Chapter – 4

Trigonometry

Ex 4.1

Question 1.
Convert the following degree measure into radian measure
(i) 60°
(ii) 150°
(iii) 240°
(iv) -320°

Solutions:

(i) $1° = \frac{\pi}{180}$ radians

$\therefore 60° = \frac{\pi}{180} \times 60$ radians $= \frac{\pi}{3}$ radians.

(ii) $150° = \frac{\pi}{180} \times 150$ radians $= \frac{5\pi}{6}$ radians.

(iii) $240° = \frac{\pi}{180} \times 240$ radians $= \frac{4\pi}{3}$ radians.

(iv) $-320° = \frac{\pi}{180} \times -320 = \frac{-16\pi}{9}$ radians

Question 2.
Find the degree measure corresponding to the following radian measure.

(i) $\frac{\pi}{8}$

(ii) $\frac{9\pi}{5}$

(iii) -3

(iv) $\frac{11\pi}{18}$

Solution:

We know that, one radian = $\frac{180^\circ}{\pi}$

(i) $\frac{\pi}{8}$

$\frac{\pi}{8} = \frac{180^\circ}{\pi} \times \frac{\pi}{8}$ degrees

= $\frac{45}{2}$

= 22.5°

= 22°30′ [∵ 0.5° = (0.5 × 60)′ = 30′]

(ii) $\frac{9\pi}{5}$

$\frac{9\pi}{5} = \frac{180^\circ}{\pi} \times \frac{9\pi}{5}$ degrees

= 36 × 9 degrees

= 324°

(iii) -3

$$-3 = \frac{180^\circ}{\pi} \times -3 = \frac{180 \times -3}{\frac{22}{7}} = \frac{-180 \times 3 \times 7}{22} = \frac{-90 \times 3 \times 7}{11}$$

= -171.81°

= -171°48′ (∵ 0.8° = (0.8 × 60)′ = 48′)

(iv) $\frac{11\pi}{18}$

$\frac{11\pi}{18} = \frac{180}{\pi} \times \frac{11\pi}{18}$

= 10 × 11°

= 110°

Question 3.

Determine the quadrants in which the following degree lie.

(i) 380°

(ii) -140°

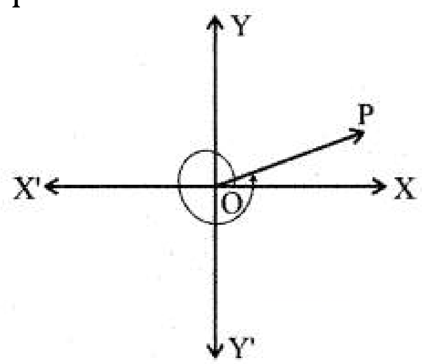
(iii) 1195°

Solution:

(i) $380° = 360° + 20°$

This is of the form $360° + \theta$

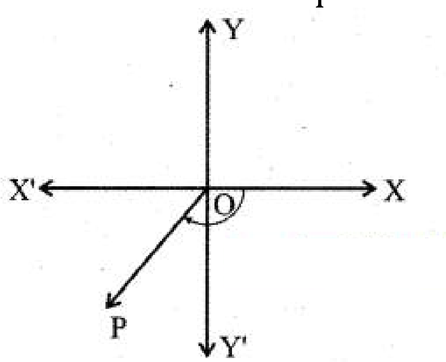
∴ After one completion of the round, the angle is 20°, 380° lies in the I quadrant.



(ii) $-140° = -90° + (-50°)$

The angle is negative it moves in the anti-clockwise direction.

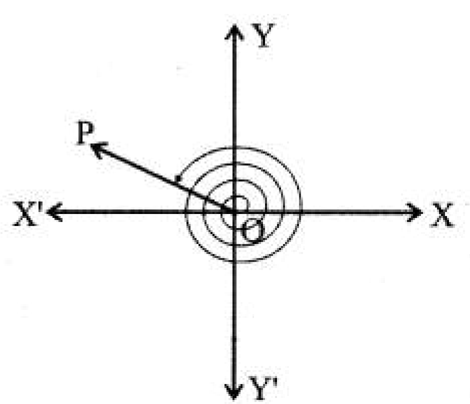
-140° lies in the III quadrants.



(iii) $1195° = (3 \times 360°) + 90° + 25°$

∴ After three completion round, the angle will lie in the II quadrant.

1195° lies in the II quadrant.



Question 4.
Find the values of each of the following trigonometric ratios.
(i) sin 300°
(ii) cos(-210°)
(iii) sec 390°
(iv) tan(-855°)
(v) cosec 1125°

Solution:
(i) sin 300° = sin(360° – 60°)
[For 360° – 60°. No change in T-ratio. 300° lies in 4th quadrant 'sin' is negative]
= -sin 60°
$$= -\frac{\sqrt{3}}{2}$$

(ii) cos(-210°) = cos 210° (∵ cos(-θ) = cos θ)
[∵ 180 + 30°. No change in T-ratio. 210° lies 3rd quadrant 'cos' is negative]
= cos(180° + 30°)
= -cos 30°
$$= -\frac{\sqrt{3}}{2}$$

(iii) sec 390° = sec(360° + 30°)
= sec 30°
$$= \frac{1}{\cos 30^\circ}$$
$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}$$
$$= \frac{2}{\sqrt{3}}$$

(iv) tan(-855°) = -tan 855° (∵ tan(-θ) = – tan θ)
[∵ Multiplies of 360° are dropped out. For 180° – 45°. No change in T-ratio. 180° – 45° lies in 2nd quadrant 'tan' is negative]
= -tan(2 × 360° + 135°)
= -tan 135°
= -tan(180° – 45°)
= -(-tan 45°)

= -(-1)
= 1

(v) cosec 1125° = cosec(3 × 360°+ 45°)

= cosec 45°

$= \frac{1}{\sin 45^\circ}$

$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)}$

= √2

Question 5.
Prove that:
(i) tan(-225°) cot(-405°) – tan(-765°) cot(675°) = 0.
(ii) $2 \sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$
(iii) $\sec\left(\frac{3\pi}{2} - \theta\right) \sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right) \tan\left(\theta - \frac{5\pi}{2}\right) = -1$

Solution:
(i) tan(-225°) = -(tan 225°)
= -(tan(180° + 45°))
= – tan 45°
= – 1
cot(-405°) = -(cot 405°)
= – cot(360° + 45°) [∵ For 360° + 45° no change in T-ratio.]
= -cot 45°
= -1
tan(-765°) = -tan 765°
= -tan(2 × 360° + 45°)
= -tan 45°
= -1
cot 675° = cot (360°+ 315°)
= cot 315°
= cot(360° – 45°)
= -cot 45°
= -1
LHS = tan(-225°) cot(-405°) – tan(-765°) cot(675°)
= (-1) (-1) – (-1) (-1)
= 1 – 1

= 0
= RHS.
Hence proved.

(ii) 2 $\sin^2 \frac{\pi}{6}$ + $\text{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3}$ = $\frac{3}{2}$

LHS = 2 $\sin^2 \frac{\pi}{6}$ + $\text{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3}$

[∵ $\frac{7\pi}{6}$ = 210°, 210° = 180° + 30°. For 180° + 30° no change in T-ratio.

210° lies in 3rd quadrant, cosec θ is negative.]

= $2\left(\sin \frac{\pi}{6}\right)^2$ + (cosec (180° + 30°))^2 $\left(\cos \frac{\pi}{3}\right)^2$

= 2 $\left(\frac{1}{2}\right)^2$ + (-cosec 30°)^2 . $\left(\frac{1}{2}\right)^2$

= $2 \times \frac{1}{4} + (-2)^2 \frac{1}{4}$

= $\frac{2}{4} + \frac{4}{4} = \frac{6}{4}$

= $\frac{6}{4}$

= $\frac{3}{2}$

= RHS

(iii) sec($\frac{3\pi}{2}$ – θ) = sec (270° – θ) = -cosec θ

[∵ For 270° – θ change T-ratio. So add 'co' infront 'sec', it becomes 'cosec']

sec(θ – $\frac{5\pi}{2}$) = $\sec\left(-\left(\frac{5\pi}{2} - \theta\right)\right)$

= sec($\frac{5\pi}{2}$ – θ) [∵ sec(-θ) = θ]

= sec(450° – θ)

= sec (360° + (90° – θ))

= sec (90° – θ)

= cosec θ

[∵ For 90° – θ change in T-ratio. So add 'co' in front of 'sec' it becomes 'cosec']

tan($\frac{5\pi}{2}$ + θ) = tan(450° + θ)

[∵ For 90° + θ, change in T-ratio. So add 'co' in front of 'tan' it becomes 'cot']

$\tan(\frac{5\pi}{2} + \theta) = \tan(450° + \theta)$

[∵ For 90° + θ, change in T-ratio. So add 'co' in front of 'tan' it becomes 'cot']

$= \tan(360° + (90° + \theta))$

$= \tan(90° + \theta)$

$= -\cot\theta$

$\tan\left(\theta - \frac{5\pi}{2}\right) = \tan\left(-\left(\frac{5\pi}{2} - \theta\right)\right)$

$= -\tan\left(\frac{5\pi}{2} - \theta\right)$ [∵ $\tan(-\theta) = -\tan\theta$]

$= -\tan(450° - \theta)$

$= -\tan(360° + (90° - \theta))$

$= -\tan(90° - \theta)$

$= -\cot\theta$

LHS $= \sec\left(\frac{3\pi}{2} - \theta\right)\sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right)\tan\left(\theta - \frac{5\pi}{2}\right)$

$= -\text{cosec}\,\theta\,(\text{cosec}\,\theta) + (-\cot\theta)(-\cot\theta)$

$= -\text{cosec}^2\theta + \cot^2\theta$

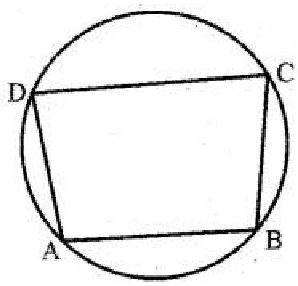
$= -(1 + \cot^2\theta) + \cot^2\theta$ [∵ $1 + \cot^2\theta = \text{cosec}^2\theta$]

$= -1$

= RHS

Question 6.
If A, B, C, D are angles of a cyclic quadrilateral, prove that: cos A + cos B + cos C + cos D = 0.

Solution:
Note: If the vertices of a quadrilateral lie on the circle then the quadrilateral is called a cyclic quadrilateral.
In a cyclic quadrilateral sum of opposite angles are 180°.



