\cos^2\frac{\theta}{2}\right)}$$

Hence, real part $=\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1+3\cos^2\frac{\theta}{2}\right)}=\frac{1}{2\left(1+3\cos^2\frac{\theta}{2}\right)}=\frac{1}{5+3\cos\theta}$.

Example: 18 The reciprocal of $3+\sqrt{7}i$ is

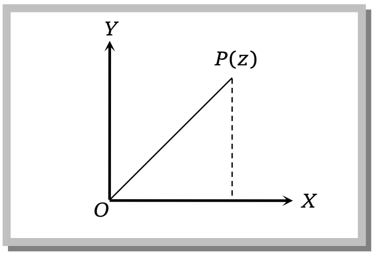
(a) $\frac{3}{4}-\frac{\sqrt{7}}{4}i$ (b) $3-\sqrt{7}i$ (c) $\frac{3}{16}-\frac{\sqrt{7}}{16}i$ (d) $\sqrt{7}+3i$

Solution: (c) $\frac{1}{3+\sqrt{7}i}=\frac{1}{3+\sqrt{7}i}.\frac{3-\sqrt{7}i}{3-\sqrt{7}i}=\frac{3-\sqrt{7}i}{9+7}=\frac{3-\sqrt{7}i}{16}=\frac{3}{16}-\frac{\sqrt{7}}{16}i.$

2.6 Modulus of a Complex Number

Modulus of a complex number $z=a+ib$ is defined by a positive real number given by $|z|=\sqrt{a^2+b^2}$, where *a, b* real numbers. Geometrically $|z|$ represents the distance of point *P* (represented by *z*) from the origin,

i.e. $|z| = OP$.



If $|z| = 0$, then *z* is known as zero modular complex number and is used to represent the origin of reference plane.

If $|z| = 1$ the corresponding complex number is known as **unimodular complex number.** Clearly *z* lies on a circle of unit radius having centre (0, 0).

Note : ❑ In the set *C* of all complex numbers, the order relation is not defined. As such $z_1 > z_2 >$ or $z_1 < z_2$ has no meaning. But $|z_1|>|z_2|$ or $|z_1|<|z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.

Properties of modulus

(i) $|z|\geq 0 \Rightarrow |z|=0$ *iff* $z=0$ and $|z|>0$ *iff* $z\neq 0$.

(ii) $-|z|\leq \text{Re}(z)\leq |z|$ and $-|z|\leq \text{Im}(z)\leq |z|$

(iii) $|z|=|\bar{z}|=|-z|=|-\bar{z}|=|zi|$

(iv) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(v) $|z_1 z_2| = |z_1||z_2|$. In general $|z_1 z_2 z_3 ...... z_n| = |z_1||z_2||z_3|....|z_n|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$

(vii) $|z^n| = |z|^n, n \in N$

(viii) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2) = |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$ or $|z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1\bar{z}_2)$

(ix) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary or $\text{Re}\left(\frac{z_1}{z_2}\right) = 0$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$ (Law of parallelogram)

(xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)$, where $a, b \in R$.

Important Tips

☞ *Modulus of every complex number is a non-negative real number.*

☞ $|z| = 0$ *iff* $z = 0$ *i.e.,* $\text{Re}(z) = \text{Im}(z) = 0$

☞ $|z| \geq |\text{Re}(z)| \geq \text{Re}(z)$ *and* $|z| \geq |\text{Im}(z)| \geq \text{Im}(z)$

☞ $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$

☞ $\left|\frac{z}{\bar{z}}\right| = 1$

☞ $\frac{z}{|\bar{z}|}$ *is always a unimodular complex number if* $z \neq 0$

☞ $\frac{z}{\bar{z}}$ *is always a unimodular complex number if* $z \neq 0$

☞ $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$

☞ $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Thus $|z_1| + |z_2|$ *is the greatest possible value of* $|z_1 + z_2|$ *and* $||z_1| - |z_2||$ *is the least possible value of* $|z_1 + z_2|$

☞ *If* $\left|z + \frac{1}{z}\right| = a$, *the greatest and least values of* $|z|$ *are respectively* $\frac{a + \sqrt{a^2 + 4}}{2}$ *and* $\frac{-a + \sqrt{a^2 + 4}}{2}$

☞ $|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_1 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2|$

Example: 19 $\left|(1+i)\frac{(2+i)}{(3+i)}\right| =$ **[MP PET 1995, 99; Rajasthan PET 1998]**

(a) – 1/2 (b) 1/2 (c) 1 (d) – 1

Solution (c) $z = \frac{(1+i)(2+i)}{(3+i)} = \frac{1+3i}{3+i} \times \frac{3-i}{3-i} = \frac{3+4i}{5} \Rightarrow |z| = 1$

Trick : $|z| = \frac{|z_1||z_2|}{|z_3|} = \frac{\sqrt{2}.\sqrt{5}}{\sqrt{10}} = 1$

Example: 20 If α and β are different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ is equal to

(a) 0 (b) 1/2 (c) 1 (d) 2

Solution (c) $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{\beta\bar{\beta} - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{\beta(\bar{\beta} - \bar{\alpha})}\right|, = \frac{1}{|\beta|}\left|\frac{\beta - \alpha}{\overline{(\beta - \alpha)}}\right| = \frac{1}{|\beta|} = 1$ $\{\because |z| = |\bar{z}|\}$

Example: 21 For any complex number z, maximum value of $|z|-|z-1|$ is

(a) 0 (b) 1 (c) 3/2 (d) None of these

Solution (b) We know that $|z_1-z_2|\geq |z_1|-|z_2|$

$\therefore |z|-|z-1|\leq |z-(z-1)|$ or $|z|-|z-1|\leq 1$, $\therefore$ Maximum value of $|z|-|z-1|$ is 1.

Example 22 If $z=x+iy$ and $iz^2-\bar{z}=0$, then $|z|$ is equal to **[Bihar CEE 1998]**

(a) 1 (b) 0 or 1 (c) 1 or 2 (d) 2

Solution: (b) $iz^2=\bar{z}\Rightarrow |iz^2|=|\bar{z}|\Rightarrow |z|^2=|z|\Rightarrow |z|(|z|-1)=0\Rightarrow |z|=0$ or $|z|=1$

Example: 23 For $x_1,x_2,y_1,y_2\in R$, if $0<x_1<x_2, y_1=y_2$ and $z_1=x_1+iy_1, z_2=x_2+iy_2$, and $z_3=\frac{1}{2}(z_1+z_2)$, then z_1,z_2 and z_3 satisfy **[Roorkee 1991]**

(a) $|z_1|=|z_2|=|z_3|$ (b) $|z_1|<|z_2|<|z_3|$ (c) $|z_1|>|z_2|>|z_3|$ (d) $|z_1|<|z_3|<|z_2|$

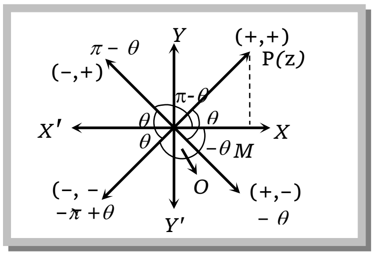
Solution: (d) $0<x_1<x_2, y_1=y_2$ (Given)

$$|z_1|=\sqrt{x_1^2+y_1^2}, |z_2|=\sqrt{x_2^2+y_2^2}\Rightarrow |z_2|>|z_1|\Rightarrow |z_3|=\frac{|z_1+z_2|}{2}=\sqrt{\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2}$$

$\sqrt{\left(\frac{x_1+x_2}{2}\right)^2+y_1^2}<|z_2|>|z_1|$. Hence, $|z_1|<|z_3|<|z_2|$

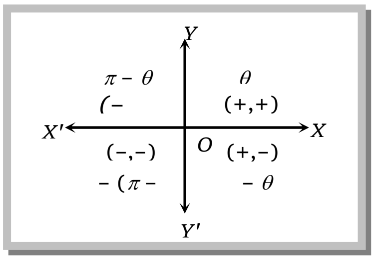
2.7 Argument of a Complex Number

Let $z=a+ib$ be any complex number. If this complex number is represented geometrically by a point P, then the angle made by the line OP with real axis is known as argument or amplitude of z and is expressed as

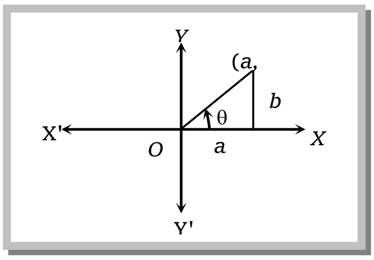


$arg\ (z)=\theta=\tan^{-1}\left(\frac{b}{a}\right)$, $\theta=\angle POM$. Also, argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi+\theta$, where $n\in I$.

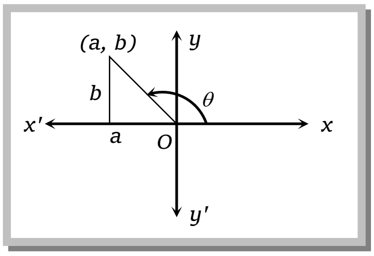
(1) **Principal value of *arg* (z) :** The value θ of the argument, which satisfies the inequality $-\pi<\theta\leq\pi$ is called the principal value of argument. Principal values of argument z will be $\theta, \pi-\theta, -\pi+\theta$ and $-\theta$ according as the point z lies in the 1st, 2nd, 3rd and 4th quadrants respectively, where $\theta=\tan^{-1}\left|\frac{b}{a}\right|=\alpha$ (acute angle). Principal value of argument of any complex number lies between $-\pi<\theta\leq\pi$.



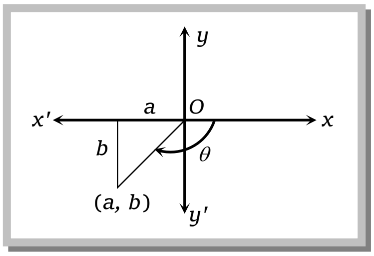
(i) $a,b\in$ First quadrant $a>0, b>0$. $arg\ (z)=\theta=\tan^{-1}\left(\frac{b}{a}\right)$. It is an acute angle and positive.



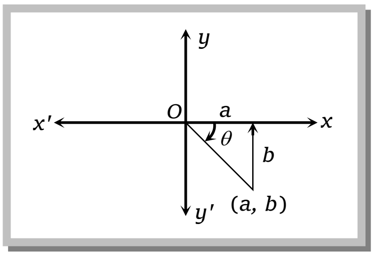
(ii) $(a,b)\in$ Second quadrant, $a<0, b>0$, *arg* $(z)=\theta=\pi-\tan^{-1}\left(\frac{b}{|a|}\right)$. It is an obtuse angle and positive.



(iii) $(a,b)\in$ Third quadrant $a<0, b<0$, *arg* $(z)=\theta=-\pi+\tan^{-1}\left(\frac{b}{a}\right)$. It is an obtuse angle and negative.



(iv) $(a,b) \in$ Fourth quadrant $a > 0, b < 0$, *arg* $(z) = \theta = -\tan^{-1}\left(\frac{|b|}{a}\right)$. It is an acute angle and negative.



Quadrant	*x*	*y*	*arg(z)*	Interval of θ
I	+	+	θ	$0 < \theta < \pi/2$
II	–	+	$\pi - \theta$	$\pi/2 < \theta < \pi$
III	–	–	$-(\pi - \theta)$	$-\pi < \theta < -\pi/2$
IV	+	–	$-\theta$	$-\pi/2 < \theta < 0$

Note : ❑ Argument of the complex number 0 is not defined.

❑ Principal value of argument of a purely real number is 0 if the real number is positive and is π if the real number is negative.

❑ Principal value of argument of a purely imaginary number is $\pi/2$ if the imaginary part is positive and is $-\pi/2$ if the imaginary part is negative.

(2) **Properties of arguments**

(i) $arg\ (z_1 z_2) = arg(z_1) + arg(z_2) + 2k\pi$, $(k = 0$ or 1 or – 1)

In general $arg\,(z_1 z_2 z_3 ....z_n) = arg(z_1) + arg(z_2) + arg(z_3) + ... + arg(z_n) + 2k\pi$, $(k = 0$ or 1 or $-1)$

(ii) $arg\,(z_1 \bar{z}_2) = arg(z_1) - arg(z_2)$

(iii) $arg\left(\frac{z_1}{z_2}\right) = arg\,z_1 - arg\,z_2 + 2k\pi$, $(k = 0$ or 1 or – 1)

(iv) $arg\left(\frac{z}{\bar{z}}\right) = 2arg\,z + 2k\pi$, $(k = 0$ or 1 or – 1)

(v) $arg(z^n) = n\,arg\,z + 2k\pi$, $(k = 0$ or 1 or – 1)

(vi) If $arg\left(\frac{z_2}{z_1}\right) = \theta$, then $arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

(vii) $arg\,\bar{z} = -arg\,z = arg\frac{1}{z}$

(viii) $arg\,(z - \bar{z}) = \pm\pi/2$

(ix) $arg\,(-z) = arg\,(z) \pm \pi$

(x) $arg\,(z) + arg(\bar{z}) = 0$ or $arg(z) = -arg(\bar{z})$

(xi) $arg(z) - arg(\bar{z}) = \pm\pi$

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$, where $\theta_1 = arg(z_1)$ and $\theta_2 = arg(z_2)$

Note : ❑ Proper value of k must be chosen so that R.H.S. of (i), (ii), (iii) and (iv) lies in $(-\pi, \pi)$

❑ The property of argument is same as the property of logarithm.

If *arg* (*z*) lies between $-\pi$ and $\pi(\pi$ inclusive), then this value itself is the principal value of *arg* (*z*). If not, see whether *arg* (*z*) $>\pi$ or $\leq -\pi$. If $arg(z) > \pi$, go on subtracting 2π until it lies between $-\pi$ and $\pi(\pi$ inclusive). The value thus obtained will be the principal value of *arg* (*z*).

❑ The general value of $arg(\bar{z})$ is $2n\pi - arg(z)$.

Important Tips

- *If* $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ *and arg* z_1 = *arg* z_2.
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ *arg* $(z_1) =$ *arg* (z_2) *i.e., z_1 and z_2 are parallel.*
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ *arg* $(z_1) -$ *arg* $(z_2) = 2n\pi$, *where n is some integer.*
- $|z_1 - z_2| = ||z_1| - |z_2|| \Leftrightarrow arg(z_1) - arg(z_2) = 2n\pi$, *where n is some integer.*
- $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow$ *arg* $(z_1) -$ *arg* $(z_2) = \pi/2$.
- *If* $|z_1| \leq 1, |z_2| \leq 1$ *then* *(i)* $|z_1 + z_2|^2 \leq (|z_1| - |z_2|)^2 + (arg(z_1) - arg(z_2))^2$ *(ii)* $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - (arg(z_1) - arg(z_2))^2$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.
- $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$.
- *If* $|z_1| = |z_2|$ *and* $amp(z_1) + amp(z_2) = 0$, *then* $z_1 + z_2$ *are conjugate complex numbers of each other.*
- $z \neq 0$, $amp(z + \bar{z}) = 0$ *or* $\pi; amp(z\bar{z}) = 0; amp(z - \bar{z}) = \pm\pi/2$.
- *arg* $(1) = 0$, *arg* $(-1) = \pi$; *arg* $(i) = \pi/2$, *arg* $(-i) = -\pi/2$.
- *arg* $(z) = \frac{\pi}{4} \Rightarrow \text{Re}(z) = \text{Im}(z)$.
- *Amplitude of complex number in I and II quadrant is always positive and in III[rd] and IV[th] quadrant is always negative.*
- *If a complex number multiplied by i (Iota) its amplitude will be increased by* $\pi/2$ *and will be decreased by* $\pi/2$, *if multiplied by –i, i.e.* $arg(iz) = \frac{\pi}{2} + arg(z)$ *and* $arg(-iz) = arg(z) - \frac{\pi}{2}$.

Complex number	**Value of argument**
+*ve* Re (*z*)	0
–*ve* Re (*z*)	π
+*ve* Im (*z*)	$\pi/2$
–*ve* Im (*z*)	$3\pi/2$ *or* $-\pi/2$
– (z)	$\|\theta \pm \pi\|$, *if* θ *is* $-ve$ *and* $+ve$ respectively
(iz)	$\left\{\frac{\pi}{2} + arg(z)\right\}$
–(iz)	$\left\{arg(z) - \frac{\pi}{2}\right\}$

(z^n)	$n.\ arg\ (z)$
$(z_1.z_2)$	$arg\ (z_1) + arg\ (z_2)$
$\left(\frac{z_1}{z_2}\right)$	$arg\ (z_1) - arg\ (z_2)$

Example: 24 Amplitude of $\left(\frac{1-i}{1+i}\right)$ is **[Rajasthan PET 1996]**

(a) $\frac{-\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

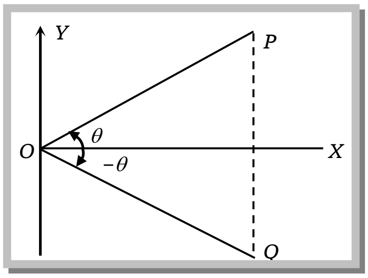
Solution: (a) $z = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = -i$

$\Rightarrow \theta = \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$ (Since z lies on negative imaginary axis)

Example: 25 If $|z_1| = |z_2|$ and $amp\ z_1 + amp\ z_2 = 0$, then **[MP PET 1999]**

(a) $z_1 = z_2$ (b) $\bar{z}_1 = z_2$ (c) $z_1 + z_2 = 0$ (d) $\bar{z}_1 = \bar{z}_2$

Solution: (b) Let $|z_1| = OP, |z_2| = OQ$



Since $amp\ (z_1) = \theta \Rightarrow amp\ (z_2) = -\theta$

$\because$ Q is point image of P

$\therefore$ $\overline{z_1} = z_2$

Trick : $arg\, z + arg\, \bar{z} = 0$, $\therefore \bar{z}_1$ must be equal to z_2.

Example: 26 Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and $arg\ zw = \pi$, then $arg\ z$ equals **[AIEEE 2004]**

(a) $5\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) $\pi/4$

Solution: (d) $\bar{z} + i\bar{w} = 0$

$\therefore z = iw \Rightarrow \theta + (\pi/2 + \theta) = \pi$, $\therefore \theta = \pi/4$.

Example: 27 The amplitude of $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$ **[Karnataka CET 2003]**

(a) $\pi/5$ (b) $2\pi/5$ (c) $\pi/10$ (d) $\pi/15$

Solution: (c) $\sin\frac{\pi}{5} + i(1 - \cos\frac{\pi}{5}) = 2\sin\frac{\pi}{10}.\cos\frac{\pi}{10} + i\,2\sin^2\frac{\pi}{10} = 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

For amplitude, $\tan\theta = \frac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}} = \tan\frac{\pi}{10} \Rightarrow \theta = \frac{\pi}{10}$.

Example: 28 If $|z| = 4$ and $arg\ z = \frac{5\pi}{6}$, then $z =$ **[MP PET 1987]**

(a) $2\sqrt{3} - 2i$ (b) $2\sqrt{3} + 2i$ (c) $-2\sqrt{3} + 2i$ (d) $-\sqrt{3} + i$

Solution: [c] $|z| = 4$ and $arg\ z = \frac{5\pi}{6} = 150^\circ$, Let $z = x + iy$, then $|z| = r = \sqrt{x^2 + y^2} = 4$ and $\theta = \frac{5\pi}{6} = 150°$

$\therefore x = r \cos\theta = 4\cos 150° = -2\sqrt{3}$ and $y = r\sin\theta = 4\sin 150° = 4 \cdot \frac{1}{2} = 2$.

$\therefore z = x + iy = -2\sqrt{3} + 2i$.

Trick: Since $arg\ z = \frac{5\pi}{6} = 150°$, here the complex number must lie in second quadrant, so (a) and (b) rejected. Also $|z| = 4$, which satisfies (c) only.

Example: 29 If z and ω are to non-zero complex numbers such that $|z\omega| = 1$ and $arg\ (z) - arg\ (\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to
[AIEEE 2003]

(a) 1 (b) – 1 (c) i (d) – i

Solution: (d) $|z||\omega| = 1$ .....(i) and $arg\left(\frac{z}{\omega}\right) = \frac{\pi}{2} \Rightarrow \frac{z}{\omega} = i \Rightarrow \left|\frac{z}{\omega}\right| = 1$ .....(ii)

From equation (i) and (ii),

$|z| = |\omega| = 1$ and $\frac{z}{\omega} + \frac{\bar{z}}{\bar{\omega}} = 0$; $z\bar{\omega} + \bar{z}\omega = 0 \Rightarrow \bar{z}\omega = -z\bar{\omega} = \frac{-z}{\omega}\bar{\omega}\omega \Rightarrow \bar{z}\omega = -i|\omega|^2 = -i$.

2.8 Square Root of a Complex Number

Let $a + ib$ be a complex number such that $\sqrt{a + ib} = x + iy$, where x and y are real numbers. Then

$\sqrt{a + ib} = x + iy \Rightarrow a + ib = (x + iy)^2 \Rightarrow a + ib = (x^2 - y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = a$ .....(i)

and $2xy = b$ .....(ii) [On equating real and imaginary parts]

Solving, $x = \pm\sqrt{\left(\frac{\sqrt{a^2 + b^2} + a}{2}\right)}$ and $y = \pm\sqrt{\left(\frac{\sqrt{a^2 + b^2} - a}{2}\right)}$

$$\therefore \sqrt{a + ib} = \pm\left[\sqrt{\left(\frac{\sqrt{a^2 + b^2} + a}{2}\right)} - i\sqrt{\left(\frac{\sqrt{a^2 + b^2} - a}{2}\right)}\right]$$

Therefore $\sqrt{a + ib} = \pm\left[\sqrt{\frac{|z| + a}{2}} + i\sqrt{\frac{|z| - a}{2}}\right]$ for $b>0$ $= \pm\left[\sqrt{\frac{|z| + a}{2}} - i\sqrt{\frac{|z| - a}{2}}\right]$ for $b<0$.

Note : ❑ To find the square root of $a - ib$, replace i by – i in the above results.

❑ The square root of i is $\pm\left(\frac{1 + i}{\sqrt{2}}\right)$, [Here b = 1]

❑ The square root of – i is $\pm\left(\frac{1 - i}{\sqrt{2}}\right)$, [Here b = –1]

Alternative method for finding the square root

(i) If the imaginary part is not even then multiply and divide the given complex number by 2. *e.g.* z = 8 – 15i here imaginary part is not even so write $z = \frac{1}{2}$ (16 – 30i) and let $a+ib$ = 16 – 30 i .

(ii) Now divide the numerical value of imaginary part of $a+ib$ by 2 and let quotient be P and find all possible two factors of the number P thus obtained and take that pair in which difference of squares of the numbers is equal to the real part of $a+ib$ *e.g.,* here numerical value of Im(16 – 30i) is 30. Now 30 = 2×15 . All possible way to express 15 as a product of two are 1×15 , 3×5 etc. here 5^2-3^2 = 16 = Re (16– 30i) so we will take 5, 3.

(iii) Take i with the smaller or the greater factor according as the real part of a + ib is positive or negative and if real part is zero then take equal factors of P and associate i with any one of them *e.g.,* Re(16 – 30i) > 0, we will take i with 3. Now complete the square and write down the square root of z.

e.g., $z=\frac{1}{2}[16-30i]=\frac{1}{2}\left[5^2+(3i)^2-2\times 5\times 3i\right]=\frac{1}{2}[5-3i]^2 \Rightarrow \sqrt{z}=\pm\frac{1}{\sqrt{2}}(5-3i)$

Example: 30 The square root of $3-4i$ are

(a) $\pm(2-i)$ (b) $\pm(2+i)$ (c) $\pm(\sqrt{3}-2i)$ (d) $\pm(\sqrt{3}+2i)$

Solution: (a) $|z|=5$,$\therefore \sqrt{3-4i}=\pm\left(\sqrt{\frac{5+3}{2}}-i\sqrt{\frac{5-3}{2}}\right)=\pm\ (2-i)$

Example: 31 $\sqrt{2i}$ equals **[Roorkee 1989]**

(a) $1+i$ (b) $1-i$ (c) $-\sqrt{2}i$ (d) None of these

Solution: (a) $z=2i=a+bi \Rightarrow a=0, b=2, |z|=2$

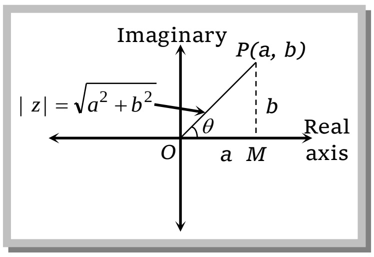
$\therefore \sqrt{z}=\pm\left(\sqrt{\frac{2+0}{2}}+i\sqrt{\frac{2-0}{2}}\right)=\pm(1+i)$

Trick: It is always better to square the options rather than finding the square root.

2.9 Representation of Complex Number

A complex number can be represented in the following from:

(1) **Geometrical representation (Cartesian representation):** The complex number $z=a+ib=(a,b)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called argand plane or argand diagram or complex plane or gaussian



plane.

Note : ❑ Distance of any complex number from the origin is called the modules of complex number and is denoted by $|z|$, *i.e.,* $|z| = \sqrt{a^2 + b^2}$

❑ Angle of any complex number with positive direction of x– axis is called amplitude or argument of z. *i.e.,* $amp\,(z) = arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$

(2) **Trigonometrical (Polar) representation :** In Δ *OPM*, let $OP = r$, then $a = r\cos\theta$ and $b = r\sin\theta$. Hence z can be expressed as $z = r(\cos\theta + i\sin\theta)$

where $r = |z|$ and θ = principal value of argument of z.

For general values of the argument $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$

Note : ❑ Sometimes $(\cos\theta + i\sin\theta)$ is written in short as $cis\theta$.

(3) **Vector representation :** If P is the point (a, b) on the argand plane corresponding to the complex number $z = a + ib$.

Then $\overrightarrow{OP} = a\hat{i} + b\hat{j}$, $\therefore |\overrightarrow{OP}| = \sqrt{a^2 + b^2} = |z|$ and *arg z* = direction of the vector $\overrightarrow{OP} = \tan^{-1}\left(\frac{b}{a}\right)$

Therefore, complex number z can also be represented by $\overrightarrow{OP}$.

