# CENTRE OF MASS AND ROTATORY MOTION OF A RIGID BODIES

## System

A collection of particles, interacting with one another, is termed as a system.

**Centre of mass :** The centre of mass of a system of particles is defined in terms of a single particle within the system, which moves in a similar way in which a single particle having the total mass of the system and subject to the same external force would move.

**Centre of mass for a '*n*' particle system :** The centre of mass for a '*n*' particle system is defined by a position vector,

$$\vec{R}_{c.m} \quad \frac{m_1\vec{r}_1 \quad m_2\vec{r}_2 \quad m_3\vec{r}_3 \quad .... \quad m_n\vec{r}_n}{M}$$

where M = $m_1 + m_2 + m_3 + .... + m_n$ and $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...., \vec{r}_n$ are the position vectors of '$n$' particles of masses $m_1$, $m_2$, $m_3$, ...., $m_n$ respectively.

Hence, $\vec{R}_{c.m}$ is the **weighted average** of all the position vectors $\vec{r}_1, \vec{r}_2, ...., \vec{r}_n$ of the system of '$n$' particles. If $m_1 = m_2 = m_3 = ...., m_n$, *then*

$$\vec{R}_{c.m} = \frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + .... + \vec{r}_n}{n}$$

**Centre of mass for a two particle system :** The centre of mass of two particle system of masses $m_1$ and $m_2$ is defined by a position vector.

$$\vec{R}_{c.m}(t) = \frac{m_1\vec{r}_1(t) + m_2\vec{r}_2(t)}{m_1 + m_2}$$

where $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are the position vectors of the two particles at any instant $t$.

If $m_1 = m_2$, then

$$\vec{R}_{c.m}(t) = \frac{\vec{r}_1(t) + \vec{r}_2(t)}{2}$$

Hence, the centre of mass of a given two particle system is a point defined mathematically where if a point mass (exactly equal to the total mass of the system) is supposed to be located, the overall motion of the system can be described with the help of this hypothetical point of mass.

**Coordinates of the centre of mass :** Cartesian coordinates of the centre of mass are given by,

$$x = \frac{1}{M}(m_1x_1 + m_2x_2 + .... + m_nx_n)$$

$$= \frac{1}{M}\sum_{i=1}^{n} m_ix_i$$

$$y = \frac{1}{M}(m_1y_1 + m_2y_2 + .... + m_ny_n)$$

$$= \frac{1}{M}\sum_{i=1}^{n} m_iy_i$$

$$z = \frac{1}{M}(m_1z_1 + m_2z_2 + .... + m_nz_n)$$

$$= \frac{1}{M}\sum_{i=1}^{n} m_iz_i$$

## Rigid Body

A rigid body means a system of point masses contained in a fixed a geometrical shape and size, i.e., the distances between every pair of a particles of a rigid body remain constant during its motion.

**Centre of mass of a Rigid Body :** The common centre about which the constituent particles of a rigid body execute internal rotational motion, is the centre of mass of the rigid body.

The position of the centre of mass is fixed with respect to the body as a whole and it may or may not lie within the body under consideration, depending upon the shape of the body.

**Torque :** Torque can be defined as the turning effect of a force about a given centre of rotation. It can be obtained by the product of the force and its level arm. i.e.,

$$\tau = \mathrm{F}d$$

Torque can also be obtained by the product of the radial distance of the point of application of the force from the centre of rotation and the tangential or angular component of the force. i.e.,

$$\tau = r\mathrm{F}_o$$

It is a vector quantity. Its SI unit are Nm.

**Torque and power :**

$$\text{Power (P)} = \frac{\Delta \mathrm{W}}{\Delta t} = \frac{\tau \Delta \theta}{\Delta t} = \tau\omega$$

Hence, power = Torque × angular velocity

## Positions of Centre of Mass of Some Symmetrical Rigid Bodies

| Shape of the rigid body | Position of centre of mass |
|---|---|
| 1. Cube | Point of intersection of the diagonals |
| 2. Sphere | Centre of the sphere |
| 3. Solid cone | On the line joining the apex of the cone to the centre of its base, at a distance equal to $\frac{1}{4}$th of its height from the base. |
| 4. Uniform rod | Middle point of the rod. |
| 5. Circular ring | Centre of the ring |
| 6. Circular disc | Centre of the disc |
| 7. Right circular cylinder | Middle point of the axis of the cylinder |

**Angular velocity :** Average angular velocity of a particle is defined as the ratio of the angular displacement to the time interval, i.e.,

Average angular velocity

$$= \frac{\text{Angular displacement}}{\text{time interval}}$$

or $$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular velocity of the particle is given by

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Its SI unit is rad $s^{-1}$.

