7

Congruence and Inequalities of Triangles

7.1 Introduction

Earlier, we have studied about triangles and their properties. In this chapter, we will study about the congruence rules of triangles and some other properties of triangles and inequalities in triangles.

7.2 Congruence of triangles

You have ever made several copies of your photograph of same size from a photographer. Similarly, you have seen bangles of same size in the wrist of your mother and seen postal stamps with same photo. Such figures are identical. If you choose any two such figures out of these and placed upon each other, then they exactly coincide.

Do you know, in geometry these figures are known by which name? These are called congruent figures. Congruent mean identically equal, i.e., figures with same shape and same size.

Thus, two triangles are congruent if and only if one of them can be made to uperpose on the other, so as to cover it exactly.

Axiom : (1)

If two sides and one included angle of a triangle are equal to the corresponding two sides and included angle of the other triangle, then the two triangles are congruent. (SAS rule of congruence)

Theorem 7.1. Angle-Side-Angle Rule (ASA Rule)

If two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the othe triangle, then the triangles are congruent.

Given : ABC and DEF are two triangles, in which $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BC = EF$

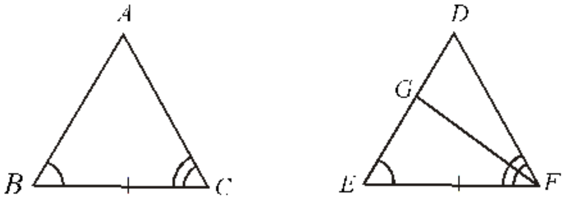

Fig. 7.01

To prove : $\Delta ABC \cong \Delta DEF$

Proof : Here, comparing the length of sides AB and DE of ΔABC and ΔDEF , following three conditions are possible :

(i) $AB = DE$ (ii) $AB < DE$ (iii) $AB > DE$

Condition (i) : If $AB = DE$, then in ΔABC and ΔDEF

$AB = DE$ (Say)

$\angle ABC = \angle DEF$ (Given)

$BC = EF$ (Given)

Thus, ΔABC and ΔDEF are congruent by Side-Angle Side rule.

i.e., $\Delta ABC \cong \Delta DEF$

Condition (ii) : When $AB < DE$, then take a point G on side DE such that $AB = GE$ and join GF (Fig. 7.01)

For ΔABC and ΔGEF

$AB = GE$ (Say)

$BC = EF$ (Given)

$\angle ABC = \angle GEF$ (Given) $[\because \angle GEF = \angle DEF]$

i.e., by SAS rule $\Delta ABC \cong \Delta GEF$

Thus, $\angle ACB = \angle GFE$ ...(1)

and $\angle ACB = \angle DFE$ (Given) ...(2)

From (1) and (2) $\angle GFE = \angle DFE$ is impossible, unless GF and DF do not coincide. It means points G and D coincide.

$\therefore$ $AB = DE$

Thus, by SAS rule,

$\Delta ABC \cong \Delta DEF$.

Condition (iii) : When $AB > DE$ then according to Fig. 7.02 take a point G on side AB of ΔABC such that $BG = ED$

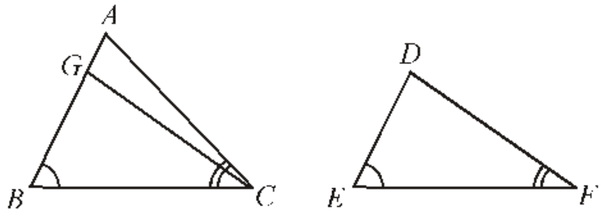

Fig. 7.02

Here, according to condition (ii), we can prove that point G will coincide point A i.e., $AB = DE$ and by Side-Angle-Side rule, $\Delta\, ABC \cong \Delta\, DEF$.

Thus, in all three condition, $\Delta\, ABC \cong \Delta\, DEF$. Hence proved.

Note : We know that the sum of three interior angles of a triangle is $180°$. Therefore, when two angles of a triangle are equal to two angles of another triangle, then their third angles will automatically by same. We will prove the following corollary on the basis of this law.

Corollary : Angle-Angle-Side Rule (AAS Rule)

If two angles and one side of one triangle are equal to the corresponding two angles and one side of the other triangle, then the two triangles are congruent.

Given : In $\Delta\, ABC$ and $\Delta\, DEF$ $\angle B = \angle E; \angle A = \angle D$ and side BC = side EF

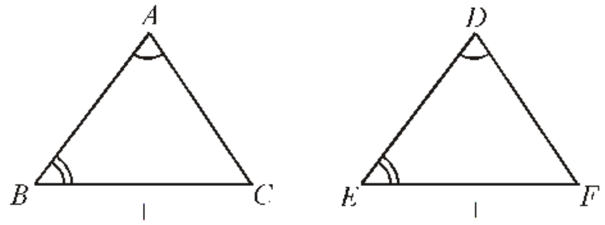

Fig. 7.03

To prove : $\Delta\, ABC \cong \Delta\, DEF$

Proof: We know that the sum of three interior angles of a triangle is $180°$.

$$\angle A + \angle B + \angle C = 180° \quad \ldots(1)$$

$$\angle D + \angle E + \angle F = 180° \quad \ldots(2)$$

From (1) and (2) $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F \quad \ldots(3)$

Given that $\angle B = \angle E$ $\qquad \angle A = \angle D$

Thus, $\angle C = \angle F$ [From (3)] $\ldots(4)$

Now, in $\Delta\, ABC$ and $\Delta\, DEF$

$\angle B = \angle E$ (Given)

$BC = EF$ (given)

$\angle C = \angle F$ [From (4)]

By Angle-Side-Angle rule $\Delta\, ABC \cong \Delta\, DEF$. Hence proved

Illustrative Examples

Example 1. In Fig. 7.04, C is the mid-point of AB, $\angle BCD = \angle ACE$ and $\angle DAB = \angle EBA$ then show that :

(i) $\Delta DAC \cong \Delta EBC$

(ii) $DA = EB$.