(4) **Eulerian representation (Exponential form) :** Since we have $e^{i\theta} = \cos\theta + i\sin\theta$ and thus z can be expressed as $z = re^{i\theta}$, where $|z| = r$ and $\theta = arg\,(z)$

Note : ❑ $e^{-i\theta} = (\cos\theta - i\sin\theta)$

❑ $e^{i\theta} + e^{-i\theta} = 2\cos\theta, e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Example: 32 $\frac{1+7i}{(2-i)^2} =$ **[Roorkee 1998]**

(a) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (b) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (c) $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (d) None of these

Solution: (a) $\frac{1+7i}{(2-i)^2} = \frac{(1+7i)(3+4i)}{(3-4i)(3+4i)} = \frac{-25+25i}{25} = -1+i$

Let $z = x + iy = -1 + i$, $\therefore x = r\cos\theta = -1$ and $y = r\sin\theta = 1$

$\therefore \theta = \frac{3\pi}{4}$ and $r = \sqrt{2}$, Thus $\frac{1+7i}{(2-i)^2} = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

Alternative method: $\left|\frac{1+7i}{(2-i)^2}\right| = \left|\frac{1+7i}{3-4i}\right| = \sqrt{2}$ and *arg* $\left(\frac{1+7i}{3-4i}\right) = \tan^{-1}7 - \tan^{-1}\left(\frac{-4}{3}\right) = \tan^{-1}7 + \tan^{-1}\frac{4}{3} = \frac{3\pi}{4}$

$\therefore \frac{1+7i}{(2-i)^2} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Example: 33 If $-1 + \sqrt{-3} = re^{i\theta}$, then θ is equal to **[Rajasthan PET 1989; MP PMT 1999]**

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{2\pi}{3}$

Solution: (c) Here $-1 + \sqrt{-3} = re^{i\theta} \Rightarrow -1 + i\sqrt{3} = re^{i\theta} = r\cos\theta + ir\sin\theta$

Equating real and imaginary part, we get $r\cos\theta = -1$ and $r\sin\theta = \sqrt{3}$

Hence $\tan\theta = -\sqrt{3} \Rightarrow \tan\theta = \tan\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$.

Example: 34 Real part of $e^{e^{i\theta}}$ is **[Rajasthan PET 1995]**

(a) $e^{\cos\theta}[\cos(\sin\theta)]$ (b) $e^{\cos\theta}[\cos(\cos\theta)]$ (c) $e^{\sin\theta}[\sin(\cos\theta)]$ (d) $e^{\sin\theta}[\sin(\sin\theta)]$

Solution: (a) $e^{e^{i\theta}} = e^{(\cos\theta + i\sin\theta)} = e^{\cos\theta}.e^{i\sin\theta} = e^{\cos\theta}[\cos(i\sin\theta) + i\sin(\sin\theta)]$

$\therefore$ Real part of $e^{e^{i\theta}}$ is $e^{\cos\theta}[\cos(\sin\theta)]$.

Example: 35 If $\frac{1}{x} + x = 2\cos\theta$, then $x^n + \frac{1}{x^n}$ is equal to **[UPSEAT 2001]**

(a) $2\cos n\theta$ (b) $2\sin n\theta$ (c) $\cos n\theta$ (d) $\sin n\theta$

Solution: (a) Let $x=\cos\theta+i\sin\theta=e^{i\theta}$ then $x^n+\frac{1}{x^n}=e^{in\theta}+\frac{1}{e^{in\theta}}=e^{in\theta}+e^{-in\theta}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta=2\cos n\theta.$

2.10 Logarithm of a Complex Number

Let $z=x+iy$ and

$\log_e(x+iy)=a+ib$ .....(i)

$x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta}$ .....(ii)

then $x=r\cos\theta, y=r\sin\theta$, clearly $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

From equation (ii), $\log(x+iy)=\log_e(re^{i\theta})=\log r+\log_e e^{i\theta}=\log_e r+i\theta=\log_e\sqrt{(x^2+y^2)}+i\tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\log_e(z)=\log_e|z|+i\,amp\,z}$$

Obviously, the general value is $Log(z)=\log_e(z)+2\pi ni \quad (-\pi<amp(z)<\pi)$

Example: 36 i^i is equal to **[EAMCET 1995]**

(a) $e^{\pi/2}$ (b) $e^{-\pi/2}$ (c) $-\pi/2$ (d) None of these

Solution: (b) Let $A=i^i$ then $\log A=\log i^i=i\log i \Rightarrow \log A=i\log(0+i) \Rightarrow \log A=i[\log 1+i\pi/2]$ $(\because |i|=1$ and $arg(i)=\pi/2)$

$\log A=i[0+i\pi/2]=-\pi/2 \Rightarrow A=e^{-\pi/2}$.

Example: 37 $i\log\left(\frac{x-i}{x+i}\right)$ is equal to **[Rajasthan PET 2000]**

(a) $\pi+2\tan^{-1}x$ (b) $\pi-2\tan^{-1}x$ (c) $-\pi+2\tan^{-1}x$ (d) $-\pi-2\tan^{-1}x$

Solution: (b) Let $z=i\log\left(\frac{x-i}{x+i}\right) \Rightarrow \frac{z}{i}=\log\left(\frac{x-i}{x+i}\right) \Rightarrow \frac{z}{i}=\log\left[\frac{x-i}{x+i}\times\frac{x-i}{x-i}\right]=\log\left[\frac{x^2-1-2ix}{x^2+1}\right]$

$\Rightarrow \frac{z}{i}=\log\left[\frac{x^2-1}{x^2+1}-i\frac{2x}{x^2+1}\right]$ .....(i)

$\because \log(a+ib)=\log(re^{i\theta})=\log r+i\theta=\log\sqrt{a^2+b^2}+i\tan^{-1}\left(\frac{b}{a}\right)$

Hence $\frac{z}{i}=\log\sqrt{\left(\frac{x^2-1}{x^2+1}\right)^2+\left(\frac{-2x}{x^2+1}\right)^2}+i\tan^{-1}\left(\frac{-2x}{x^2-1}\right)$ (By equation (i))

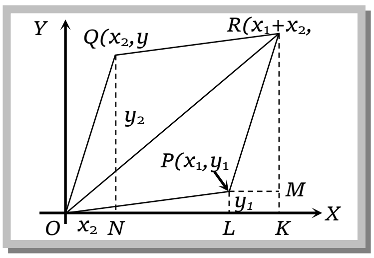
$\frac{z}{i}=\log\frac{\sqrt{x^4+1-2x^2+4x^2}}{(x^2+1)^2}+i\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\log 1+i(2\tan^{-1}x)=0+i(\tan^{-1}x)$

$\therefore z=i^2 2\tan^{-1}x=-2\tan^{-1}x=\pi-2\tan^{-1}x$.

2.11 Geometry of Complex Numbers

(1) **Geometrical representation of algebraic operations on complex numbers**

(i) **Sum:** Let the complex numbers $z_1=x_1+iy_1=(x_1,y_1)$ and $z_2=x_2+iy_2=(x_2,y_2)$ be represented by the points P and Q on the argand plane.



Then sum of z_1 and z_2 *i.e.,* z_1+z_2 is represented by the point R.

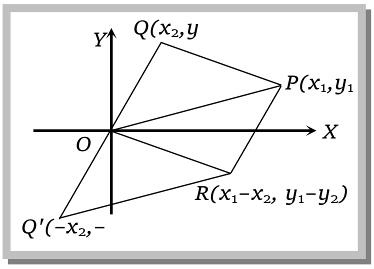
Complex number z can be represented by $\overrightarrow{OR}$.

$=(x_1+x_2)+i(y_1+y_2)=(x_1+iy_1)+(x_2+iy_2)=(z_1+z_2)=(x_1,y_1)+(x_2,y_2)$

In vector notation, we have $z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$

(ii) **Difference :** We first represent $-z_2$ by Q', so that QQ' is bisected at O.

The point R represents the difference $z_1 - z_2$.

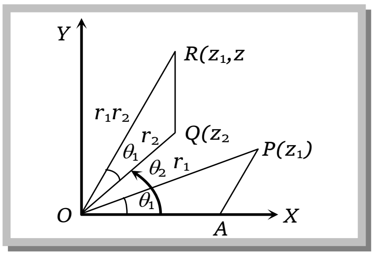


In parallelogram $ORPQ$, $\overrightarrow{OR} = \overrightarrow{QP}$

We have in vectorial notation $z_1 - z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{QO}$

$$= \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} = \overrightarrow{QP}.$$

(iii) **Product :** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$



$\therefore |z_1| = r_1$ and *arg* $(z_1) = \theta_1$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore |z_2| = r_2$ and *arg* $(z_2) = \theta_2$

Then, $z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$$

$\therefore \quad |z_1 z_2| = r_1 r_2$ and *arg* $(z_1 z_2) = \theta_1 + \theta_2$

R is the point representing product of complex numbers z_1 and z_2.

Important Tips

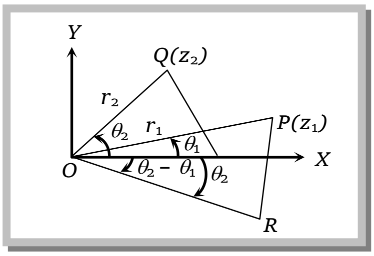
☞ *Multiplication of i : Since* $z = r(\cos\theta + i\sin\theta)$ *and* $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ *then* $iz = \left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right]$

Hence, multiplication of z with i then vector for z rotates a right angle in the positive sense.

i.e., To multiply a vector by –1 is to turn it through two right angles.

i.e., To multiply a vector by $(\cos\theta + i\sin\theta)$ *is to turn it through the angle* θ *in the positive sense.*

(iv) **Division :** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$



$\therefore |z_1| = r_1$ and *arg* $(z_1) = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

$\therefore |z_2| = r_2$ and *arg* $(z_2) = \theta_2$

Then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\dfrac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \qquad (z_2 \neq 0,\ r_2 \neq 0)$

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}, arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

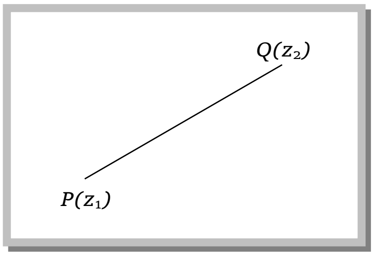
Note : ❑ The vertical angle R is $-(\theta_2 - \theta_1)$ *i.e.,* $\theta_1 - \theta_2$.

- If θ_1 and θ_2 are the principal values of z_1 and z_2 then $\theta_1+\theta_2$ and $\theta_1-\theta_2$ are not necessarily the principal value of $\arg(z_1 z_2)$ and $\arg(z_1/z_2)$.

2.12 Use of Complex Numbers in Co-ordinate Geometry

(1) **Distance formula :** The distance between two points $P(z_1)$ and $Q(z_2)$ is given by

$PQ = |z_2 - z_1|$ = |*affix of Q – affix of P*|



Note : The distance of point z from origin $|z-0| = |z| = |z-(0+i0)|$. Thus, modulus of a complex number z represented by a point in the argand plane is its distance from the origin.

- Three points $A(z_1), B(z_2)$ and $C(z_3)$ are collinear then $AB + BC = AC$

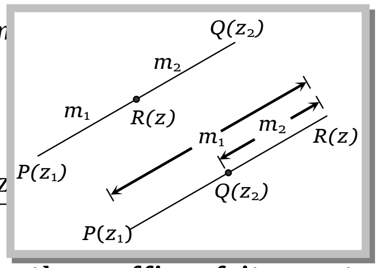
  i.e., $|z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|$.

(2) **Section formula :** If $R(z)$ divides the joining of $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$

(i) If $R(z)$ divides the segment PQ internally in the ratio of $m_1 : m_2$ then $z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$

(ii) If $R(z)$ divides the segment PQ externally in the ratio of m

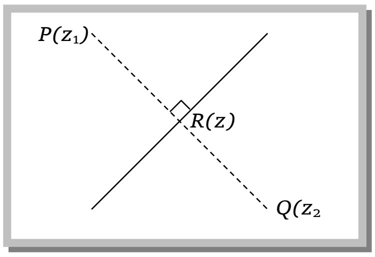
then $z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$



Note : If $R(z)$ is the mid point of PQ then affix of R is $\frac{z_1 + z}{2}$

- If z_1, z_2, z_3 are affixes of the vertices of a triangle, then affix of its centroid is $\frac{z_1 + z_2 + z_3}{3}$.

(3) **Equation of the perpendicular bisector :** If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point such that it is always at equal distance from $P(z_1)$ and



i.e., $PR = QR$ or $|z - z_1| = |z - z_2|$

$\Rightarrow |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow (z - z_1)\overline{(z - z_1)} = (z - z_2)\overline{(z - z_2)}$

$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow \bar{z}\ z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = z_1\bar{z}_1 - z_2\bar{z}_2$ $\quad\Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$

Hence, z lies on the perpendicular bisector of z_1 and z_2.

(4) **Equation of a straight line**

(i) **Parametric form :** Equation of a straight line joining the point having affixes z_1 and z_2 is $z = t z_1 + (1-t) z_2$, when $t \in R$

(ii) **Non parametric form :** Equation of a straight line joining the points having affixes z_1 and z_2 is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$.

Note : ❑ Three points z_1, z_2 and z_3 are collinear $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(iii) **General equation of a straight line:** The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b = 0$, where *a* is complex number and *b* is real number.

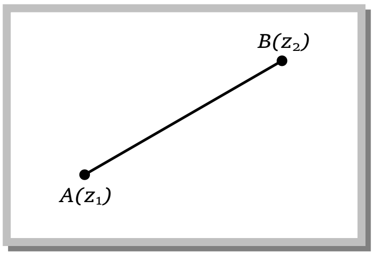
(iv) **Slope of a line :** The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}} = -\frac{\text{coeff. of } \bar{z}}{\text{coeff. of } z}$ and real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{\text{Re}(a)}{\text{Im}(a)} = -i\frac{(a+\bar{a})}{(a-\bar{a})}$.

Note : ❑ If α_1 and α_2 are the are the complex slopes of two lines on the argand plane, then

(i) If lines are perpendicular then $\alpha_1 + \alpha_2 = 0$ (ii) If lines are parallel then $\alpha_1 = \alpha_2$

❑ If lines $a\bar{z} + \bar{a}z + b = 0$ and $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ are the perpendicular or parallel, then $\left(\frac{-a}{a}\right) + \left(\frac{-a_1}{\bar{a}_1}\right) = 0$ or $\frac{-a}{\bar{a}} = \frac{-a_1}{\bar{a}_1} \Rightarrow a\bar{a}_1 + a_1\bar{a} = 0$ or $a\bar{a}_1 - \bar{a}a_1 = 0$, where a, a_1 are the complex numbers and $b, b_1 \in R$.

(v) **Slope of the line segment joining two points :** If $A(z_1)$ and $B(z_2)$ represent two points in the argand plane then the complex slope of AB is defined by $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.



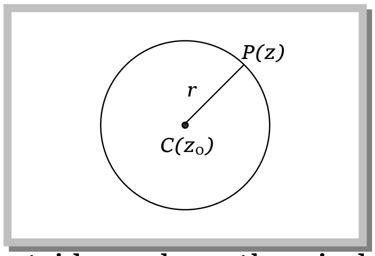
Note : ❑ If three points $A(z_1), B(z_2), C(z_3)$ are collinear then slope of AB = slope of BC = slope of AC

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

(vi) **Length of perpendicular :** The length of perpendicular from a point z_1 to the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{|a| + |\bar{a}|}$ or $\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$

(5) **Equation of a circle :** The equation of a circle whose centre is at point having affix z_o and radius *r* is $|z - z_o| = r$

Note : ❑ If the centre of the circle is at origin and radius r, then its equation is $|z| = r$.



❑ $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represent exterior of the circle $|z - z_o| = r$. Similarly, $|z - z_0| > r$ is the set of all points lying outside the circle and $|z - z_0| \geq r$ is the set of all points lying outside and on the circle $|z - z_0| = r$.

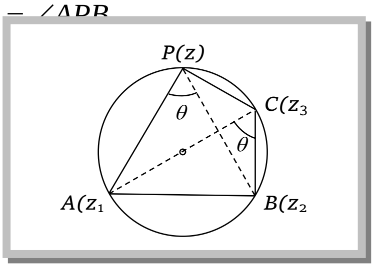
(i) **General equation of a circle :** The general equation of the circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where a is complex number and $b \in R$.

$\therefore$ Centre and radius are – a and $\sqrt{|a|^2 - b}$ respectively.

Note : ❑ Rule to find the centre and radius of a circle whose equation is given:

- Make the coefficient of $z\bar{z}$ equal to 1 and right hand side equal to zero.
- The centre of circle will be = – a $= -\text{coefficent of } \bar{z}$
- Radius $= \sqrt{|a|^2 - \text{constant term}}$

(ii) **Equation of circle through three non-collinear points :** Let $A(z_1), B(z_2), C(z_3)$ are three points on the circle and $P(z)$ be any point on the circle, then $\angle ACB = \angle APB$



Using coni method

In ΔACB, $\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{BC}{CA} e^{i\theta}$ .....(i)

In $\Delta\, APB$, $\dfrac{z_2 - z}{z_1 - z} = \dfrac{BP}{AP} e^{i\theta}$ .....(ii)

From (i) and (ii) we get

$\dfrac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$ = Real .....(iii)

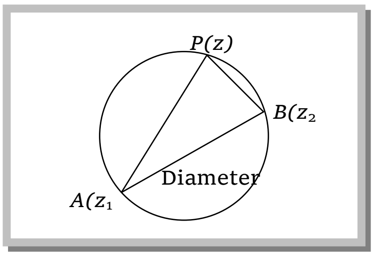
(iii) **Equation of circle in diametric form :** If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on circle then, $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

which is required equation of circle in diametric form.

(iv) **Other forms of circle :** (a) Equation of all circle which are orthogonal to $|z - z_1| = r_1$ and $|z - z_2| = r_2$. Let the circle be $|z - \alpha| = r$ cut given circles orthogonally

$\Rightarrow r^2 + r_1^2 = |\alpha - z_1|^2$ ......(i) and $r^2 + r_2^2 = |\alpha - z_2|^2$ .....(ii)

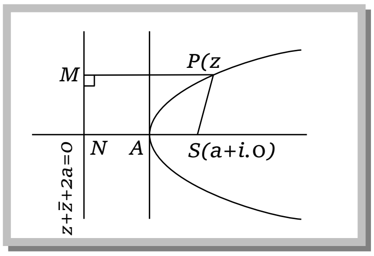
on solving $r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$ and let $\alpha = a + ib$



(b) $\left|\frac{z-z_1}{z-z_2}\right| = k$ is a circle if $k \neq 1$ and a line if $k = 1$.

(c) The equation $|z-z_1|^2 + |z-z_2|^2 = k,$ will represent a circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$

(6) **Equation of parabola :** Now for parabola $SP = PM$



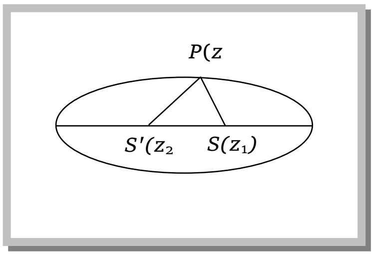
$$|z-a| = \frac{|z+\bar{z}+2a|}{2}$$

or $z\bar{z} - 4a(z+\bar{z}) = \frac{1}{2}\{z^2 + (\bar{z})^2\}$

where $a \in R$ (focus)

Directrix is $z + \bar{z} + 2a = 0$

(7) **Equation of ellipse :** For ellipse $SP + S'P = 2a$

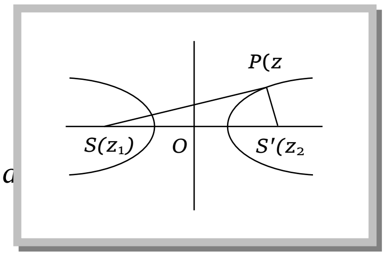


$\Rightarrow |z - z_1| + |z - z_2| = 2a$

where $2a > |z_1 - z_2|$ (since eccentricity <1)

Then point z describes an ellipse having foci at z_1 and z_2 and $a \in$

(8) **Equation of hyperbola :** For hyperbola $SP - S'P = 2a$

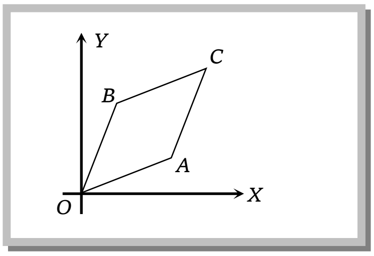


$\Rightarrow |z - z_1| - |z - z_2| = 2a$

where $2a < |z_1 - z_2|$ (since eccentricity >1)

Then point z describes a hyperbola having foci at z_1 and z_2 and a

Example: 38 If in the adjoining diagram, A and B represent complex number z_1 and z_2 respectively, then C represents



(a) $z_1 + z_2$

(b) $z_1 - z_2$

(c) $z_1.z_2$

(d) z_1 / z_2

Solution: (a) It is a fundamental concept.

Example: 39 If centre of a regular hexagon is at origin and one of the vertex on argand diagram is $1+2i,$ then its perimeter is

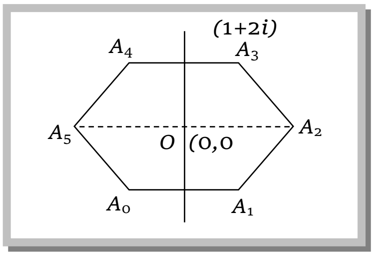
[Rajasthan PET 1999; Himachal CET 2002]

(a) $2\sqrt{5}$ (b) $6\sqrt{2}$ (c) $4\sqrt{5}$ (d) $6\sqrt{5}$

Solution: (d) Let the vertices be $z_0, z_1, \ldots\ldots, z_5$ *w.r.t.* centre O and $|z_0| = \sqrt{5}$

$\Rightarrow \quad A_0A_1 = |z_1 - z_0| = |z_0 e^{i\theta} - z_0| \quad = |z_0||\cos\theta + i\sin\theta - 1| \quad = \sqrt{5}\sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$

$\Rightarrow \quad = \sqrt{5}\sqrt{2(1-\cos\theta)} \quad = \sqrt{5}.2\sin(\theta/2)$



$\Rightarrow A_0A_1 = \sqrt{5}.2\sin(\pi/6) \ = \sqrt{5} \quad \left(\because \theta = \frac{2\pi}{6} = \frac{\pi}{3}\right)$ .....(i)

Similarly, $A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_0 = \sqrt{5}$

Hence, the perimeter of regular polygon is

$= A_0A_1 + A_1A_2 + A_2A_3 + A_3A_4 + A_4A_5 + A_5A_0 = 6\sqrt{5}$.

Example: 40 The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

[IIT Screening2001]

(a) Of area zero (b) Right-angled isosceles (c) Equilateral (d)

Solution: (b) Taking mod of both sides of given relation $\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

So, $|z_1 - z_3| = |z_2 - z_3|$. Also, $amp\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ or $amp\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{3}$ or $\angle z_2 z_3 z_1 = 60°$

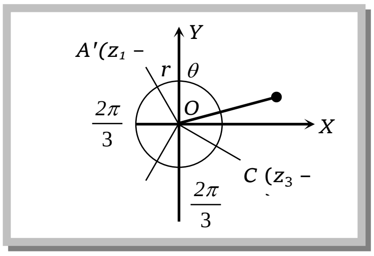
$\therefore$ The triangle has two sides equal and the angle between the equal sides $= 60°$. So it is equilateral.

Example: 41 Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 =$

(a) z_0^2 (b) $-z_0^2$ (c) $3z_0^2$ (d) $-3z_0^2$

Solution: (c) Let r be the circum-radius of the equilateral triangle and ω the cube root of unity.

Let ABC be the equilateral triangle with z_1, z_2 and z_3 as its vertices A, B and C respectively with circumcentre $O'(z_0)$. The vectors $O'A, O'B, O'C$ are equal and parallel to OA', OB', OC' respectively.

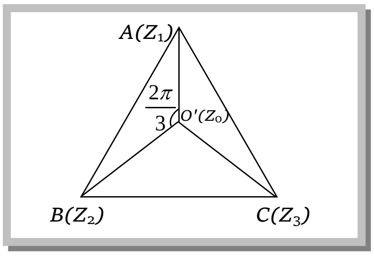


Then the vectors $\overrightarrow{OA'} = z_1 - z_0 = re^{i\theta}$

$\Rightarrow \overrightarrow{OB'} = z_2 - z_0 = re^{i(\theta + 2\pi/3)} = r\omega e^{i\theta}$

$\Rightarrow \overrightarrow{OC'} = z_3 - z_0 = re^{i(\theta + 4\pi/3)} = r\omega^2 e^{i\theta}$

$\therefore \ z_1 = z_0 + re^{i\theta}, z_2 = z_0 + r\omega e^{i\theta}, z_3 = z_0 + r\omega^2 e^{i\theta}$



Squaring and adding, we get,

$z_1^2 + z_2^2 + z_3^2 = 3z_0^2 + 2(1 + \omega + \omega^2)z_0 re^{i\theta} + (1 + \omega^2 + \omega^4)r^2 e^{i2\theta} = 3z_0^2,$

since $1 + \omega + \omega^2 = 0 = 1 + \omega^2 + \omega^4$.

Example: 42 The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if

[IIT 1981,83]

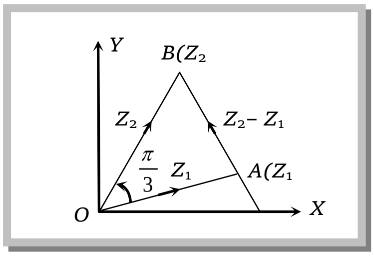
(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$ (c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

Solution: (b) Diagonals of a parallelogram $ABCD$ are bisected each other at a point *i.e.*, $\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$ $\Rightarrow$ $z_1 + z_3 = z_2 + z_4$.

Example: 43 If the complex number z_1, z_2 and the origin form an equilateral triangle then $z_1^2 + z_2^2 =$

(a) $z_1 z_2$ (b) $z_1 \bar{z}_2$ (c) $\bar{z}_2 z_1$ (d) $|z_1|^2 = |z_2|^2$

Solution: (a) Let *OA, OB* be the sides of an equilateral $\Delta\, OAB$ and *OA, OB* represent the complex numbers or vectors z_1, z_2 respectively.



From the equilateral $\Delta\, OAB$, $\overrightarrow{AB} = Z_2 - Z_1$

$\therefore\ arg\left(\frac{z_2 - z_1}{z_2}\right) = arg(z_2 - z_1) - arg z_2 = \frac{\pi}{3}$ and $arg\left(\frac{z_2}{z_1}\right) = arg(z_2) - arg(z_1) = \frac{\pi}{3}$

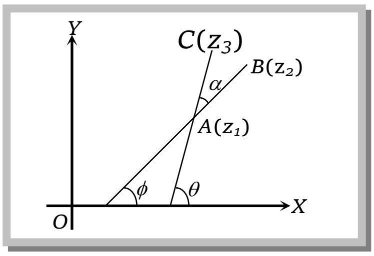
Also, $\left|\frac{z_2 - z_1}{z_2}\right| = 1 = \left|\frac{z_2}{z_1}\right|$, since triangle is equilateral.