Since A, B, C, D are angles of cyclic quadrilateral
A + C = 180° and B + D = 180°
LHS = cos A + cos B + cos C + cos D
= cos A + cos B + cos(180° – A) + cos(180° – B)
= cos A + cos B – cos A – cos B
= 0
= RHS

Question 7.
Prove that

$$\textit{(i)}\ \frac{\sin(180^\circ-\theta)\cos(90^\circ+\theta)\tan(270^\circ-\theta)\cot(360^\circ-\theta)}{\sin(360^\circ-\theta)\cos(360^\circ+\theta)\sin(270^\circ-\theta)\operatorname{cosec}(-\theta)}=-1.$$

(ii) sin θ . cos{sin(π/2 – θ) . cosec θ + cos(π/2 – θ) . sec θ} = 1

Solution:

$$\textit{(i)}\ \frac{\sin(180^\circ-\theta)\cos(90^\circ+\theta)\tan(270^\circ-\theta)\cot(360^\circ-\theta)}{\sin(360^\circ-\theta)\cos(360^\circ+\theta)\sin(270^\circ-\theta)\operatorname{cosec}(-\theta)}=-1.$$

$$\text{LHS}=\frac{\sin(180^\circ-\theta)\cos(90^\circ+\theta)\tan(270^\circ-\theta)\cot(360^\circ-\theta)}{\sin(360^\circ-\theta)\cos(360^\circ+\theta)\sin(270^\circ-\theta)\operatorname{cosec}(-\theta)}$$

$$=\frac{(\sin\theta)(-\sin\theta)(\cot\theta)(-\cot\theta)}{(-\sin\theta)(\cos\theta)(-\cos\theta)(-\operatorname{cosec}\theta)}$$

$$=\frac{-\sin\theta\times\cot\theta\cot\theta}{\cos\theta\times\cos\theta\ \operatorname{cosec}\theta}$$

$$=\frac{-\sin\theta\times\frac{\cos\theta}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}}{\cos\theta\times\cos\theta\times\frac{1}{\sin\theta}}$$

$$=-1\times\frac{\sin\theta}{\sin\theta}=-1=\text{RHS}$$

(ii) $\sin\theta \,.\, \cos\{\sin(\frac{\pi}{2}-\theta) \,.\, \text{cosec}\,\theta + \cos(\frac{\pi}{2}-\theta) \,.\, \sec\theta\} = 1$

$$\text{LHS} = \sin\theta \cdot \cos\theta\left\{\sin\left(\frac{\pi}{2}-\theta\right)\cdot \text{cosec}\,\theta + \cos\left(\frac{\pi}{2}-\theta\right)\sec\theta\right\}$$

$$= \sin\theta \cdot \cos\theta\left\{\cos\theta\frac{1}{\sin\theta} + \sin\theta\cdot\frac{1}{\cos\theta}\right\}$$

$$= \sin\theta \cdot \cos\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \cos^2\theta + \sin^2\theta = 1 = \text{RHS} \qquad [\text{since } \sin^2\theta + \cos^2\theta = 1]$$

Question 8.
Prove that: cos 510° cos 330° + sin 390° cos 120° = -1.

Solution:
LHS = cos 510° cos 330° + sin 390° cos 120°
= cos(360° + 150°) cos(360° – 30°) + sin(360° + 30°) × cos(180° – 60°)
= cos 150° cos 30° + sin 30° (-cos 60°)
= cos(180° – 30°) cos 30° + sin 30° cos 60°

$$= -\cos 30° \cos 30° + \frac{1}{2} \times \left(\frac{-1}{2}\right)$$

$$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2}$$

$$= -\frac{3}{4} - \frac{1}{4}$$

$$= \frac{-3-1}{4}$$

$$= -1$$

Question 9.
Prove that:
(i) tan(π + x) cot(x – π) – cos(2π – x) cos(2π + x) = $\sin^2$ x.

$$\textit{(ii)}\ \frac{\sin(180°+A)\cos(90°-A)\tan(270°-A)}{\sec(540°-A)\cos(360°+A)\text{cosec}(270°+A)} = -\sin A\cos^2 A.$$

Solution:
(i) tan(π + x) cot(x – π) – cos(2π – x) cos(2π + x) = (tan x) (-cot(π – x) – cos x cos x
[∵ cot(x – π) = cot(-(π – x)) = -cot(π – x) = cot x]
= tan x cot x – $\cos^2$ x

$= 1 - \cos^2 x$
$= \sin^2 x$ [$\because \sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = (1 - \cos^2 x)$]

(ii) $\dfrac{\sin(180° + A)\cos(90° - A)\tan(270° - A)}{\sec(540° - A)\cos(360° + A)\text{cosec}(270° + A)} = -\sin A \cos^2 A.$

$$\text{LHS} = \frac{\sin(180° + A)\cos(90° - A)\tan(270° - A)}{\sec(540° - A)\cos(360° + A)\text{cosec}(270° + A)}$$

$$= \frac{(-\sin A)(\sin A)(\cot A)}{(-\sec A)\cos A(-\sec A)}$$

$$= \frac{-\sin A \sin A \frac{\cos A}{\sin A}}{-\frac{1}{\cos A}\cos A - \frac{1}{\cos A}}$$

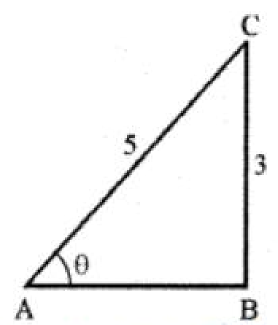
$$= -\sin A \times \cos A \times \cos A$$

$$= -\sin A \cos^2 A = \text{RHS}$$

$\sec(540° - A)$
$= \sec(360° + 180° - A)$
$= \sec(180° - A)$
$= (-\sec A)$

Question 10.
If $\sin \theta = 3/5$, $\tan \varphi = 1/2$ and $\pi/2 < \theta < \pi < \varphi < 3\pi/2$, then find the value of $8 \tan \theta - \sqrt{5} \sec \varphi$.

Solution:



Given that $\sin \theta = \frac{3}{5} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

$\because AB = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$

Here θ lies in second quadrant [$\because \frac{\pi}{2} < \theta < \pi$]

$\because$ tan θ is negative.

$\tan \theta = -\frac{3}{4}$

Also given that tan Φ = $\frac{1}{2} = \frac{\text{Opposite side}}{\text{Adjacent side}}$

∴ PR = $\sqrt{PQ^2 + QP^2} = \sqrt{4+1} = \sqrt{5}$

Here Φ lies in third quadrant (∵ π < Φ < $\frac{3\pi}{2}$)

∴ sec Φ is negative.

$\sec\phi = \frac{1}{\cos\phi} = -\frac{1}{\left(\frac{2}{\sqrt{5}}\right)} = -\frac{\sqrt{5}}{2}$

Now 8 tan θ – √5 sec Φ = $8\left(-\frac{3}{4}\right) - \sqrt{5}\left(\frac{-\sqrt{5}}{2}\right)$

= 2 × (-3) + $\frac{5}{2}$

= -6 + $\frac{5}{2}$

= $\frac{-12+5}{2}$

= $\frac{-7}{2}$

Ex 4.2

Question 1.
Find the values of the following:
(i) cosec 15°
(ii) sin (-105°)
(iii) cot 75°

Solution:

(i) cosec 15° = $\frac{1}{\sin 15^\circ}$

Consider sin 15° = sin(45° – 30°)

= sin 45° cos 30° – cos 45° sin 30°

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

cosec 15° = $\frac{1}{\sin 15^\circ} = \frac{2\sqrt{2}}{\sqrt{3}-1}$

(ii) sin (-105°) = -sin (105°) ($\because$ sin (-θ) = – sin θ)

= -[sin(60° + 45°)]

= -[sin 60° cos 45° + cos 60° sin 45°]

$$= -\left[\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}\right] = -\left[\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\right] = -\left[\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$

(iii) cot 75° = $\frac{1}{\tan 75^\circ}$

Consider tan 75° = tan (30° + 45°)

$$= \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}$$

$$= \frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)\times 1} = \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)}{\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)}$$

$$= \frac{1+\sqrt{3}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

cot 75° = $\frac{1}{\tan 75^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Question 2.

Find the values of the following:

(i) sin 76° cos 16° – cos 76° sin 16°

(ii) sin π/4 cos π/12 +cos π/4 sin π/12

(iii) cos 70° cos 10° – sin 70° sin 10°

(iv) $\cos^2$ 15° – $\sin^2$ 15°

Solution:

(i) Given that, sin 76° cos 16° – cos 76° sin 16° ($\therefore$ This is of the form sin(A – B))

= sin(76° – 16°)

= sin 60°

$= \frac{\sqrt{3}}{2}$

(ii) This is of the form sin(A + B) = $\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)$

$= \sin\left(\frac{3\pi+\pi}{12}\right)$

$= \sin \frac{4\pi}{12}$

$= \sin \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$ ($\because$ sin 60° = $\frac{\sqrt{3}}{2}$)

(iii) Given that cos 70° cos 10° – sin 70° sin 10°

(This is of the form of cos (A + B), A = 70°, B = 10°)

= cos (70° + 10°)

= cos 80°

(iv) $\cos^2 15° - \sin^2 15°$

[$\because \cos 2A = \cos^2 A - \sin^2 A$, Here A = 15°]

= cos (2 × 15°)

= cos 30°

$= \frac{\sqrt{3}}{2}$

Question 3.

If sin A = 3/5, 0 < A < π/2 and cos B = −12/13, π < B < 3π/2, find the values of the following:

(i) cos(A + B)

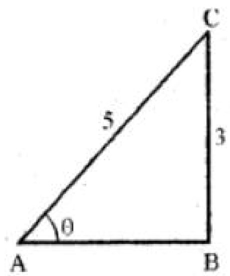
(ii) sin(A – B)

(iii) tan(A – B)

Solution:

Given that sin A = 3/5, 0 < A < π/2 (i.e., A lies in first quadrant)

Since A lies in first quadrant cos A is positive.



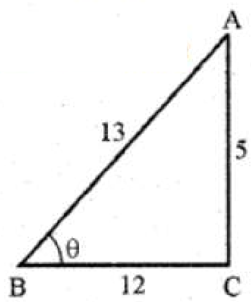
cos A = $\frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{4}{5}$

tan A = $\frac{3}{4}$

AB = $\sqrt{5^2 - 3^2}$ = 4

Also given that cos B = $\frac{-12}{13}$, π < B < $\frac{3\pi}{2}$ (i.e., B lies in third quadrant)

Now sin B lies in third quadrant. sin B is negative.



CA = $\sqrt{13^2 - 12^2}$ = 5

sin B = $\frac{-\text{Opposite side}}{\text{Hypotenuse}} = \frac{-5}{13}$

tan B = $\frac{-\text{Opposite side}}{\text{Adjacent}} = \frac{5}{12}$ [B lies in 3rd quadrant. tan B is positive.]