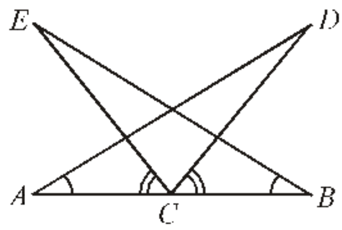

Fig. 7.04

Sol : Given : In Fig. 7.04,

$AC = BC$, $\angle DAB = \angle EBA$ and $\angle BCD = \angle ACE$

To prove : (i) $\Delta DAC \cong \Delta EBC$ (ii) $DA = EB$.

Proof : It is given that C, is the mid-point of side AB.

So, $AC = BC$ ...(1)

and $\angle BCD = \angle ACE$ (Given) ...(2)

Adding $\angle DCE$ both sides,

$$\angle BCD + \angle DCE = \angle ACE + \angle DCE$$

or $\angle ECB = \angle DCA$ ...(3)

Now, in ΔDAC and ΔEBC

$\angle DAC = \angle EBC$ (Given that)

$AC = BC$ [From (1)]

$\angle DCA = \angle ECB$ [From (3)]

By Angle-Side-Angle rule,

$$\Delta DAC \cong \Delta EBC$$

By the property of congruence, corresponding sides of two triangles are same.

Thus, $DA = EB$ Hence proved.

Example 2. In fig. 7.05, in a quadrilateral $ABCD$ $BC = AD$ and $\angle ADC = \angle BCD$ then prove that :

(i) $AC = BD$ (ii) $\angle ACD = \angle CDB$.

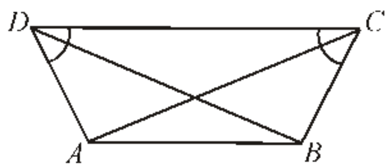

Fig. 7.05

Sol : According to Fig. 7.05, it is given that :

$BC = AD$ and $\angle ADC = \angle BCD$

So, in ΔADC and ΔBCD

$AD = BC$ (Given)

$CD = CD$ (Common side)

$\angle ADC = \angle BCD$ (Given)

So, by Side-Angle-Side rule, $\Delta ADC \cong \Delta BCD$

Since, corresponding sides and corresponding angles of congruent triangles are same.

Therefore, $AC = BD$ and $\angle ACD = \angle CDB$. or $\angle ACD = \angle CDB$. Hence proved.

Example 3. AB **is a line segment and line** ℓ **is its perpendicular bisector. If** P **is any point on** ℓ **, then show that** P **is equidistant from points** A **and** B

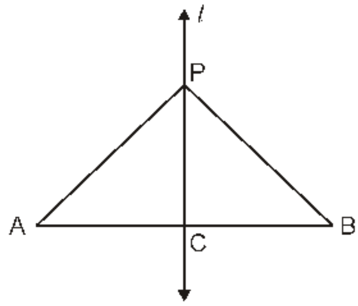

Fig. 7.06

Sol. : AB is a line segment and line ℓ passes through the mid-point C of AB (see fig. 7.06). We have to show that $PA = PB$. For this, think about ΔPCA and ΔPCB . It is given that

$AC = BC$ (C is the mid-point of AB)

$\angle PCA = \angle PCB = 90°$ (Given)

$PC = PC$ (Common)

So, $\Delta PCA \cong \Delta PCB$ (SAS Rule)

Therefore, $PA = PB$ (Corresponding sides of congruent triangles) Hence proved

Example 4. In Fig. 7.07, $AE = EC$ **and** $DE = BE$ **then show that :**

(i) $\Delta AED \cong \Delta CEB$ **(ii)** $\angle A = \angle C$.

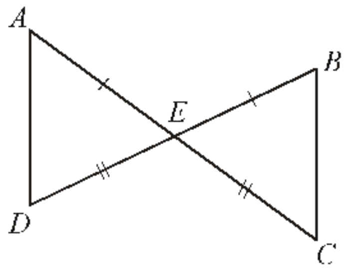

Fig. 7.07

Sol. : According Fig. 7.07, it is given that

$AE = EC$

$DE = BE$ ... (1)

Now, for ΔAED and ΔBEC

$AE = EC$ (given)

$\angle AED = \angle CEB$ (vertically opposite angles)

$DE = EB$ (given)

Therefore, by Side-Angle-Side rule $\Delta AED \cong \Delta CEB$ and their corresponding angles $\angle A = \angle C$ $\angle D = \angle B$ Hence proved

Example 5. In Fig. 7.08, $AD = BC$ **and** $BD = CA$ **then show that :**

(i) $\angle ADB = \angle BCA$

(ii) $\angle DAB = \angle CBA$.

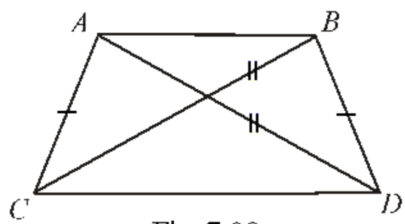

Fig. 7.08

Sol. : In Fig. 7.08 $AD = BC$ and $BD = CA$

So, in ΔABD and ΔABC

$$\left.\begin{aligned} AD &= BC \\ BD &= CA \end{aligned}\right\} \quad \text{(Given)}$$

$AB = AB$ (Common)

So, by Side-Side-Side rule

$\Delta ABD \cong \Delta ABC$

So, corresponding angles

(i) $\angle ADB = \angle BCA$ (ii) $\angle DAB = \angle CBA$ Hence proved

Exercise 7.1

1. In ABC and PQR $\angle A = \angle Q$ and $\angle B = \angle R$. Which side of ΔPQR should be equal to the side AB of ΔABC , so that two triangles become congruent? Given reason to your answer.
2. In triangles ABC and PQR $\angle A = \angle Q$ and $\angle B = \angle R$. Which side of ΔPQR should be equal to side BC of ΔABC ,so that two triangles are congruent? Give reason to your answer.
3. If two sides and one angle of a triangle is equal to the two sides and one angle of other triangle, then two triangles should be congruent. Is this statement true? Why?
4. If two angles and one side of a triangle is equal to the two angles and one side of other triangle then triangles sure should be congruent. Is this statement true? Why?
5. It is given that $\Delta ABC \cong \Delta RPQ$. Is $BC = QR$ true? Why?
6. If $\Delta PQR \cong \Delta EDF$, then is this true $PR = EF$? Given reason your answer.
7. In fig. 7.09, diagonal AC of quadrilateral $ABCD$, is the bisector of $\angle A$ and $\angle C$? Prove that : $AB = AD$ and $CB = CD$.