Thus the vectors $\frac{z_2 - z_1}{z_2}$ and $\frac{z_2}{z_1}$ have same modulus and same argument, which implies that the vectors are equal, that is $\frac{z_2 - z_1}{z_2} = \frac{z_2}{z_1} \Rightarrow z_1 z_2 - z_1^2 = z_2^2 \Rightarrow z_1^2 + z_2^2 = z_1 z_2$.

2.13 Rotation Theorem

Rotational theorem *i.e.,* angle between two intersecting lines. This is also known as coni method.

Let z_1, z_2 and z_3 be the affixes of three points *A, B* and *C* respectively taken on argand plane.



Then we have $\overrightarrow{AC} = z_3 - z_1$ and $\overrightarrow{AB} = z_2 - z_1$

and let $arg\ \overrightarrow{AC} = arg\ (z_3 - z_1) = \theta$ and $\overrightarrow{AB} = arg(z_2 - z_1) = \phi$

Let $\angle CAB = \alpha$, $\because \angle CAB = \alpha = \theta - \phi$

$= arg\ \overrightarrow{AC} - arg\ \overrightarrow{AB} = arg(z_3 - z_1) - arg(z_2 - z_1) = arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

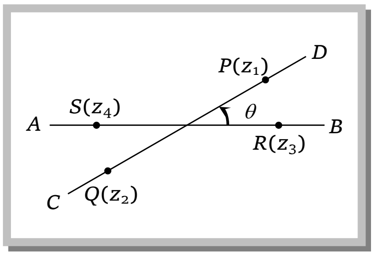
or angle between *AC* and *AB* = $arg\left(\frac{\text{affix } of\ C - \text{affix } of\ A}{\text{affix } of\ B - \text{affix } of\ A}\right)$

For any complex number *z* we have $z = |z|\, e^{i(arg z)}$

Similarly, $\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \left|\left(\frac{z_3 - z_1}{z_2 - z_1}\right)\right| e^{i\left(arg\frac{z_3 - z_1}{z_2 - z_1}\right)}$ or $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i(\angle CAB)} = \frac{AC}{AB} e^{i\alpha}$

Note : ❑ Here only principal values of the arguments are considered.

❑ $arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \theta$, if *AB* coincides with *CD*, then $arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0$ or $\pm\pi$, so that $\frac{z_1 - z_2}{z_3 - z_4}$ is real. It follows that if $\frac{z_1 - z_2}{z_3 - z_4}$ is real, then the points *A, B, C, D* are collinear.

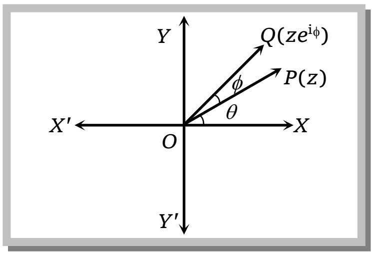


❑ If *AB* is perpendicular to *CD,* then *arg*

$\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\pi/2$, so $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary. It follows that if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k purely imaginary number, then AB and CD are perpendicular to each other.

(1) **Complex number as a rotating arrow in the argand plane :** Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ .....(i)

$r.e^{i\theta}$ be a complex number representing a point P in the argand plane.



Then $OP = |z| = r$ and $\angle POX = \theta$

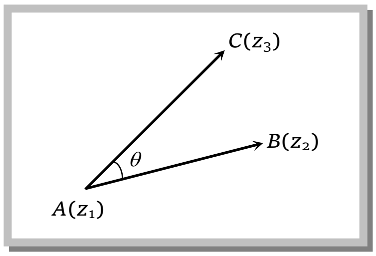
Now consider complex number $z_1 = ze^{i\phi}$

or $z_1 = re^{i\theta}.e^{i\phi} = re^{i(\theta+\phi)}$ {from (i)}

Clearly the complex number z_1 represents a point Q in the argand plane, when $OQ = r$ and $\angle QOX = \theta + \phi$.

Clearly multiplication of z with $e^{i\phi}$ rotates the vector $\overrightarrow{OP}$ through angle ϕ in anticlockwise sense. Similarly multiplication of z with $e^{-i\phi}$ will rotate the vector $\overrightarrow{OP}$ in clockwise sense.

Note : ❑ If z_1, z_2 and z_3 are the affixes of the points *A,B* and *C* such that $AC = AB$ and $\angle CAB = \theta$. Therefore, $\overrightarrow{AB} = z_2 - z_1$, $\overrightarrow{AC} = z_3 - z_1$.



Then $\overrightarrow{AC}$ will be obtained by rotating $\overrightarrow{AB}$ through an angle θ in anticlockwise sense, and therefore,

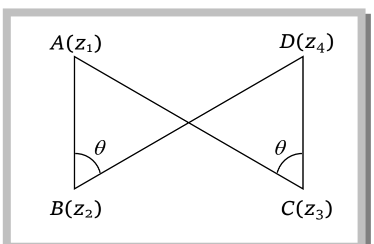
$\overrightarrow{AC} = \overrightarrow{AB}e^{i\theta}$ or $(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$ or $\frac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$

❑ If *A, B* and *C* are three points in argand plane such that $AC = AB$ and $\angle CAB = \theta$ then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$ use coni method.

❑ Let z_1 and z_2 be two complex numbers represented by point P and Q in the argand plane such that $\angle POQ = \theta$. Then, $z_1e^{i\theta}$ is a vector of magnitude $|z_1| = OP$ along $\overrightarrow{OQ}$ and $\frac{z_1e^{i\theta}}{|z_1|}$ is a unit vector along $\overrightarrow{OQ}$. Consequently, $|z_2|.\frac{z_1e^{i\theta}}{|z_1|}$ is a vector of magnitude $|z_2| = OQ$ along OQ *i.e.,* $z_2 = \frac{|z_2|}{|z_1|}.z_1e^{i\theta} = z_2 = \left|\frac{z_2}{z_1}\right|$.

(2) **Condition for four points to be concyclic :** If points *A,B,C* and *D* are concyclic $\angle ABD = \angle ACD$

Using rotation theorem



In $\Delta ABD \frac{(z_1 - z_2)}{|z_1 - z_2|} = \frac{z_4 - z_2}{|z_4 - z_2|} e^{i\theta}$ .....(i)

In $\Delta ACD \frac{(z_1 - z_3)}{|z_1 - z_3|} = \frac{z_4 - z_3}{|z_4 - z_3|} e^{i\theta}$ .....(ii)

From (i) and (ii)

$$\frac{(z_1 - z_2)}{|z_1 - z_3|} \frac{(z_4 - z_3)}{|z_4 - z_2|} = \left|\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}\right| = \text{Real}$$

So if z_1, z_2, z_3 and z_4 are such that $\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}$ is real, then these four points are concyclic.

Example: 44 If complex numbers z_1, z_2 and z_3 represent the vertices *A, B* and *C* respectively of an isosceles triangle *ABC* of which $\angle C$ is right angle, then correct statement is

(a) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 z_3$ (b) $(z_3 - z_1)^2 = z_3 - z_2$

(c) $(z_1 - z_2)^2 = (z_1 - z_3)(z_3 - z_2)$ (d) $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Solution: (d) $BC = AC$ and $\angle C = \pi / 2$

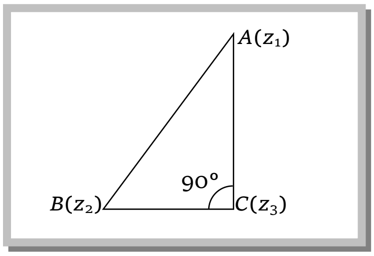
By rotation about *C* in anticlockwise sense $CB = CAe^{i\pi/2}$

$\Rightarrow (z_2 - z_3) = (z_1 - z_3)e^{i\pi/2} = i(z_1 - z_3)$

$\Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2 \Rightarrow z_2^2 + z_3^2 - 2z_2 z_3 = -z_1^2 - z_3^2 + 2z_1 z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1 z_2 = 2z_1 z_3 + 2z_2 z_3 - 2z_3^2 - 2z_1 z_2$

$\Rightarrow (z_1 - z_2)^2 = 2[(z_1 z_3 - z_3^2) - (z_1 z_2 - z_2 z_3)] \Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.



Example: 45 In the argand diagram, if *O,P* and *Q* represents respectively the origin, the complex numbers *z* and $z + iz$, then the angle $\angle OPQ$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

Solution: (c) It is a fundamental concept.

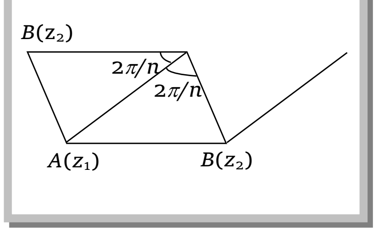
Example: 46 The centre of a regular polygon of *n* sides is located at the point $z = 0$ and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

(a) $z_1\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$ (b) $z_1\left(\cos\frac{\pi}{n} \pm i \sin\frac{\pi}{n}\right)$ (c) $z_1\left(\cos\frac{\pi}{2n} \pm i \sin\frac{\pi}{2n}\right)$ (d) None of these

Solution: (a) Let *A* be the vertex with affix z_1. There are two possibilities of z_2 *i.e.,* z_2 can be obtained by rotating z_1 through $\frac{2\pi}{n}$ either in clockwise or in anticlockwise direction.

$\therefore \frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{\frac{i2\pi}{2}} \Rightarrow z_2 = z_1 e^{\frac{i2\pi}{2}}$ $(\because |z_2| = |z_1|)$

$\Rightarrow z_2 = z_1\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$



Example: 47 Let z_1, z_2, z_3 be three vertices of an equilateral triangle circumscribing the circle $|z| = \frac{1}{2}$. If $z_1 = \frac{1}{2} + \frac{\sqrt{3}i}{2}$ and z_1, z_2, z_3 are in anticlockwise sense then z_2 is

(a) $1 + \sqrt{3}i$ (b) $1 - \sqrt{3}\,i$ (c) 1 (d) – 1

Solution: (d) $z_2 = z_1 e^{i2\pi/3} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right) = \frac{-3}{4} - \frac{1}{4} = -1.$

2.14 Triangle Inequalities

In any triangle, sum of any two sides is greater than the third side and difference of any two side is less than the third side. By applying this basic concept to the set of complex numbers we are having the following results.

(1) $|z_1 + z_2| \le |z_1| + |z_2|$ (2) $|z_1 - z_2| \le |z_1| + |z_2|$

(3) $|z_1 + z_2| \ge ||z_1| - |z_2||$ (4) $|z_1 - z_2| \ge ||z_1| - |z_2||$

Note :
- ❑ In a complex plane $|z_1 - z_2|$ is the distance between the points z_1 and z_2.
- ❑ The equality $|z_1 + z_2| = |z_1| + |z_2|$ holds only when *arg* (z_1) = *arg* (z_2) i.e., z_1 and z_2 are parallel.
- ❑ The equality $|z_1 - z_2| = ||z_1| - |z_2||$ holds only when *arg* (z_1) – *arg* $(z_2) = \pi$ *i.e.*, z_1 and z_2 are antiparallel.
- ❑ In any parallelogram sum of the squares of its sides is equal to the sum of the squares of its diagonals *i.e.* $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- ❑ Law of polygon *i.e.*, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$

Important Tips

- ☞ *The area of the triangle whose vertices are z, iz and z + iz is* $\frac{1}{2}|z|^2$.
- ☞ *If* z_1, z_2, z_3 *be the vertices of a triangle then the area of the triangle is* $\frac{\sum (z_2 - z_3)|z_1|^2}{4iz_1}$.
- ☞ *Area of the triangle with vertices* z, wz *and* $z + wz$ *is* $\frac{\sqrt{3}}{4}|z^2|$.
- ☞ *If* z_1, z_2, z_3 *be the vertices of an equilateral triangle and* z_o *be the circumcentre, then* $z_1^2 + z_2^2 + z_3^2 + = 3z_0^2$.
- ☞ *If* $z_1, z_2, z_3 ..... z_n$ *be the vertices of a regular polygon of n sides and* z_0 *be its centroid, then* $z_1^2 + z_2^2 + ... + z_n^2 = nz_0^2$.
- ☞ *If* z_1, z_2, z_3 *be the vertices of a triangle, then the triangle is equilateral iff* $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$ *or* $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ *or* $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.
- ☞ *If* $z_1, z_2 z_3$ *are the vertices of an isosceles triangle, right angled at* z_2 *then* $z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.
- ☞ *If* z_1, z_2, z_3 *are the vertices of right-angled isosceles triangle, then* $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.
- ☞ *If one of the vertices of the triangle is at the origin i.e.,* $z_3 = 0$, *then the triangle is equilateral iff* $z_1^2 + z_2^2 - z_1z_2 = 0$.

☞ *If* z_1, z_2, z_3 *and* z_1', z_2', z_3' *are the vertices of a similar triangle, then* $\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$.

☞ *If* z_1, z_2, z_3 *be the affixes of the vertices A, B, C respectively of a triangle ABC, then its orthocentre is* $\frac{a(\sec A)z_1 + b(\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$.

Example: 48 The points $1+3i, 5+i$ and $3+2i$ in the complex plane are

(a) Vertices of a right angled triangle (b) Collinear

(c) Vertices of an obtuse angled triangle (d) Vertices of an equilateral triangle

Solution: (b) Let $z_1 = 1+3i, z_2 = 5+i$ and $z_3 = 3+2i$. Then area of triangle $A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$, Hence z_1, z_2 and z_3 are collinear.

Example: 49 If $z = x+iy$, then area of the triangle whose vertices are points z, iz and $z+iz$ is

[IIT 1986; MP PET 1997, 2001; DCE 1997; AMU 2000; UPSEAT 2002]

(a) $2|z|^2$ (b) $\frac{1}{2}|z|^2$ (c) $|z|^2$ (d) $\frac{3}{2}|z|^2$

Solution: (b) Let $z = x+iy$, $z+iz = (x-y)+i(x+y)$ and $iz = -y+ix$

If A denotes the area of the triangle formed by $z, z+iz$ and iz, then

$A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x-y & x+y & 1 \\ -y & x & 1 \end{vmatrix}$ (Applying transformation $R_2 \to R_2 - R_1 - R_3$)

We get $A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 0 & -1 \\ -y & x & 0 \end{vmatrix} = \frac{1}{2}(x^2+y^2) = \frac{1}{2}|z|^2$.

Example: 50 $|z_1 + z_2| = |z_1| + |z_2|$ is possible if **[MP PET 1999]**

(a) $z_2 = \bar{z}_1$ (b) $z_2 = \frac{1}{z_1}$ (c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

Solution: (c) Squaring both sides, we get

$|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow 2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2| \Rightarrow \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0^o \Rightarrow \theta_1 = \theta_2$

Hence *arg* $(z_1) =$ *arg* (z_2)

Trick: Let z_1 and z_2 are the two sides of a triangle. By applying triangle inequality $(z_1 + z_2)$ is the third side. Equality holds only when $\theta_1 = \theta_2$ *i.e.*, z_1 and z_2 are parallel.

2.15 Standard Loci in the Argand Plane

(1) If z is a variable point in the argand plane such that *arg* $(z) = \theta$, then locus of z is a straight line (excluding origin) through the origin inclined at an angle θ with x–axis.

(2) If z is a variable point and z_1 is a fixed point in the argand plane such that *arg* $(z - z_1) = \theta$, then locus of z is a straight line passing through the point representing z_1 and inclined at an angle θ with x-axis. Note that the point z_1 is excluded from the locus.

(3) If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

(i) $|z - z_1| = |z - z_2|$ $\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2

(ii) $|z - z_1| + |z - z_2| = \text{constant}\ (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is an ellipse

(iii) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is the line segment joining z_1 and z_2

(iv) $|z - z_1| - |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(v) $|z - z_1| - |z - z_2| = \text{constant}\ (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is a hyperbola.

(vi) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

(vii) $|z - z_1| = k|z - z_2|\ \ k \neq 1$ $\Rightarrow$ Locus of z is a circle.

(viii) $arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha\ (\text{fixed})$ $\Rightarrow$ Locus of z is a segment of circle.

(ix) $arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\pi/2$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

(x) $arg\left(\dfrac{z - z_1}{z - z_2}\right) = 0$ or π $\Rightarrow$ Locus z is a straight line passing through z_1 and z_2.

(xi) The equation of the line joining complex numbers z_1 and z_2 is given by $\dfrac{z - z_1}{z_2 - z_1} = \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

or $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

Example: 51 The locus of the points z which satisfy the condition $arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$ is **[Rajasthan PET 2000,2002; MP PET 20**

(a) A straight line (b) A circle (c) A parabola (d) None of these

Solution:(c) We have $\dfrac{z-1}{z+1} = \dfrac{x+iy-1}{x+iy+1} = \dfrac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$

$\Rightarrow arg\ \dfrac{z-1}{z+1} = \tan^{-1}\dfrac{2y}{x^2+y^2-1}$

Hence $\tan^{-1}\dfrac{2y}{x^2+y^2-1} = \dfrac{\pi}{3}$

$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3} \Rightarrow x^2+y^2-1 = \frac{2}{\sqrt{3}}y \Rightarrow x^2+y^2-\frac{2}{\sqrt{3}}y-1=0$, which is obviously a circle.

Example: 52 If $\left|z^2-1\right| = |z|^2+1$, then z lies on **[AIEEE 2004]**

(a) An ellipse (b) The imaginary axis (c) A circle (d) The real axis

Solution: (b) $|z^2-1| = |z|^2+1$

$\Rightarrow |z-1|^2|z+1|^2 = (z\bar{z}+1)^2 \Rightarrow (z-1)(\bar{z}-1)(z+1)(\bar{z}+1) = (z\bar{z}+1)^2 \Rightarrow z+\bar{z}=0$

$\therefore$ z lies on imaginary axis.

Example: 53 The locus of the point z satisfying $arg\left(\frac{z-1}{z+1}\right) = k$. (where k is non-zero) is

(a) Circle with centre on y–axis (b) Circle with centre on x–axis

(b) A straight line parallel to x–axis (d) A straight line making an angle 60° with x–axis

Solution: (a) $arg\left(\frac{z-1}{z+1}\right) = k \Rightarrow arg\left[\frac{(x-1)+iy}{(x+1)+iy}\right] = k \Rightarrow arg[(x-1)+iy] - arg[(x+1)+iy] = k$

$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = k \Rightarrow \tan^{-1}\left[\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y^2}{x^2-1}}\right] = k \Rightarrow \tan k = \frac{y(x+1)-y(x-1)}{x^2+y^2-1} = \frac{2y}{x^2+y^2-1}$

$\Rightarrow \frac{2y}{\tan k} = x^2+y^2-1 \Rightarrow x^2+y^2-\frac{2y}{\tan k}-1=0$

It is an equation of circle whose centre is $(-g,-f) = (0, \cot k)$ on y–axis.

Example: 54 The locus of z satisfying the inequality $\log_{1/3}|z+1| > \log_{1/3}|z-1|$ is

(a) $R(z) < 0$ (b) $R(z) > 0$ (c) $I(z) < 0$ (d) None of these

Solution: (a) $\log_{1/3}|z+1| > \log_{1/3}|z-1|$

$\Rightarrow |z+1| < |z-1| \Rightarrow x^2+1+2x+y^2 < x^2+1-2x+y^2 \Rightarrow x<0 \Rightarrow \text{Re}(z) < 0.$

Example: 55 If $\alpha + i\beta = \tan^{-1}(z), z = x+iy$ and α is constant, the locus of 'z' is **[EAMCET 1995; KCET 1996]**

(a) $x^2+y^2+2x\cot 2\alpha = 1$ (b) $\cot 2\alpha(x^2+y^2) = 1+x$ (c) $x^2+y^2+2y\tan 2\alpha = 1$ (d) $x^2+y^2+2x\sin 2\alpha = 1$

Solution: (a) $\tan(\alpha+i\beta) = x+iy$

$\therefore \tan(\alpha-i\beta) = x-iy$ (conjugate), α is a constant and β is known to be eliminated

$\tan 2\alpha = \tan(\overline{\alpha+i\beta}+\overline{\alpha-i\beta}) \Rightarrow \tan 2\alpha = \frac{x+iy+x-iy}{1-(x^2+y^2)} \Rightarrow 1-(x^2+y^2) = 2x\cot 2\alpha$

$\therefore x^2+y^2+2x\cot 2\alpha = 1.$

2.16 De' Moivre's Theorem

(1) If n is any rational number, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(2) If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3).....(\cos\theta_n + i\sin\theta_n)$

then $z = \cos(\theta_1+\theta_2+\theta_3+.....+\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+.....+\theta_n)$, where $\theta_1, \theta_2, \theta_3.....\theta_n \in R$.

(3) If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then $z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$,

where $k = 0, 1, 2, 3, ...(n-1)$.

(4) If $p, q \in z$ and $q \neq 0$, then $(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$,

where $k = 0, 1, 2, 3.....(q-1)$.

Deductions: If $n \in Q$, then

(i) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ (ii) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

(iii) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$ (iv) $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

Applications

(i) In finding the expansions of trigonometric functions *i.e.* $\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - ......$

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - ....$

(ii) In finding the roots of complex numbers.

(iii) In finding the complex solution of algebraic equations.

Note : ❑ This theorem is not valid when *n* is not a rational number or the complex number is not in the form of $\cos\theta + i\sin\theta$.

Powers of complex numbers : Let $z = x + iy = r(\cos\theta + i\sin\theta)$

$\therefore\ z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$

Number	*x + iy* form	Standard complex form	General
1	1+*i*0	$\cos 0 + i\sin 0$	$\cos 2n\pi + i\sin 2n\pi$
– 1	– 1+*i*0	$\cos\pi + i\sin\pi$	$\cos(2n+1)\pi + i\sin(2n+1)\pi$
i	0 +*i*(1)	$\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$	$\cos(4n+1)\frac{\pi}{2} + i\sin(4n+1)\frac{\pi}{2}$
– *i*	0 +*i*(–1)	$\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$	$\cos(4n+1)\frac{\pi}{2} - i\sin(4n+1)\frac{\pi}{2}$

Example: 56 If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then

(a) $a = 2, b = -1$ (b) $a = 1, b = 0$ (c) $a = 0, b = 1$ (d) $a = -1, b = 2$

Solution: (b) $\frac{1-i}{1+i}\times\frac{1-i}{1-i}=-i=\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\Rightarrow(-i)^{100}=\cos(-50\pi)+i\sin(-50\pi)=1+i(0)\quad\Rightarrow a=1,\ b=0$

Example: 57 If $x_r=\cos\left(\frac{\pi}{2^r}\right)+i\sin\left(\frac{\pi}{2^r}\right)$, then $x_1.x_2.x_3..........\infty$ is

[Rajasthan PET 1990, 2000; Karnataka CET 2000; UPSEAT 1990; Haryana CEE 1998; BIT Ranchi 1996]

(a) –3 (b) –2 (c) –1 (d) 0

Solution: (c) $x_1.x_2.x_3.....$ upto $\infty=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)..................$

$$=\cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+...\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+....\right)=\cos\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right)+i\sin\left(\frac{\frac{\pi}{2}}{1-\frac{1}{2}}\right)=\cos\pi+i\sin\pi=-1$$

Example: 58 If $z_r=\cos\frac{r\alpha}{n^2}+i\sin\frac{r\alpha}{n^2}$, where $r=1,2,3,.....,n$, then $\lim_{n\to\infty}z_1z_2z_3.....z_n$ is equal to **[UPSEAT 2001]**

(a) $\cos\alpha+i\sin\alpha$ (b) $\cos(\alpha/2)-i\sin(\alpha/2)$ (c) $e^{i\alpha/2}$ (d) $\sqrt[3]{e^{i\alpha}}$

Solution: (c) $z_r=\cos\frac{r\alpha}{n^2}+i\sin\frac{r\alpha}{n^2}\Rightarrow z_1=\cos\frac{\alpha}{n^2}+i\sin\frac{\alpha}{n^2}$;

$z_2=\cos\frac{2\alpha}{n^2}+i\sin\frac{2\alpha}{n^2}$;..................

$$\Rightarrow z_n=\cos\frac{n\alpha}{n^2}+i\sin\frac{n\alpha}{n^2}\Rightarrow\lim_{n\to\infty}(z_1,z_2,z_3,.....z_n)=\lim_{n\to\infty}\left[\cos\left\{\frac{\alpha}{n^2}(1+2+3+.....+n)\right\}+i\sin\left\{\frac{\alpha}{n^2}(1+2+3+.....+n)\right\}\right]$$

$$=\lim_{n\to\infty}\left[\cos\frac{\alpha\,n(n+1)}{2n^2}+i\sin\frac{\alpha\,n(n+1)}{2n^2}\right]=\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}=e^{i\alpha/2}.$$

Example: 59 $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=$ **[Kerala (Engg.) 2002]**

(a) $\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$ (b) $\cos\left(\frac{n\pi}{2}+n\theta\right)+i\sin\left(\frac{n\pi}{2}+n\theta\right)$

(c) $\sin\left(\frac{n\pi}{2}-n\theta\right)+i\cos\left(\frac{n\pi}{2}-n\theta\right)$ (d) $\cos n\left(\frac{n\pi}{2}+n\theta\right)+i\sin n\left(\frac{n\pi}{2}+n\theta\right)$

Solution: (a) $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\left(\frac{1+\cos\alpha+i\sin\alpha}{1+\cos\alpha-i\sin\alpha}\right)^n$ $\left(\text{where }\alpha=\frac{\pi}{2}-\theta\right)$

$$=\left(\frac{2\cos^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}-2i\sin\frac{\alpha}{2}.\cos\frac{\alpha}{2}}\right)^n=\left(\frac{\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-i\sin\frac{\alpha}{2}}\right)^n=\left[\frac{cis\left(\frac{\alpha}{2}\right)}{cis\left(-\frac{\alpha}{2}\right)}\right]^n=\left\{cis\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right)\right\}^n=cis(n\alpha)$$

$$=cis\,n\left(\frac{\pi}{2}-\theta\right)=cis\left(\frac{n\pi}{2}-\theta\right)=\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right).$$

2.17 Roots of a Complex Number

(1) **n^{th} roots of complex number** ($z^{1/n}$) **:** Let $z=r(\cos+i\sin\theta)$ be a complex number. To find the roots of a complex number, first we express it in polar form with the general value of its

amplitude and use the De' moivre's theorem. By using De'moivre's theorem n^{th} roots having n distinct values of such a complex number are given by

$$z^{1/n}=r^{1/n}\left[\cos\frac{2m\pi+\theta}{n}+i\sin\frac{2m\pi+\theta}{n}\right], \text{where } m=0,1,2,.....,(n-1).$$

Properties of the roots of $z^{1/n}$:

(i) All roots of $z^{1/n}$ are in geometrical progression with common ratio $e^{2\pi i/n}$.