Also, Linear velocity = Radius of the circular path × Angular velocity

or $v = r\omega$

**Angular Acceleration :** The average angular acceleration of a rotating body is defined as the ratio of the change in the angular velocity to the time interval i.e.,

Average angular acceleration

$$= \frac{\text{change in angular velocity}}{\text{time interval}}$$

## Comparison of linear and rotational motions

| | Linear motion | | Rotational motion | |
|---|---|---|---|---|
| | Concept | Representation | Concept | Representation |
| 1. | Position | $x$ | Angle | $\theta$ |
| 2. | Displacement | $\Delta x$ | Angular Displacement | $\Delta\theta$ |
| 3. | Velocity | $v = \frac{dx}{dt}$ | Angular Velocity | $\omega = \frac{d\theta}{dt}$ |
| 4. | Acceleration | $a = \frac{dv}{dt}$ | Angular Acceleration | $\alpha = \frac{d\omega}{dt}$ |
| 5. | Power | $Fv$ | Power | $\tau\omega$ |
| 6. | Work | $F.\Delta x$ | Work | $\tau.\Delta\theta$ |
| 7. | Mass | $m$ | Moment of inertia | $I$ |
| 8. | Momentum | $p = mv$ | Angular momentum | $L = I\omega$ |
| 9. | Force | $F$<br>$F = ma$<br>$F = \frac{dp}{dt}$ | Torque | $\tau$<br>$\tau = I\alpha$<br>$\tau = \frac{dL}{dt}$ |
| .0. | Kinetic Energy | $\frac{1}{2}mv^2$ | Kinetic energy | $\frac{1}{2}I\omega^2$ |

or $\vec{\alpha} = \dfrac{\omega_2 - \omega_1}{t_2 - t_1} = \dfrac{\Delta\omega}{\Delta t}$

Instantaneous angular acceleration is given by

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

SI unit of angular acceleration is rad $s^{-1}$.

Also, linear acceleration = Radius of the circular path × angular acceleration

$$\text{or } a = r\alpha.$$

**Angular Momentum (L) :** Angular momentum can be understand as the twisting or turning effect linked with the momentum vector. Due to this reason, angular momentum is generally called as the moment of momentum, just like as torque is called moment of force. It is given as the product of linear momentum ($p_\theta$) and the lever arm for momentum ($r$) i.e.,

$$L = r.p_\theta = r.mv_\theta = rm.r\omega = mr^2\omega$$

$$\text{Also, } \tau = \frac{dL}{dt}$$

or torque = rate of change of angular momentum.

**Law of Conservation of Angular Momentum :** According to this law, if the total external torque acting on a particle moving about a fixed point vanishes, the angular momentum of the particle about that point is conserved.

**Areal velocity and Angular momentum :**

The ratio $\frac{\Delta A}{\Delta t}$ gives the average rate of sweeping out of the area in time $\Delta t$ and is known as the average areal velocity of the moving particle.

$$\text{Also, } \frac{\Delta A}{\Delta t} = \frac{1}{2} r . v_{\theta}$$

i.e., the areal velocity of a moving particle is equal to half the product of its radial distance and the angular component of its velocity.

$$\text{Also, } L = 2m . \frac{\Delta A}{\Delta t}$$

i.e., the angular momentum of a moving particle about a given point is equal to twice the product of its mass and areal velocity about that point.

**Centripetal force :** Centripetal force is that external force due to which any particle moves along a circle, the force always acts radially inwards.

Centripetal force, $F = \frac{mv^2}{\ } = mr\omega^2$

where, $m$ = mass of the particle
$v$ = linear velocity of the particle
$\omega$ = angular velocity of the particle
$r$ = radius of the circle along which the particle moves.

**Note :**

*(i)* Centripetal force always acts on the particle performing circular motion.

*(ii)* Without centripetal force there can be no circular motion.

*(iii)* Its magnitude is constant but the direction is variable, therefore it is a variable force.

*(iv)* Centripetal force is always a real force. It may be mechanical, electrical, magnetic, etc.

**Centrifugal Force :** The pseudo force that balances the centripetal force in uniform circular motion is called centrifugal force.

**Note :**

*(i)* Centrifugal force always act on the centre.

*(ii)* It is directed always from the centre, along the radius.

*(iii)* Although it is equal and opposite to centripetal force, yet it is not the reaction of centripetal force because reaction can't exist without action while centrifugal force can exist without centripetal force.

**Rotational Kinetic Energy :**

$$\text{Rotational K.E.} = \frac{1}{2} I\omega^2$$

If $\omega = 1$, then I = 2 × rotational K.E.