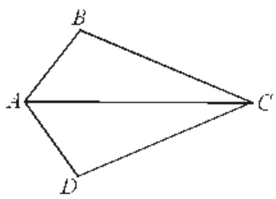

Fig. 7.09

8. In Fig. 7.10, in quadrilateral $ADBC$, $\angle ABC = \angle ABD$ and $BC = BD$, then prove that $\Delta ABC \cong \Delta ABD$.

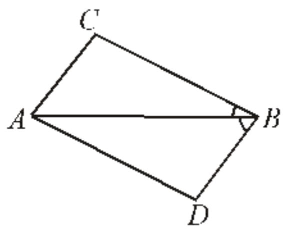

Fig. 7.10

9. According to Fig. 7.11 $AB \parallel DC$ and $AD \parallel BC$ then prove that : $\Delta ADB \cong \Delta CBD$.

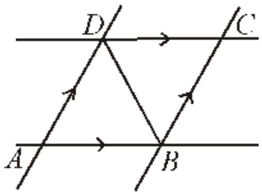

Fig. 7.11

10. In Fig. 7.12 , if $AB \parallel DC$ and E is the mid-point of side AC , then prove that E is the mid-point of side BD

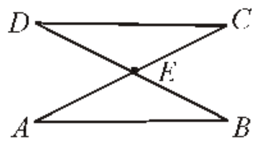

Fig. 7.12

7.3 Special Properties of Triangle

You have already studied about two conditions of congruence of triangles. Now, we will use thier results to prove the theorems related to an isosceles triangle and remaining theorems of congruence of triangles.

7.4 Isosceles Triangle

A triangle with two equal sides is called an isosceles triangle.

Theorem 7.3 : If two sides of a triangle are equal, then their opposite angles are also equal.

or

In an isosceles triangle, angles opposite to equal sides are equal.

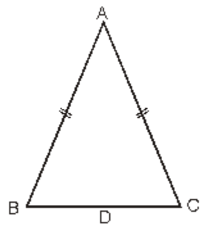

Fig. 7.13

Given : $\Delta\ ABC$ is an isosceles triangle

Where, $AB = AC$

To prove : $\angle B = \angle C$

Construction : Draw bisector AD of $\angle A$, which meets BC at D.

Proof : In $\Delta\ ABD$ and $\Delta\ ACD$,

$AB = AC$ (Given)

$\angle BAD = \angle CAD$ (By construction)

$AD = AD$ (Common side)

By Side Angle Side rule, $\Delta\ ABD \cong \Delta\ ACD$

Since, corresponding angles of congruent triangles are equal.

$\angle B = \angle C$ Hence proved.

Theorem 7.4 : If two angles in a triangle are equal, then their opposite sides will be also equal.

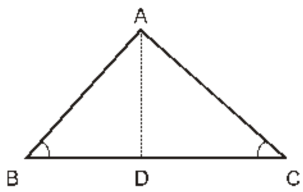

Fig. 7.14

Given : $\Delta\, ABC$ in which $\angle B = \angle C$

To prove : $AB = AC$

Construction : Draw AD, the bisector of $\angle BAC$

Proof : In $\Delta\, ABD$ and $\Delta\, ACD$

$\angle B = \angle C$ (Given)

$AD = AD$ (common side)

$\angle BAD = \angle CAD$ (by construction)

By Angle Side Angle rule,

$\Delta\, ABD \cong \Delta\, ACD$

Thus, corresponding sides $AB = AC$ Hence proved.

Illustrative Examples

Example 6. In ΔABC, the bisector AD of $\angle A$, is perpendicular to side BC. Show that ΔABC is an isosceles triangle.

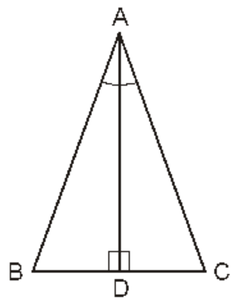

Fig. 7.15

Sol : In **ΔABD** and **ΔACD**

$\angle BAD = \angle CAD$ (given that AD is bisector of $\angle A$)

AD = AD (Common side)

$\angle ADB = \angle ADC = 90^\circ$ (Given)

ΔABD ≅ ΔACD (By ASA rule)

Therefore, AB = AC

Thus, ΔABC is an isosceles triangle.

Example 7. According to Fig. 7.16, $ABCD$ is a square and ΔCDE is an equilateral triangle, then prove that $AE = BE$.

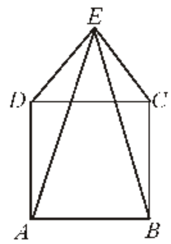

Fig. 7.16

Sol : Given, $ABCD$ is a square and $\Delta\, CDE$ is an equilateral triangle.

To prove : $AE = BE$

Proof : $\Delta\, CDE$ is an equilateral triangle

Thus, $CD = DE = CE$ ... (1)

$\angle DEC = \angle EDC = \angle DCE = 60^\circ$ ... (2)

and $ABCD$ is a square, so

$\angle ADC = \angle BCD = 90^\circ$

Adding $\angle EDC$ to both sides

$\angle ADC + \angle EDC = \angle BCD + \angle EDC$

$\Rightarrow \quad \angle EDA = \angle ECB \quad \ldots(3)$

Now, in $\Delta\, ADE$ and $\Delta\, BCE$

$AD = BC$ (sides of square)

$\angle EDA = \angle ECB$ [From (3)]

$DE = EC$ [From (1)]

By Side-Angle-Side rule $\Delta\, ADE \cong \Delta\, BCE.$

Thus, corresponding sides $AE = BE$ Hence proved

Example 8. Prove that the medians, which bisects equal sides of an isosceles triangle, are equal.