(ii) Sum of all roots of $z^{1/n}$ is always equal to zero.

(iii) Product of all roots of $z^{1/n}=(-1)^{n-1}z$.

(iv) Modulus of all roots of $z^{1/n}$ are equal and each equal to $r^{1/n}$ or $|z|^{1/n}$.

(v) Amplitude of all the roots of $z^{1/n}$ are in A.P. with common difference $\frac{2\pi}{n}$.

(vi) All roots of $z^{1/n}$ lies on the circumference of a circle whose centre is origin and radius equal to $|z|^{1/n}$. Also these roots divides the circle into n equal parts and forms a polygon of n sides.

(2) **The n^{th} roots of unity :** The n^{th} roots of unity are given by the solution set of the equation

$$x^n=1=\cos 0+i\sin 0=\cos 2k\pi+i\sin 2k\pi$$

$$x=[\cos 2k\pi+i\sin 2k\pi]^{1/n}$$

$$x=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, \text{ where } k=0,1,2,.....,(n-1).$$

Properties of n^{th} roots of unity

(i) Let $\alpha=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}=e^{i(2\pi/n)}$, the n^{th} roots of unity can be expressed in the form of a series *i.e.,* $1,\alpha,\alpha^2,.....\alpha^{n-1}$. Clearly the series is G.P. with common difference α *i.e.,* $e^{i(2\pi/n)}$.

(ii) The sum of all n roots of unity is zero *i.e.,* $1+\alpha+\alpha^2+.....+\alpha^{n-1}=0$.

(iii) Product of all n roots of unity is $(-1)^{n-1}$.

(iv) Sum of p^{th} power of n roots of unity

$$1+\alpha^p+\alpha^{2p}+.....+\alpha^{(n-1)p}=\begin{cases}0, \text{when } p \text{ is not multiple of } n\\ n, \text{when } p \text{ is a multiple of } n\end{cases}$$

(v) The n, n^{th} roots of unity if represented on a complex plane locate their positions at the vertices of a regular plane polygon of n sides inscribed in a unit circle having centre at origin, one vertex on positive real axis.

Note : ❑ $x^n-1=(x-1)(x^{n-1}+x^{n-2}+.....+x+1)$

❑ $(\sin\theta+i\cos\theta)=-i^2\sin\theta+i\cos\theta=i(\cos\theta-i\sin\theta)$

(3) **Cube roots of unity :** Cube roots of unity are the solution set of the equation $x^3 - 1 = 0 \Rightarrow$ $x = (1)^{1/3}$ $\Rightarrow$ $x = (\cos 0 + i \sin 0)^{1/3}$ $\Rightarrow$ $x = \cos\frac{2k\pi}{3} + i \sin\left(\frac{2k\pi}{3}\right)$, where $k = 0,1,2$

Therefore roots are $1, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ or $1, e^{2\pi i/3}, e^{4\pi i/3}$.

Alternative : $x = (1)^{1/3} \Rightarrow x^3 - 1 = 0 \Rightarrow (x-1)(x^2 + x + 1) = 0$

$$x = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

If one of the complex roots is ω, then other root will be ω^2 or vice-versa.

Properties of cube roots of unity

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) $1 + \omega^r + \omega^{2r} = \begin{cases} 0, \text{ if } r \text{ not a multiple of 3} \\ 3, \text{ if } r \text{ is a multiple of 3} \end{cases}$

(iv) $\overline{\omega} = \omega^2$ and $(\overline{\omega})^2 = \omega$ and $\omega . \overline{\omega} = \omega^3$.

(v) Cube roots of unity from a G.P.

(vi) Imaginary cube roots of unity are square of each other *i.e.,* $(\omega)^2 = \omega^2$ and $(\omega^2)^2 = \omega^3 . \omega = \omega$.

(vii) Imaginary cube roots of unity are reciprocal to each other *i.e.,* $\frac{1}{\omega} = \omega^2$ and $\frac{1}{\omega^2} = \omega$.

(viii) The cube roots of unity by, when represented on complex plane, lie on vertices of an equilateral triangle inscribed in a unit circle having centre at origin, one vertex being on positive real axis.

(ix) A complex number $a + ib$, for which $|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$, can always be expressed in terms of i, ω, ω^2.

Note : ❑ If $\omega = \frac{-1 + i\sqrt{3}}{2} = e^{2\pi i/3}$, then $\omega^2 = \frac{-1 - i\sqrt{3}}{2} = e^{-4\pi i/3} = e^{-2\pi i/3}$ or vice-versa $\omega . \overline{\omega} = \omega^3$.

❑ $a + b\omega + c\omega^2 = 0 \Rightarrow a = b = c$, if *a, b,* c are real.

❑ Cube root of – 1 are $-1, -\omega, -\omega^2$.

Important Tips

- $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
- $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
- $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- $x^2 + y^2 = (x + iy)(x - iy)$
- $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
- $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
- $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Fourth roots of unity : The four, fourth roots of unity are given by the solution set of the equation $x^4 - 1 = 0. \Rightarrow (x^2-1)(x^2+1) = 0 \Rightarrow x = \pm 1, \pm i$

Note : ❑ Sum of roots = 0 and product of roots =−1.

❑ Fourth roots of unity are vertices of a square which lies on coordinate axes.

Continued product of the roots

If $z = r(\cos\theta + i\sin\theta)$ *i.e.*, $|z| = r$ and $amp\,(z) = \theta$ then continued product of roots of $z^{1/n}$ is

$$= r(\cos\phi + i\sin\phi), \text{ where } \phi = \sum_{m=0}^{n-1} \frac{2m\pi + \theta}{n} = (n-1)\pi + \theta.$$

Thus continued product of roots of $z^{1/n} = r[\cos\{(n-1)\pi + \theta\} + i\sin\{(n-1)\pi+\theta\}] = \begin{cases} z, \text{if } n \text{ is odd} \\ -z, \text{if } n \text{ is even}\end{cases}$

Similarly, the continued product of values of $z^{m/n}$ is $= \begin{cases} z^m, & \text{if } n \text{ is odd} \\ (-z)^m, & \text{if } n \text{ is even}\end{cases}$

Important Tips

☞ *If* $x + \frac{1}{x} = 2\cos\theta$ *or* $x - \frac{1}{x} = 2i\sin\theta$ *then* $x = \cos\theta + i\sin\theta, \frac{1}{x} = \cos\theta - i\sin\theta, x^n + \frac{1}{x^n} = 2\cos n\theta, x^n - \frac{1}{x^n} = 2i\sin n\theta.$

☞ *If n be a positive integer then ,* $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$.

☞ *If z is a complex number, then* e^z *is periodic.*

☞ n^{th} *root of −1 are the solution of the equation* $z^n + 1 = 0$

$z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2).....(z-\alpha^{n-1})$, *where* $\alpha = n^{th}$ *root of unity*

$z^n - 1 = (z-1)(z+1)\prod_{r=1}^{(n-2)/2}(z^2 - 2z\cos\frac{2r\pi}{n} + 1)$, *if n is even.*

$$z^n + 1 = \begin{cases} \prod_{r=0}^{(n-2)/2}\left[z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right) + 1\right], \text{if } n \text{ is even.} \\ (z+1)\prod_{r=0}^{(n-3)/2}\left[z^2 - 2z\cos\left(\frac{(2r+1)\pi}{n}\right)+1\right], \text{if } n \text{ is odd.}\end{cases}$$

☞ *If* $x = \cos\alpha + i\sin\alpha, y = \cos\beta + i\sin\beta, z = \cos\gamma + i\sin\gamma$ *and given,* $x + y + z = 0$, *then*

(i) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ *(ii)* $yz + zx + xy = 0$ *(iii)* $x^2 + y^2 + z^2 = 0$ *(iv)* $x^3 + y^3 + z^3 = 3xyz$

then, putting, values if x, y, z *in these results*

$x + y + z = 0 \Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma \Rightarrow yz + zx + xy = 0 \Rightarrow \begin{cases}\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0 \\ \sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0\end{cases}$

$x^2 + y^2 + z^2 = 0 \Rightarrow \begin{cases}\sum\cos 2\alpha = 0 \\ \sum \sin 2\alpha = 0,\end{cases}$ *the summation consists 3 terms*

$x^3 + y^3 + z^3 = 3xyz$, *gives similarly*

$\sum\cos 3\alpha = 3\cos(\alpha+\beta+\gamma) \Rightarrow \sum\sin 3\alpha = 3\sin(\alpha+\beta+\gamma)$

If the condition given be $x+y+z=xyz$, *then* $\sum \cos\alpha = \cos(\alpha+\beta+\gamma)$ *etc.*

Example: 60 If the cube roots of unity be $1,\omega,\omega^2$, then the roots of the equation $(x-1)^3+8=0$ are **[DCE 2000; IIT 1979; UPSEAT 1986]**

(a) $-1,1+2\omega,1+2\omega^2$ (b) $-1,1-2\omega,1-2\omega^2$ (c) $-1,1,-1$ (d) None of these

Solution: (c) $(x-1)^3=-8 \Rightarrow x-1=(-8)^{1/3} \Rightarrow x-1=-2,-2\omega,-2\omega^2 \Rightarrow x=-1,1-2\omega,1-2\omega^2$

Example: 61 ω is an imaginary cube root of unity. If $(1+\omega^2)^m=(1+\omega^4)^m$, then least positive integral value of *m* is **[IIT Screening 2004]**

(a) 6 (b) 5 (c) 4 (d) 3

Solution: (d) The given equation reduces to $(-\omega)^m=(-\omega^2)^m \Rightarrow \omega^m=1 \Rightarrow m=3$.

Example: 62 If ω is the cube root of unity, then $(3+5\omega+3\omega^2)^2+(3+3\omega+5\omega^2)^2=$ **[MP PET 1999]**

(a) 4 (b) 0 (c) – 4 (d) None of these

Solution: (c) $(3+5\omega+3\omega^2)^2+(3+3\omega+5\omega^2)^2=(3+3\omega+3\omega^2+2\omega)^2+(3+3\omega+3\omega^2+2\omega^2)^2$

$=(2\omega)^2+(2\omega^2)^2=4\omega^2+4\omega^4=4(-1)=-4$ $(\because 1+\omega+\omega^2=0,\omega^3=1)$

Example: 63 If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to **[IIT 1999]**

(a) $1-i\sqrt{3}$ (b) $-1+i\sqrt{3}$ (c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

Solution: (c) Given equation is $4+5\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{365}$

$=4+5\omega^{334}+3\omega^{365}=4+5\omega+3\omega^2=1+2\omega=1+2\left(\frac{-1+i\sqrt{3}}{2}\right)=i\sqrt{3}$

Example: 64 Let ω is an imaginary cube root of unity then the value of

$2(\omega+1)(\omega^2+1)+3(2\omega+1)(2\omega^2+1)+.....+(n+1)(n\omega+1)(n\omega^2+1)$ is **[Orissa JEE 2002]**

(a) $\left[\frac{n(n+1)}{2}\right]^2+n$ (b) $\left[\frac{n(n+1)}{2}\right]^2$ (c) $\left[\frac{n(n+1)}{2}\right]^2-n$ (d) None of these

Solution: (a) $2(\omega+1)(\omega^2+1)+3(2\omega+1)(2\omega^2+1)+.....+(n+1)(n\omega+1)(n\omega^2+1)=\sum_{r=1}^{n}(r+1)(r\omega+1)(r\omega^2+1)$

$=\sum_{r=1}^{n}(r+1)(r^2\omega^3+r\omega+r\omega^2+1)=\sum_{r=1}^{n}(r+1)(r^2-r+1)=\sum_{r=1}^{n}(r^3-r^2+r+r^2-r+1)=\sum_{r=1}^{n}(r^3)+\sum_{r=1}^{n}(1)=\left[\frac{n(n+1)}{2}\right]^2+n.$

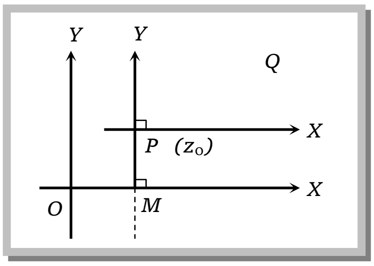
Example: 65 The roots of the equation $x^4-1=0$, are **[MP PET 1986]**

(a) $1,1,i,-i$ (b) $1,-1,i,-i$ (c) $1,-1,\omega,\omega^2$ (d) None of these

Solution: (b) Given equation $x^4-1=0 \Rightarrow (x^2-1)(x^2+1)=0 \Rightarrow x^2=1$ and $x^2=-1 \Rightarrow x=\pm1,\pm i$

2.18 Shifting the Origin in Case of Complex Numbers

Let *O* be the origin and *P* be a point with affix z_0. Let a point *Q* has affix *z* with respect to the co-ordinate system passing through *O*.



When origin is shifted to the point $P(z_0)$ then the new affix *Z* of the point *Q* with respect to new origin *P* is given by $Z=z-z_0$ *i.e.,* to shift the origin at z_0 we should replace *z* by $Z+z_0$.

Example: 66 If z_1, z_2, z_3 are the vertices of an equilateral triangle with z_0 as its circumcentre then changing origin to z_0, then (where z_1, z_2, z_3 are new complex numbers of the vertices)

(a) $z_1^2 + z_2^2 + z_3^2 = 0$ (b) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$ (c) Both (a) and (b) (d) None of these

Solution: (a) In an equilateral triangle the circumcentre and the centroid are the same point. So,

$$z_0 = \frac{z_1 + z_2 + z_3}{3} \Rightarrow z_1 + z_2 + z_3 = 3z_0 \qquad \text{..... (i)}$$

To shift the origin at z_0, we have to replace z_1, z_2, z_3 and z_0 by $z_1 + z_0, z_2 + z_0, z_3 + z_0$ and $0 + z_0$ then equation (i) becomes $(z_1 + z_0) + (z_2 + z_0) + (z_3 + z_0) = 3(0 + z_0) \Rightarrow z_1 + z_2 + z_3 = 0$

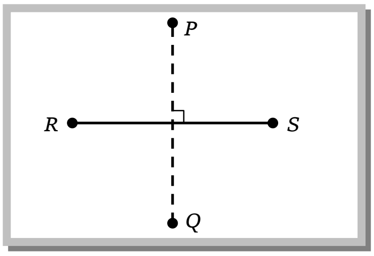
On squaring $z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_2 z_3 + z_3 z_1) = 0$ ..... (ii)

But triangle with vertices z_1, z_2 and z_3 is equilateral, then $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ .....(iii)

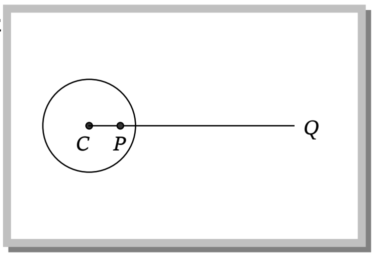
From (ii) and (iii) we get, $3(z_1^2 + z_2^2 + z_3^2) = 0$. Therefore, $z_1^2 + z_2^2 + z_3^2 = 0$.

2.19 Inverse Points

(1) **Inverse points with respect to a line :** Two points *P* and *Q* are said to be the inverse points with respect to the line *RS*. If *Q* is the image of *P* in *RS, i.e.,* if the line *RS* is the right bisector of *PQ*.



(2) **Inverse points with respect to a circle :** If *C* is the centre of the circle and *P,Q* are the inverse points with respect to the circle then three points *C,P,Q* are collinear, and also *CP* . *CQ* $= r^2$, where *r* is the radius of the c



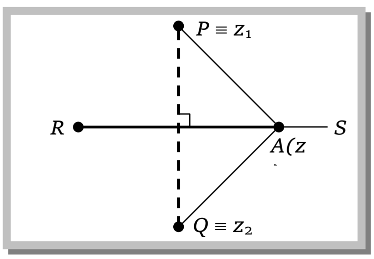
Example: 67 z_1, z_2, are the inverse points with respect to the line $z\bar{a} + a\bar{z} = b$ if

(a) $z_1 a + z_2 a = b$ (b) $z_1 \bar{a} + a\overline{z_2} = b$ (c) $z_1 \bar{a} - a\overline{z_2} = b$ (d) None of these

Solution: (b) Let *RS* be the line represented by the equation $z\bar{a} + a\bar{z} = b$

Let *P* and *Q* are the inverse points with respect to the line *RS*.

The point *Q* is the reflection (inverse) of the point *P* in the line *RS* if the line *RS* is the right bisector of *PQ*. Take any point *z* in the line *RS,* then lines joining *z* to *P* and *z* to *Q* are equal.



i.e., $|z-z_1| = |z-z_2|$ or $|z-z_1|^2 = |z-z_2|^2$

i.e., $(z-z_1)(\bar{z}-\bar{z_1}) = (z-z_2)(\bar{z}-\bar{z_2}) \Rightarrow z(\overline{z_2}-\overline{z_1}) + \bar{z}(z_2-z_1) + (z_1\overline{z_1} - z_2\overline{z_2}) = 0$ .....(ii)

Hence, equations (i) and (ii) are identical, therefore comparing coefficients, we get

$\frac{\bar{a}}{\overline{z_2}-\overline{z_1}} = \frac{a}{z_2-z_1} = \frac{-b}{z_1\overline{z_1}-z_2\overline{z_2}}$ So that, $\frac{z_1\bar{a}}{z_1(\overline{z_2}-\overline{z_1})} = \frac{a\overline{z_2}}{\overline{z_2}(z_2-z_1)} = \frac{-b}{z_1\overline{z_1}-z_2\overline{z_2}} = \frac{z_1\bar{a}+a\overline{z_2}-b}{0}$

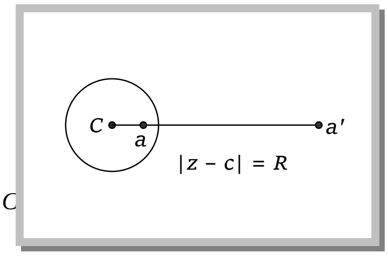
(By ratio and proportion rule)

Hence, $z_1\bar{a} + a\overline{z_2} - b = 0$ or $z_1\bar{a} + a\overline{z_2} = b$.

Example: 68 Inverse of a point a with respect to the circle $|z-c| = R$ (a and c are complex numbers, centre C and radius R) is the point $c + \frac{R^2}{\bar{a}-\bar{c}}$

(a) $c + \frac{R^2}{\bar{a}-\bar{c}}$ (b) $c - \frac{R^2}{\bar{a}-\bar{c}}$ (c) $c + \frac{R}{\bar{c}-\bar{a}}$ (d) None of these

Solution: (a) Let a' be the inverse point of a with respect to the circle $|z-c| = R$, then by definition the points c, a, a' are collinear.



We have, $arg(a'-c) = arg(a-c) = -arg(\bar{a}-\bar{c})$ $(\because arg\bar{z} = -arg z)$

$\Rightarrow arg(a'-c) + arg(\bar{a}-\bar{c}) = 0 \Rightarrow arg\{(a'-c)(\bar{a}-\bar{c})\} = 0$

$\therefore$ $(a'-c)(\bar{a}-\bar{c})$ is purely real and positive.

By definition $|a'-c||a-c| = R^2$ $(\because CP.C$

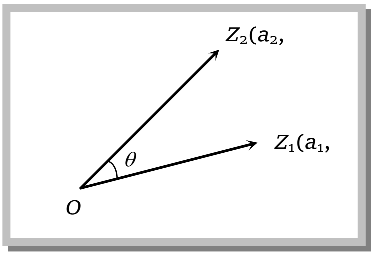
$\Rightarrow |a'-c||\bar{a}-\bar{c}| = R^2$ $(\because |z| = |\bar{z}|)$

$\Rightarrow |(a'-c)(\bar{a}-\bar{c})| = R^2 \Rightarrow (a'-c)(\bar{a}-\bar{c}) = R^2$ $\{\because (a'-c)(\bar{a}-\bar{c})$ is purely real and positive$\}$

$\Rightarrow a'-c = \frac{R^2}{\bar{a}-\bar{c}}$. Therefore, the inverse point a' of a point a, $a' = c + \frac{R^2}{\bar{a}-\bar{c}}$.

2.20 Dot and Cross Product

Let $z_1 = a_1 + ib_1 \equiv (a_1, b_1)$ and $z_2 = a_2 + ib_2 \equiv (a_2, b_2)$ be two complex numbers.



If $\angle POQ = \theta$ then from coni method $\frac{z_2-0}{z_1-0} = \frac{|z_2|}{|z_1|}e^{i\theta}$

$\Rightarrow \frac{z_2\bar{z}_1}{z_1\bar{z}_1} = \frac{|z_2|}{|z_1|}e^{i\theta} \Rightarrow \frac{z_2\bar{z}_1}{|z_1|^2} = \frac{|z_2|}{|z_1|}e^{i\theta} \Rightarrow z_2\bar{z}_1 = |z_1||z_2|e^{i\theta}$

$\Rightarrow z_2\bar{z}_1 = |z_1||z_2|(\cos\theta + i\sin\theta)$

$\Rightarrow \text{Re}(z_2\bar{z}_1) = |z_1||z_2|\cos\theta$ .....(i) and $\text{Im}(z_2\bar{z}_1) = |z_1||z_2|\sin\theta$ .....(ii)

The dot product z_1 and z_2 is defined by $z_1 o z_2 = |z_1||z_2|\cos\theta = \text{Re}(\bar{z}_1 z_2) = a_1a_2 + b_1b_2$ (From(i))

Cross product of z_1 and z_2 is defined by $z_1 \times z_2 = |z_1||z_2|\sin\theta = \text{Im}(\bar{z}_1 z_2) = a_1b_2 - a_2b_1$ (From(ii))

Hence, $z_1 o z_2 = a_1a_2 + b_1b_2 = \text{Re}(\bar{z}_1 z_2)$ and $z_1 \times z_2 = a_1b_2 - a_2b_1 = \text{Im}(\bar{z}_1 z_2)$

Important Tips

☞ *If z_1 and z_2 are perpendicular then $z_1 o z_2 = 0$*

☞ *If z_1 and z_2 are parallel then $z_1 \times z_2 = 0$*

- ☞ *Projection of* z_1 *on* $z_2 = (z_1 o z_2)/|z_2|$
- ☞ *Projection of* z_2 *on* $z_1 = (z_1 o z_2)/|z_1|$
- ☞ *Area of triangle if two sides represented by* z_1 *and* z_2 *is* $\frac{1}{2}|z_1 \times z_2|$
- ☞ *Area of a parallelogram having sides* z_1 *and* z_2 *is* $|z_1 \times z_2|$
- ☞ *Area of parallelogram if diagonals represents by* z_1 *and* z_2 *is* $\frac{1}{2}|z_1 \times z_2|$

Example: 69 If $z_1 = 2 + 5i, z_2 = 3 - i$ then projection of z_1 on z_2 is

(a) 1/10 (b) $1/\sqrt{10}$ (c) $-7/10$ (d) None of these

Solution: (b) Projection of z_1 on z_2 $= \frac{z_1 o z_2}{|z_2|}$ $= \frac{a_1a_2 + b_1b_2}{\sqrt{a_2^2 + b_2^2}} = \frac{1}{\sqrt{10}}$.