(i) cos(A + B) = cos A cos B – sin A sin B

$$= \frac{4}{5}\left(\frac{-12}{13}\right) - \frac{3}{5} \times \left(\frac{-5}{13}\right) = \frac{-48}{65} + \frac{15}{65} = \frac{-33}{65}$$

(ii) sin(A – B) = sin A cos B – cos A sin B

$$= \frac{3}{5}\left(\frac{-12}{13}\right) - \frac{4}{5} \times \left(\frac{-5}{13}\right) = \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}$$

(iii) tan(A – B)

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{\frac{3}{4}-\left(\frac{5}{12}\right)}{1+\frac{3}{4}\times\left(\frac{5}{12}\right)}=\frac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}}=\frac{\frac{9-5}{12}}{1+\frac{5}{4\times4}}$$

$$=\frac{\frac{4}{12}}{\frac{21}{16}}=\frac{4}{12}\times\frac{16}{21}=\frac{4\times4}{3\times21}=\frac{16}{63}$$

Question 4.
If cos A = 13/14 and cos B = 1/7 where A, B are acute angles prove that A – B = π/3

Solution:
cos A = 13/14, cos B = 1/7

$$\sin A=\sqrt{1-\cos^2 A}$$

$$=\sqrt{1-\left(\frac{13}{14}\right)^2}=\sqrt{1-\frac{169}{196}}=\sqrt{\frac{196-169}{196}}=\sqrt{\frac{27}{14}}=\frac{3\sqrt{3}}{14}$$

$$\sin B=\sqrt{1-\cos^2 B}$$

$$=\sqrt{1-\frac{1}{49}}=\sqrt{\frac{48}{49}}=\frac{\sqrt{48}}{7}=\frac{4\sqrt{3}}{7}$$

cos(A – B) = cos A cos B + sin A sin B

$$=\frac{13}{14}\times\frac{1}{7}+\frac{3\sqrt{3}}{14}\times\frac{4\sqrt{3}}{7}=\frac{13}{98}+\frac{36}{98}=\frac{49}{98}=\frac{1}{2}$$

cos(A – B) = cos 60°
A – B = 60° = π/3

Question 5.
Prove that 2 tan 80° = tan 85° – tan 5°.

Solution:
Consider tan 80° = tan(85° – 5°)

$$=\frac{\tan 85^\circ-\tan 5^\circ}{1+\tan 85^\circ\tan 5^\circ}=\frac{\tan 85^\circ-\tan 5^\circ}{1+\tan 85^\circ\tan(90^\circ-85^\circ)}$$

$$= \frac{\tan 85° - \tan 5°}{1 + \tan 85° \times \cot 85°} = \frac{\tan 85° - \tan 5°}{1+1}$$

$$= \frac{\tan 85° - \tan 5°}{2}$$

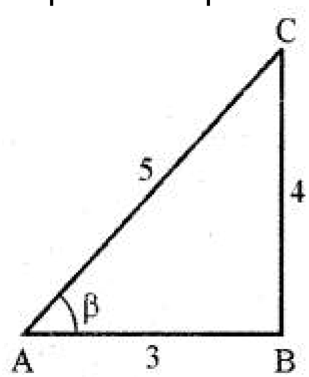
∴ 2 tan 80° = tan 85° – tan 5°
Hence Proved.

Question 6.
If cot α = 1/2, sec β = −5/3, where π < α < 3π/2 and π/2 < β < π, find the value of tan(α + β). State the quadrant in which α + β terminates.

Solution:
Given that cot α = 1/2 where π < α < 3π/2 (i.e,. α lies in third quadrant)

$\tan\alpha = \frac{1}{\frac{1}{2}}$ = 2 [∵ In 3rd quadrant tan α is positive]

Also given that sec β = −5/3 where π/2 < β < π (i.e., β lies in second quadrant cos β and tan β are negative)



$$BC = \sqrt{5^2 - 3^2} = 4$$

$$\text{Now } \cos\beta = \frac{1}{\sec\beta} = \frac{-3}{5}$$

$$\therefore \tan\beta = \frac{-\text{Opposite side}}{\text{Hypotenuse}} = -\frac{4}{3}$$

$$\text{Consider } \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{2+\left(\frac{-4}{3}\right)}{1-2\left(\frac{-4}{3}\right)} = \frac{\frac{(2\times3-4)}{3}}{1+\frac{8}{3}} = \frac{\frac{2}{3}}{\frac{11}{3}} = \frac{2}{11}$$

tan (α + β) = 211 which is positive.
α + β terminates in first quadrant.

Question 7.
If A + B = 45°, prove that (1 + tan A) (1 + tan B) = 2 and hence deduce the value of tan $22\frac{1}{2}$.

Solution:
Given A + B = 45°
tan (A + B) = tan 45°

$$\frac{\tan A+\tan B}{1-\tan A\tan B} = 1$$

tan A + tan B = 1 – tan A . tan B

tan A + tan B + tan A tan B = 1

Add 1 on both sides we get,

(1 + tan A) + tan B + tan A tan B = 2

1(1+ tan A) + tan B (1 + tan A) = 2

(1 + tan A) (1 + tan B) = 2 ……. (1)

Put A = B = $22\frac{1}{2}$ in (1) we get

(1 + tan $22\frac{1}{2}$) (1 + tan $22\frac{1}{2}$) = 2

⇒ $(1 + \tan 22\frac{1}{2})^2 = 2$

⇒ 1 + tan $22\frac{1}{2}$ = ±√2

⇒ tan $22\frac{1}{2}$ = ±√2 – 1

Since $22\frac{1}{2}$ is acute, tan $22\frac{1}{2}$ is positive and therefore tan $22\frac{1}{2}$ = √2 – 1

Question 8.
Prove that
(i) sin(A + 60°) + sin(A – 60°) = sin A.
(ii) tan 4A tan 3A tan A + tan 3A + tan A – tan 4A = 0

Solution:
(i) LHS = sin (A + 60°) + sin (A – 60°)
= sin A cos 60° + cos A sin 60° + sin A cos 60° – cos A sin 60°
= 2 sin A cos 60°
= 2 sin A (1/2)
= sin A
= RHS

(ii) 4A = 3A + A

tan 4A = tan (3A + A)

tan 4A = $\frac{\tan 3A + \tan A}{1 - \tan 3A \tan A}$

on cross multiplication we get,

tan 3A + tan A = tan 4A (1 – tan 3A tan A) = tan 4A – tan 4A tan 3A tanA

i.e., tan 4A tan 3A tan A + tan 3A + tan A = tan 4A

(or) tan 4A tan 3A tan A + tan 3A + tan A – tan 4A = 0

Question 9.
(i) If tan θ = 3 find tan 3θ
(ii) If sin A = 12/13, find sin 3A.

Solution:
(i) tan θ = 3

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{3(3) - (3)^3}{1 - 3(3)^2}$$

$$= \frac{9 - 27}{1 - 27} = \frac{-18}{-26} = \frac{9}{13}$$

(ii) If sin A = 12/13

We know that sin 3A = 3 sin A – 4 $\sin^3$ A

$$= 3\left(\frac{12}{13}\right) - 4\left(\frac{12}{13}\right)^3 = \frac{12}{13}\left[3 - 4 \times \frac{12}{13} \times \frac{12}{13}\right]$$

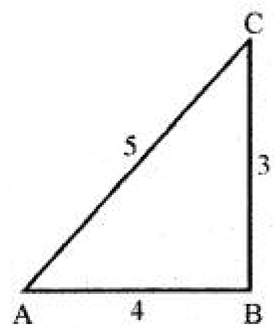
$$= \frac{12}{13}\left[3 - \frac{576}{169}\right] = \frac{12}{13}\left[\frac{507 - 576}{169}\right] = \frac{12}{13}\left[\frac{-69}{169}\right]$$

$$= \frac{-828}{2197}$$

Question 10.

If sin A = 3/5, find the values of cos 3A and tan 3A.

Solution:



Given sin A = $\frac{3}{5}$

cos A = $\frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{4}{5}$

and tan A = $\frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{3}{4}$

$$= 4\left(\frac{4}{5}\right)^3 - 3\left(\frac{4}{5}\right)$$

$$= \left(\frac{4}{5}\right)\left[4 \times \left(\frac{4}{5}\right)^2 - 3\right] = \frac{4}{5}\left[4 \times \frac{16}{25} - 3\right] = \frac{4}{5}\left[\frac{64 - 3 \times 25}{25}\right]$$

$$= \frac{4}{5}\left(\frac{64 - 75}{25}\right) = \frac{4}{5} \times \frac{-11}{25} = \frac{-44}{125}$$

$$\tan 3A = \frac{3\tan A - 4\tan^3 A}{1 - 3\tan^2 A} = \frac{3\left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^3}{1 - 3\left(\frac{3}{4}\right)^2}$$

$$= \frac{\frac{3}{4}\left[3 - \left(\frac{3}{4}\right)^2\right]}{1 - 3 \times \frac{9}{16}} = \frac{\frac{3}{4}\left[\frac{48-9}{16}\right]}{\frac{(16-27)}{16}} = \frac{3}{4}\left[\frac{39}{16} \times \frac{16}{-11}\right] = \frac{-117}{44}$$

Question 11.

Prove that $\frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B} = 0$

Solution:

Consider $\frac{\sin(B-C)}{\cos B \cos C}$

$= \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C}$

$= \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C}$

= tan B – tan C .... (1)

Similarly we can prove $\frac{\sin(C-A)}{\cos C \cos A}$ = tan C – tan A ....(2)

and $\frac{\sin(A-B)}{\cos A \cos B}$ = tan A – tan B .... (3)

Add (1), (2) and (3) we get

$\frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B} = 0$

Question 12.

If tan A – tan B = x and cot B – cot A = y prove that cot(A – B) = $\frac{1}{x} + \frac{1}{y}$.

Solution:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{\tan A - \tan B} + \frac{1}{\cot B - \cot A}$$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\frac{1}{\tan B} - \frac{1}{\tan A}}$$

$$= \frac{1}{\tan A - \tan B} + \frac{1}{\left(\frac{\tan A - \tan B}{\tan A \tan B}\right)}$$

$$= \frac{1}{\tan A - \tan B} + \frac{\tan A \tan B}{\tan A - \tan B} = \frac{1 + \tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1}{\tan(A - B)} \qquad \left(\because \tan(A - B) = \frac{\tan A - \tan B}{1 - \tan A \tan B}\right)$$

$$= \cot(A - B) = \text{LHS}$$

Hence proved.