Hence, the moment of inertia of a rigid body about a given axis of rotation is numerically equal to twice its rotational kinetic energy, when the body is rotating with unit angular velocity.

**Moment of Inertia :** Moment of inertia of a particle about a fixed axis is defined as the product of its mass and square of its distance from the axis of rotation. i.e.,

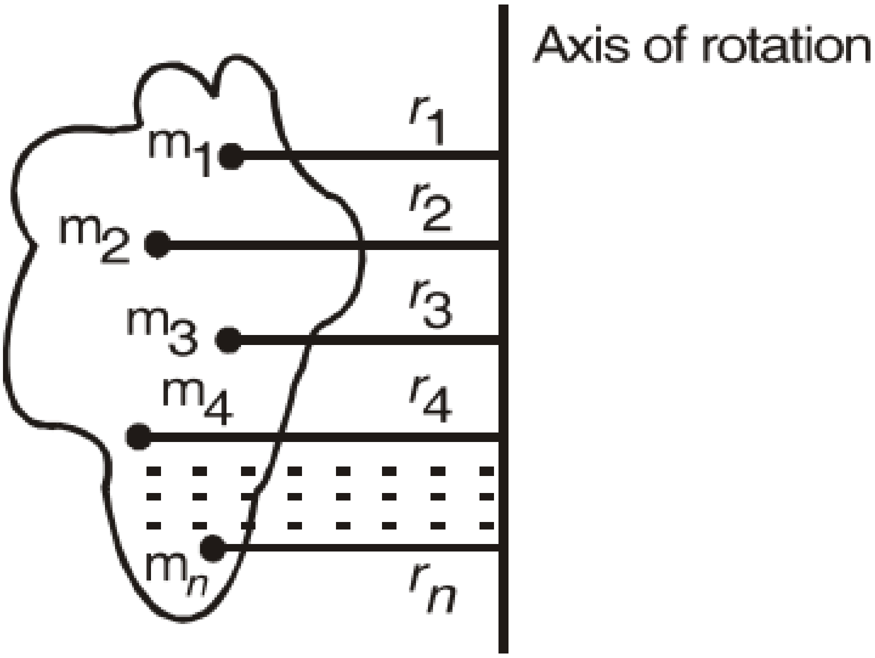
$$I = mr^2$$

If a rigid body consists of a number of particles of masses $m_1, m_2, m_3, ...., m_n$ distant $r_1, r_2, r_3, ...., r_n$ from the axis of rotation, then moment of inertia of the rigid body about a fixed axis is defined as the sum of the products of the masses of all the particles of the body and the squares of their respective distances from the axis of rotation, i.e.,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + .... + m_n r_n^2$$

$$= \sum_{i=1}^{n} m_i r_i^2$$

Its SI units are kg $m^2$ and it is a scalar quantity.




**Note :** Moment of inertia of a body depends upon

*(a)* the mass of the body

*(b)* the position of the axis of rotation

*(c)* the distribution of mass about the axis of rotation.

**Radius of gyration :** The radius of gyration of a body is the distance from the axis of rotation of a point at which, if the total mass of the body is supposed to be concentrated, its moment of inertia about the axis will be the same as that determined by the actual distribution of mass.

Hence, $I = MK^2$, where K is the radius of gyration.

$$\text{Also, } K = \sqrt{\frac{r_1^2 + r_2^2 + .... + r_n^2}{n}}$$

i.e., radius of gyration of a body about a given axis of rotation is equal to the root mean square distance of its individual particles about the same axis of rotation.

It depends upon *(a)* the position and direction of the axis of rotation, and *(b)* the distribution of mass about the axis of rotation. Its SI unit is metre (m).

**Theorems on moment of inertia :** There are two basic theorem on moment of inertia.

*(i)* Theorem of parallel axis.

*(ii)* Theorem of perpendicular axis.

***(i)* Theorem of parallel axis :** "The moment of inertia of a plane lamina about any axis in its own plane is equal to the sum of its moment of inertia about a parallel axis passing through its centre of mass and the product of the mass of the lamina and the square of the distance between the two axes."

i.e., $I = I_G + Mr^2$

where $I_G$ = moment of inertia of the plane lamina about an axis passing through is centre of mass.

I = moment of inertia of the plane lamina about a parallel axis in its own plane.

M = mass of the plane lamina

$r$ = distance between the two parallel axes.