***

MATHEMATICS

BY KAPIL KANT SHARMA

Assignment

Introduction, Integral Power of Iota

Basic Level

1. $\sqrt{-2}\sqrt{-3}=$ **[Roorkee 1978]**

(a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $i\sqrt{6}$ (d) None of these

2. The value of $(1+i)^5 \times (1-i)^5$ is **[Karnataka CET 1992]**

(a) – 8 (b) 8i (c) 8 (d) 32

3. $(1+i)^4+(1-i)^4=$ **[Karnataka CET 2001]**

(a) 8 (b) – 8 (c) 4 (d) – 4

4. The value of $(1+i)^8+(1-i)^8$ is **[Rajasthan PET 2001]**

(a) 16 (b) – 16 (c) 32 (d) – 32

5. The value of $(1+i)^6+(1-i)^6$ is **[Rajasthan PET 2002]**

(a) 0 (b) 2^7 (c) 2^6 (d) None of these

6. $(1+i)^{10}$, where $i^2=-1$, is equal to **[AMU 2001]**

(a) 32 i (b) 64 + i (c) 24 i – 32 (d) None of these

7. If $i=\sqrt{-1}$, then $1+i^2+i^3-i^6+i^8$ is equal to **[Rajsthan PET 1995]**

(a) 2 – i (b) 1 (c) 3 (d) – 1

8. The value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1=$

(a) – 1 (b) – 2 (c) – 3 (d) – 4

9. If $i^2=-1$, then sum $i+i^2+i^3+....\text{to } 1000$ terms is equal to **[Kerala (Engg.) 2002]**

(a) 1 (b) – 1 (c) i (d) 0

10. If $(1-i)^n=2^n$, then n **[Rajasthan PET 1990]**

(a) 1 (b) 0 (c) – 1 (d) None of these

11. If $\left(\frac{1+i}{1-i}\right)^m = 1,$ then the least integral value of m is **[IIT 1982; MNR 1984; UPSEAT 2001; MP PET 2002]**

(a) 2 (b) 4 (c) 8 (d) None of these

12. The least positive integer n which will reduce $\left(\frac{i-1}{i+1}\right)^n$ to a real number, is **[Roorkee 1998]**

(a) 2 (b) 3 (c) 4 (d) 5

13. $i^2 + i^4 + i^6 + ...$ upto $(2n+1)$ terms = **[EAMCET 1980; DCE 2000]**

(a) i (b) $-i$ (c) 1 (d) -1

14. The value of the sum $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals **[IIT 1998]**

(a) i (b) $i-1$ (c) $-i$ (d) 0

15. The value of $i^{1+3+5+...+(2n+1)}$ is **[AMU 1999]**

(a) i if n is even, $-i$ if n is odd (b) 1 if n is even, -1 if n is odd

(c) 1 if n is odd, i if n is even (d) i if n is even, -1 if n is odd

16. $i^{57} + \frac{1}{i^{125}}$, when simplified has the value **[Roorkee 1993]**

(a) 0 (b) $2i$ (c) $-2i$ (d) 2

17. The number $\frac{(1-i)^3}{1-i^3}$ is equal to **[Pb. CET 1991, Karnataka CET 1998]**

(a) i (b) -1 (c) 1 (d) -2

18. $(1+i)^6 + (1-i)^3 =$ **[Karnataka CET 1997; Kurukshetra CEE 1995]**

(a) $2+i$ (b) $2-10i$ (c) $-2+i$ (d) $-2-10i$

19. If $(a+ib)^5 = \alpha + i\beta$ then $(b+ia)^5$ is equal to

(a) $\beta + i\alpha$ (b) $\alpha - i\beta$ (c) $\beta - i\alpha$ (d) $-\alpha - i\beta$

20. For a positive integer n, the expression $(1-i)^n\left(1-\frac{1}{i}\right)^n$ equals **[AMU 1992]**

(a) 0 (b) $2i^n$ (c) 2^n (d) 4^n

21. The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = -1$ is **[Roorkee 1992]**

(a) 1 (b) 2 (c) 3 (d) 4

22. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer is **[Kurukshetra CEE 1992]**

(a) 2 (b) 4 (c) 8 (d) 16

Real and imaginary parts of complex numbers, Algebraic operations, Equality of two Complex numbers

Basic Level

23. The statement $(a+ib) < (c+id)$ is true for **[Rajasthan PET 2002]**

(a) $a^2 + b^2 = 0$ (b) $b^2 + c^2 = 0$ (c) $a^2 + c^2 = 0$ (d) $b^2 + d^2 = 0$

24. The true statement is **[Roorkee 1989]**

(a) $1 - i < 1+i$ (b) $2i + 1 > -2i + 1$ (c) $2i > 1$ (d) None of these

25. The complex number $\frac{1+2i}{1-i}$ lies in which quadrant of the complex plane **[MP PET 2001]**

(a) First (b) Second (c) Third (d) Fourth

26. If $|z|=1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), then $\text{Re}(\omega)$ is **[IIT Screening 2003; Rajasthan PET 1997]**

(a) 0 (b) $-\frac{1}{|z+1|^2}$ (c) $\left|\frac{z}{z+1}\right| . \frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

27. $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if θ = **[IIT 1976]**

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $n\pi + \frac{\pi}{3}$ (c) $n\pi \pm \frac{\pi}{3}$ (d) None of these

[Where n is an integer]

28. If $z \neq 0$ is a complex numbers, then

(a) $\text{Re}(z)=0 \Rightarrow \text{Im}(z^2)=0$ (b) $\text{Re}(z^2)=0 \Rightarrow \text{Im}(z^2)=0$ (c) $\text{Re}(z)=0 \Rightarrow \text{Re}(z^2)=0$ (d) None of these

29. If z_1 and z_2 be two complex numbers, then Re $(z_1 z_2)$ =

(a) Re (z_1). Re (z_2) (b) Re (z_1). Im (z_2) (c) Im (z_1). Re (z_2) (d) None of these

30. The real part of $\frac{1}{1-\cos\theta + i\sin\theta}$ is equal to **[Karnataka CET 2001]**

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\tan\frac{\theta}{2}$ (d) $\frac{1}{1-\cos\theta}$

31. The multiplication inverse of a number is the number itself, then its initial value is **[Rajasthan PET 2003]**

(a) i (b) -1 (c) 2 (d) $-i$

32. If $z = 1+i$, then the multiplicative inverse of z^2 is (where $i = \sqrt{-1}$) **[Karnataka CET 1999]**

(a) $2i$ (b) $1 - i$ (c) $-\frac{i}{2}$ (d) $\frac{i}{2}$

33. If $a = \cos\theta + i\sin\theta$, then $\frac{1+a}{1-a}$ =

(a) $\cot\theta$ (b) $\cot\frac{\theta}{2}$ (c) $i\cot\frac{\theta}{2}$ (d) $i\tan\frac{\theta}{2}$

34. If $z = x - iy$ and $z^{\frac{1}{3}} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{q}\right) \Big/ (p^2 + q^2)$ is equal to **[AIEEE 2004]**

(a) -2 (b) -1 (c) 2 (d) 1

35. If $(x+iy)^{1/3} = a+ib$, then $\frac{x}{a} + \frac{y}{b}$ is equal to **[IIT 1982; Karnataka CET 2000]**

(a) $4(a^2+b^2)$ (b) $4(a^2-b^2)$ (c) $4(b^2-a^2)$ (d) None of these

36. If $\sqrt{3} + i = (a+ib)(c+id)$, then $\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{d}{c}\right)$ has the value

(a) $\frac{\pi}{3} + 2n\pi,\ n \in I$ (b) $n\pi + \frac{\pi}{6},\ n \in I$ (c) $n\pi - \frac{\pi}{3},\ n \in I$ (d) $2n\pi - \frac{\pi}{3},\ n \in I$

37. Additive inverse of $1 - i$ is

(a) $0 + 0i$ (b) $-1 - i$ (c) $-1 + i$ (d) None of these

38. If $a^2 + b^2 = 1$, then $\frac{1+b+ia}{1+b-ia} =$

(a) 1 (b) 2 (c) $b + ia$ (d) $a + ib$

39. $\left|(1+i)\frac{(2+i)}{(3+i)}\right| =$ **[MP PET 1995, 99]**

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) -1

40. $\left\{\frac{2i}{1+i}\right\}^2 =$ **[BIT Ranchi 1992]**

(a) 1 (b) $2i$ (c) $1 - i$ (d) $1 - 2i$

41. If $Z_1 = (4, 5)$ and $Z_2 = (-3, 2)$, then $\frac{Z_1}{Z_2}$ equals **[Rajasthan PET 1996]**

(a) $\left(\frac{-23}{12}, \frac{-2}{13}\right)$ (b) $\left(\frac{2}{13}, \frac{-23}{13}\right)$ (c) $\left(\frac{-2}{13}, \frac{-23}{13}\right)$ (d) $\left(\frac{-2}{13}, \frac{23}{13}\right)$

42. If $x + \frac{1}{x} = 2\cos\theta$, then x is equal to **[Rajasthan PET 2001]**

(a) $\cos\theta + i\sin\theta$ (b) $\cos\theta - i\sin\theta$ (c) $\cos\theta \pm i\sin\theta$ (d) $\sin\theta \pm i\cos\theta$

43. The number of real values of a satisfying the equation $a^2 - 2a\sin x + 1 = 0$ is

(a) Zero (b) One (c) Two (d) Infinite

44. Solving $3 - 2yi = 9^x - 7i$, where $i^2 = -1$, for real x and y, we get **[AMU 2000]**

(a) $x = 0.5$, $y = 3.5$ (b) $x = 5$, $y = 3$ (c) $x = \frac{1}{2}$, $y = 7$ (d) $x = 0, y = \frac{3+7i}{2i}$

45. $\frac{1-i}{1+i}$ is equal to **[Rajasthan PET 1984]**

(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$ (c) $\sin\frac{\pi}{2} + i\cos\frac{\pi}{2}$ (d) None of these

46. The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$, are **[IIT 1980; MNR 1987, 88]**

(a) $x = -1, y = 3$ (b) $x = 3, y = -1$ (c) $x = 0, y = 1$ (d) $x = 1, y = 0$

47. If $x + iy = \frac{3}{2+\cos\theta + i\sin\theta}$, then $x^2 + y^2$ is equal to

(a) $3x - 4$ (b) $4x - 3$ (c) $4x + 3$ (d) None of these

48. If $\frac{5(-8+6i)}{(1+i)^2} = a + ib$, then (a, b) equals **[Rajasthan PET 1986]**

(a) (15, 20) (b) (20, 15) (c) (–15, 20) (d) None of these

49. If $x = -5 + 2\sqrt{-4}$, then the value of the expression $x^4 + 9x^3 + 35x^2 - x + 4$ is **[IIT 1972]**

(a) 160 (b) -160 (c) 60 (d) -60

50. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then (x, y) is **[MP PET 2000]**

(a) (3, 1) (b) (1, 3) (c) (0, 3) (d) (0, 0)

51. If $\left(\frac{1-i}{1+i}\right)^{100}=a+ib,$ then **[MP PET 1998]**

(a) $a=2, b=-1$ (b) $a=1,\ b=0$ (c) $a=0,\ b=1$ (d) $a=-1,\ b=2$

52. The real values of x and y for which the equation $(x+iy)(2-3i)=4+i$ is satisfied, are **[Roorkee 1978]**

(a) $x=\frac{5}{13}, y=\frac{8}{13}$ (b) $x=\frac{8}{13}, y=\frac{5}{13}$ (c) $x=\frac{5}{13}, y=\frac{14}{13}$ (d) None of these

53. The solution of the equation $|z|-z=1+2i$ is **[MP PET 1993, Kurukshetra CEE 1999]**

(a) $2-\frac{3}{2}i$ (b) $\frac{3}{2}+2i$ (c) $\frac{3}{2}-2i$ (d) $-2+\frac{3}{2}i$

54. Which of the following is not applicable for a complex number **[Kerala (Engg.) 1993; Assam JEE 1998; DCE 1999]**

(a) Addition (b) Subtraction (c) Division (d) Inequality

55. Multiplicative inverse of the non-zero complex number $x + iy$ $(x, y \in R)$ is

(a) $\frac{x}{x+y}-\frac{y}{x+y}i$ (b) $\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i$ (c) $-\frac{x}{x^2+y^2}+\frac{y}{x^2+y^2}i$ (d) $\frac{x}{x+y}+\frac{y}{x+y}i$

56. The real value of α for which the expression $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real, is **[Kurukshetra CEE 1995]**

(a) $(n+1)\frac{\pi}{2}$, where n is an integer (b) $(2n+1)\frac{\pi}{2}$, where n is an integer

(c) $n\pi$, where n is an integer (d) None of these

57. The real value of θ for which the expression $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number is **[Pb. CET 2000; IIIT Kolkata 2001]**

(a) $n\pi+\frac{\pi}{4}$ (b) $n\pi+(-1)^n\frac{\pi}{4}$ (c) $2n\pi\pm\frac{\pi}{2}$ (d) None of these

58. If $z(2-i)=3+i,$ then $z^{20}=$ **[Karnataka CET 2002]**

(a) $1 - i$ (b) $- 1024$ (c) 1024 (d) $1 + i$

59. If $z_1=\sqrt{3}+i\sqrt{3}$ and $z_2=\sqrt{3}+i,$ then the complex number $\left(\frac{z_1}{z_2}\right)$ lies in the quadrant number **[AMU 1991]**

(a) I (b) II (c) III (d) IV

60. If $\text{Re}\left(\frac{z-8i}{z+6}\right)=0,$ then z lies on the curve

(a) $x^2+y^2+6x-8y=0$ (b) $4x-3y+24=0$ (c) $x^2+y^2-8=0$ (d) None of these

Advance Level

61. If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1+z_2}{z_1-z_2}\right|=1,$ then $\frac{z_1}{z_2}$ is a number which is

(a) Positive real (b) Negative real (c) Zero or purely imaginary (d) None of these

62. If $z(1+a)=b+ic$ and $a^2+b^2+c^2=1,$ then $\frac{1+iz}{1-iz}=$

(a) $\frac{a+ib}{1+c}$ (b) $\frac{b-ic}{1+a}$ (c) $\frac{a+ic}{1+b}$ (d) None of these

63. Given that the equation $z^2+(p+iq)z+r+is=0$, where, p, q, r, s are real and non-zero has a real root, then **[DCE 1992]**

(a) $pqr=r^2+p^2s$ (b) $prs=q^2+r^2p$ (c) $qrs=p^2+s^2q$ (d) $pqs=s^2+q^2r$

64. If $\sum_{k=0}^{100} i^k=x+iy$, then the value of x and y are

(a) $x=-1, y=0$ (b) $x=1, y=1$ (c) $x=1, y=0$ (d) $x=0, y=1$

65. Let $\frac{1-ix}{1+ix}=a-ib$ and $a^2+b^2=1$, where a and b are real, then $x=$

(a) $\frac{2a}{(1+a)^2+b^2}$ (b) $\frac{2b}{(1+a)^2+b^2}$ (c) $\frac{2a}{(1+b)^2+a^2}$ (d) $\frac{2b}{(1+b)^2+a^2}$

66. If $\frac{(p+i)^2}{2p-i}=\mu+i\lambda$, then $\mu^2+\lambda^2$ is equal to

(a) $\frac{(p^2+1)^2}{4p^2-1}$ (b) $\frac{(p^2-1)^2}{4p^2-1}$ (c) $\frac{(p^2-1)^2}{4p^2+1}$ (d) $\frac{(p^2+1)^2}{4p^2+1}$

67. If $(1+i)(1+2i)(1+3i)......(1+ni)=a+ib$, then $2 \,.\, 5 \,.\, 10.........(1+n^2)$ is equal to **[Karnataka CET 2002; Kerala (Engg.) 2002]**

(a) a^2-b^2 (b) a^2+b^2 (c) $\sqrt{a^2+b^2}$ (d) $\sqrt{a^2-b^2}$

68. Given $z=\frac{q+ir}{1+p}$, then $\frac{p+iq}{1+r}=\frac{1+iz}{1-iz}$ if

(a) $p^2+q^2+r^2=1$ (b) $p^2+q^2+r^2=2$ (c) $p^2+q^2-r^2=1$ (d) None of these

Conjugate of a Complex Number

Basic Level

69. Conjugate of $1+i$ is **[Rajasthan PET 2003]**

(a) i (b) 1 (c) $1-i$ (d) $1+i$

70. The conjugate of the complex number $\frac{2+5i}{4-3i}$ is **[MP PET 1994]**

(a) $\frac{7-26i}{25}$ (b) $\frac{-7-26i}{25}$ (c) $\frac{-7+26i}{25}$ (d) $\frac{7+26i}{25}$

71. The conjugate of $\frac{(2+i)^2}{3+i}$, in the form of $a+ib$, is **[Karnataka CET 2001]**

(a) $\frac{13}{2}+i\left(\frac{15}{2}\right)$ (b) $\frac{13}{10}+i\left(\frac{-15}{2}\right)$ (c) $\frac{13}{10}+i\left(\frac{-9}{10}\right)$ (d) $\frac{13}{10}+i\left(\frac{9}{10}\right)$

72. If $x+iy=\sqrt{\frac{a+ib}{c+id}}$, then $(x^2+y^2)^2=$ **[IIT 1979; Rajasthan PET 1997; Karnataka CET 1999; BIT Ranchi 1993]**

(a) $\frac{a^2+b^2}{c^2+d^2}$ (b) $\frac{a+b}{c+d}$ (c) $\frac{c^2+d^2}{a^2+b^2}$ (d) $\left(\frac{a^2+b^2}{c^2+d^2}\right)^2$

73. If $(a+ib)(c+id)(e+if)(g+ih)=A+iB$, then $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)$ is equal to **[MNR 1989]**

(a) A^2+B^2 (b) A^2-B^2 (c) A^2 (d) B^2

74. If z is a complex number, then $z\,.\,\bar{z}=0$ if and only if

(a) $z = 0$ (b) Re $(z) = 0$ (c) Im $(z) = 0$ (d) None of these

75. Let z_1, z_2 be two complex numbers such that z_1+z_2 and z_1z_2 both are real, then **[Rajasthan PET 1996]**

(a) $z_1 = -z_2$ (b) $z_1 = \bar{z}_2$ (c) $z_1 = -\bar{z}_2$ (d) $z_1 = z_2$

76. For any complex number z, $\bar{z} = \left(\frac{1}{z}\right)$ if and only if **[Rajasthan PET 1985]**

(a) z is a pure real number (b) $|z| = 1$

(c) z is a pure imaginary number (d) $z = 1$

77. If $\frac{c+i}{c-i} = a + ib$, where a, b, c are real, then $a^2 + b^2 =$ **[MP PET 1996]**

(a) 1 (b) – 1 (c) c^2 (d) $-c^2$

78. If $z = 3 + 5i$, then $z^3 + \bar{z} + 198 =$ **[EAMCET 2002]**

(a) $-3-5i$ (b) $-3+5i$ (c) $3+5i$ (d) $3-5i$

79. If a complex number lies in the IIIrd quadrant then its conjugate lies in quadrant number **[AMU 1986, 89]**

(a) I (b) II (c) III (d) IV

80. If $z = x + iy$ lies in IIIrd quadrant then $\frac{\bar{z}}{z}$ also lies in the IIIrd quadrant if **[AMU 1990; Kurukshetra CEE 1993]**

(a) $x > y > 0$ (b) $x < y < 0$ (c) $y < x < 0$ (d) $y > x > 0$

81. If $(1+i)z = (1-i)\bar{z}$ then z is

(a) $t(1-i), t \in R$ (b) $t(1+i), t \in R$ (c) $\frac{t}{1+i}, t \in R$ (d) None of these

82. The value of $(z+3)(\bar{z}+3)$ is equivalent to **[JMIEE 2000]**

(a) $|z+3|^2$ (b) $|z-3|$ (c) $z^2 + 3$ (d) None of these

83. The set of values of $a \in R$ for which $x^2 + i(a-1)x + 5 = 0$ will have a pair of conjugate complex roots is

(a) R (b) $\{1\}$ (c) $\{a|\ a^2 - 2a + 21 > 0\}$ (d) None of these

Advance Level

84. The equation $z^2 = \bar{z}$ has **[DCE 1995]**

(a) No solution (b) Two solutions

(c) Four solutions (d) An infinite number of solutions

85. If $z_1 = 9y^2 - 4 - 10ix, z_2 = 8y^2 - 20i$, where $z_1 = \bar{z}_2$, then $z = x + iy$ is equal to

(a) $-2+2i$ (b) $-2 \pm 2i$ (c) $-2 \pm i$ (d) None of these

86. If α is a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root then

(a) $\alpha + \bar{\alpha} = 1$ (b) $\alpha + \bar{\alpha} = 0$

(c) $\alpha + \bar{\alpha} = -1$ (d) The absolute value of the real root is 1

Modulus of Complex Numbers

Basic Level

87. The value of $|z-5|$, if $z = x + iy$ is **[Rajasthan PET 1995]**

(a) $\sqrt{(x-5)^2+y^2}$ (b) $x^2+\sqrt{(y-5)^2}$ (c) $\sqrt{(x-y)^2+5^2}$ (d) $\sqrt{x^2+(y-5)^2}$

88. Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is **[Rajasthan PET 1996]**

(a) 1 (b) 1/2 (c) 2 (d) $\sqrt{2}$

89. The product of two complex numbers each of unit modulus is also a complex number, of

(a) Unit modulus (b) Less than unit modulus (c) Greater than unit modulus (d) None of these

90. The moduli of two complex numbers are less than unity, then the modulus of the sum of these complex numbers

(a) Less than unity (b) Greater than unity (c) Equal to unity (d) Any

91. If z is a complex number, then which of the following is not true **[MP PET 1987]**

(a) $|z^2| = |z|^2$ (b) $|z^2| = |\bar{z}|^2$ (c) $z = \bar{z}$ (d) $\bar{z}^2 = \overline{z^2}$

92. The values of z for which $|z+i| = |z-i|$ are **[Bihar CEE 1994]**

(a) Any real number (b) Any complex number (c) Any natural number (d) None of these

93. If z is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then **[MP PET 1998, 2002]**

(a) $|z|=0$ (b) $|z|=1$ (c) $|z|>1$ (d) $|z|<1$

94. The minumum value of $|2z-1|+|3z-2|$ is **[Rajasthan PET 1997]**

(a) 0 (b) 1/2 (c) 1/3 (d) 2/3

95. If z_1 and z_2 are any two complex numbers then $|z_1+z_2|^2+|z_1-z_2|^2$ is equal to **[MP PET 1993]**

(a) $2|z_1|^2|z_2|^2$ (b) $2|z_1|^2+2|z_2|^2$ (c) $|z_1|^2+|z_2|^2$ (d) $2|z_1||z_2|$

96. If $\frac{2z_1}{3z_2}$ is a purely imaginary number, then $\left|\frac{z_1-z_2}{z_1+z_2}\right|$ is equal to **[MP PET 1993]**

(a) 3/2 (b) 1 (c) 2/3 (d) 4/9

97. If z_1, z_2, z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then $|z_1+z_2+z_3|$ is **[IIT Screening 2000]**

(a) Equal to 1 (b) Less than 1 (c) Greater than 3 (d) Equal to 3

98. If z_1 and z_2 are any two complex numbers, then $|z_1+\sqrt{z_1^2-z_2^2}|+|z_1-\sqrt{z_1^2-z_2^2}|$ is equal to

(a) $|z_1|$ (b) $|z_2|$ (c) $|z_1+z_2|$ (d) $|z_1+z_2|+|z_1-z_2|$

99. Find the complex number z satisfying the equations $\left|\frac{z-12}{z-8i}\right|=\frac{5}{3}, \left|\frac{z-4}{z-8}\right|=1$ **[Roorkee 1993]**

(a) 6 (b) $6\pm 8i$ (c) $6+8i, 6+17i$ (d) None of these

100. A real value of x will satisfy the equation $\left(\frac{3-4ix}{3+4ix}\right)=\alpha-i\beta\,(\alpha,\beta \text{ real})$, if **[Orissa JEE 2003]**

(a) $\alpha^2-\beta^2=-1$ (b) $\alpha^2-\beta^2=1$ (c) $\alpha^2+\beta^2=1$ (d) $\alpha^2-\beta^2=2$

101. The inequality $|z-4|<|z-2|$ represents the region given by **[IIT 1982; Rajasthan PET 1995, 98; AIEEE 2002; DCE 2002]**

(a) Re $(z)>0$ (b) Re $(z)<0$ (c) Re $(z)>2$ (d) None of these

102. If $z = 1 + i\tan\alpha$, where $\pi < \alpha < \frac{3\pi}{2}$, then | z | is equal to

(a) $\sec\alpha$ (b) $-\sec\alpha$ (c) $\text{cosec}\,\alpha$ (d) None of these

103. If z is a non-zero complex number then $\left|\frac{|\bar{z}|^2}{z\bar{z}}\right|$ is equal to

(a) $\left|\frac{\bar{z}}{z}\right|$ (b) 1 (c) $|\bar{z}|$ (d) None of these

104. If z is a complex number, then **[DCE 1997, Kurukshetra CEE 1995]**

(a) $|z^2| > |z|^2$ (b) $|z^2| = |z|^2$ (c) $|z^2| < |z|^2$ (d) $|z^2| \geq |z|^2$

105. If $z_1 \neq -z_2$ and $|z_1 + z_2| = \left|\frac{1}{z_1} + \frac{1}{z_2}\right|$ then

(a) At least one of z_1, z_2 is unimodular (b) Both z_1, z_2 are unimodular

(c) $z_1 . z_2$ is unimodular (d) None of these

106. Let z be a complex number of constant modulus such that z^2 is purely imaginary then the number of possible values of z is

(a) 2 (b) 1 (c) 4 (d) Infinite

107. Number of solutions of the equation $z^2 + |z|^2 = 0$ where $z \in C$ is **[Karnataka CET 1997; Pb. CET 2001]**

(a) 1 (b) 2 (c) 3 (d) Infinitely many

108. If | z | = Max. { | z – 2 |, | z + 2 | }, then

(a) $|z + \bar{z}| = 1$ (b) $z + \bar{z} = 2^2$ (c) $|z + \bar{z}| = 2$ (d) None of these

109. The modulus of $\sqrt{2i} - \sqrt{-2i}$ is **[EAMCET 1995]**

(a) 2 (b) $\sqrt{2}$ (c) 0 (d) $2\sqrt{2}$

Advance Level

110. If z is a complex number, then the minimum value of | z | + | z – 1 | is **[Roorkee 1992]**

(a) 1 (b) 0 (c) 1/2 (d) None of these

111. The maximum value of | z | where z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$ (c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

112. If $|z + 4| \leq 3$, then the greatest and the least value of | z + 1 | are **[Rajasthan PET 2002; Karnataka CET 1995; DCE 1999]**

(a) 6, – 6 (b) 6, 0 (c) 7, 2 (d) 0, – 1

113. Let z be a complex number, then the equation $z^4 + z + 2 = 0$ cannot have a root, such that

(a) | z | < 1 (b) | z | = 1 (c) | z | > 1 (d) None of these

114. Let z and w be two complex numbers such that $|z| \leq 1$, $|w| \leq 1$ and $|z + iw| = |z - i\bar{w}| = 2$. Then z is equal to **[IIT 1995]**

(a) 1 or i (b) i or – i (c) 1 or – 1 (d) i or – 1

115. If $|z_1| = |z_2| = \ldots\ldots\ldots\ldots = |z_n| = 1$, then the value of $|z_1 + z_2 + z_3 + \ldots\ldots\ldots\ldots\ldots + z_n| =$

(a) 1 (b) $|z_1| + |z_2| + ... + |z_n|$ (c) $\left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right|$ (d) None of these

116. If z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{(z_1 + z_2)}{(z_1 - z_2)}$ may be **[IIT 1986]**

(a) Purely imaginary (b) Real and positive (c) Real and negative (d) None of these

117. For any two complex numbers z_1 and z_2 and any real numbers a and b; $|(az_1 - bz_2)|^2 + |(bz_1 + az_2)|^2 =$ **[IIT 1988]**

(a) $(a^2 + b^2)(|z_1| + |z_2|)$ (b) $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$ (c) $(a^2 + b^2)(|z_1|^2 - |z_2|^2)$ (d) None of these

118. If $|a_k| < 1, \lambda_k \geq 0$ for k = 1, 2, ... n and $\lambda_1 + \lambda_2 + ... + \lambda_n = 1$, then the value of $|\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_n a_n|$ is

(a) Equal to one (b) Greater than one (c) Zero (d) Less than one

119. If z_1, z_2, z_3, z_4 are roots of the equation $a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$,

where a_0, a_1, a_2, a_3 and a_4 are real, then

(a) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of the equation (b) z_1 is equal to at least one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$

(c) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of the equation (d) None of these

120. If z satisfies $|z + 1| < |z - 2|$, then $w = 3z + 2 + i$ **[MP PET 1998]**

(a) $|w + 1| < |w - 8|$ (b) $|w + 1| < |w - 7|$ (c) $w + \bar{w} > 7$ (d) $|w + 5| < |w - 4|$

121. $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$ then $|z_1 + z_2 + z_3|$

(a) Is less than 6 (b) Is more than 3 (c) Is less than 12 (d) Lies between 6 and 12

122. If $|z - 4 + 3i| \leq 1$ and m and n be the least and greatest values of $|z|$ and K be the least value of $\frac{x^4 + x^2 + 4}{x}$ on the interval $(0, \infty)$, then K =