Sol. Given : In an Isosceles $\Delta ABC, D$ and E are mid points of equal sides AB and AC.

To prove : $BE = CD$

Proof : $\Delta\, ABC$ is an isosceles triangle whose sides AB and AC are equal.

$AB = AC \quad \ldots(1)$

and $\angle ABC = \angle ACB \quad \ldots(2)$

and D and E are the mid points of sides AB and AC .

Thus, $DB = DA = EC = AE \quad \ldots(3)$

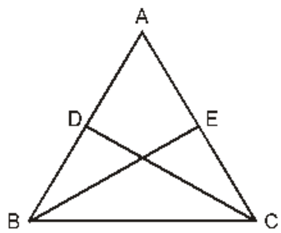

Fig. 7.17

Now, in $\Delta\, BCD$ and $\Delta\, BCE$

$BC = BC$ (Common side)

$\angle DBC = \angle ECB$ [From (2)]

$BD = CE$ [From (3)]

By Side-Angle-Side rule $\Delta\, BCD \cong \Delta\, BCE$

Corresponding sides will be equal i.e.,

$CD - BE$ or $BE = CD$ Hence proved.

Example 9. In Fig. 7.18, $AB = AC$ and D is a point in ΔABC such that $\angle DBC = \angle DCB.$ Prove that AD bisects $\angle BAC$.

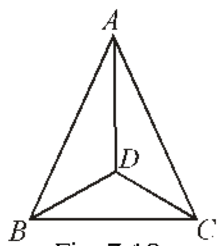

Fig. 7.18

Sol. Given : In $\Delta\, ABC$, $AB = AC$ and $\angle DBC = \angle DCB.$

To prove : AD is the bisector of $\angle BAC.$

i.e. $\angle BAD = \angle CAD$

Proof : In $\Delta\, BDC$, $\angle DBC = \angle DCB$ so their opposite sides will be same.

$CD = BD \quad \ldots(1)$

Now, in $\Delta\, ABD$ and $\Delta\, ACD$

$BD = CD$ [From (1)]

$AD = AD$ [Common side]

$AB = AC$ (Given)

By Side-Side-Side rule, $\Delta ABD \cong \Delta ACD$.

So, corresponding angles will be same. i.e., $\angle BAD = \angle CAD$.

Thus, AD is the bisector of $\angle BAC$. Hence proved.

Example 10. If perpendiculars drawn from mid-point of a side of a triangle to the other two sides are equal, then prove that triangle will be isosceles.

Sol. Given : D is the mid point of side BC of ΔABC. DE and DF are the perpendiculars on AC and AB respectively, and $DE = DF$.

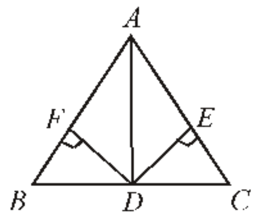

Fig. 7.19

To prove : ΔABC is an isosceles triangle, i.e., $AB = AC$

Construction : Join AD

Proof : In ΔBDF and ΔCDE

Hypotenuse $BD =$ Hypotenuse CD (Given)

$\angle DFB = \angle DEC = 90°$

and $DF = DE$ (Given)

By Right Angle Hypotenuse Side rule

$\Delta BDF \cong \Delta CDE$.

Thus, corresponding angles $\angle B = \angle C$ and oppostie sides to two equal angles will be equal

i.e., $AB = AC$. Hence proved

Example 11. In an isosceles triangle ABC, $AB = AC$ and D, E, F are mid-points of sides BC, AC and AB, then prove that $DE = DF$.

Sol. : According to Fig. 7.20 in ΔABC

$AB = AC$ ...(1)

And D, E, F are the mid-points of sides BC, AC and AB respectively

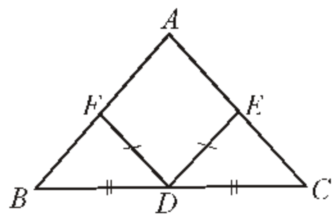

Fig. 7.20

So, in ΔBDF and ΔCDE

$BD = CD$ [D is the mid-point of side BC]

$CE = BF$ [Given $AB = AC$]

and $\angle B = \angle C$ [Angles opposite to equal sides are equal]

By Side-Angle-Side-rule

$\Delta BDF \cong \Delta CDE$

Thus, or $DE = DF$ Hence Proved

Example 12. In Fig. 7.21, ABC is a right-angled triangle, in which $\angle B = 90°$, such that $\angle BCA = 2\angle BAC$. Show that hypotenuse AC = 2 BC.

Sol. Given : A right angled ABC such that $\angle B = 90°$ and $\angle BCA = 2\angle BAC$

To prove : $AC = 2BC$

Construction : Produce **CB** upto **D** such that **BC = BD** and join **AD**

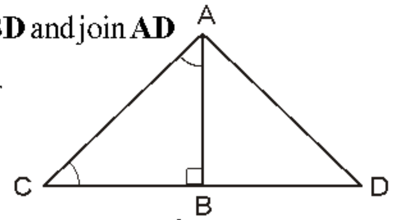

Fig. 7.21

Proof : Let $\angle BAC = x$, then $\angle BCA = 2\angle BAC = 2x$.