Question 13.
If sin α + sin β = a and cos α + cos β = b, then prove that cos(α – β) = $\frac{a^2+b^2-2}{2}$

Solution:
Consider $a^2 + b^2 = \sin^2\alpha + \sin^2\beta + 2 \sin \alpha \sin \beta + \cos^2\alpha + \cos^2\beta + 2 \cos \alpha \cos \beta$

$a^2 + b^2 = (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + 2[\cos \alpha \cos \beta + \sin \alpha \sin \beta]$

$a^2 + b^2 = 1 + 1 + 2 \cos(\alpha - \beta)$

$\therefore \cos(\alpha - \beta) = \frac{a^2+b^2-2}{2}$

Question 14.
Find the value of tan π/8.

Solution:
Method 1:

$\frac{\pi}{8} = \frac{180^\circ}{8} = \frac{45^\circ}{2} = 22\frac{1}{2}$

We know that tan 2A = $\frac{2 \tan A}{1-\tan^2 A}$

Put A = $22\frac{1}{2}$ in the above formula

We get $\tan 2(22\frac{1}{2}^\circ) = \frac{2 \tan 22\frac{1}{2}^\circ}{1 - \tan^2 22\frac{1}{2}^\circ}$

$$\tan 45^\circ = \frac{2\tan 22\frac{1}{2}^\circ}{1-\tan^2 22\frac{1}{2}^\circ}$$

$$1 = \frac{2\tan 22\frac{1}{2}^\circ}{1-\tan^2 22\frac{1}{2}^\circ}$$

On cross multiplication we get

$$1-\tan^2 22\frac{1}{2}^\circ = 2\tan 22\frac{1}{2}^\circ$$

$$\text{(or) } \tan^2 22\frac{1}{2}^\circ + 2\tan 22\frac{1}{2}^\circ - 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\tan 22\frac{1}{2}^\circ = \frac{-2 \pm \sqrt{4-4\times1\times(-1)}}{2\times1}$$

Here a = 1, b = 2, c = -1

$$= \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2} = 2\left[\frac{-1 \pm \sqrt{2}}{2}\right] = -1 \pm \sqrt{2}$$

Since $22\frac{1}{2}$ is acute $\tan 22\frac{1}{2}$ is positive $\tan 22\frac{1}{2} = \tan\frac{\pi}{8}$

$= -1 + \sqrt{2}$

$= \sqrt{2} - 1$

Method 2:

$$\frac{\pi}{8} = \frac{180^\circ}{8} = \frac{45^\circ}{2} = 22\frac{1}{2}^\circ$$

$$\text{Consider } \tan^2\frac{A}{2} = \frac{\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}} = \frac{\left(\frac{1-\cos A}{2}\right)}{\frac{1+\cos A}{2}}$$

$$\left(\because \sin^2 A = \frac{1-\cos 2A}{2}; \cos^2 A = \frac{1+\cos 2A}{2}\right)$$

$$\tan^2 \frac{A}{2} = \frac{1-\cos A}{1+\cos A}$$

Put A = 45°, we get

$$\tan^2 \frac{45^\circ}{2} = \frac{1-\cos 45^\circ}{1+\cos 45^\circ}$$

$$\tan^2 \frac{A}{2} = \frac{1-\cos A}{1+\cos A}$$

Put A = 45°, we get

$$\tan^2 \frac{45^\circ}{2} = \frac{1-\cos 45^\circ}{1+\cos 45^\circ}$$

$$= \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)}{\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)} = \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \frac{(\sqrt{2}-1)^2}{(\sqrt{2})^2 - 1^2}$$

$$\tan^2 \frac{45^\circ}{2} = \frac{(\sqrt{2}-1)^2}{1}$$

$\therefore \tan^2 22\frac{1}{2} = (\sqrt{2}-1)^2$

Taking square root, $\tan^2 22\frac{1}{2}$ = ±(√2 – 1)

But $22\frac{1}{2}$ lies in first quadrant, tan $22\frac{1}{2}$ is positive.

∴ tan $22\frac{1}{2}$ = √2 – 1

Method 3:

consider tan A = $\frac{\sin 2A}{1+\cos 2A}$

Put A = $22\frac{1}{2}$

$$\left[\because \frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin A}{\cos A} = \tan A\right]$$

$$\tan 22\frac{1}{2}^\circ = \frac{\sin\left(2\times 22\frac{1}{2}^\circ\right)}{1+\cos\left(2\times 22\frac{1}{2}^\circ\right)} = \frac{\sin 45^\circ}{1+\cos 45^\circ} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}+1}$$

$$= \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{(\sqrt{2})^2 - 1^2} = \frac{\sqrt{2}-1}{2-1}$$

$\tan 22\frac{1}{2} = \sqrt{2} - 1$

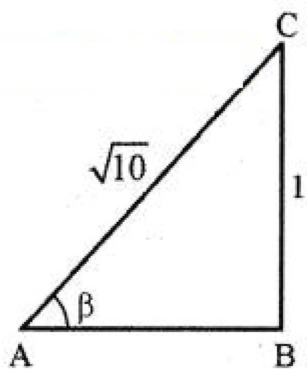
Question 15.

If $\tan \alpha = \frac{1}{7}$, $\sin \beta = \frac{1}{\sqrt{10}}$. Prove that $\alpha + 2\beta = \frac{\pi}{4}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.

Solution:

Given that $\tan \alpha = 1/7$

We wish to find $\tan(\alpha + 2\beta)$



$$AB^2 = AC^2 - BC^2$$
$$= 10 - 1 = 9$$
$$AB = 3$$

$$\sin \beta = \frac{1}{\sqrt{10}} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\tan \beta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{1}{3} \text{ (Here } \beta \text{ is an acute angle)}$$

$$\text{Now } \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}$$

$$= \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

$$\text{Consider } \tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha \tan 2\beta} = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}}$$

$$= \frac{\frac{1\times4+3\times7}{28}}{1-\frac{3}{28}} = \frac{\frac{25}{28}}{\frac{25}{28}} = 1$$

$$\tan(\alpha + 2\beta) = \tan\frac{\pi}{4} \quad \left(\because \tan\frac{\pi}{4} = 1\right)$$

$$\therefore \alpha + 2\beta = \frac{\pi}{4}$$

Ex 4.3

Question 1.
Express each of the following as the sum or difference of sine or cosine:

(i) $\sin\frac{A}{8} \sin\frac{3A}{8}$

(ii) cos(60° + A) sin(120° + A)

(iii) $\cos\frac{7A}{3} \sin\frac{5A}{3}$

(iv) cos 7θ sin 3θ

Solution:

(i) $\sin\frac{A}{8} \sin\frac{3A}{8}$

$$\sin\frac{A}{8} \sin\frac{3A}{8} = \frac{1}{2}\left(2\sin\frac{A}{8}\sin\frac{3A}{8}\right)$$

[∵ 2 sin A sin B = cos(A – B) – cos(A + B)

$$= \frac{1}{2}\left[\cos\left(\frac{A}{8}-\frac{3A}{8}\right)-\cos\left(\frac{A}{8}+\frac{3A}{8}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{A-3A}{8}\right)-\cos\left(\frac{A+3A}{8}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{-2A}{8}\right)-\cos\left(\frac{4A}{8}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{-A}{4}\right)-\cos\left(\frac{A}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{A}{4}-\cos\frac{A}{2}\right]$$

[$\because$ cos(-θ) = cos θ]

(ii) cos(60° + A) sin(120° + A) = $\frac{1}{2}$ [2 cos(60° + A) sin(120° + A)] [Multiply and divide by 2]

= $\frac{1}{2}$ [sin((60° + A) + (120° + A))] – sin((60° + A) – (120° + A))]

[$\because$ 2 cos A sin B = sin(A + B) – sin(A – B)]

= $\frac{1}{2}$ [sin(180° + 2A) – sin(60° + A – 120° – A)]

= $\frac{1}{2}$ [(-sin 2A) – sin(-60°)]

= $\frac{1}{2}$ [-sin 2A + sin 60°]

= $\frac{1}{2}$ [-sin 2A + $\frac{\sqrt{3}}{2}$]

(iii) cos $\frac{7A}{3}$ sin $\frac{5A}{3}$

$$= \frac{1}{2}\left[2\cos\frac{7A}{3}\sin\frac{5A}{3}\right]$$ [Multiply and divide by 2]

$$= \frac{1}{2}\left[\sin\left(\frac{7A}{3}+\frac{5A}{3}\right)-\sin\left(\frac{7A}{3}-\frac{5A}{3}\right)\right]$$

$$= \frac{1}{2}\left[\sin\frac{12A}{3}-\sin\frac{7A-5A}{3}\right]$$

$$= \frac{1}{2}\left[\sin 4A-\sin\frac{2A}{3}\right]$$

(iv) cos 7θ sin 3θ = 1/2 [sin(7θ + 3θ) – sin(7θ – 3θ)]
= 1/2 (sin 10θ – sin 4θ)

Question 2.
Express each of the following as the product of sine and cosine
(i) sin A + sin 2A
(ii) cos 2A + cos 4A
(iii) sin 6θ – sin 2θ
(iv) cos 2θ – cos θ

Solution:

(i) sin A + sin 2A = 2 sin$(\frac{A+2A}{2})$ cos$(\frac{A-2A}{2})$
[∵ sin C + sin D = sin$(\frac{C+D}{2})$ cos$(\frac{C-D}{2})$)]
= 2 sin $\frac{3A}{2}$ cos $\frac{A}{2}$ [∵ cos(-θ) = cos θ]

(ii) cos 2A + cos 4A = 2 cos$(\frac{2A+4A}{2})$ cos$(\frac{2A-4A}{2})$
[∵ cos C + cos D = 2 cos$(\frac{C+D}{2})$ cos$(\frac{C-D}{2})$
= 2 cos$(\frac{6A}{2})$ cos$(\frac{6-2A}{2})$
= 2 cos(3A) cos (-A) [∵ cos(-θ) = cos θ]
= 2 cos 3A cos A

(iii) sin 6θ – sin 2θ = 2 cos$(\frac{6\theta+2\theta}{2})$ cos$(\frac{6\theta-2\theta}{2})$
[∵ sin C – sin D = 2 cos$(\frac{C+D}{2})$ sin$(\frac{C-D}{2})$
= 2 cos$(\frac{8\theta}{2})$ sin$(\frac{4\theta}{2})$
= 2 cos 4θ sin 2θ

(iv) cos 2θ – cos θ = -2 sin$(\frac{2\theta+\theta}{2})$ sin$(\frac{2\theta-\theta}{2})$
[∵ cos C – cos D = -2 sin$(\frac{C+D}{2})$ sin$(\frac{C-D}{2})$
= -2 sin$(\frac{3\theta}{2})$ sin$(\frac{\theta}{2})$

Question 3.
Prove that
(i) $\cos 20° \cos 40° \cos 80° = 1/8$
(ii) $\tan 20° \tan 40° \tan 80° = \sqrt{3}$.