(a) n (b) m (c) $m + n$ (d) None of these

123. The system of equations $|z + 1 - i| = \sqrt{2}$ and $|z| = 3$ has

(a) No solution (b) One solution (c) Two solutions (d) None of these

Amplitude (Argument) of Complex Numbers

Basic Level

124. The amplitude of 0 is **[Rajasthan PET 2000]**

(a) 0 (b) $\pi/2$ (c) π (d) None of these

125. The argument of the complex number $-1 + i\sqrt{3}$ is **[MP PET 1994]**

(a) – 60° (b) 60° (c) 120° (d) – 120°

126. Argument of $-1 - i\sqrt{3}$ is **[Rajasthan PET 2003]**

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$

127. The amplitude of $\frac{1 + \sqrt{3}\, i}{\sqrt{3} + i}$ is **[DCE 1999]**

(a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) None of these

128. The amplitude of $\frac{1 + \sqrt{3}\, i}{\sqrt{3} + 1}$ is **[Karnataka CET 1992]**

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

129. The argument of the complex number $\frac{13-5i}{4-9i}$ is **[MP PET 1997]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{5}$ (d) $\frac{\pi}{6}$

130. If $z=\frac{-2}{1+\sqrt{3}\,i}$ then the value of $arg(z)$ is **[Orissa JEE 2002]**

(a) π (b) $\pi/3$ (c) $2\pi/3$ (d) $\pi/4$

131. If $z=\frac{1-i\sqrt{3}}{1+i\sqrt{3}}$, then $arg\ (z)$ = **[Roorkee 1990]**

(a) 60° (b) 120° (c) 240° (d) 300°

132. The amplitude of $\frac{1+\sqrt{3}\,i}{\sqrt{3}-i}$ is **[Rajasthan PET 2001]**

(a) 0 (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/2$

133. If $z=1-\cos\alpha+i\sin\alpha$, then *amp z* =

(a) $\frac{\alpha}{2}$ (b) $-\frac{\alpha}{2}$ (c) $\frac{\pi}{2}+\frac{\alpha}{2}$ (d) $\frac{\pi}{2}-\frac{\alpha}{2}$

134. If $z=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$, then **[AMU 2002]**

(a) $|z|=1,\ arg\,z=\frac{\pi}{4}$ (b) $|z|=1,\ arg\,z=\frac{\pi}{6}$ (c) $|z|=\frac{\sqrt{3}}{2}, arg\,z=\frac{5\pi}{24}$ (d) $|z|=\frac{\sqrt{3}}{2}, arg\,z=\tan^{-1}\frac{1}{\sqrt{2}}$

135. Argument and modulus of $\frac{1+i}{1-i}$ are respectively **[Rajasthan PET 1984; MP PET 1987; Karnataka CET 2001]**

(a) $-\frac{\pi}{2}$ and 1 (b) $\frac{\pi}{2}$ and $\sqrt{2}$ (c) 0 and $\sqrt{2}$ (d) $\frac{\pi}{2}$ and 1

136. If $arg\ (z)=\theta$, then $arg\ (\bar{z})$ = **[MP PET 1995]**

(a) θ (b) $-\theta$ (c) $\pi-\theta$ (d) $\theta-\pi$

137. If $arg\,z<0$ then $arg(-z)-arg(z)$ is equal to **[IIT Screening 2000]**

(a) π (b) $-\pi$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

138. Let z and w be the two non-zero complex numbers such that $|z|=|w|$ and $arg\ z + arg\ w = \pi$. Then z is equal to

[IIT 1995; AIEEE 2002]

(a) w (b) $-w$ (c) $\overline{w}$ (d) $-\overline{w}$

139. If z is a complex number, then the principal value of $arg\ (z)$ lies between

(a) $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ (b) $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (c) $-\pi$ and π (d) None of these

140. The principal value of the argument of the complex number $-3i$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) $-\frac{\pi}{2}$ (d) None of these

141. If $|z_1+z_2|=|z_1-z_2|$, then the difference in the amplitudes of z_1 and z_2 is **[EAMCET 1985]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) 0

142. If z_1 and z_2 are two non-zero complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$, then *arg* (z_1) – *arg* (z_2) is equal to

[IIT 1979, 87; EAMCET 1986; Rajasthan PET 1997; MP PET 1999, 2001]

(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) 0

143. If $z_1, z_2 \ldots\ldots z_n = z$, then $arg z_1 + arg z_2 + \ldots\ldots + arg z_n$ and *arg z* differ by a

(a) Multiple of π (b) Multiple of $\frac{\pi}{2}$ (c) Greater than π (d) Less than π

144. If *z* is a purely real number such that Re (*z*) < 0, then *arg* (*z*) is equal to

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $-\frac{\pi}{2}$

145. Let *z* be a purely imaginary number such that Im (*z*) < 0. then *arg* (*z*) is equal to

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $-\frac{\pi}{2}$

146. If $\bar{z}$ be the conjugate of the complex number *z*, then which of the following relations is false **[MP PET 1987]**

(a) $|z|=|\bar{z}|$ (b) $z.\bar{z}=|\bar{z}|^2$ (c) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$ (d) *arg z* = *arg* $\bar{z}$

147. Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha+\beta>\pi$, then principal *arg* ($z_1 z_2$) is given by **[Roorkee 1989]**

(a) $\alpha+\beta+\pi$ (b) $\alpha+\beta-\pi$ (c) $\alpha+\beta-2\pi$ (d) $\alpha+\beta$

148. If *z* = –1, then the principal value of the *arg* $\left(z^{2/3}\right)$ is equal to **[IIT 1991, Kurukshetera CEE 1998]**

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{10\pi}{3}$ (d) π

149. If *z* is any complex number satisfying | *z* – 1 | = 1, then which of the following is correct **[EAMCET 1999]**

(a) *arg* (*z* – 1) = 2 *arg z* (b) $2arg(z)=\frac{2}{3}arg(z^2-z)$ (c) $arg(z-1)=arg(z+1)$ (d) $arg z=2\,arg(z+1)$

150. If *z* = *x* + *iy* satisfies *amp* (*z* –1) = *amp* (*z* + 3*i*) then the value of (*x* – 1) : *y* is equal to

(a) 2 : 1 (b) 1 : 3 (c) – 1 : 3 (d) None of these

151. If $z(2-i2\sqrt{3})^2=i(\sqrt{3}+i)^4$ then amplitude of *z* is

(a) $\frac{5\pi}{6}$ (b) $-\frac{\pi}{6}$ (c) $\frac{\pi}{6}$ (d) $\frac{7\pi}{6}$

Advance Level

152. If complex number $z=x+iy$ is taken such that the amplitude of fraction $\frac{z-1}{z+1}$ is always $\frac{\pi}{4}$, then

[UPSEAT 1999]

(a) $x^2+y^2+2y=1$ (b) $x^2+y^2-2y=0$ (c) $x^2+y^2+2y=-1$ (d) $x^2+y^2-2y=1$

153. If $z_1=10+6i$, $z_2=4+6i$ and *z* is a complex number such that *amp* $\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{4}$, then the value of | *z* – 7 – 9*i* | is equal to

[IIT 1990]

(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $2\sqrt{3}$

154. If $z_1 = 8 + 4i$, $z_2 = 6 + 4i$ and $arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$, then z satisfies **[IIT 1993]**

(a) $|z - 7 - 4i| = 1$ (b) $|z - 7 - 5i| = \sqrt{2}$ (c) $|z - 4i| = 8$ (d) $|z - 7i| = \sqrt{18}$

155. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $arg\left(\frac{z_1}{z_4}\right) + arg\left(\frac{z_2}{z_3}\right)$ equals

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

156. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $R(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies **[IIT 1985; UPSEAT 1996]**

(a) $|w_1| = 1$ (b) $|w_2| = 1$ (c) $R(w_1 \bar{w}_2) = 0$ (d) All the above

157. If z_1, z_2, z_3 be three non-zero complex numbers, such that $z_2 \neq z_1$, $a = |z_1|$, $b = |z_2|$ and $c = |z_3|$.

Suppose that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then $arg\left(\frac{z_3}{z_2}\right)$ is equal to

(a) $arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right)^2$ (b) $arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right)$ (c) $arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$ (d) $arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

158. If $amp\ \frac{z - 2}{2z + 3i} = 0$ and $z_0 = 3 + 4i$ then

(a) $z_0\bar{z} + \bar{z}_0 z = 12$ (b) $z_0 z + \bar{z}_0 \bar{z} = 12$ (c) $z_0\bar{z} + \bar{z}_0 z = 0$ (d) None of these

159. The principal value of the $arg(z)$ and $|z|$ of the complex number $z = 1 + \cos\left(\frac{11\pi}{9}\right) + i\sin\frac{11\pi}{9}$ are respectively

(a) $\frac{11\pi}{8}, 2\cos\left(\frac{\pi}{18}\right)$ (b) $-\frac{7\pi}{18}, -2\cos\left(\frac{11\pi}{18}\right)$ (c) $\frac{2\pi}{9}, 2\cos\left(\frac{7\pi}{18}\right)$ (d) $-\frac{\pi}{9}, -2\cos\left(\frac{\pi}{18}\right)$

160. If $amp\ (z_1 z_2) = 0$ and $|z_1| = |z_2| = 1$ then

(a) $z_1 + z_2 = 0$ (b) $z_1 z_2 = 1$ (c) $z_1 = \bar{z}_2$ (d) None of these

161. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then

(a) $\frac{z_1}{z_2}$ is purely real (b) $\frac{z_1}{z_2}$ is purely imaginary (c) $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$ (d) $amp\ \frac{z_1}{z_2} = \frac{\pi}{2}$

162. Let $z_1 = \frac{(\sqrt{3} + i)^2.(1 - \sqrt{3}\,i)}{1 + i}, z_2 = \frac{(1 + \sqrt{3}i)^2.(\sqrt{3} - i)}{1 - i}$. Then

(a) $|z_1| = |z_2|$ (b) $amp\ z_1 + amp\ z_2 = 0$ (c) $3|z_1| = |z_2|$ (d) $3\ amp\ z_1 + amp\ z_2 = 0$

163. If z_1 and z_2 both satisfy $z + \bar{z} = 2|z - 1|$ and $arg\ (z_1 - z_2) = \frac{\pi}{4}$, then the imaginary part of $(z_1 + z_2)$ is

(a) 0 (b) 1 (c) 2 (d) None of these

164. If $z=\frac{(z_1+\bar{z}_2)z_1}{z_2\bar{z}_1}$, where $z_1=1+2i$ and $z_2=1-i$, then

(a) $|z|=\frac{1}{2}\sqrt{26}$, $\arg z=-\pi+\tan^{-1}\frac{19}{17}$ (b) $|z|=\frac{1}{2}\sqrt{26}$, $\arg z=\tan^{-1}\frac{19}{17}$

(c) $|z|=\frac{1}{2}\sqrt{15}$, $\arg z=\tan^{-1}\frac{19}{17}$ (d) $\arg z=-\pi+\tan^{-1}\frac{19}{17}; |z|=\frac{1}{3}\sqrt{26}$

165. If $(a_1+ib_1)(a_2+ib_2)....(a_n+ib_n)=A+iB$, then $\sum_{i=1}^{n}\tan^{-1}\left(\frac{b_i}{a_i}\right)$ is equal to

(a) $\frac{B}{A}$ (b) $\tan\left(\frac{B}{A}\right)$ (c) $\tan^{-1}\left(\frac{B}{A}\right)$ (d) $\tan^{-1}\left(\frac{A}{B}\right)$

Square Root of Complex Numbers

Basic Level

166. A square root of $2i$ is

(a) $\sqrt{2}\,i$ (b) $\sqrt{2}(1+i)$ (c) $1 + i$ (d) None of these

167. If $\sqrt{-8-6i}=$ **[Roorkee 1979; Rajasthan PET 1992]**

(a) $1\pm 3i$ (b) $\pm(1-3i)$ (c) $\pm(1+3i)$ (d) $\pm(3-i)$

168. If $\sqrt{a+ib}=x+iy$, then possible value of $\sqrt{a-ib}$ is **[Kerala (Engg.) 2002]**

(a) x^2+y^2 (b) $\sqrt{x^2+y^2}$ (c) $x+iy$ (d) $x-iy$

169. If $(-7-24i)^{1/2}=x-iy$, then $x^2+y^2=$ **[Rajasthan PET 1989]**

(a) 15 (b) 25 (c) – 25 (d) None of these

170. If $\sqrt{x+iy}=\pm(a+ib)$, then $\sqrt{-x-iy}$ is equal to

(a) $\pm(b+ia)$ (b) $\pm(a-ib)$ (c) $\pm(b-ia)$ (d) None of these

171. A value of $\sqrt{i}+\sqrt{-i}$ is **[AMU 1985]**

(a) 0 (b) $\sqrt{2}$ (c) $-i$ (d) i

172. Given that the real parts of $\sqrt{5+12i}$ and $\sqrt{5-12i}$ are negative. Then the number $z=\frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$ reduces to **[Roorkee 1989]**

(a) $\frac{3}{2}i$ (b) $-\frac{3}{2}i$ (c) $-3+\frac{2}{5}i$ (d) None of these

Representation of Complex Numbers

Basic Level

173. If $x+\frac{1}{x}=\sqrt{3}$, then $x=$ **[Rajasthan PET 2002]**

(a) $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$ (b) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ (c) $\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}$ (d) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

174. $\sqrt{3}+i=$ **[MP PET 1999]**

(a) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ (b) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ (c) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (d) None of these

175. If $(1+i\sqrt{3})^9=a+ib,$ then b is equal to **[Rajasthan PET 1995]**

(a) 1 (b) 256 (c) 0 (d) 9^3

176. If $x=\cos\theta+i\sin\theta$ and $y=\cos\phi+i\sin\phi$, then $x^m y^n+x^{-m}y^{-n}$ is equal to

(a) $\cos(m\theta+n\phi)$ (b) $\cos(m\theta-n\phi)$ (c) $2\cos(m\theta+n\phi)$ (d) $2\cos(m\theta-n\phi)$

177. If $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then **[MP PET 1997]**

(a) Re (z) = 0 (b) Im (z) = 0 (c) Re (z) > 0, Im (z) > 0 (d) Re (z) > 0, Im (z) < 0

178. If $z=re^{i\theta}$, then $|e^{iz}|=$

(a) $e^{r\sin\theta}$ (b) $e^{-r\sin\theta}$ (c) $e^{-r\cos\theta}$ (d) $e^{r\cos\theta}$

179. $\left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n=$

(a) $\cos n\phi-i\sin n\phi$ (b) $\cos n\phi+i\sin n\phi$ (c) $\sin n\phi+i\cos n\phi$ (d) $\sin n\phi-i\cos n\phi$

180. If n is a positive integer, then $(1+i)^n+(1-i)^n$ is equal to **[Orissa JEE 2003]**

(a) $(\sqrt{2})^{n-2}\cos\left(\frac{n\pi}{4}\right)$ (b) $(\sqrt{2})^{n-2}\sin\left(\frac{n\pi}{4}\right)$ (c) $(\sqrt{2})^{n+2}\cos\left(\frac{n\pi}{4}\right)$ (d) $(\sqrt{2})^{n+2}\sin\left(\frac{n\pi}{4}\right)$

181. If $y=\cos\theta+i\sin\theta$, then the value of $y+\frac{1}{y}$ is **[Rajasthan PET 1995]**

(a) $2\cos\theta$ (b) $2\sin\theta$ (c) $2\,\text{cosec}\,\theta$ (d) $2\tan\theta$

182. The polar form of the complex number $(i^{25})^3$ is **[Tamilnadu Engg. 2002]**

(a) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ (b) $\cos\pi+i\sin\pi$ (c) $\cos\pi-i\sin\pi$ (d) $\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$

Advance Level

183. The amplitude of $e^{e^{-i\theta}}$ is equal to **[Rajasthan PET 1997]**

(a) $\sin\theta$ (b) $-\sin\theta$ (c) $e^{\cos\theta}$ (d) $e^{\sin\theta}$

184. The real part of $\sin^{-1}(e^{i\theta})$ is

(a) $\cos^{-1}(\sqrt{\sin\theta})$ (b) $\sinh^{-1}(\sqrt{\sin\theta})$ (c) $\sin^{-1}(\sqrt{\sin\theta})$ (d) $\sin^{-1}(\sqrt{\cos\theta})$

Logarithm of Complex Number

Basic Level

185. The real part of $(1-i)^{-i}$ is **[Rajasthan PET 1999]**

(a) $e^{-\pi/4}\cos\left(\frac{1}{2}\log 2\right)$ (b) $-e^{-\pi/4}\sin\left(\frac{1}{2}\log 2\right)$ (c) $e^{\pi/4}\cos\left(\frac{1}{2}\log 2\right)$ (d) $e^{-\pi/4}\sin\left(\frac{1}{2}\log 2\right)$

186. If $z = i\log(2-\sqrt{3})$, then $\cos z =$ **[Rajasthan PET 2001; Karnataka CET 2002; EAMCET 1991]**

(a) i (b) $2i$ (c) 1 (d) 2

187. The imaginary part of $\tan^{-1}\left(\frac{5i}{3}\right)$ is **[Rajasthan PET 1997]**

(a) 0 (b) ∞ (c) $\log 2$ (d) $\log 4$

188. The expression $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]$ reduces to

(a) $\frac{ab}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$ (c) $\frac{ab}{a^2-b^2}$ (d) $\frac{2ab}{a^2+b^2}$

189. If $\log_{\tan 30^o}\left(\frac{2|z|^2+2|z|-3}{|z|+1}\right) < -2$, then

(a) $|z| < 3/2$ (b) $|z| > 3/2$ (c) $|z| < 2$ (d) $|z| > 2$

190. If $\sin(\log i^i) = a + ib$, then a and b are respectively

(a) – 1, 0 (b) 0, –1 (c) 1, 0 (d) 0, 1

191. The general value of $\log_2 (5i)$ is

(a) $\left\{\log 5 + 2\pi ni + \frac{i\pi}{2}\right\}$ (b) $\frac{1}{\log 2}\left\{\log 5 + 2\pi ni + \frac{i\pi}{2}\right\}$ (c) $-\frac{1}{\log 2}\left\{\log 5 + 2\pi ni - \frac{i\pi}{2}\right\}$ (d) None of these

Geometry of Complex Numbers, Rotation Theorem

Basic Level

192. $R(z^2) = 1$ is represented by

(a) The parabola $x^2 + y^2 = 1$ (b) The hyperbola $x^2 - y^2 = 1$

(c) Parabola or a circle (d) All the above

193. If $z = x + iy$ and $w = \frac{1-iz}{z-i}$, then $|w| = 1$ implies that **[Rajasthan PET 1985, 97; IIT 1983; DCE 2000, 01; UPSEAT 2003]**

(a) z lies on the imaginary axis (b) z lies on the real axis

(c) z lies on the unit circle (d) None of these

194. If $|z| = 2$, then the points representing the complex numbers $-1 + 5z$ will lie on a

(a) Circle (b) Straight line (c) Parabola (d) None of these

195. The equation $\bar{b}z + b\bar{z} = c$, where b is a non-zero complex constant and c is real, represents

(a) A circle (b) A straight line (c) A parabola (d) None of these

196. If $z = x + iy$ and $|z - zi| = 1$, then **[Rajasthan PET 1988, 1991]**

(a) z lies on x – axis (b) z lies on y – axis (c) z lies on circle (d) None of these

197. If three complex numbers are in A.P., then they lie on **[IIT 1985; DCE 1994, 2001]**

(a) A circle in the complex plane (b) A straight line in the complex plane

(c) A parabola in the complex plane (d) None of these

198. Length of the line segment joining the points – 1 – i and 2 + 3i is

(a) – 5 (b) 15 (c) 5 (d) 25

199. The equation $z\bar{z}+a\bar{z}+\bar{a}z+b=0,\ b\in R$ represents a circle if

(a) $|a|^2=b$ (b) $|a|^2>b$ (c) $|a|^2<b$ (d) None of these

200. If the complex numbers z_1,z_2,z_3 represent the vertices of an equilateral triangle such that $|z_1|=|z_2|=|z_3|$, then $z_1+z_2+z_3=$

[IIT 1984]

(a) 0 (b) 1 (c) – 1 (d) None of these

201. If z_1,z_2,z_3 are affixes of the vertices *A, B* and *C* respectively of a triangle *ABC* having centroid at *G* such that *z* = 0 is the mid point of *AG,* then

(a) $z_1+z_2+z_3=0$ (b) $z_1+4z_2+z_3=0$ (c) $z_1+z_2+4z_3=0$ (d) $4z_1+z_2+z_3=0$

202. For all complex numbers z_1,z_2 satisfying $|z_1|=12$ *and* $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is**[IIT Screening 200**

(a) 0 (b) 2 (c) 7 (d) 17

203. If the points z_1,z_2,z_3 are the vertices of an equilateral triangle in the complex plane, then the value of $z_1^2+z_2^2+z_3^2$ is equal to

(a) $\frac{z_1}{z_2}+\frac{z_2}{z_3}+\frac{z_3}{z_1}$ (b) $z_1z_2+z_2z_3+z_3z_1$ (c) $z_1z_2-z_2z_3-z_3z_1$ (d) $\frac{-z_1}{z_2}-\frac{z_2}{z_3}-\frac{z_3}{z_1}$

204. Let *z* be a complex number. Then the angle between vectors *z* and – *iz* is

(a) π (b) 0 (c) $-\frac{\pi}{2}$ (d) None of these

205. If $e^{i\theta}=\cos\theta+i\sin\theta$ then for the $\Delta ABC, e^{iA}.e^{iB}.e^{iC}$ is

(a) – i (b) 1 (c) – 1 (d) None of these

206. If $\text{Re}\left(\frac{z+4}{2z-i}\right)=\frac{1}{2}$, then *z* is represented by a point lying on

(a) A circle (b) An ellipse (c) A straight line (d) None of these

207. Let z_1 and z_2 be two complex numbers such that $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$. then **[Kurukshetra CEE 1997]**

(a) z_1,z_2 are collinear (b) z_1,z_2 and the origin form a right angled triangle

(c) z_1,z_2 and the origin form an equilateral triangle (d) None of these

208. The equation not representing a circle is given by **[IIT 1991; DCE 1993]**

(a) $\text{Re}\left(\frac{1+z}{1-z}\right)=0$ (b) $z\bar{z}+iz-i\bar{z}+1=0$ (c) *arg* $\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}$ (d) $\left|\frac{z-1}{z+1}\right|=1$

209. Let z_1 and z_2 be two non-real complex cube roots of unity and $|z-z_1|^2+|z-z_2|^2=\lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter then the value of λ is

(a) 4 (b) 3 (c) 2 (d) $\sqrt{2}$

210. Let α and β be two fixed non-zero complex numbers and 'z' a variable complex number. If the lines $\alpha\bar{z}+\bar{\alpha}z+1=0$ and $\beta\bar{z}+\bar{\beta}z-1=0$ are mutually perpendicular, then

(a) $\alpha\beta+\bar{\alpha}\bar{\beta}=0$ (b) $\alpha\beta-\bar{\alpha}\bar{\beta}=0$ (c) $\bar{\alpha}\beta-\alpha\bar{\beta}=0$ (d) $\alpha\bar{\beta}+\bar{\alpha}\beta=0$

211. If P, P' represent the complex number z_1 and its additive inverse respectively, then the complex equation of the circle with PP' as a diameter is

(a) $\frac{z}{z_1}=\left(\frac{\bar{z}_1}{z}\right)$ (b) $z\bar{z}+z_1\bar{z}_1=0$ (c) $z\bar{z}_1+\bar{z}z_1=0$ (d) None of these

212. The triangle formed by the points 1, $\frac{1+i}{\sqrt{2}}$ and i as vertices in the Argand diagram is **[EAMCET 1995]**

(a) Scalene (b) Equilateral (c) Isosceles (d) Right-angled

213. If P, Q, R, S are represented by the complex numbers $4+i$, $1+6i$, $-4+3i$, $-1-2i$ respectively, then $PQRS$ is a **[Orissa JEE 2003]**

(a) Rectangle (b) Square (c) Rhombus (d) Parallelogram

214. Let A, B and C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then the orthocentre is represented by the complex number

(a) $z_1+z_2-z_3$ (b) $z_2+z_3-z_1$ (c) $z_3+z_1-z_2$ (d) $z_1+z_2+z_3$

215. Multiplying a complex numbers by i rotates the vector representing the complex number through an angle of

(a) 180° (b) 90° (c) 60° (d) 360°

Advance Level

216. Let z be a complex number satisfying $|z-5i|\le 1$ such that *amp z* is minimum. Then z is equal to

(a) $\frac{2\sqrt{6}}{5}+\frac{24i}{5}$ (b) $\frac{24}{5}+\frac{2\sqrt{6}i}{5}$ (c) $\frac{2\sqrt{6}}{5}-\frac{24i}{5}$ (d) None of these

217. If ω is a complex number satisfying $\left|\omega+\frac{1}{\omega}\right|=2,$ then maximum distance of ω from origin is

(a) $2+\sqrt{3}$ (b) $1+\sqrt{2}$ (c) $1+\sqrt{3}$ (d) None of these

218. If $|z-25i|\le 15$, then $|\max.\ amp\,(z)-\min.\ amp\,(z)|=$

(a) $\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\pi-2\cos^{-1}\left(\frac{3}{5}\right)$ (c) $\frac{\pi}{2}+\cos^{-1}\left(\frac{3}{5}\right)$ (d) $\sin^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{3}{5}\right)$

219. If z_1, z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$ and $iz_1=kz_2$, where $k\in R,$ then the angle between z_1-z_2 and z_1+z_2 is

(a) $\tan^{-1}\left(\frac{2k}{k^2+1}\right)$ (b) $\tan^{-1}\left(\frac{2k}{1-k^2}\right)$ (c) $-2\tan^{-1}k$ (d) $2\tan^{-1}k$

220. If at least one value of the complex number $z=x+iy$ satisfy the condition $|z+\sqrt{2}|=a^2-3a+2$ and the inequality $|z+i\sqrt{2}|<a^2$, then

(a) $a>2$ (b) $a=2$ (c) $a<2$ (d) None of these

221. The maximum distance from the origin of coordinates to the point z satisfying the equation $\left|z+\frac{1}{z}\right|=a$ is

(a) $\frac{1}{2}(\sqrt{a^2+1}+a)$ (b) $\frac{1}{2}(\sqrt{a^2+2}+a)$ (c) $\frac{1}{2}(\sqrt{a^2+4}+a)$ (d) None of these

222. Let a be a complex number such that $|a| < 1$ and z_1, z_2 .... be vertices of a polygon such that $z_k = 1 + a + a^2 + ... + a^{k-1}$. Then the vertices of the polygon lie within a circle

(a) $|z - a| = a$ (b) $\left|z - \frac{1}{1-a}\right| = |1-a|$ (c) $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$ (d) $|z - (1-a)| = |1-a|$

223. If a, b, c and u, v, w are complex numbers representing the vertices of two traingles such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where r is a complex number, then the two triangles