In right triangle ABC

$\angle BAC + \angle BCA + \angle ABC = 180°$

$\Rightarrow \quad x + 2x + 90° = 180°$

$\Rightarrow \quad 3x = 180° - 90°$

$\Rightarrow \quad x = 30°$

$\therefore \quad \angle BCA = 2 \times 30° = 60°$ and $\angle BAC = 30°$ ...(i)

In ΔABC and ΔABD

BC = BD (By construction)

$\angle ABC = \angle ABD$ (Each is $90°$)

AB = AB [Common side]

$\therefore \quad \Delta ABC \cong \Delta ABD$ (By SAS rule)

$\Rightarrow \quad \angle CAB = \angle DAB$

and AC = AD

Now $\angle BAC = \angle BAD = 30°$ [Using (1)]

$\therefore \quad \angle A = 2 \times 30° = 60°$

and AC = AD

$\Rightarrow \quad \angle C = \angle D$ [Angles opp. to equal sides are equal]

$\Rightarrow \quad \angle C = \angle D = 60°$

In ΔACD,

$\angle A = \angle C = \angle D = 60°$

$\Rightarrow \quad \Delta ACD$ is an equilateral triangle.

$\Rightarrow$ AC = CD = AD ...(2)

But BC = BD

$\therefore$ CD = BC + BD.

$\Rightarrow$ CD = BC + BC

$\Rightarrow$ CD = 2 BC

$\Rightarrow$ AC = 2 BC. [Using (2)] Hence Proved.

Exercise 7.2

1. In Fig. 7.22 $AB = AC$ and $\angle B = 58°$ then find the value of $\angle A$

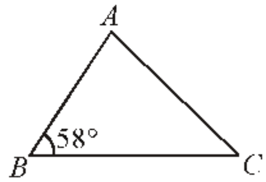

Fig. 7.22

2. In Fig. 7.23 $AD=BD$ and $\angle C = \angle E$, then prove that $BC = AE$.

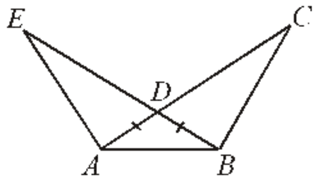

Fig. 7.23

3. If AD is the median of an isosceles triangle ABC and $\angle A = 120°$ and $AB = AC$, then find the value of $\angle ADB$

4. If the bisector of any angle of a triangle also bisects the opposite side then prove that the triangle is an isosceles triangle.

5. In Fig. 7.24, $AB = AC$ and $BE = CD$, then prove that : $AD = AE$.

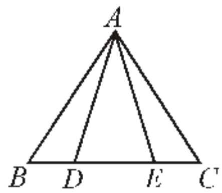

Fig. 7.24

6. E and F are two points on sides AD and BC respectively of square ABCD, such that $AF = BE$. Show that :

   (i) $\angle BAF = \angle ABE$ (ii) $BF = AE$

7. AD and BC are two equal perpendiculars on a line segment AB (see fig. 7.25). Show that CD bisects line segment AB.

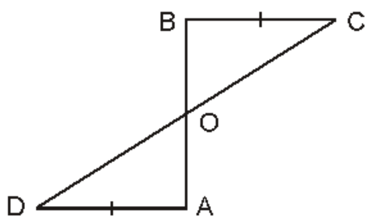

Fig. 7.25

8. The bisectors of angles B and C of an isosceles triangle with AB = AC, intersect each other at point O. Produce BO upto point M. Prove that : $\angle MOC = \angle ABC$

9. Line ℓ bisects angle A and B is any point on line ℓ. BP and BQ are the perpendiculars drawn on the sides of angle A from point B (See Fig. 7.26)

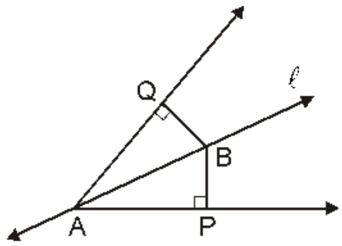

Fig. 7.26

(i) $\Delta APB \cong \Delta AQB$

(ii) BP = BQ it means point B is equidistant from the sides of $\angle A$.

11. In Fig. 7.27 AC = AE, AB = AD and $\angle BAD = \angle EAC$. Show that : BC = DE

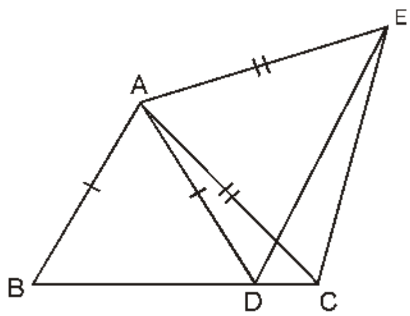

Fig. 7.27

12. In right triangle ABC, angle C is right angle M is the mid-point of hypotenuse AB Join C to M and produce it upto D such that DM = CM. Join point D and B (see fig. 7.28). Show that :

(i) $\Delta AMC \cong \Delta BMD$

(ii) $\angle DBC$ is a right angle

(iii) $\Delta DBC \cong \Delta ACB$

(iv) $CM = \frac{1}{2} AB$

7.5 Some other concepts for the Congruence of Triangles

If three angles of a triangle are equal to three angles of another triangle, then it is not necessary that these two triangles are congruent.

If three sides of triangle are equal to three sides of another triangle then according to you whether these triangles be congruent? Definitely, they will be congruent. Now, we prove this theorem by using obtained result.

Theorem 7.5 : Side, Side, Side-Rule (S S S Rule)

If three sides of a triangle are equal to the corresponding three sides of other triangle, then two triangles are congruent.

Given : Corresponding sides of Δ ABC and $\Delta\, DEF$ are same, *i.e.*,

$AB - DE; BC - EF$ and AC = DF

To prove : $\Delta ABC \cong \Delta DEF$

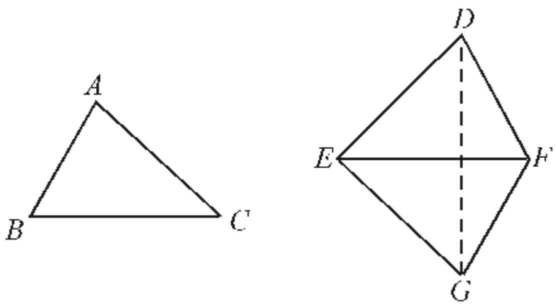

Fig. 7.29

Construction : Draw a line segment EG in the opposite side of $\Delta\, DEF$ such that $EG - AB$ and $\angle ABC = \angle FEG$. Join GE and DG.