Solution:

(i) $\cos 20° \cos 40° \cos 80° = \left(\frac{2 \sin 20°}{2 \sin 20°}\right) \cos 20° \cos 40° \cos 80°$

[multiply and divide by 2 sin 20°]

$$= \frac{(2\sin 20° \cos 20°)\cos 40° \cos 80°}{2\sin 20°}$$

$$= \frac{\sin(2 \times 20°)\cos 40° \cos 80°}{2\sin 20°}$$

$$= \frac{\sin 40° \cos 40° \cos 80°}{2\sin 20°}$$

(Multiply and divide by 2)

$$= \frac{1}{2} \times \frac{(2\sin 40° \cos 40°)}{2\sin 20°} \cos 80°$$

$$= \frac{1}{2} \times \frac{(\sin 2 \times 40°)\cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \times \frac{\sin 80° \cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \times \frac{1}{2} \frac{(2\sin 80° \cos 80°)}{2\sin 20°}$$

$$= \frac{1}{8} \times \frac{\sin 160°}{\sin 20°} = \frac{1}{8} \times \frac{\sin(180° - 20°)}{\sin 20°}$$

$$= \frac{1}{8} \times \frac{\sin 20°}{\sin 20°} = \frac{1}{8} \times 1 = \frac{1}{8}$$

[$\because \sin(180° - \theta) = \sin \theta$]

(ii) $\tan 20° \tan 40° \tan 80°$

$$= \frac{\sin 20°}{\cos 20°} \times \frac{\sin 40°}{\cos 40°} \times \frac{\sin 80°}{\cos 80°}$$

$$= \frac{\sin 20° \times \sin 40° \times \sin 80°}{\cos 20° \cos 40° \cos 80°}$$

Consider $\sin 20° \times \sin 40° \sin 80°$

$= \sin 20° \sin (60° - 20°) \sin (60° + 20°)$

$= \sin 20° [\sin^2 60° - \sin^2 20°]$

$$= \sin 20° \left[\frac{3}{4} - \sin^2 20°\right]$$

$$= \sin 20° \left[\frac{3 - 4\sin^2 20°}{4}\right]$$

$$= \frac{3\sin 20° - 4\sin^3 20°}{4}$$

$$= \frac{\sin 60°}{4} = \frac{\frac{\sqrt{3}}{2}}{4} = \frac{\sqrt{3}}{8}$$

$\cos 20° \times \cos 40° \cos 80° = \frac{1}{8}$ [∵ from (i)] ...... (2)

divide (1) by (2) we get, $\tan 20° \tan 40° \tan 80° = \frac{\frac{\sqrt{3}}{8}}{\frac{1}{8}} = \sqrt{3}$

Question 4.
Prove that

(i) $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \sin^2 \left(\frac{\alpha - \beta}{2}\right)$

(ii) $\sin A \sin(60° + A) \sin(60° - A) = \sin 3A$

Solution:

(i) LHS = $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$

$$= \left(-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right)^2 + \left(2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right)^2$$

$$= 4\sin^2\frac{\alpha+\beta}{2}\ \sin^2\frac{\alpha-\beta}{2} + 4\cos^2\frac{\alpha+\beta}{2}\ \sin^2\frac{\alpha-\beta}{2}$$

$$= 4\sin^2\frac{\alpha-\beta}{2}\left[\sin^2\frac{\alpha+\beta}{2} + \cos^2\frac{\alpha+\beta}{2}\right]$$

$$= 4\sin^2\frac{\alpha-\beta}{2} = \text{RHS}$$

(ii) LHS = 4 sin A sin (60° + A) . sin (60° – A)
= 4 sin A {sin (60° + A) . sin (60° – A)}
= 4 sin A {$\sin^2$ 60° – $\sin^2$ A}
= 4 sin A {3/4 – $\sin^2$ A}
= 3 sin A – 4 $\sin^3$ A
= sin 3A
= RHS

Question 5.
Prove that
(i) sin (A – B) sin C + sin (B – C) sin A + sin(C – A) sin B = 0
(ii) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

Solution:
Consider sin (A – B) sin C
= (sin A cos B – cos A sin B) sin C
= sin A cos B sin C – cos A sin B sin C …….. (1)

Similarly sin(B – C) sin A = sin B cos C sin A – cos B sin C sin A …….. (2)
[Replace A by B, B by C, C by A in (1)]
and sin(C – A) sin B [Replace A by B, B by C, C by A in (2)]
= sin C cos A sin B – cos C sin A sin B …….. (3)

Adding (1), (2) and (3) we get
sin (A – B) sin C + sin (B – C) sin A + sin(C – A) sin B = 0

(ii) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

$$\text{LHS}=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\left(\cos\frac{\left(\frac{3\pi}{13}+\frac{5\pi}{13}\right)}{2}\right)\times\left(\cos\frac{\left(\frac{3\pi}{13}-\frac{5\pi}{13}\right)}{2}\right)$$

[∵ $\cos C+\cos D=2\cos(\frac{C+D}{2})\cos(\frac{C-D}{2})$]

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\left(\cos\frac{\frac{8\pi}{13}}{2}\right)\times\left(\cos\frac{-\frac{2\pi}{13}}{2}\right)$$

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\left(\frac{-\pi}{13}\right)$$

[∵ $\cos(-\theta)=\cos\theta$]

$$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$$

$$=2\cos\frac{\pi}{13}\left(\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right)$$

[take 2 cos $\frac{\pi}{3}$ as commom)

$$=2\cos\frac{\pi}{13}\left(2\cos\frac{\left(\frac{9\pi+4\pi}{13}\right)}{2}\cos\frac{\left(\frac{9\pi-4\pi}{13}\right)}{2}\right)$$

$$=2\cos\frac{\pi}{13}\left(2\cos\frac{13\pi}{13\times2}\cos\frac{5\pi}{13\times2}\right)$$

$$=2\cos\frac{\pi}{13}\left(2\cos\frac{\pi}{2}\cos\frac{5\pi}{2}\right)$$

$=0=\text{RHS}$

Hence proved.

Question 6.
Prove that

(i) $\frac{\cos 2A-\cos 3A}{\sin 2A+\sin 3A}=\tan\frac{A}{2}$

(ii) $\frac{\cos 7\mathbf{A}+\cos 5\mathbf{A}}{\sin 7\mathbf{A}-\sin 5\mathbf{A}}=\cot\mathbf{A}$

Solution:

(i) $\frac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} = \tan \frac{A}{2}$

$$\text{LHS} = \frac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} \qquad [\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)]$$

$$= \frac{-2\sin\left(\frac{2A+3A}{2}\right)\sin\left(\frac{2A-3A}{2}\right)}{2\sin\left(\frac{2A+3A}{2}\right)\cos\left(\frac{2A-3A}{2}\right)} \qquad [\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)]$$

$$= \frac{-2\sin\left(\frac{5A}{2}\right)\sin\left(\frac{-A}{2}\right)}{2\sin\left(\frac{5A}{2}\right)\cos\left(\frac{-A}{2}\right)} = \frac{2\sin\left(\frac{5A}{2}\right)\sin\left(\frac{A}{2}\right)}{2\sin\left(\frac{5A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \tan\left(\frac{A}{2}\right) = \text{RHS}$$

(ii) $\frac{\cos 7\mathbf{A} + \cos 5\mathbf{A}}{\sin 7\mathbf{A} - \sin 5\mathbf{A}} = \cot \mathbf{A}$

$$\text{LHS} = \frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A}$$

$$= \frac{2\cos\left(\frac{7A+5A}{2}\right)\cos\left(\frac{7A-5A}{2}\right)}{2\cos\left(\frac{7A+5A}{2}\right)\sin\left(\frac{7A-5A}{2}\right)}$$

$$[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)]$$

$$[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)]$$

$$= \frac{2\cos 6A \cos A}{2\cos 6A \sin A}$$

$$= \frac{\cos A}{\sin A} = \cot A = \text{RHS}$$

Hence proved.

Question 7.
Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = 1/16$.

Solution:

LHS = $\cos 20° \cos 40° \cos 60° \cos 80°$

$= \cos 20° \cos 40° \left(\frac{1}{2}\right) \cos 80°$ [$\because \cos 60° = \frac{1}{2}$]

$= \frac{1}{2} (\cos 20° \cos 40° \cos 80°)$

$= \frac{1}{2}\left(\frac{2 \sin 20°}{2 \sin 20°}\right) (\cos 20° \cos 40° \cos 80°)$

[multiply and divide by 2 sin 20°]

$$= \frac{1}{2} \frac{(2\sin 20° \cos 20°)\cos 40° \cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \frac{\sin(2 \times 20°)\cos 40° \cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \frac{\sin 40° \cos 40° \cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \frac{1}{2} \times \frac{(2\sin 40° \cos 40°)}{2\sin 20°} \cos 80°$$

[multiply and divide by 2]

$$= \frac{1}{2} \frac{1}{2} \times \frac{(\sin 2 \times 40°)\cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \frac{1}{2} \times \frac{\sin 80° \cos 80°}{2\sin 20°}$$

$$= \frac{1}{2} \frac{1}{2} \times \frac{1}{2} \frac{(2\sin 80° \cos 80°)}{2\sin 20°}$$

$$= \frac{1}{2} \frac{1}{8} \times \frac{\sin 160°}{\sin 20°} = \frac{1}{8} \times \frac{\sin(180° - 20°)}{\sin 20°}$$

$$= \frac{1}{2} \frac{1}{8} \times \frac{\sin 20°}{\sin 20°} = \frac{1}{2} \frac{1}{8} \times 1 = \frac{1}{2}\left(\frac{1}{8}\right) = \frac{1}{16}$$

Hence Proved.

Question 8.
Evaluate:
(i) cos 20° + cos 100° + cos 140°
(ii) sin 50° – sin 70° + sin 10°

Solution:
(i) LHS = (cos 20° + cos 100°) + cos 140°

$= 2\cos\left(\frac{20^\circ+100^\circ}{2}\right)\cos\left(\frac{20^\circ-100^\circ}{2}\right) + \cos 140^\circ$

$[\because \cos C + \cos D = 2\cos(\frac{C+D}{2})\cos(\frac{C-D}{2})]$

= 2 cos 60° cos(-40°) + cos 140°

$= 2 \times \frac{1}{2} \times \cos(-40^\circ) + \cos(180^\circ - 140^\circ)$

$[\because \cos(-\theta) = \cos\theta, \cos 60^\circ = \frac{1}{2}$

= cos 40° – cos 40°

= 0

Hence Proved.

(ii) LHS = (sin 50° – sin 70°) + sin 10°

$= 2\cos\left(\frac{50^\circ+70^\circ}{2}\right)\sin\left(\frac{50^\circ-70^\circ}{2}\right) + \sin 10^\circ$

$[\because \sin C - \sin D = 2\cos(\frac{C+D}{2})\sin(\frac{C-D}{2})]$

= 2 cos 60° sin(-10°) + sin 10°

$= 2 \times \frac{1}{2}(-\sin 10^\circ) + \sin 10^\circ$ $[\because \sin(-\theta) = -\sin\theta]$

= -sin 10° + sin 10°

= 0

= RHS

Question 9.