(a) Have the same area (b) Are similar (c) Are congruent (d) None of these

224. If z_1, z_2, z_3, z_4 are the affixes of four points in the Argand plane and z is the affix of a point such that $|z - z_1| = |z - z_2| = |z - z_3| = |z - z_4|$, then z_1, z_2, z_3, z_4 are

(a) Concyclic (b) Vertices of a parallelogram (c) Vertices of a rhombus (d) In a straight line

225. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represents the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex number

(a) $3 - \frac{1}{2}i$ or $1 - \frac{3}{2}i$ (b) $\frac{3}{2} - i$ or $\frac{1}{2} - 3i$ (c) $\frac{1}{2} - i$ or $1 - \frac{1}{2}i$ (d) None of these

226. Suppose Z_1, Z_2, Z_3 are the vertices of an equilateral triangle inscribed in the circle $|Z| = 2$. If $Z_1 = 1 + i\sqrt{3}$, then values of Z_3 and Z_2 are respectively **[IIT 1994]**

(a) $-2, 1 - i\sqrt{3}$ (b) $2, 1 + i\sqrt{3}$ (c) $1 + i\sqrt{3}, -2$ (d) None of these

227. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i, z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

[IIT 1989]

(a) $a = b = 2 + \sqrt{3}$ (b) $a = b = 2 - \sqrt{3}$ (c) $a = 2 - \sqrt{3}, b = 2 + \sqrt{3}$ (d) None of these

228. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that origin, z_1 and z_2 form an equilateral triangle. Then **[AIEEE 2003]**

(a) $a^2 = b$ (b) $a^2 = 2b$ (c) $a^2 = 3b$ (d) $a^2 = 4b$

229. If z_1, z_2, z_3, z_4 are represented by the vertices of a rhombus taken in the anticlockwise order then

(a) $z_1 - z_2 + z_3 - z_4 = 0$ (b) $z_1 + z_2 = z_3 + z_4$ (c) $amp \frac{z_2 - z_4}{z_1 - z_3} = \frac{\pi}{2}$ (d) $amp \frac{z_1 - z_2}{z_3 - z_4} = \frac{\pi}{2}$

230. The join of $z_1 = a + ib$ and $z_2 = \frac{1}{-a + ib}$ passes through

(a) Origin (b) $z = 1 + i\,0$ (c) $z = 0 + i$ (d) $z = 1 + i$

231. If A, B, C are three points in the Argand plane representing the complex numbers z_1, z_2, z_3 such that $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then the distance of A from the line BC is

(a) λ (b) $\frac{\lambda}{\lambda + 1}$ (c) 1 (d) 0

232. The roots of the equation $1 + z + z^3 + z^4 = 0$ are represented by the vertices of

(a) A square (b) An equilateral triangle (c) A rhombus (d) None of these

233. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C, then

[DCE 1998]

(a) $(z_1 - z_3)^2 = 2(z_1 - z_2)(z_3 - z_2)$ (b) $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

(c) $(z_1 + z_2)^2 = 2(z_1 - z_2)(z_3 + z_2)$ (d) $(z_1 + z_3)^2 = 2(z_1 + z_2)(z_3 + z_2)$

234. *ABCD* is a square, vertices being taken in the anticlockwise sense. If *A* represents the complex number *z* and the intersection of the diagonals is the origin then

(a) *B* represents the complex number *iz* (b) *D* represents the complex number $i\bar{z}$

(c) *B* represents the complex number $i\bar{z}$ (d) *D* represents the complex number $-iz$

235. The angle that the vector representing the complex number $\frac{1}{(\sqrt{3}-i)^{25}}$ makes with the positive direction of the real axis is

(a) $\frac{2\pi}{3}$ (b) $-\frac{\pi}{6}$ (c) $\frac{5\pi}{6}$ (d) $\frac{\pi}{6}$

236. If z_0, z_1 represent points *P, Q* on the locus $|z-1| = 1$ and the line segment *PQ* subtends an angle $\pi/2$ at the point $z = 1$ then z_1 is equal to

(a) $1 + i(z_0 - 1)$ (b) $\frac{i}{z_0 - 1}$ (c) $1 - i(z_0 - 1)$ (d) $i(z_0 - 1)$

237. If $z^n \sin\theta_0 + z^{n-1}\sin\theta_1 + z^{n-2}\sin\theta_2 + ... + z\sin\theta_{n-1} + \sin\theta_n = 2,$ then all the roots of the equation lies

(a) Outside the circle $|z| = \frac{1}{2}$ (b) Inside the circle $|z| = \frac{1}{2}$ (c) On the circle $|z| = \frac{1}{2}$ (d)

238. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle circumscribing the circle $|z| = 1$. If $z_1 = 1 + \sqrt{3}\,i$ and z_1, z_2, z_3 are in the anticlockwise sense, then z_2 is

(a) $1 - \sqrt{3}i$ (b) 2 (c) $\frac{1}{2}(1 - \sqrt{3}i)$ (d) None of these

239. In the Argand plane, the vector $z = 4 - 3i$ is turned in the clockwise sense through 180° and streatched three times. The complex number represented by the new vector is

(a) $12 + 9i$ (b) $12 - 9i$ (c) $-12 - 9i$ (d) $-12 + 9i$

240. The vector $z = 3 - 4i$ is turned anticlockwise through an angle of 180° and stretched 2.5 times. The complex number corresponding to the newly obtained vector is

(a) $\frac{15}{2} - 10i$ (b) $\frac{-15}{2} + 10i$ (c) $\frac{-15}{2} - 10i$ (d) None of these

Triangle Inequilities, Area of Triangle and Collinearity

Basic Level

241. If z_1 and z_2 are any two complex numbers, then which of the following is true

[Rajasthan PET 1985; MP PET 1987; Kerala (Engg.) 2002]

(a) $|z_1 + z_2| = |z_1| + |z_2|$ (b) $|z_1 - z_2| = |z_1| - |z_2|$ (c) $|z_1 + z_2| \le |z_1| + |z_2|$ (d) $|z_1 - z_2| \le |z_1| - |z_2|$

242. Which of the following are correct for any two complex numbers z_1 and z_2 **[MP PET 1994; Roorkee 1998]**

(a) $|z_1 z_2| = |z_1|\ |z_2|$ (b) $arg(z_1 z_2) = (arg z_1)(arg z_2)$ (c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 - z_2| \ge |z_1| - |z_2|$

243. If $z_1, z_2 \in C,$ then **[MP PET 1995]**

(a) $|z_1 + z_2| \ge |z_1| + |z_2|$ (b) $|z_1 - z_2| \ge |z_1| + |z_2|$ (c) $|z_1 - z_2| \le |z_1| - |z_2|$ (d) $|z_1 + z_2| \ge |z_1| - |z_2|$

244. Which one of the following statement is true **[Rajasthan PET 2002]**

(a) $|x - y| = |x| - |y|$ (b) $|x + y| \le |x| - |y|$ (c) $|x - y| \ge |x| - |y|$ (d) $|x + y| \ge |x| - |y|$

245. The value of $|z_1+z_2|^2+|z_1-z_2|^2$ is **[Rajasthan PET 1997]**

(a) $\frac{1}{2}[|z_1|^2+|z_2|^2]$ (b) $2[|z_1|^2+|z_2|^2]$ (c) $2[|z_1|^2-|z_2|^2]$ (d) $\frac{1}{2}[|z_1|^2-|z_2|^2]$

246. If z, iz and $z+iz$ are the vertices of a triangle whose area is 2 units, then the value of $|z|$ is **[Rajasthan PET 2000]**

(a) – 2 (b) 2 (c) 4 (d) 8

247. If the area of the triangle formed by the points $z, z+iz$ and iz on the complex plane is 18, then the value of $|z|$ is **[MP PET 2001]**

(a) 6 (b) 9 (c) $3\sqrt{2}$ (d) $2\sqrt{3}$

248. If *A, B, C* are represented by $3+4i, 5-2i, -1+16i$, then *A, B, C* are **[Rajasthan PET 1986]**

(a) Collinear (b) Vertices of equilateral triangle

(c) Vertices of isosceles triangle (d) Vertices of right angled triangle

249. If $z_1=1+i, z_2=-2+3i$ and $z_3=ai/3$, where $i^2=-1$, are collinear then the value of *a* is **[AMU 2001]**

(a) – 1 (b) 3 (c) 4 (d) 5

250. The area of the triangle whose vertices are the points, represented by the complex numbers z_1, z_2, z_3 on the Argand diagram is

[DCE 1997]

(a) $\frac{\sum |z_2-z_3|\ |z_1|^2}{4iz_1}$ (b) $\frac{1}{2}|z_1|\ |z_2|$ (c) $\frac{1}{3}|z_1|^2$ (d) $\sum \frac{z_1-z_3}{4iz_1}$

251. Area of the triangle formed by 3 complex numbers 1 + *i*, *i* – 1, 2*i* in the Argand plane is **[EAMCET 1993]**

(a) 1/2 (b) 1 (c) $\sqrt{2}$ (d) 2

252. The area of the triangle whose vertices are represented by the complex numbers 0, *z*, $ze^{i\alpha}$, $(0<\alpha<\pi)$ equals **[AMU 20**

(a) $\frac{1}{2}|z|^2\cos\alpha$ (b) $\frac{1}{2}|z|^2\sin\alpha$ (c) $\frac{1}{2}|z|^2\sin\alpha\cos\alpha$ (d) $\frac{1}{2}|z|^2$

253. If the roots of $z^3+iz^2+2i=0$ represent the vertices of a ΔABC in the argand plane, then the area of the triangle is

(a) $\frac{3\sqrt{7}}{2}$ (b) $\frac{3\sqrt{7}}{4}$ (c) 2 (d) None of these

254. If $2z_1-3z_2+z_3=0$ then z_1, z_2, z_3 are represented by

(a) Three vertices of a triangle (b) Three collinear points (c) Three vertices of a rhombus (d) None of these

Standard Loci in the Argand Plane

Basic Level

255. The complex numbers $z=x+iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right|=1$ lie on **[IIT 1982; Pb. CET 1998]**

(a) Real axis (*x*-axis) (b) The line $y=5$

(c) A circle passing through the origin (d) None of these

256. If $z = x + iy$ is a complex number satisfying $\left|z + \frac{i}{2}\right|^2 = \left|z - \frac{i}{2}\right|^2$, then the locus of z is **[EAMCET 2002]**

(a) $2y = x$ (b) $y = x$ (c) y-axis (d) x-axis

257. If *arg* $(z - a) = \frac{\pi}{4}$, where $a \in R$, then the locus of $z \in C$ is a **[MP PET 1997]**

(a) Hyperbola (b) Parabola (c) Ellipse (d) Straight line

258. The locus of z given by $\left|\frac{z-1}{z-i}\right| = 1$, is **[Roorkee 1990]**

(a) A circle (b) An ellipse (c) A straight line (d) A parabola

259. Locus of the point z satisfying the equation $|iz - 1| + |z - i| = 2$ is **[Roorkee 1999]**

(a) A straight line (b) A circle (c) An ellipse (d) A pair of straight lines

260. If the imaginary part of $\frac{2z+1}{iz+1}$ is – 2, then the locus of the point representing z in the complex plane is **[DCE 2001]**

(a) A circle (b) A straight line (c) A parabola (d) None of these

261. The locus represented by $|z - 1| = |z + i|$ is **[EAMCET 1991]**

(a) A circle of radius 1 (b) An ellipse with foci at (1, 0) and (0, –1)

(c) A straight line through the origin (d) A circle on the line joining (1, 0), (0, 1) as diameter

262. If $z^2 + z|z| + |z|^2 = 0$, then the locus of z is

(a) A circle (b) A straight line (c) A pair of straight lines (d) None of these

263. If $z = x + iy$ and $|z - 2 + i| = |z - 3 - i|$, then locus of z is **[Rajasthan PET 1999]**

(a) $2x + 4y - 5 = 0$ (b) $2x - 4y - 5 = 0$ (c) $x + 2y = 0$ (d) $x - 2y + 5 = 0$

264. If the amplitude of $z - 2 - 3i$ is $\pi/4$, then the locus of $z = x + iy$ is **[EAMCET 2003]**

(a) $x + y - 1 = 0$ (b) $x - y - 1 = 0$ (c) $x + y + 1 = 0$ (d) $x - y + 1 = 0$

265. If $z = x + iy$ and $arg\left|\frac{z-2}{z+2}\right| = \frac{\pi}{6}$, then locus of z is **[Rajasthan PET 2002]**

(a) A straight line (b) A circle (c) A parabola (d) An ellipse

266. If $z = (\lambda + 3) + i\sqrt{5 - \lambda^2}$, then the locus of z is a

(a) Circle (b) Straight line (c) Parabola (d) None of these

267. A complex number z is such that $arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3}$. The points representing this complex number will lie on **[MP PET 2001**

(a) An ellipse (b) A parabola (c) A circle (d) A straight line

268. The equation $|z - 5i| \div |z + 5i| = 12$, where $z = x + iy$, represents a/an **[AMU 1999]**

(a) Circle (b) Ellipse (c) Parabola (d) No real curve

269. If $\frac{|z-2|}{|z-3|} = 2$ represents a circle, then its radius is equal to **[Karnataka CET 1990; Kurukshetra CEE 1998]**

(a) 1 (b) 1/3 (c) 3/4 (d) 2/3

270. A point z moves on Argand diagram in such a way that $|z - 3i| = 2$, then its locus will be **[Rajasthan PET 1992; MP PET 20**

(a) y – axis (b) A straight line (c) A circle (d) None of these

271. A circle whose radius is r and centre z_0, then the equation of the circle is **[Rajasthan PET 2000]**

(a) $z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$ (b) $z\bar{z} + z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$

(c) $z\bar{z} - z\bar{z}_0 + \bar{z}z_0 - z_0\bar{z}_0 = r^2$ (d) None of these

88 Complex Numbers

272. If $|z+\bar{z}|+|z-\bar{z}|=2,$ then z lies on

(a) A straight line (b) A square (c) A circle (d) None of these

273. If $z=x+iy,$ then $z\bar{z}+2(z+\bar{z})+c=0$ implies **[Rajasthan PET 1998; Pb. CET 2002]**

(a) A circle (b) Straight line (c) Parallel (d) Point

274. The equation $|z+1-i|=|z+i-1|$ represents **[EAMCET 1996]**

(a) A straight line (b) A circle (c) A parabola (d) A hyperbola

275. The equation $z\bar{z}+(2-3i)z+(2+3i)\bar{z}+4=0$ represents a circle of radius **[Kurukshetra CEE 1996]**

(a) 2 (b) 3 (c) 4 (d) 6

276. In the Argand diagram all the complex number z satisfying $|z-4i|+|z+4i|=10$ lie on a **[EAMCET 1996]**

(a) Straight line (b) Circle (c) Ellipse (d) Parabola

Advance Level

277. When $\frac{z+i}{z+2}$ is purely imaginary, the locus described by the point z in the Argand diagram is a

(a) Circle of radius $\frac{\sqrt{5}}{2}$ (b) Circle of radius $\frac{5}{4}$ (c) Straight line (d) Parabola

278. If $\log_{\sqrt{3}}\left(\frac{|z|^2-|z|+1}{2+|z|}\right)<2,$ then the locus of z is **[Karnataka CET 1996]**

(a) $|z|=5$ (b) $|z|<5$ (c) $|z|>5$ (d) None of these

279. The region of Argand plane defined by $|z-1|+|z+1|\le 4$ is

(a) Interior of an ellipse (b) Exterior of a circle

(c) Interior and boundary of an ellipse (d) None of these

280. The equation $|z+i|-|z-i|=k$ represent a hyperbola if

(a) $-2<k<2$ (b) $k>2$ (c) $0<k<2$ (d) None of these

281. The equation $|z-i|-|z+i|=k$, $k>0$, can represent an ellipse if k is

(a) 1 (b) 2 (c) 4 (d) None of these

282. If $|z|=2$ and locus of $5z-1$ is the circle having radius a and $z_1^2+z_2^2-2z_1z_2\cos\theta=0,$ then $|z_1|:|z_2|=$

(a) $a:1$ (b) $2a:1$ (c) $a:10$ (d) None of these

283. The locus of the complex number z in an argand plane satisfying the inequality $\log_{\left(\frac{1}{2}\right)}\left(\frac{|z-1|+4}{3|z-1|-2}\right)>1$ is, (where $|z-1|\ne\frac{2}{3}$)

(a) A circle (b) An interior of a circle (c) The exterior of the circle (d) None of these

284. Let $z=1-t+i\sqrt{t^2+t+2},$ where t is a real parameter. The locus of z in the Argand plane is

(a) A hyperbola (b) An ellipse (c) A straight line (d) None of these

285. The locus of the centre of a circle which touches the circle $|z-z_1|=a$ and $|z-z_2|=b$ externally (z, z_1 and z_2 are complex numbers) will be **[AIEEE 2002]**

(a) An ellipse (b) A hyperbola (c) A circle (d) None of these

De' Moivre's Theorem

Basic Level

286. The value of $i^{1/3}$ is **[UPSEAT 2002]**

(a) $\frac{\sqrt{3}+i}{2}$ (b) $\frac{\sqrt{3}-i}{2}$ (c) $\frac{1+i\sqrt{3}}{2}$ (d) $\frac{1-i\sqrt{3}}{2}$

287. Given $z=(1+i\sqrt{3})^{100}$, then $\frac{\text{Re}(z)}{\text{Im}(z)}$ equals **[AMU 2002]**

(a) 2^{100} (b) 2^{50} (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

288. $(-1+i\sqrt{3})^{20}$ is equal to **[Rajasthan PET 2003]**

(a) $2^{20}(-1+i\sqrt{3})^{20}$ (b) $2^{20}(1-i\sqrt{3})^{20}$ (c) $2^{20}(-1-i\sqrt{3})^{20}$ (d) None of these

289. $(-\sqrt{3}+i)^{53}$ where $i^2=-1$ is equal to **[AMU 2000]**

(a) $2^{53}(\sqrt{3}+2i)$ (b) $2^{52}(\sqrt{3}+i)$ (c) $2^{53}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ (d) $2^{53}(\sqrt{3}-i)$

290. If $z=\frac{\sqrt{3}+i}{2}$, then the value of z^{69} is **[Rajasthan PET 2002]**

(a) $-i$ (b) i (c) 1 (d) -1

291. If $a=\sqrt{2i}$, then which of the following is correct **[Roorkee 1989]**

(a) $a=1+i$ (b) $a=1-i$ (c) $a=-(\sqrt{2})i$ (d) None of these

292. If $z=\cos\theta+i\sin\theta$ then the value of $z^n+\frac{1}{z^n}$ is

(a) $\cos 2n\theta$ (b) $2\cos n\theta$ (c) $2\sin n\theta$ (d) None of these

293. The value of $(-i)^{1/3}$ is **[Roorkee 1995]**

(a) $\frac{1+\sqrt{3}\,i}{2}$ (b) $\frac{1-\sqrt{3}\,i}{2}$ (c) $\frac{-\sqrt{3}-i}{2}$ (d) $\frac{\sqrt{3}-i}{2}$

294. $(\sin\theta+i\cos\theta)^n$ is equal to **[Rajasthan PET 2001]**

(a) $\cos n\theta+i\sin n\theta$ (b) $\sin n\theta+i\cos n\theta$

(c) $\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$ (d) None of these

295. The product of all the roots of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4}$ is **[MNR 1984; EAMCET 1985]**

(a) -1 (b) 1 (c) $\frac{3}{2}$ (d) $-\frac{1}{2}$

296. $\left[\frac{1+\cos(\pi/8)+i\sin(\pi/8)}{1+\cos(\pi/8)-i\sin(\pi/8)}\right]^8$ is equal to **[Rajasthan PET 2001]**

(a) -1 (b) 0 (c) 1 (d) 2

297. $\left(\frac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4$ equals **[Rajasthan PET 1996]**

(a) $\sin 8\theta - i\cos 8\theta$ (b) $\cos 8\theta - i\sin 8\theta$ (c) $\sin 8\theta + i\cos 8\theta$ (d) $\cos 8\theta + i\sin 8\theta$

298. $\frac{(\cos\theta+i\sin\theta)^4}{(\sin\theta+i\cos\theta)^5}$ is equal to **[MNR 1985; UPSEAT 2000]**

(a) $\cos\theta - i\sin\theta$ (b) $\cos 9\theta - i\sin 9\theta$ (c) $\sin\theta - i\cos\theta$ (d) $\sin 9\theta - i\cos 9\theta$

299. $\frac{(\cos\alpha+i\sin\alpha)^4}{(\sin\beta+i\cos\beta)^5} =$ **[Rajasthan PET 1992, 96, 2002; UPSEAT 2000]**

(a) $\cos(4\alpha+5\beta)+i\sin(4\alpha+5\beta)$ (b) $\cos(4\alpha+5\beta)-i\sin(4\alpha+5\beta)$

(c) $\sin(4\alpha+5\beta)-i\cos(4\alpha+5\beta)$ (d) None of these

300. We express $\frac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$ in the form of $x + iy$, we get **[Karnataka CET 2001]**

(a) $\cos 49\theta - i\sin 49\theta$ (b) $\cos 23\theta - i\sin 23\theta$ (c) $\cos 49\theta + i\sin 49\theta$ (d) $\cos 21\theta + i\sin 21\theta$

301. If $\left(\frac{1+\cos\theta+i\sin\theta}{i+\sin\theta+i\cos\theta}\right)^4 = \cos n\theta + i\sin n\theta$, then n is equal to **[EAMCET 1986]**

(a) 1 (b) 2 (c) 3 (d) 4

302. The value of $\frac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)}{(\cos\gamma+i\sin\gamma)(\cos\delta+i\sin\delta)}$ is **[Rajasthan PET 2001]**

(a) $\cos(\alpha+\beta-\gamma-\delta)-i\sin(\alpha+\beta-\gamma-\delta)$ (b) $\cos(\alpha+\beta-\gamma-\delta)+i\sin(\alpha+\beta-\gamma-\delta)$

(c) $\sin(\alpha+\beta-\gamma-\delta)-i\cos(\alpha+\beta-\gamma-\delta)$ (d) $\sin(\alpha+\beta-\gamma-\delta)+i\cos(\alpha+\beta-\gamma-\delta)$

303. The value of $\left[\frac{1-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right]^{10} =$ **[Karnataka CET 2001]**

(a) 0 (b) – 1 (c) 1 (d) 2

304. If $z = \frac{1+i\sqrt{3}}{\sqrt{3}+i}$, then $(\bar{z})^{100}$ lies in **[AMU 1999]**

(a) I quadrant (b) II quadrant (c) III quadrant (d) IV quadrant

305. The following in the form of $A + iB$

$(\cos 2\theta + i\sin 2\theta)^{-5}(\cos 3\theta - i\sin 3\theta)^6(\sin\theta - i\cos\theta)^3$ is **[MNR 1991]**

(a) $(\cos 25\theta + i\sin 25\theta)$ (b) $i(\cos 25\theta + i\sin 25\theta)$ (c) $i(\cos 25\theta - i\sin 25\theta)$ (d) $(\cos 25\theta - i\sin 25\theta)$

306. $A+iB$ form of $\frac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cot u+i)(1+i\tan v)}$ is **[Roorkee 1980]**

(a) $\sin u\cos v[\cos(x+y-u-v)+i\sin(x+y-u-v)]$ (b) $\sin u\cos v[\cos(x+y+u+v)+i\sin(x+y+u+v)]$

(c) $\sin u\cos v[\cos(x+y+u+v)-i\sin(x+y+u+v)]$ (d) None of these

307. The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is **[IIT 1987; DCE 2000; Karnataka CET 2002]**

(a) – 1 (b) 0 (c) $-i$ (d) i

308. If $x_n = \cos\left(\frac{\pi}{3^n}\right) + i\sin\left(\frac{\pi}{3^n}\right)$, then $x_1.x_2.x_3....x_\infty$ is equal to **[Rajasthan PET 2002; Kurukshetra CEE 2002]**

(a) 1 (b) – 1 (c) i (d) $-i$

309. If $x_n = \cos\left(\frac{\pi}{4^n}\right) + i\sin\left(\frac{\pi}{4^n}\right)$, then $x_1.x_2.x_3....\infty =$ **[EAMCET 2002]**

(a) $\frac{1+i\sqrt{3}}{2}$ (b) $\frac{-1+i\sqrt{3}}{2}$ (c) $\frac{1-i\sqrt{3}}{2}$ (d) $\frac{-1-i\sqrt{3}}{2}$

310. The value of infinite product $(\cos\theta + i\sin\theta)\ (\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})\ (\cos\frac{\theta}{2^2} + i\sin\frac{\theta}{2^2})....$ is **[Rajasthan PET 1999]**

(a) $\cos 2\theta - i\sin 2\theta$ (b) $\cos 2\theta + i\sin 2\theta$ (c) $\sin 2\theta - i\cos 2\theta$ (d) $\sin 2\theta + i\cos 2\theta$

311. The value of expression $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)$..... to ∞ is **[Kurukshetra CEE 1998]**

(a) – 1 (b) 1 (c) 0 (d) 2

312. If $z_i = \cos\frac{i\pi}{10} + i\sin\frac{i\pi}{10}$, then $z_1z_2z_3z_4$ is equal to **[DCE 1998]**

(a) – 1 (b) 1 (c) – 2 (d) 2

313. If $2\cos\alpha = a + \frac{1}{a}$ and $2\cos\beta = b + \frac{1}{b}$, then the value of $ab + \frac{1}{ab}$ is **[Rajasthan PET 1992, Pb. CET 2002]**

(a) $2\cos(\alpha+\beta)$ (b) $2\sin(\alpha+\beta)$ (c) $2\cos(\alpha-\beta)$ (d) $4\cos\alpha\cos\beta$

314. $\frac{(\sin\pi/8 + i\cos\pi/8)^8}{(\sin\pi/8 - i\cos\pi/8)^8} =$ **[EAMCET 1994]**

(a) – 1 (b) 0 (c) 1 (d) $2i$

Advance Level

315. If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)........(\cos n\theta + i\sin n\theta) = 1$, then the value of θ is **[Karnataka CET 1992; Kurukshetra CEE 2002]**

(a) $4m\pi$ (b) $\frac{2m\pi}{n(n+1)}$ (c) $\frac{4m\pi}{n(n+1)}$ (d) $\frac{m\pi}{n(n+1)}$

316. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, then $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$ equals to **[Karnataka CET 2000]**

(a) 0 (b) $\cos(\alpha+\beta+\gamma)$ (c) $3\cos(\alpha+\beta+\gamma)$ (d) $3\sin(\alpha+\beta+\gamma)$

317. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ equals **[Rajasthan PET 2000]**

(a) $2\cos(\alpha+\beta+\gamma)$ (b) $\cos 2(\alpha+\beta+\gamma)$ (c) 0 (d) 1

318. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is **[Rajasthan PET 1999]**