Proof: In $\Delta\, ABC$ and ΔGEF

$AB - GE$ (By construction)

$\angle\, ABC = \angle\, GEF$ (By construction)

$BC - EF$ (Given)

By Side-Angle-Side rule, $\Delta ABC \cong \Delta DEF$

So, corresponding angles and corresponding sides of congruent triangles are equal.

$\angle$ A = $\angle$ G; AB = GF ...(1)

Now, $AB - EG$ (By construction, and $AB - DE$ (Given)

So $EG - DE$ ...(2)

Similarly, $AC = GF$ from equation (1) and $AC - DF$ (Given)

$\therefore \qquad GF = DF \qquad ...(3)$

$\Rightarrow$ In ΔEDG, angles opposite to equal sides EG and DE are same.

$$\angle EDG = \angle EGD \qquad ...(4)$$

Similarly, in ΔFDG, angles opposite to equal sides GF and DF are same.

$$\angle GDF = \angle DGF \qquad ...(5)$$

Adding equations (4) and (5)

$$\angle EDG + \angle GDF = \angle EGD + \angle DGF$$

$\Rightarrow \qquad \angle D = \angle G \qquad ...(6)$

But from equation (1)

$$\angle A = \angle G \qquad ...(7)$$

From equation (6) and (7)

$$\angle A = \angle D \qquad ...(8)$$

Thus, in ΔABC and ΔDEF

$AB = DE$ (Given)

$\angle A = \angle D$ [From (8)]

$AC = DF$ (Given)

By SAS rule.

$\Delta \ \Delta ABC \cong \Delta DEF$ Hence Proved

Now, we try to verify this theorem by the following activity :

Construct two triangles each of which have sides 4 cm, 3.5 cm and 3 cm.

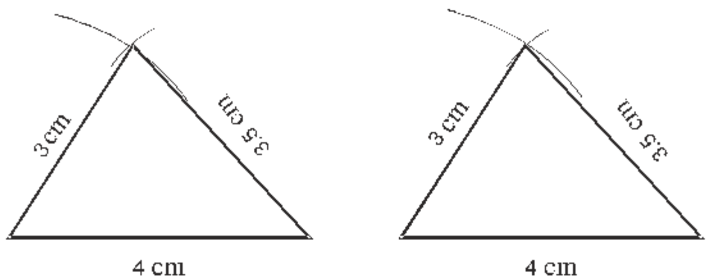

Fig. 7.30

Now cut them and place one of them on the other. What do you see? By keeping equal sides in mind when we placed one upon the other triangle they cover each other completely. This is possible only when two triangles are congruent.

It means two triangles are congruent.

You have already observed SAS rule of congruence. By SAS rule of congruence pair of equal angles may be between pair of corresponding sides. If not so, then two triangles may not be congruent.

Let verify it by an activity.

Draw two right angled triangles in which each one of hypotenuse is 5 cm and a side is 4 cm (See Fig. 7.31).

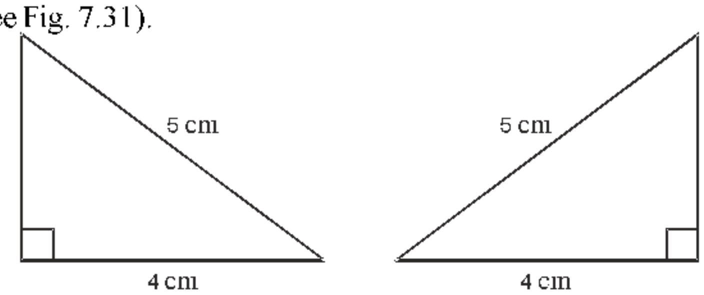

Fig. 7.31

Cut them and place one on the other such that equal sides coincide. If necessary, rotate them. What do you see? You see, on placing one on the other, they exactly cover each other so they are congruent. Repeat this activity by taking differe of right angled triangles. What do you see? You will see that if their hypotenuse and one pair of side are equal, then two right angled triangles will be congruent.

Note that, in this condition right angle is not the angle included in the hypotenuse and side.

In this way we conclude an important fact for right angled triangles which can be proved by theorem.

Theorem 7.6. Right Hypotenuse Side Rule (RHS congruence rule) :

Two right triangles are congruent if and only if the hypotenuse and a side of one triangle are equal to the hypotenuse and the corresponding side of the other triangle.

Given : In two right triangles, ABC and DEF

$$\angle B = \angle E = 90°$$

Hypotenuse AC – hypotenuse DF

and side AB – side DE

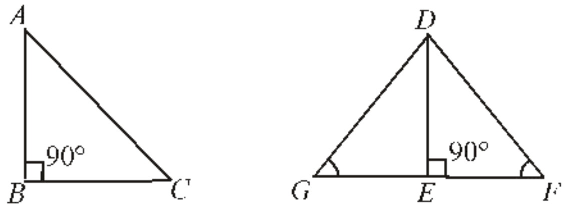

Fig. 7.32

To prove : $\Delta ABC \cong \Delta DEF$

Construction : In ΔDEF; produce E upto G such that $GE - BC$ and join G to D.

Proof : Here, $\angle DEF = 90°$

$\therefore$ and $\angle$ DEG = 90° ...(1)

Now, in ΔABC and Δ DEG

$AB - DE$	(Given)
$BC - GE$	(By construction)
$\angle ABC - \angle DEC - 90°$	[From (i)]

By Side-Angle-Side rule, ΔABC and ΔDEG are congruent. So, their corresponding sides and corresponding angles will be equal.