If $\cos A + \cos B = \frac{1}{2}$ and $\sin A + \sin B = \frac{1}{4}$, prove that $\tan\left(\frac{\mathbf{A+B}}{2}\right) = \frac{\mathbf{1}}{\mathbf{2}}$

Solution:

Given that $\cos A + \cos B = \frac{1}{2}$

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \frac{1}{2} \quad \text{........ (1)}$$

Also given that $\sin A + \sin B = \frac{1}{4}$

$$2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \frac{1}{4} \quad \text{....... (2)}$$

$\frac{(2)}{(1)}$ gives

$$\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \frac{\frac{1}{4}}{\frac{1}{2}}$$

$$\tan\left(\frac{A+B}{2}\right) = \frac{2}{4}$$

$$\text{thus } \tan\left(\frac{A+B}{2}\right) = \frac{1}{2}$$

Question 10.
If sin(y + z – x), sin(z + x – y), sin(x + y – z) are in A.P, then prove that tan x, tan y and tan z are in A.P.

Solution:
In A.P. common difference are equal, namely $t_2 - t_1 = t_3 - t_2$
sin(z + x – y) – sin(y + z – x) = sin(x + y – z) – sin(z + x – y)

$$\Rightarrow 2\cos\left(\frac{(z+x-y)+(y+z-x)}{2}\right)\sin\left(\frac{(z+x-y)-(y+z-x)}{2}\right)$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= 2\cos\left(\frac{(x+y-z)+(z+x-y)}{2}\right)\sin\left(\frac{(x+y-z)-(z+x-y)}{2}\right)$$

$$= 2\cos\left(\frac{z+x-y+z-x}{2}\right)\sin\left(\frac{z+x-y-y-z+x}{2}\right)$$

$$= 2\cos\frac{x+y-z+z+x-y}{2}\sin(x+y-z-z-x+y)$$

$$2\cos\left(\frac{2z}{2}\right)\sin\left(\frac{2x-2y}{2}\right) = 2\cos\left(\frac{2x}{2}\right)\sin\left(\frac{2y-2z}{2}\right)$$

cos z sin (x – y) = cos x sin (y – z)

cos z (sin x cos y – cos x sin y) = cos x (sin y cos z – cos y sin z)

Divide both sides by cos x cos y cos z we get

$$\frac{\cos z(\sin x\cos y-\cos x\sin y)}{\cos x\cos y\cos z} = \frac{\cos x(\sin y\cos z-\cos y\sin z)}{\cos x\cos y\cos z}$$

$$\frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y} = \frac{\sin y\cos z-\cos y\sin z}{\cos y\cos z}$$

$$\frac{\sin x\cos y}{\cos x\cos y}-\frac{\cos x\sin y}{\cos x\cos y} = \frac{\sin y\cos z}{\cos y\cos z}-\frac{\cos y\sin z}{\cos y\cos z}$$

$$\frac{\sin x}{\cos x}-\frac{\sin y}{\cos y} = \frac{\sin y}{\cos y}-\frac{\sin z}{\cos z}$$

tan x – tan y = tan y – tan z

Multiply both sides by (-1) we get,

tan y – tan x = tan z – tan y

This means tan x, tan y, and tan z are in A.P.

Hence proved.

Question 11.

If cosec A + sec A = cosec B + sec B prove that cot(A+B/2) = tan A tan B.

Solution:

Given that cosec A + sec A = cosec B + sec B

$$\frac{1}{\sin A}+\frac{1}{\cos A}=\frac{1}{\sin B}+\frac{1}{\cos B}$$

$$\frac{1}{\sin A}-\frac{1}{\sin B}=\frac{1}{\cos B}-\frac{1}{\cos A}$$

Arrange T-ratios of the sine and cosine in the separate side

$$\frac{\sin B-\sin A}{\sin A\sin B}=\frac{\cos A-\cos B}{\cos A\cos B}$$

$$\frac{\sin B-\sin A}{\sin A\sin B}=\frac{\cos A-\cos B}{\cos A\cos B}$$

$$\frac{\sin B-\sin A}{\cos A-\cos B}=\tan A\tan B$$

$[\because \sin C-\sin D=2\cos(\frac{C+D}{2})\sin(\frac{C-D}{2})]$

$$\frac{2\cos\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}=\tan A\tan B$$

$$\frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(-\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}=\tan A\tan B$$

$$\frac{-2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}=\tan A\tan B$$

$$\cot\left(\frac{A+B}{2}\right)=\tan A\tan B$$

Ex 4.4

Question 1.
Find the principal value of the following:

(i) $\sin^{-1}(-\frac{1}{2})$

(ii) $\tan^{-1}(-1)$

(iii) $\text{cosec}^{-1}(2)$

(iv) $\sec^{-1}(-\sqrt{2})$

Solution:

(i) $\sin^{-1}\left(-\frac{1}{2}\right)$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$

[where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$]

$-\frac{1}{2} = \sin y$

$\sin y = -\frac{1}{2}$ ($\because \sin\frac{\pi}{6} = \frac{1}{2}$)

$\sin y = \sin\left(-\frac{\pi}{6}\right)$ [$\because \sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$]

$\therefore y = -\frac{\pi}{6}$

$\therefore$ The principal value of sin-1 $\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$

(ii) $\tan^{-1}(-1) = y$

$(-1) = \tan y$ where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

(or) $\tan y = -1$

$\tan y = \tan\left(-\frac{\pi}{4}\right)$ ($\because \tan\frac{\pi}{4} = 1$)

$\therefore y = -\frac{\pi}{4}$ [$\because \tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$]

$\therefore$ The principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

(iii) Let $\text{cosec}^{-1}(2) = y$

$2 = \text{cosec } y$

(or) $\text{cosec } y = 2$

$\Rightarrow \frac{1}{\sin y} = 2$

$\Rightarrow \sin y = \frac{1}{2}$ (Take reciprocal)

$\Rightarrow \sin y = \sin\left(\frac{\pi}{6}\right)$

$\Rightarrow y = \frac{\pi}{6}$

The principal value of $\text{cosec}^{-1}(-1)$ is $\frac{\pi}{6}$.

(iii) Let cosec^{-1} (2) = y

2 = cosec y

(or) cosec y = 2

$\Rightarrow \frac{1}{\sin y} = 2$

$\Rightarrow$ sin y = $\frac{1}{2}$ (Take reciprocal)

$\Rightarrow$ sin y = $\sin\left(\frac{\pi}{6}\right)$

$\Rightarrow$ y = $\frac{\pi}{6}$

The principal value of cosec^{-1} (-1) is $\frac{\pi}{6}$.

(iv) Let sec^{-1} (-√2) = y

-√2 = sec y

sec y = -√2

$\frac{1}{\cos y}$ = -√2

Taking reciprocal cos y = $\frac{-1}{\sqrt{2}}$ [where 0 ≤ y ≤ π]

$$\cos y = \cos\left(\pi - \frac{\pi}{4}\right) \quad \left[\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos(180° - \theta) = -\cos\theta\right]$$

$$= \cos\left(\frac{4\pi - \pi}{4}\right) = \cos\frac{3\pi}{4}$$

∴ The principal value of sec^{-1} (-√2) is $\frac{3\pi}{4}$

Question 2.
Prove that

(i) 2 tan^{-1} (x) = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(ii) $\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

Solution:
(i) Let tan^{-1} x = θ

$x = \tan\theta$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2x}{1+x^2}$$

$$2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\therefore 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \text{RHS}$$

(ii) $\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

$$\text{LHS} = \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}-\frac{1}{7}}{1+\frac{4}{3}\cdot\frac{1}{7}}\right) = \tan^{-1}\left(\frac{28-3}{21+4}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS}$$

Question 3.

Show that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Solution:

We know that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Now LHS = $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{11+4}{22}}{1-\frac{1}{11}}\right) = \tan^{-1}\left(\frac{\frac{15}{22}}{\frac{10}{11}}\right)$$

$$= \tan^{-1}\left(\frac{15}{22}\times\frac{11}{10}\right) = \tan^{-1}\left(\frac{3\times1}{2\times2}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) = \text{RHS}$$

Question 4.
Solve: $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$.

Solution:
Given $\tan^{-1} (2x) + \tan^{-1} (3x) = \pi/4$

$$\tan^{-1}\left[\frac{2x+3x}{1-(2x)(3x)}\right] = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{5x}{1-6x^2}\right] = \frac{\pi}{4}$$

$$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4}$$

$$\frac{5x}{1-6x^2} = 1$$

$\Rightarrow 5x = 1(1 - 6x^2)$
$\Rightarrow 6x^2 + 5x - 1 = 0$
$\Rightarrow (x + 1)(6x - 1) = 0$
$\Rightarrow x + 1 = 0$ (or) $6x - 1 = 0$
$\Rightarrow x = -1$ (or) $x = 1/6$
$x = -1$ is rejected. It doesn't satisfies the question.
Note: Put $x = -1$ in the given question.

$$\tan^{-1}(-2x) + \tan^{-1}(-3x) = \frac{\pi}{4}$$

$$-\tan^{-1}(2x) + (-\tan^{-1} 3x) = \frac{\pi}{4}$$

$$-[\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

$$\tan^{-1}(2x) + \tan^{-1}(3x) = -\frac{\pi}{4}$$

$$\tan^{-1}(-2x) + \tan^{-1}(-3x) = \frac{\pi}{4}$$

$$-\tan^{-1}(2x) + (-\tan^{-1} 3x) = \frac{\pi}{4}$$

$$-[\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

$$\tan^{-1}(2x) + \tan^{-1}(3x) = -\frac{\pi}{4}$$

$$\therefore x = \frac{1}{6}$$

So the question changes.

Question 5.