(a) 2/3 (b) 3/2 (c) 1/2 (d) 1

319. If $a = \cos(2\pi/7) + i\sin(2\pi/7)$, then the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$ is

[Rajasthan PET 2000]

(a) $x^2 - x + 2 = 0$ (b) $x^2 + x - 2 = 0$ (c) $x^2 - x - 2 = 0$ (d) $x^2 + x + 2 = 0$

320. If $x^2 - x + 1 = 0$ then the value of $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$ is

(a) 8 (b) 10 (c) 12 (d) None of these

321. If n_1, n_2 are positive integers, then $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$ is a real number iff **[IIT 1996]**

(a) $n_1 = n_2 + 1$ (b) $n_1 + 1 = n_2$

(c) $n_1 = n_2$ (d) n_1, n_2 are any two +ve integers

322. If $a=\cos\alpha+i\sin\alpha,\ b=\cos\beta+i\sin\beta,\ c=\cos\gamma+i\sin\gamma$ and $\frac{b}{c}+\frac{c}{a}+\frac{a}{b}=1,$ then $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)$ is equal to

[Rajasthan PET 1993, 2001]

(a) 3/2 (b) – 3/2 (c) 0 (d) 1

323. If $\cos A+\cos B+\cos C=0,\ \ \sin A+\sin B+\sin C=0$ and $A+B+C=180^o$, then the value of $\cos 3A+\cos 3B+\cos 3C$ is

[EAMCET 1995]

(a) 3 (b) – 3 (c) $\sqrt{3}$ (d) 0

324. The value of z satisfying the equation $\log z+\log z^2+.........+\log z^n=0$ is

(a) $\cos\frac{4m\pi}{n(n+1)}+i\sin\frac{4m\pi}{n(n+1)},\ m=1,2,..$

(b) $\cos\frac{4m\pi}{n(n+1)}-i\sin\frac{4m\pi}{n(n+1)},\ m=1,2,..$

(c) $\sin\frac{4m\pi}{n}+i\cos\frac{4m\pi}{n},m=1,2,..$

(d) 0

Cube Roots of Unity, n^{th} Roots of Unity

Basic Level

325. The product of cube roots of –1 is equal to

(a) 0 (b) 1 (c) – 1 (d) None of these

326. One of the cube roots of unity is **[MP PET 1994, 2003]**

(a) $\frac{-1+i\sqrt{3}}{2}$ (b) $\frac{1+i\sqrt{3}}{2}$ (c) $\frac{1-i\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}-i}{2}$

327. The two numbers such that each one is square of the other, are **[MP PET 1987]**

(a) ω,ω^3 (b) – i, i (c) –1, 1 (d) ω,ω^2

328. If $1,\omega,\omega^2$ are the cube roots of unity, then their product is **[Karnataka CET 1999, 2001]**

(a) 0 (b) ω (c) – 1 (d) 1

329. The value of $(8)^{1/3}$ is **[Rajasthan PET 2003]**

(a) $-1+i\sqrt{3}$ (b) $-1-i\sqrt{3}$ (c) 2 (d) All of these

330. If $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^n$ is an integer, then n is **[UPSEAT 2002]**

(a) 1 (b) 2 (c) 3 (d) 4

331. $\left(\frac{\sqrt{3}+i}{2}\right)^6+\left(\frac{i-\sqrt{3}}{2}\right)^6$ is equal to **[Rajasthan PET 1997]**

(a) – 2 (b) 0 (c) 2 (d) 1

332. If $\frac{1+\sqrt{3}\,i}{2}$ is a root of equation $x^4-x^3+x-1=0$, then its real roots are **[EAMCET 2002]**

(a) 1, 1 (b) – 1, – 1 (c) 1, – 1 (d) 1, 2

333. If $z=\frac{\sqrt{3}+i}{-2}$, then z^{69} is equal to **[Rajasthan PET 2001]**

(a) 1 (b) – 1 (c) i (d) – i

334. If ω is a complex cube root of unity, then for positive integral value of n, the product of $\omega.\omega^2.\omega^3.........\omega^n$ will be **[Roorkee 1991]**

(a) $\frac{1-i\sqrt{3}}{2}$ (b) $-\frac{1-i\sqrt{3}}{2}$ (c) 1 (d) (b) and (c) both

335. If $\omega(\neq 1)$ is a cube root of unity and $(1+\omega)^7=A+B\omega,$ then A and B are respectively, the numbers **[IIT 1995]**

(a) 0, 1 (b) 1, 0 (c) 1, 1 (d) – 1, 1

336. If ω is a cube root of unity, then $(1+\omega-\omega^2)(1-\omega+\omega^2)=$ **[MNR 1990; UPSEAT 1999; MP PET 1993, 02]**

(a) 1 (b) 0 (c) 2 (d) 4

337. If cube root of 1 is ω, then the value of $(3+\omega+3\omega^2)^4$ is **[MP PET 2001]**

(a) 0 (b) 16 (c) $16\,\omega$ (d) $16\,\omega^2$

338. If $1,\,\omega\,\omega^2$ are the three cube roots of unity, then $(3+\omega^2+\omega^4)^6=$ **[MP PET 1995]**

(a) 64 (b) 729 (c) 2 (d) 0

339. The value of $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$ will be **[Ranchi BIT 1989; Orissa JEE 2003]**

(a) 1 (b) – 1 (c) 2 (d) – 2

340. If ω is a non real cube root of unity, then $(a+b)(a+b\omega)(a+b\omega^2)$ is **[Kerala (Engg.) 2002]**

(a) a^3+b^3 (b) a^3-b^3 (c) a^2+b^2 (d) a^2-b^2

341. If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^7$ equals **[IIT 1998; MP PET 2000]**

(a) $128\,\omega$ (b) $-128\,\omega$ (c) $128\,\omega^2$ (d) $-128\,\omega^2$

342. If ω is the cube root of unity, then $(3+5\omega+3\omega^2)^2+(3+3\omega+5\omega^2)^2=$ **[MP PET 1999]**

(a) 4 (b) 0 (c) – 4 (d) None of these

343. If ω is an imaginary cube root of unity, then the value of $\sin\left[(\omega^{10}+\omega^{23})\pi-\frac{\pi}{4}\right]$ is **[IIT Screening 1994]**

(a) $-\sqrt{3}/2$ (b) $-1/\sqrt{2}$ (c) $1/\sqrt{2}$ (d) $\sqrt{3}/2$

344. If $1,\omega,\,\omega^2$ are three cube roots of unity, then $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$ is equal to, if $a+b+c=0$ **[WB JEE 1992]**

(a) $27abc$ (b) 0 (c) $3abc$ (d) None of these

345. The value of $(1-\omega+\omega^2)(1-\omega^2+\omega)^6$, where ω,ω^2 are cube roots of unity **[DCE 2001]**

(a) 128ω (b) $-128\omega^2$ (c) -128ω (d) $128\omega^2$

346. If ω is a cube root of unity, then the value of $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5=$ **[IIT 1965; MP PET 1997; Rajasthan PET 1997]**

(a) 16 (b) 32 (c) 48 (d) – 32

347. If ω is a complex cube root of unity, then $225+(3\omega+8\omega^2)^2+(3\omega^2+8\omega)^2=$ **[EAMCET 2003]**

(a) 72 (b) 192 (c) 200 (d) 248

348. If $x=a,\ y=b\omega,\ z=c\omega^2$, where ω is a complex cube root of unity, then $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=$ **[AMU 1983]**

(a) 3 (b) 1 (c) 0 (d) None of these

349. If $x=a+b, y=a\omega+b\omega^2, z=a\omega^2+b\omega,$ then the value of $x^3+y^3+z^3$ is equal to **[Roorkee 1977; IIT 1970]**

(a) a^3+b^3 (b) $3(a^3+b^3)$ (c) $3(a^2+b^2)$ (d) None of these

350. If ω is an *nth* root of unity, other than unity, then the value of $1+\omega+\omega^2+...+\omega^{n-1}$ is **[Karnataka CET 1999]**

(a) 0 (b) 1 (c) – 1 (d) None of these

351. If ω is a complex cube root of unity, then $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)......$ to $2n$ factors = **[AMU 2000]**

(a) 0 (b) 1 (c) – 1 (d) None of these

352. Find the value of $(1+2\omega+\omega^2)^{3n}-(1+\omega+2\omega^2)^{3n}$ is **[UPSEAT 2002]**

(a) 0 (b) 1 (c) ω (d) ω^2

353. If α and β are imaginary cube roots of unity, then the value of $\alpha^4+\beta^{28}+\frac{1}{\alpha\beta}$, is **[MP PET 1998]**

(a) 1 (b) – 1 (c) 0 (d) None of these

354. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x-2)^3+27=0$ are **[Kurukshetra CEE 1998]**

(a) – 1, – 1, – 1 (b) $-1,-\omega,-\omega^2$ (c) $-1, 2+3\omega,\ 2+3\omega^2$ (d) $-1, 2-3\omega,\ 2-3\omega^2$

355. If α, β are non-real cube roots of unity, then $\alpha\beta+\alpha^5+\beta^5$ is equal to **[Kurukshetra CEE 1999]**

(a) 1 (b) 0 (c) – 1 (d) 3

356. If α and β are imaginary cube roots of unity, then $\alpha^4+\beta^4+\frac{1}{\alpha\beta}=$ **[IIT 1977]**

(a) 3 (b) 0 (c) 1 (d) 2

357. If ω is a cube root of unity, then a root of the equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}=0$ is **[MNR 1990; MP PET 1999, 2002]**

(a) $x=1$ (b) $x=\omega$ (c) $x=\omega^2$ (d) $x=0$

358. If $1, \omega, \omega^2$ are the cube roots of unity, then $\Delta=\begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$ is equal to **[AIEEE 2003]**

(a) 0 (b) 1 (c) ω (d) ω^2

359. If $\omega(\neq 1)$ is a cube root of unity, then $\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$ is equal to **[IIT 1995]**

(a) 0 (b) 1 (c) ω (d) i

360. If ω is a complex root of the equation $z^3=1$, then $\omega+\omega^{\left(\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+\frac{27}{128}+...\right)}$ is equal to **[Roorkee 2000]**

(a) – 1 (b) 0 (c) 9 (d) i

361. The product of n, nth roots of unity is

(a) 1 (b) – 1 (c) $(-1)^{n-1}$ (d) $(-1)^n$

362. Let $\omega_n=\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right), i^2=-1$, then $(x+y\omega_3+z\omega_3{}^2)(x+y\omega_3{}^2+z\omega_3)$ is equal to **[AMU 2001]**

(a) 0 (b) $x^2+y^2+z^2$

(c) $x^2+y^2+z^2-yz-zx-xy$ (d) $x^2+y^2+z^2+yz+zx+xy$

363. If p is not a multiple of n, then the sum of pth powers of nth roots of unity is

(a) 0 (b) 1 (c) n (d) p

364. If n is a positive integer greater than unity and z is a complex number satisfying the equation $z^n=(z+1)^n$, then

(a) Re (z) < 0 (b) Re (z) > 0 (c) Re (z) = 0 (d) None of these

365. If z_1, z_2, z_3, z_4 are the roots of the equation $z^4=1$, then the value of $\sum_{i=1}^{4} z_i^3$ is **[Kurukshetra CEE 1996]**

(a) 0 (b) 1 (c) i (d) 1 + i

366. If α is an imaginary cube root of unity, then for $n \in N$, the value of $\alpha^{3n+1}+\alpha^{3n+3}+\alpha^{3n+5}$ is **[MP PET 1996]**

(a) – 1 (b) 0 (c) 1 (d) 3

367. If $\alpha \neq 1$ is any n^{th} root of unity, then $S=1+3\alpha+5\alpha^2+.....$ upto n terms, is equal to

(a) $\frac{2n}{1-\alpha}$ (b) $-\frac{2n}{1-\alpha}$ (c) $\frac{n}{1-\alpha}$ (d) $-\frac{n}{1-\alpha}$

368. The common roots of the equations $x^{12}-1=0$, $x^4+x^2+1=0$ are **[EAMCET 1989]**

(a) $\pm\omega$ (b) $\pm\omega^2$ (c) $\pm\omega, \pm\omega^2$ (d) None of these

369. Which of the following is a fourth root of $\frac{1}{2}+\frac{i\sqrt{3}}{2}$ **[Karnataka CET 2003]**

(a) $cis\left(\frac{\pi}{2}\right)$ (b) $cis\left(\frac{\pi}{12}\right)$ (c) $cis\left(\frac{\pi}{6}\right)$ (d) $cis\left(\frac{\pi}{3}\right)$

370. If ω is a complex root of unity, then **[T.S. Rajendra 1991, Kurukshetra CEE 2000]**

(a) $\omega^4=1$ (b) $\omega^{14}=\omega^2$ (c) $\omega^6=\omega$ (d) $\omega^5=1$

371. If ω is an imaginary cube root of unity, then the value of $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}$ is **[Karnataka CET 1998]**

(a) – 2 (b) – 1 (c) 1 (d) 0

372. The value of $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6+\left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6$ is **[Kurukshetra CEE 1994, EAMCET 1995]**

(a) 2 (b) – 2 (c) 1 (d) 0

373. If the roots of the equation $x^3-1=0$ are 1, ω and ω^2, then the value of $(1-\omega)(1-\omega^2)$ is **[MNR 1992]**

(a) 0 (b) 1 (c) 2 (d) 3

374. If $i=\sqrt{-1}$, ω = non-real cube root of unity then $\frac{(1+i)^{2n}-(1-i)^{2n}}{(1+\omega^4-\omega^2)(1-\omega^4+\omega^2)}$ is equal to

(a) 0 if n is even (b) 0 for all $n\in Z$ (c) $2^{n-1}-i$ for all $n\in N$ (d) None of these

375. If $z+z^{-1}=1$, then $z^{100}+z^{-100}$ is equal to **[UPSEAT 2001]**

(a) i (b) $-i$ (c) 1 (d) – 1

376. If α is nonreal and $\alpha=\sqrt[5]{1}$ then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to

(a) 4 (b) 2 (c) 1 (d) None of these

377. Which of the following statements are true **[Tamilnadu Engg. 2002]**

(1) The amplitude of the product of complex numbers is equal to the product of their amplitudes

(2) For any polynomial $f(x)=0$ with real co-efficients, imaginary roots occur in conjugate pairs.

(3) Order relation exists in complex numbers whereas it does not exist in real numbers.

(4) The values of ω used as a cube root of unity and as a fourth root of unity are different

(a) (1) and (2) only (b) (2) and (4) only (c) (3) and (4) only (d) (1), (2) and (4) only

378. If $x=a+b$, $y=a\alpha+b\beta$ and $z=a\beta+b\alpha$, where α and β are complex cube roots of unity, then xyz =

[IIT 1978; Roorkee 1989; Rajasthan PET 1997]

(a) a^2+b^2 (b) a^3+b^3 (c) a^3b^3 (d) a^3-b^3

Advance Level

379. $\frac{(-1+i\sqrt{3})^{15}}{(1-i)^{20}}+\frac{(-1-i\sqrt{3})^{15}}{(1+i)^{20}}$ is equal to **[AMU 2000]**

(a) – 64 (b) – 32 (c) – 16 (d) $\frac{1}{16}$

380. $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$.......... .. to $2n$ factors is **[EAMCET 1988; AMU 1997]**

(a) 2^n (b) 2^{2n} (c) 0 (d) 1

381. If $1,\omega,\omega^2,\omega^3$......, ω^{n-1} are the n, n^{th} roots of unity, then $(1-\omega)(1-\omega^2)......(1-\omega^{n-1})$ equals **[MNR 1992; IIT 1984; DCE 2001]**

(a) 0 (b) 1 (c) n (d) n^2

382. The value of the expression $1.(2-\omega)(2-\omega^2)+2.(3-\omega)(3-\omega^2)+$.......... .. $+(n-1).(n-\omega).(n-\omega^2)$, where ω is an imaginary cube root of unity, is **[IIT 1996]**

(a) $\frac{1}{2}(n-1)n(n^2+3n+4)$ (b) $\frac{1}{4}(n-1)n(n^2+3n+4)$ (c) $\frac{1}{2}(n+1)n(n^2+3n+4)$ (d) $\frac{1}{4}(n+1)n(n^2+3n+4)$

383. If α,β,γ are the cube roots of $p\,(p<0)$, then for any x, y and z, $\frac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha}=$ **[IIT 1989]**

(a) $\frac{1}{2}(-1+i\sqrt{3})$ (b) $\frac{1}{2}(1+i\sqrt{3})$ (c) $\frac{1}{2}(1-i\sqrt{3})$ (d) None of these

384. Common roots of the equations $z^3+2z^2+2z+1=0$ and $z^{1985}+z^{100}+1=0$ are

(a) ω,ω^2 (b) ω,ω^3 (c) ω^2,ω^3 (d) None of these

385. If $z_1,z_2,z_3,$.......... ...z_n are n, nth roots of unity, then for $k=1,2,$.......... , n

(a) $|z_k|=k|z_{k+1}|$ (b) $|z_{k+1}|=k|z_k|$ (c) $|z_{k+1}|=|z_k|+|z_{k+1}|$ (d) $|z_k|=|z_{k+1}|$

386. Let z_1 and z_2 be n^{th} roots of unity which are ends of a line segment that subtend a right angle at the origin. Then n must be of the form **[IIT Screening 2001; Karnataka CEE 2002]**

(a) $4k + 1$ (b) $4k + 2$ (c) $4k + 3$ (d) $4k$

387. The cube roots of unity when represented on the Argand plane form the vertices of an **[IIT 1988]**

(a) Equilateral triangle (b) Isosceles triangle (c) Right angled triangle (d) None of these

388. If $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=\frac{2}{\omega}$, where a, b, c are real and ω is a non-real cube root of unity, then

(a) $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=-\frac{2}{\omega^2}$ (b) $abc+bcd+abd+acd=4$

(c) $a+b+c+d=-2abcd$ (d) $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}=2$

389. If z is a complex number satisfying $z+z^{-1}=1$ then z^n+z^{-n}, $n \in N$, has the value

(a) $2(-1)^n$, when n is a multiple of 3 (b) $(-1)^{n-1}$, when n is not a multiple of 3

(c) $(-1)^{n+1}$, when n is a multiple of 3 (d) 0, when n is not a multiple of 3

390. If z be a complex number satisfying $z^4+z^3+2z^2+z+1=0$ then $|z|$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) 1 (d) None of these

391. If the fourth roots of unity are z_1, z_2, z_3, z_4 then $z_1^2+z_2^2+z_3^2+z_4^2$ is equal to

(a) 1 (b) 0 (c) i (d) None of these

392. If $1, \alpha, \alpha^2, \ldots\ldots, \alpha^{n-1}$ are the nth roots of unity, then $\sum_{i=1}^{n-1}\frac{1}{2-\alpha^i}$ is equal to

(a) $(n-2).2^n$ (b) $\frac{(n-2)\,2^{n-1}+1}{2^n-1}$ (c) $\frac{(n-2)\,2^{n-1}}{2^n-1}$ (d) None of these

393. If $z_1+z_2+z_3=A$, $z_1+z_2\omega+z_3\omega^2=B$, $z_1+z_2\omega^2+z_3\omega=C$, where $1, \omega, \omega^2$ are the three cube roots of unity, then $|A|^2+|B|^2+|C|^2=$

(a) $3(|z_1|^2+|z_2|^2+|z_3|^2)$ (b) $2(|z_1|^2+|z_2|^2+|z_3|^2)$

(c) $|z_1|^2+|z_2|^2+|z_3|^2$ (d) None of these

394. For complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, if $\sin\theta=\frac{x_1y_2-x_2y_1}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}$ where θ is the angle between z_1 and z_2, then the angle between the roots of the equation $z^2+2z+3=0$ is

(a) $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ (b) $\sin^{-1}\left(\frac{2}{3}\right)$ (c) $\sin^{-1}\left(\frac{1}{3}\right)$ (d) None of these

Miscellaneous Problems

Basic Level

395. $\sinh ix$ is **[EAMCET 2002]**

98 Complex Numbers

(a) $i\sin(ix)$ (b) $i\sin x$ (c) $-i\sin x$ (d) $\sin(ix)$

396. The value of $\sec h(i\pi)$ is **[Rajasthan PET 1999]**

(a) – 1 (b) *i* (c) 0 (d) 1

397. The imaginary part of $\cosh(\alpha+i\beta)$ is **[Rajasthan PET 2000]**

(a) $\cosh\alpha\cos\beta$ (b) $\sinh\alpha\sin\beta$ (c) $\cos\alpha\cosh\beta$ (d) $\cos\alpha\cos\beta$

398. $\cosh(\alpha+i\beta)-\cosh(\alpha-i\beta)$ is equal to **[Rajasthan PET 2000]**

(a) $2\sinh\alpha\sinh\beta$ (b) $2\cosh\alpha\cosh\beta$ (c) $2i\sinh\alpha\sin\beta$ (d) $2\cosh\alpha\cos\beta$

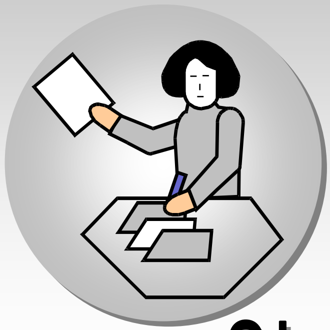
399. If $\cos(u+iv)=\alpha+i\beta,$ then $\alpha^2+\beta^2+1$ equals **[Rajasthan PET 1999]**

(a) $\cos^2 u+\sinh^2 v$ (b) $\sin^2 u+\cosh^2 v$ (c) $\cos^2 u+\cosh^2 v$ (d) $\sin^2 u+\sinh^2 v$

400. If $\tan^{-1}(\alpha+i\beta)=x+iy,$ then $x=$ **[Rajasthan PET 2002]**

(a) $\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$ (b) $\frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1+\alpha^2+\beta^2}\right)$ (c) $\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$ (d) None of these

***



Answer Sheet

Complex Numbers ***Assignment (Basic and Advance Level)***

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	d	b	c	a	a	a	b	d	b	b	a	d	b	c	a	d	d	a	c
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
b	c	d	d	b	a	c	a	d	b	b	c	c	a	b	b	c	c	c	b
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
c	c	c	a	b	b	b	a	b	d	b	c	c	d	b	c	c	b	a	a
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
c	a	d	c	b	d	b	a	c	b	c	a	a	a	b	b	a	c	b	c
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**
a	a	b	c	b	a,c,d	a	a	a	d	c	a	b	c	b	b	a	d	c	c
101	**102**	**103**	**104**	**105**	**106**	**107**	**108**	**109**	**110**	**111**	**112**	**113**	**114**	**115**	**116**	**117**	**118**	**119**	**120**
d	b	a,b	b	c	c	d	c	a	a	b	b	a	c	c	a	b	d	b	a
121	**122**	**123**	**124**	**125**	**126**	**127**	**128**	**129**	**130**	**131**	**132**	**133**	**134**	**135**	**136**	**137**	**138**	**139**	**140**
c	a	a	d	c	d	a	a	b	c	c	d	d	b	d	b	a	d	c	c
141	**142**	**143**	**144**	**145**	**146**	**147**	**148**	**149**	**150**	**151**	**152**	**153**	**154**	**155**	**156**	**157**	**158**	**159**	**160**
c	d	a	a	d	d	c	b	a	b	b	d	c	b	a	d	c	b	b	b,c
161	**162**	**163**	**164**	**165**	**166**	**167**	**168**	**169**	**170**	**171**	**172**	**173**	**174**	**175**	**176**	**177**	**178**	**179**	**180**
b,c,d	d	c	a	c	c	b	d	b	c	b	b	d	b	c	c	b	b	b	c
181	**182**	**183**	**184**	**185**	**186**	**187**	**188**	**189**	**190**	**191**	**192**	**193**	**194**	**195**	**196**	**197**	**198**	**199**	**200**
a	d	b	a	a	d	c	b	d	a	b	b	b	a	b	c	b	c	b	a
201	**202**	**203**	**204**	**205**	**206**	**207**	**208**	**209**	**210**	**211**	**212**	**213**	**214**	**215**	**216**	**217**	**218**	**219**	**220**
d	b	b	c	c	c	c	d	b	d	a	c	b	d	b	a	b	b	c	a
221	**222**	**223**	**224**	**225**	**226**	**227**	**228**	**229**	**230**	**231**	**232**	**233**	**234**	**235**	**236**	**237**	**238**	**239**	**240**
c	c	b	a	a	a	b	c	c	a	d	b	b	d	d	c	a	d	d	b
241	**242**	**243**	**244**	**245**	**246**	**247**	**248**	**249**	**250**	**251**	**252**	**253**	**254**	**255**	**256**	**257**	**258**	**259**	**260**
c	d	d	c,d	b	b	a	a	d	a	b	b	c	b	a	d	d	c	a	b
261	**262**	**263**	**264**	**265**	**266**	**267**	**268**	**269**	**270**	**271**	**272**	**273**	**274**	**275**	**276**	**277**	**278**	**279**	**280**
c	c	a	d	b	a	c	a	d	c	a	b	a	a	b	c	a	b	c	a
281	**282**	**283**	**284**	**285**	**286**	**287**	**288**	**289**	**290**	**291**	**292**	**293**	**294**	**295**	**296**	**297**	**298**	**299**	**300**
c	c	c	a	b	a	c	d	c	a	a	b	c,d	c	b	a	d	d	c	a
301	**302**	**303**	**304**	**305**	**306**	**307**	**308**	**309**	**310**	**311**	**312**	**313**	**314**	**315**	**316**	**317**	**318**	**319**	**320**
d	b	b	c	c	a	d	c	a	b	a	a	a	c	c	c	c	b	d	a
321	**322**	**323**	**324**	**325**	**326**	**327**	**328**	**329**	**330**	**331**	**332**	**333**	**334**	**335**	**336**	**337**	**338**	**339**	**340**
d	d	b	a	c	a	d	d	d	c	a	c	c	d	c	d	c	a	b	a
341	**342**	**343**	**344**	**345**	**346**	**347**	**348**	**349**	**350**	**351**	**352**	**353**	**354**	**355**	**356**	**357**	**358**	**359**	**360**
d	c	c	a	c	b	d	c	b	a	b	a	c	d	b	b	d	a	a	a
361	**362**	**363**	**364**	**365**	**366**	**367**	**368**	**369**	**370**	**371**	**372**	**373**	**374**	**375**	**376**	**377**	**378**	**379**	**380**
c	c	a	a	a	b	b	c	b	b	d	a	d	a	d	a	b	b	a	b

381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398	399	400
c	b	a	a	d	d	a	d	a	c	b	b	a	a	b	a	b	c	c	a