$\therefore$ $AC - DG$ and $\angle C = \angle G$ ...(2)

But it is given that $AC - DF$ ...(3)

From equations (2) and (3),

$DG - DF$ ...(4)

$\because$ In ΔDGF, angles opposite to equal sides ($DG - DF$) will be equal

$\angle G - \angle F$ ...(5)

From equations (2) and (5), $\angle C = \angle F$ ... (6)

Now, in ΔABC and ΔDEF

	$AB - DE$	(Given)
	$\angle C = \angle F$	[from (6)]
and	$\angle ABC - \angle DEF - 90°$	(Given)

By Angle-Angle-Side rule,

$\Delta ABC \cong \Delta DEF$ **Hence Proved**

Illustrative Examples

Example 13. ***AB*** **is a line segment and points** ***P*** **and** ***Q*** **are situated on the opposite side of AB such that each of then is equidistant from** ***A*** **and** ***B*****. (see fig. 7.33). Show that line segment** ***PQ*** **is the perpendicular bisector of the line segment AB.**

Solution : Here, $PA - PB$ and $QA - QB$ *is* given. We have to show that PQ $\perp$ AB and PQ bisects AB. Let PQ interests line segment AB at C.

You can see two congruent triangles in this figure? Let us take ΔPAQ and ΔPBQ.

AP = BP (Given)

$AQ - BQ$ (Given)

PQ = PQ (Common side)

∴ $\Delta PQA \cong \Delta PBQ$ (BySSS rule)

So, $\angle$ APQ = $\angle$ BPQ

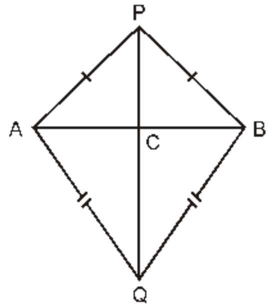

Fig. 7.33

In Δ PAC and Δ PBC, we get

AP = BP (Given)

$\angle$ APC = $\angle BPC$ ($\angle$ APQ = $\angle$ BPQ has been proved)

PC = PC (Common side)

∴ $\Delta PAC \cong \Delta PBC$

and $\angle$ ACP = $\angle$ BCP

and AC = CB

Now, $\angle$ACP + $\angle$BCP = 180° (Linear pair) ...(1)

So, 2 $\angle$ ACP = 180°

or, $\angle$ ACP = 90° ...(2)

From equations (1) and (2), we conclude that line PQ is perpendicular bisector of AB.

Note that without proving congruence of $APAQ$ and $APBQ$, we cannot show Δ APQ – Δ BPQ, whereas $AP - BP$ (given) $PC - PC$ (common) and $\angle$ PAC = $\angle PBC$.

(Opposite angles of equal sides in Δ PAB). We obtained this by SSA rule which is not always acceptable for the congruence of triangles and angle is not (included) between the equal sides. Let us take some other examples.

Example 14: ***P* is *a* point equidistant from two lines *l* and *m* intersecting at point *A* (see fig 7.34). Show that line *AP* bisects the angle between them.**

Solution : It is given that lines l and m intersect at A. Let $PB \perp l$ and $PC \perp m$.

It is given that $PB - PC$ ($\because P$ is equidistant from l and m.)

To Prove : $\angle PAB = \angle PAC$

Proof: Now, in Δ PAB and Δ PAC

$PB - PC$ (Given)

$\angle PBA - \angle PCA - 90°$ ($PB \perp l$ and $PC \perp m$)

$PA - PA$ (Common hypotenuse)

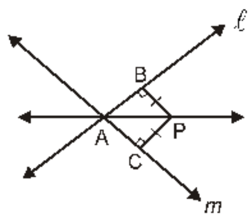

Fig. 7.34

∴ $\Delta PAB \cong \Delta PAC$ (ByRHSrule)

Therefore $\angle$ PAB = $\angle$ PAC **Hence proved**

Exercise 7.3

1. $\Delta\, ABC$ and $\Delta\, DBC$ are two isosceles triangles on the same base BC such that vertices A and D are situated on the same side of BC (see Fig. 7.35). If AD is extended to intersect BC at P, then show that

   (i) Δ ABD $\cong$ Δ ACD

   (ii) $\Delta\, ABP \cong \Delta$ ACP

   (iii) AP bisects both $\angle$ A and $\angle D$

   (iv) AP is the perpendicular bisector of line segment BC.

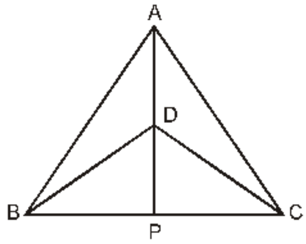

Fig. 7.35

2. AD is an altitude of an isosceles triangle ABC in which $AB - AC$.

   Show that :

   (i) AD bisects line segment BC.

   (ii) AD bisects $\angle$ A

3. Two sides AB and BC and median AM of a $\Delta\, ABC$ are respectively equal to the corresponding sides PQ and QR and median PN of other triangle (see Fig. 7.36). Show that:

   (i) $\Delta\, ABM \cong \Delta\, PQN$

   (ii) $\Delta\, ABC \cong \Delta\, PQR$

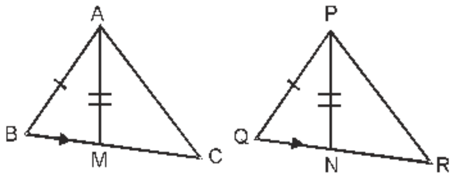

Fig. 7.36

4. BE and CF are two equal altitudes of a triangle ABC. By using RHS rule of congruence, prove that $\Delta\, ABC$ *is* an isosceles triangle.

5. ABC is an isosceles triangle in which $AB - AC$. By drawing $AP \perp BC$, show that $\angle$ B = $\angle$ C

7.8. Inequalities of a triangle :

In the previous chapter, you have studied about scalene triangles, isosceles triangle and equilateral triangle on the basis of sides of triangle and acute angled triangle, right-angled triangle and obtuse-angled triangle on the basis of angles.

Did you ever think that if measure of sides of triangle are change, then angles also change and if angles of a triangle are change then measure of sides are also change

Why?

Let us try to understand this by the following activity and theorems.

Theorem 7.7. ***If two sides of a triangle are unequal, then angle opposite to longest side is greater than the angle opposite to smaller side.***

Given : In ΔABC, $AB > AC$

To Prove $\angle C > \angle B$

Construction : Draw line CD from vertex C such that $AC = AD$.