***(ii)* Theorem of perpendicular axes :** "The sum of the moments of inertia of a plane lamina about any two mutually perpendicular axes in its own plane, is equal to its moment of inertia about a third axis which is perpendicular to the plane and passes through the point of intersection of the first two axes."

i.e., $I_z = I_x + I_y$

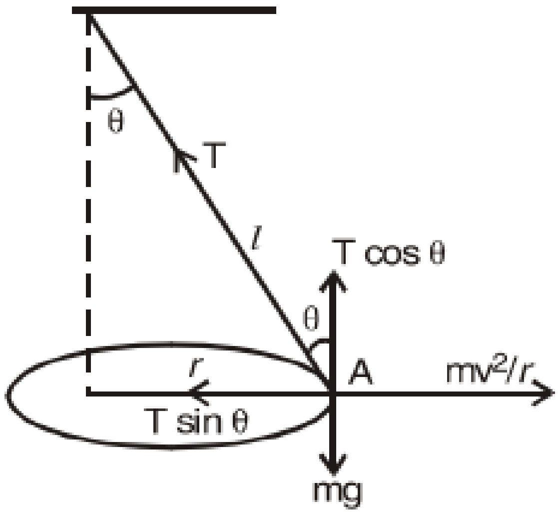
where $I_x$ = moment of inertia of the plane lamina about X-axis.

$I_y$ = moment of inertia of the plane lamina about Y-axis.

and $I_z$ = moment of inertia of the plane lamina about Z-axis.

**Conical pendulum :** A mass suspended from one end of a wire, other end of which is fixed, forms a conical pendulum. If its free end describes a circle and the wire describes the surface of the cone.

$$T \cos \theta = mg \qquad T \sin \theta = \frac{mv^2}{r}$$




**Kinetic Equations for rotational and linear motion under constant acceleration**

| | Rotational motion about fixed axis with constant acceleration ($\alpha$) | | Linear motion with constant acceleration ($a$) |
|---|---|---|---|
| *(i)* | $\omega = \dfrac{d\theta}{dt}$ | *(i)* | $v = \dfrac{dx}{dt}$ |
| *(ii)* | $\omega = \omega_0 + \alpha t$ | *(ii)* | $v(t) = v(0) + at$ |
| *(iii)* | $\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$ | *(iii)* | $v(t)^2 = v(0)^2 + 2a[x(t) - x(0)]$ |
| *(iv)* | $\theta = \theta_0 + \omega_0 t + \dfrac{1}{2}\alpha t^2$ | *(iv)* | $x(t) = x(0) + v(0)t + \dfrac{1}{2}at^2$ |

## Moment of inertia of uniform rigid bodies

| | Name of the body | | Axis of rotation | Moment of inertia |
|---|---|---|---|---|
| 1. | Circular ring of radius R and mass M. | (*a*) | Through its centre and perpendicular to its length | $MR^2$ |
| | | (*b*) | About its any diameter. | $\frac{MR^2}{2}$ |
| 2. | Circular lamina of disc of radius R and mass M. | (*a*) | Through its centre and perpendicular to its plane | $\frac{MR^2}{2}$ |
| | | (*b*) | About its any diameter. | $\frac{MR^2}{4}$ |

| | | |
|---|---|---|
| 3. Solid sphere of radius R and mass M. | About its any diameter. | $\frac{2}{5}MR^2$ |
| 4. Hallow sphere of radii R and r and mass M. | About its any diameter. | $\frac{2}{5}M\frac{(R^5 - r^5)}{(R^3 - r^3)}$ |
| 5. Thin rectangular lamina of sides of $a$ and $b$ and mass M. | (*a*) Through its centre and parallel to one side '$a$' or '$b$'. | $\frac{Mb^2}{12}$ or $\frac{Ma^2}{12}$ |
| | (*b*) About its one side. | $\frac{Mb^2}{3}$ or $\frac{Ma^2}{3}$ |

| | | |
|---|---|---|
| | (*c*) Through its centre and perpendicular to its plane. | $M\left(\frac{a^2+b^2}{12}\right)$ |
| | (*d*) Through mid point of one side (*a* or *b*) and perpendicular to its plane. | $M\left(\frac{b^2}{3}+\frac{a^2}{12}\right)$ or $M\left(\frac{a^2}{3}+\frac{b^2}{12}\right)$ |
| 6. Thin uniform rod of length L and mass M. | (*a*) Through its centre and perpendicular to its length. | $\frac{ML^2}{12}$ |

| | | |
|---|---|---|
| | (*b*) Through one end and perpendicular to its length. | $\frac{ML^2}{3}$ |
| 7. Thick uniform rectangular bar of length $a$, breadth $b$ and mass M. | Through its centre of mass and perpendicular to its length. | $\frac{M}{12}(a^2 + b^2)$ |
| 8. Hollow cylinder of radii R and $r$ and mass M. | About its own axis. | $M\frac{(R^2 + r^2)}{2}$ |
| 9. Solid cylinder of radius R and mass M. | About its own axis. | $\frac{MR^2}{2}$ |