Solve: $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\left(\frac{4}{7}\right)$

Solution:

$$\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\left(\frac{4}{7}\right)$$

$$\tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

$$\tan^{-1}\left(\frac{2x}{1-(x^2-1)}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

$$\tan^{-1}\left(\frac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

$$\tan^{-1}\left(\frac{2x}{2-x^2}\right) = \tan^{-1}\left(\frac{4}{7}\right)$$

$$\therefore \frac{2x}{2-x^2} = \frac{4}{7}$$

$$\frac{x}{2-x^2} = \frac{2}{7}$$

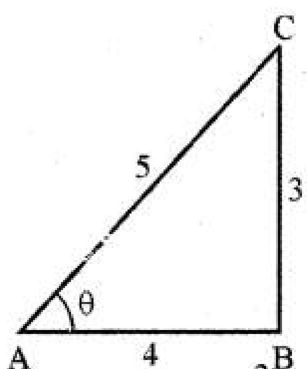
$\Rightarrow 7x = 2(2 - x^2)$

$\Rightarrow 7x = 4 - 2x^2$

$\Rightarrow 2x^2 + 7x - 4 = 0$
$\Rightarrow (x + 4)(2x - 1) = 0$
$\Rightarrow x + 4 = 0$ (or) $2x - 1 = 0$
$\Rightarrow x = -4$ (or) $x = 1/2$
$x = -4$ is rejected, since does not satisfies the question.
$\therefore x = 1/2$

Question 6.
Evaluate
(i) $\cos[\tan^{-1}(\frac{3}{4})]$
(ii) $\sin\left[\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right]$

Solution:
(i) Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta$
$\frac{3}{4} = \tan\theta$
$\tan\theta = \frac{3}{4}$
$\therefore \cos\theta = \frac{4}{5}$



$[\because \theta = \tan^{-1}\frac{3}{4}]$

Now $\cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\theta = \frac{4}{5}$

(ii) Let $\cos^{-1}\left(\frac{4}{5}\right) = A$
Then $\frac{4}{5} = \cos A$

$\cos A = \frac{4}{5}$

$$\therefore \sin\left[\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right] = \sin\left(\frac{1}{2}A\right) = \sin\frac{A}{2}$$

We know that

$$\cos A = 1 - 2\sin^2\frac{A}{2}$$

$$\frac{4}{5} = 1 - 2\sin^2\frac{A}{2}$$

$$2\sin^2\frac{A}{2} = 1 - \frac{4}{5}$$

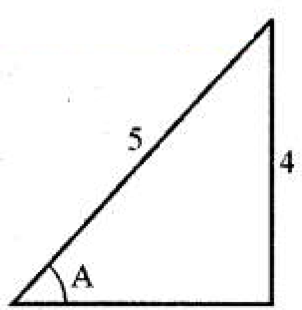
$$2\sin^2\frac{A}{2} = \frac{1}{5}$$

$$\therefore \sin^2\frac{A}{2} = \frac{1}{10} \Rightarrow \therefore \sin\frac{A}{2} = \frac{1}{\sqrt{10}}$$

Question 7.

Evaluate: $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right)$

Solution:



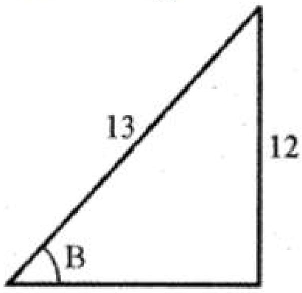
$$\sqrt{5^2 - 4^2} = 3$$

$$\cos A = \frac{\text{Adj}}{\text{Hyp}} = \frac{3}{5}$$

Let $\sin^{-1}\left(\frac{4}{5}\right)$ = A

sin A = $\frac{4}{5}$

∴ cos A = $\frac{3}{5}$



$\sqrt{169-144}=\sqrt{25}=5$

$\cos B=\frac{\text{Adj}}{\text{Hyp}}=\frac{5}{13}$

Let $\sin^{-1}\left(\frac{12}{13}\right)$ = B

$\frac{12}{13}$ = sin B

sin B = $\frac{12}{13}$

∴ cos B = $\frac{5}{13}$

Now $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{12}{13}\right)\right)$ = cos (A + B)

= cos A cos B – sin A sin B

= $\frac{3}{5}\times\frac{5}{13}-\frac{4}{5}\times\frac{12}{13}$

= $\frac{15}{65}-\frac{48}{65}$

= $-\frac{33}{65}$

Question 8.

Prove that $\tan^{-1}\left(\frac{m}{n}\right)-\tan^{-1}\left(\frac{m-n}{m+n}\right)=\frac{\pi}{4}$

Solution:

$$\text{LHS} = \tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right)$$

$$= \tan^{-1}\left(\frac{\frac{m}{n} - \frac{m-n}{m+n}}{1+\left(\frac{m}{n}\right)\left(\frac{m-n}{m+n}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{m(m+n)-n(m-n)}{n(m+n)}}{\frac{n(m+n)+m(m-n)}{n(m+n)}}\right)$$

$$= \tan^{-1}\left(\frac{m^2+mn-nm+n^2}{nm+n^2+m^2-mn}\right)$$

$$= \tan^{-1}\left(\frac{m^2+n^2}{m^2+n^2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

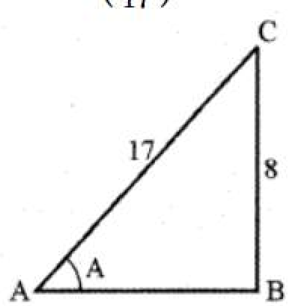
Question 9.

Show that $\sin^{-1}\left(-\frac{3}{5}\right) - \sin^{-1}\left(-\frac{8}{17}\right) = \cos^{-1}\left(\frac{84}{85}\right)$

Solution:

$\sin^{-1}\left(-\frac{3}{5}\right) - \sin^{-1}\left(-\frac{8}{17}\right) = -\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right)$

$= \sin^{-1}\left(\frac{8}{17}\right) - \sin^{-1}\left(\frac{3}{5}\right)$



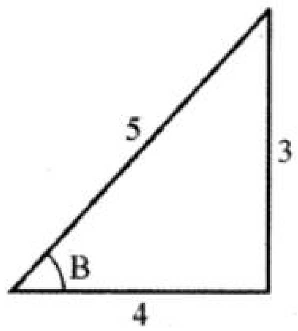
$AB = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$

Let $\sin^{-1}\left(\frac{8}{17}\right) = A$

$\frac{8}{17} = \sin A$

$\sin A = \frac{8}{17}$

$\therefore \cos A = \frac{15}{17}$



Let $\sin^{-1}\left(\frac{3}{5}\right)$ = B

sin B = $\frac{3}{5}$

∴ cos B = $\frac{4}{5}$

Consider cos(A – B) = cos A cos B + sin A sin B

$= \frac{15}{17} \times \frac{4}{5} + \frac{8}{17} \times \frac{3}{5}$

$= \frac{60}{85} + \frac{24}{85}$

cos (A – B) = $\frac{84}{85}$

∴ A – B = $\cos^{-1}\left(\frac{84}{85}\right)$

$$\text{i.e., } \sin^{-1}\left(\frac{8}{17}\right) - \sin^{-1}\frac{3}{5} = \cos^{-1}\left(\frac{84}{85}\right)$$

$$\text{i.e., } \sin^{-1}\left(\frac{-3}{5}\right) - \sin^{-1}\left(\frac{-8}{17}\right) = \cos^{-1}\left(\frac{84}{85}\right)$$

Question 10.

Express $\tan^{-1}\left[\frac{\cos x}{1-\sin x}\right]$, $-\frac{\pi}{2} < x < \frac{3\pi}{2}$ in the simplest form.

Solution:

$\tan^{-1}\left[\frac{\cos x}{1-\sin x}\right]$

$= \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$

$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$

[$\because a^2 - b^2 = (a + b)(a - b)$]

$= \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right]$

[$\because$ Divide each term by $\cos\frac{x}{2}$]

$= \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$

$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$

Ex 4.5

Question 1.
The degree measure of π/8 is:
(a) 20°60′
(b) 22°30′
(c) 22°60′

(d) 20°30′

Answer:
(b) 22°30′
Hint:

We know that, one radian = $\frac{180^\circ}{\pi}$

$\therefore \frac{\pi}{8} = \frac{180^\circ}{\pi} \times \frac{\pi}{8}$ degrees

$= \frac{45^\circ}{2}$

= 22.5°

= 22°30′

Question 2.
The radian measure of 37°30′ is:

(a) $\frac{5\pi}{24}$

(b) $\frac{3\pi}{24}$

(c) $\frac{7\pi}{24}$

(d) $\frac{9\pi}{24}$

Answer:

(a) $\frac{5\pi}{24}$

Hint:

$$37^\circ 30' = 37^\circ + \left(\frac{30}{60}\right)^\circ$$

$$= 37^\circ + 0.5^\circ = 37.5^\circ$$

$$= \frac{75^\circ}{2}$$

$$37^\circ 30' = \frac{75}{2} \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{5\pi}{2 \times 12} = \frac{5\pi}{24} \text{ radian}$$

Question 3.

If tan θ = $\frac{1}{\sqrt{5}}$ and θ lies in the first quadrant then cos θ is:

(a) $\frac{1}{\sqrt{6}}$

(b) $\frac{-1}{\sqrt{6}}$

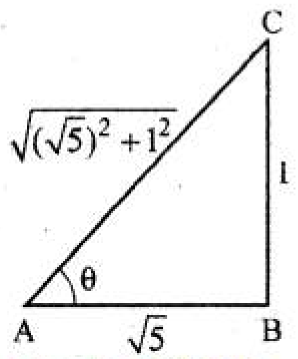
(c) $\frac{\sqrt{5}}{\sqrt{6}}$

(d) $\frac{-\sqrt{5}}{\sqrt{6}}$

Answer:

(c) $\frac{\sqrt{5}}{\sqrt{6}}$

Hint:



$$\tan\theta = \frac{1}{\sqrt{5}}$$

$$\frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{1}{\sqrt{5}}$$

$$\therefore AC = \sqrt{(\sqrt{5})^2 + 1^2} = \sqrt{6}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{\sqrt{5}}{\sqrt{6}}$$

Question 4.

The value of sin 15° is:

(a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$
(b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$
(c) $\frac{\sqrt{3}}{\sqrt{2}}$
(d) $\frac{\sqrt{3}}{2\sqrt{2}}$

Answer:

(b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

Hint:

sin 15° = sin(45° – 30°)

= sin 45° cos 30° – cos 45° sin 30°

= $\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

= $\frac{\sqrt{3}-1}{2\sqrt{2}}$

Question 5.
The value of sin(-420°)
(a) $\frac{\sqrt{3}}{2}$
(b) $-\frac{\sqrt{3}}{2}$
(c) $\frac{1}{2}$
(d) $\frac{-1}{2}$

Answer:

(b) $-\frac{\sqrt{3}}{2}$

Hint:

sin(-420°) = -sin(420°) [∵ sin(-θ) = -sin θ]

$= -\sin(360° + 60°)$

$= -\sin 60°$

$= -\frac{\sqrt{3}}{2}$

Question 6.
The value of cos(-480°) is:

(a) √3

(b) $-\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$

(d) $\frac{-1}{2}$

Answer:
(d) $\frac{-1}{2}$

Hint:
$\cos(-480°) = \cos 480°$ [$\because \cos(-\theta) = \cos \theta$]
$= \cos(360° + 120°)$
$= \cos 120°$
$= \cos(180° – 60°)$
$= -\cos 60°$
$= \frac{-1}{2}$

Question 7.
The value of sin 28° cos 17° + cos 28° sin 17°

(a) $\frac{1}{\sqrt{2}}$

(b) 1

(c) $\frac{-1}{\sqrt{2}}$

(d) 0

Answer:

(a) $\frac{1}{\sqrt{2}}$

Hint:

sin 28° cos 17° + cos 28° sin 17° = sin(28° + 17°)

This is of the form sin(A + B), A = 28°, B = 17°

= sin 45°

= $\frac{1}{\sqrt{2}}$

Question 8.

The value of sin 15° cos 15° is:

(a) 1

(b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{4}$

Answer:

(d) $\frac{1}{4}$

Hint:

sin 15° cos 15° = $\frac{1}{2}$ (2 sin 15° cos 15°)

= $\frac{1}{2}$ (sin 30°)

= $\frac{1}{2}\left(\frac{1}{2}\right)$

= $\frac{1}{4}$

Question 9.

The value of sec A $\sin(270° + A)$ is:

(a) -1

(b) $\cos^2 A$

(c) $\sec^2 A$

(d) 1

Answer:

(a) -1

Hint:
$\sec A (\sin(270° + A)) = \frac{1}{\cos A} (-\cos A) = -1$

Question 10.
If sin A + cos A = 1 then sin 2A is equal to:
(a) 1
(b) 2
(c) 0
(d) 1/2

Answer:
(c) 0
Hint:
Given sin A + cos A = 1
Squaring both sides we get
$\sin^2 A + \cos^2 A + 2 \sin A \cos A = 1$
$1 + \sin 2A = 1$
$\sin 2A = 0$

Question 11.
The value of $\cos^2 45° - \sin^2 45°$ is:
(a) $\frac{\sqrt{3}}{2}$
(b) $\frac{1}{2}$
(c) 0
(d) $\frac{1}{\sqrt{2}}$

Answer:
(c) 0
Hint:
$\cos^2 45° - \sin^2 45°$
$= \cos 2 \times 45°$ ($\because \cos^2 A - \sin^2 A = \cos 2A$)
$= \cos 90°$
$= 0$

Question 12.
The value of $1 - 2 \sin^2 45°$ is:
(a) 1
(b) $\frac{1}{2}$
(c) $\frac{1}{4}$
(d) 0

Answer:
(d) 0
Hint:
$1 - 2 \sin^2 45°$
$= \cos(2 \times 45°)$ [$\because \cos 2A = 1 - 2 \sin^2 A$]
$= \cos 90°$
$= 0$

Question 13.
The value of $4 \cos^3 40° - 3 \cos 40°$ is
(a) $\frac{\sqrt{3}}{2}$
(b) $-\frac{1}{2}$
(c) $\frac{1}{2}$
(d) $\frac{1}{\sqrt{2}}$

Answer:
(b) $-\frac{1}{2}$
Hint:
$4 \cos^3 40° - 3 \cos 40°$
$= \cos (3 \times 40°)$ [$\because \cos 3A = 4 \cos^3 A - 3 \cos A$]
$= \cos 120°$
$= \cos (180° - 60°)$
$= -\cos 60°$
$= -\frac{1}{2}$

Question 14.

The value of $\frac{2 \tan 30^\circ}{1+\tan^2 30^\circ}$ is:

(a) $\frac{1}{2}$

(b) $\frac{1}{\sqrt{3}}$

(c) $\frac{\sqrt{3}}{2}$

(d) √3

Answer:
(d) √3
Hint:

We know that sin 2A = $\frac{2 \tan A}{1+\tan^2 A}$

$\frac{2 \tan 30^\circ}{1+\tan^2 30^\circ}$ = sin(2 × 30°) = sin 60° = $\frac{\sqrt{3}}{2}$

= tan 2A

= tan 60°

= √3

Question 15.

If sin A = $\frac{1}{2}$ then 4 $\cos^3$ A – 3 cos A is:

(a) 1

(b) 0

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{\sqrt{2}}$

Answer:
(b) 0
Hint:
Given sin A = 1/2
sin A = sin 30°
∴ A = 30°

[∵ $4 \cos^3 A - 3 \cos A = \cos 3A$]
$= \cos(3 \times 30°)$
$= \cos 90°$
$= 0$

Question 16.
The value of $\frac{3 \tan 10° - \tan^3 10°}{1 - 3 \tan^2 10°}$ is:
(a) $\frac{1}{\sqrt{3}}$
(b) $\frac{1}{2}$
(c) $\frac{\sqrt{3}}{2}$
(d) $\frac{1}{\sqrt{2}}$

Answer:
(a) $\frac{1}{\sqrt{3}}$
Hint:
$\frac{3 \tan 10° - \tan^3 10°}{1 - 3 \tan^2 10°}$ = tan(3 × 10°) [∵ $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$]
= tan 30°
$= \frac{1}{\sqrt{3}}$

Question 17.
The value of $\text{cosec}^{-1} \left(\frac{2}{\sqrt{3}}\right)$ is:
(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$
(d) $\frac{\pi}{6}$

Answer:

(c) $\frac{\pi}{3}$

Hint:

Let $cosec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$\frac{2}{\sqrt{3}}$ = cosec A

cosec A = $\frac{2}{\sqrt{3}}$

sin A = $\frac{\sqrt{3}}{2}$ = sin 60°

∴ A = 60° = $\frac{\pi}{3}$

Question 18.

$\sec^{-1}\left(\frac{2}{3}\right) + cosec^{-1}\left(\frac{2}{3}\right) =$

(a) $\frac{-\pi}{2}$

(b) $\frac{\pi}{2}$

(c) π

(d) -π

Answer:

(b) $\frac{\pi}{2}$

Hint:

We know that $\sec^{-1} x + cosec^{-1} x = \frac{\pi}{2}$

∴ $\sec^{-1}\left(\frac{2}{3}\right) + cosec^{-1}\left(\frac{2}{3}\right) = \frac{\pi}{2}$

Question 19.

If α and β be between 0 and π/2 and if cos(α + β) = 12/13 and sin(α – β) = 3/5 then sin 2α is:

(a) $\frac{16}{15}$

(b) 0

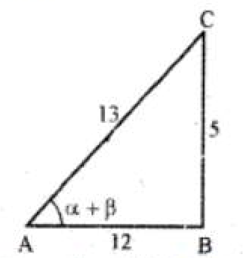
(c) $\frac{56}{65}$

(d) $\frac{64}{65}$

Answer:

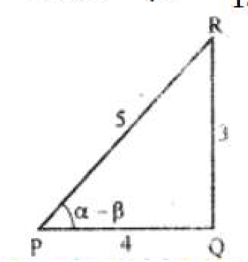
(c) $\frac{56}{65}$

Hint:



Given that cos(α + β) = $\frac{12}{13}$

∴ sin(α + β) = $\frac{5}{13}$



Also given that sin(α – β) = $\frac{3}{5}$

∴ cos(α – β) = $\frac{4}{5}$

sin 2α = sin[(α + β) + (α – β)]

= sin(α + β) cos(α – β) + cos(α + β) sin(α – β)

$= \frac{5}{13} \times \frac{4}{5} + \frac{12}{13} \times \frac{3}{5}$

$= \frac{20}{65} + \frac{36}{65}$

$= \frac{56}{65}$

Question 20.
If tan A = 1/2 and tan B = 1/3 then tan(2A + B) is equal to:
(a) 1
(b) 2
(c) 3
(d) 4

Answer:
(c) 3
Hint:
Given tan A = 1/2, tan B = 1/3

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$\tan(2A+B) = \frac{\tan 2A + \tan B}{1-\tan 2A \tan B}$$

$$= \frac{\frac{4}{3}+\frac{1}{3}}{1-\frac{4}{3}\times\frac{1}{3}} = \frac{\frac{5}{3}}{1-\frac{4}{9}} = \frac{\frac{5}{3}}{\frac{5}{9}} = 3$$

Question 21.
$\tan(\frac{\pi}{4} - x)$ is:
(a) $\left(\frac{1+\tan x}{1-\tan x}\right)$
(b) $\left(\frac{1-\tan x}{1+\tan x}\right)$
(c) 1 – tan x
(d) 1 + tan x

Answer:

(b) $\left(\frac{1-\tan x}{1+\tan x}\right)$

Hint:

$$\tan\left(\frac{\pi}{4}-x\right) = \frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x} = \frac{1-\tan x}{1+\tan x}$$

[∵ $\tan \frac{\pi}{4} = 1$]

Question 22.

$\sin\left(\cos^{-1} \frac{3}{5}\right)$ is:

(a) $\frac{3}{5}$

(b) $\frac{5}{3}$

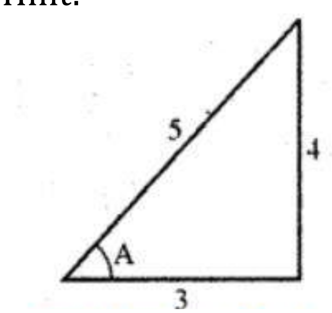
(c) $\frac{4}{5}$

(d) $\frac{5}{4}$

Answer:

(c) 4/5

Hint:



Let $\cos^{-1}\left(\frac{3}{5}\right) = A$

$\frac{3}{5} = \cos A$

$\sin A = \frac{4}{5}$

Now $\sin(\cos-1\left(\frac{3}{5}\right)) = \sin A = \frac{4}{5}$

Question 23.

The value of $\frac{1}{cosec(-45°)}$ is:

(a) $\frac{-1}{\sqrt{2}}$
(b) $\frac{1}{\sqrt{2}}$
(c) √2
(d) -√2

Answer:
(a) $\frac{-1}{\sqrt{2}}$
Hint:
$\frac{1}{cosec(-45^\circ)}$ = sin(-45°)
= -sin 45°
= $\frac{-1}{\sqrt{2}}$

Question 24.
If p sec 50° = tan 50° then p is:
(a) cos 50°
(b) sin 50°
(c) tan 50°
(d) sec 50°

Answer:
(b) sin 50°

Hint:
p sec 50° = tan 50°
$p(\frac{1}{\cos 50^\circ}) = \frac{\sin 50^\circ}{\cos 50^\circ}$
∴ p = sin 50°

Question 25.

$\left(\frac{cosx}{cosecx}\right) - \sqrt{1-\sin^2 x}\sqrt{1-\cos^2 x}$ is:

(a) $\cos^2 x - \sin^2 x$

(b) $\sin^2 x - \cos^2 x$

(c) 1

(d) 0

Answer:
(d) 0

Hint:

$\left(\frac{cosx}{cosecx}\right) - \sqrt{1-\sin^2 x}\sqrt{1-\cos^2 x}$

$= \cos x \times \sin x - \sqrt{\cos^2 x}\sqrt{\sin^2 x}$

$= \cos x \times \sin x - \cos x \times \sin x$

$= 0$