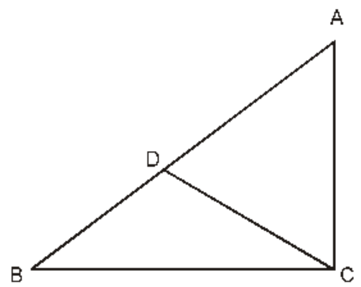

Fig. 7.37

Proof : $AC = AD$ (By construction)

$\therefore$ Their opposite angles will be same.

$\therefore$ $\angle ACD = \angle ADC$ ...(1)

$\because$ $\angle ADC$ is exterior angle of ΔBDC

$\therefore$ $\angle ADC > \angle B$ ...(2)

From equations (1) and (2) $\angle ACD > \angle B$ ...(3)

From figure $\angle ACB > \angle ACD$ ...(4)

From equations (3) and (4), $\angle ACB > \angle ACD > \angle B$

$\Rightarrow$ $\angle ACB > \angle B$

Thus, $\angle C > \angle B$ **Hence Proved**

Theorem 7.8. (Converse of Theorem 7.7)

In a triangle, side opposite to greatest angle is greater than the side opposite to smaller angle.

Given : In ΔABC, $\angle B > \angle C$

To Prove: AC > AB

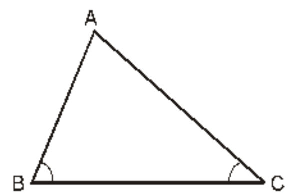

Fig. 7.38

Proof : Following are the three possiblities for sides AC and AB of Δ ABC. Out of them only one is possible.

(i) $AC = AB$

(ii) AC $\angle$ AS, and

(iii) $AC > AB$

Condition : (i) When $AC = AB$

If $AC = AB$, then in Δ ABC, angles opposite to euqal sides will be equal i.e., $\angle B = \angle C$ which is impossible because it is given that $\angle B > \angle C$

Thus, $AC \neq AB$

Condition : (ii) When $AC < AB$

We know that angle opposite to longest side is greater.

$\therefore$ $AC < AB \Rightarrow AB > AC$

$\Rightarrow$ $\angle C > \angle B$

Which is contradiction of given statement

$\therefore$ $AC < AB$

Condition : (iii) Since, AC is neither less than nor equal to AB

AC must $> AB$

Hence, $AC > AB$ is true. **Hence Proved**

Theorem 7.9 : *Sum of any two sides of a triangle is greater than its third side.*

Given : A triangle ABC

To Prove :

(i) $AB \quad BC > AC$

(ii) $BC + AC > AB$

(iii) $AC + AB > BC$

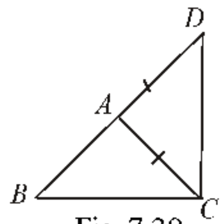

Fig. 7.39

Construction : Produce BA upto D such that $AD = AC$

Proof: In ADC

$AD - AC$ (By construction)

$\therefore$ Angle opposite to equal sides will be equal.

$\angle ACD = \angle ADC$ ...(1)

and $\angle BCD > \angle ACD$ ...(2)

From equations (1) and (2)

$\angle BCD > \angle ADC - \angle BDC$

$\therefore$ $BD > BC$ [$\because$ Side opposite to greater angle is longer]

Thus $BD > BC$

$\Rightarrow$ $BA \mid AD > BC$ [$\because BD - BA + AD$]

$\Rightarrow$ $BA + AC > BC$ [$\because AD - AC$, by construction]

Similarly, we can prove that

$AB + BC > AC$

$BC \quad AC > AB$ Hence Proved

7.7. Lines and Perpendicular Distance from an External Point:

Distance between a line and its external only point is equal to the length of perpendicular drawn from that point on the line.

Theorem 7.10. ***Out of the all line segments drawn from an external point to a straight line (line segment), then the perpendicular line segment is the shortest.***

AB is a line and C is an external point not lying on it. $CE \perp AB$ and D is any point on AB other than E.

To prove : $CE < CD$

Proof : *In* $\Delta\, CED$

$\angle CED = 90^\circ$ $\quad [\because\ CE \perp AB]$

In $\quad \Delta\, CED,\ \angle CED = 90^\circ$

$\angle CDE + \angle CDE\ 90^\circ$

$\Rightarrow \quad \angle CDE < \angle CED$

$\Rightarrow \quad \angle CED > \angle CDE$

$\Rightarrow \quad CD > CE$

[Side opposite to greater angle is longer]

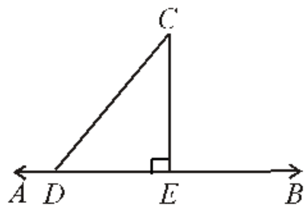

Fig. 7.40

And $\quad \angle DCE < \angle CED$

$\Rightarrow \quad \angle CED > \angle DCE$

$\Rightarrow \quad CD > DE$

It means out of all line segments drawn from an external point to a straight line, the perpendicular line segment is the shortest. **Hence proved**

Illustrative Examples

Example 15 : In Fig. 7.41, AD is the median of Δ ABC, then prove that $AB + AC >$ $2AD$

Or

Prove that the sum of two sides of a triangle is more than twice the median draw on third side.

Sol : Given : AD is the median of $\Delta\, ABC$

To Prove : $AB \quad AC > 2\,AD$

Construction : According to figure, produce AD upto E

such that $DE = AD$. Join C to E.

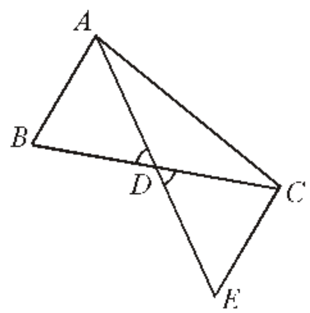

Fig. 7.41

Proof : In $\Delta\, ADB$ and $\Delta\, EDC$

$AD = DE$ (By construction)

$BD = DC$ (Given)

$\angle ADB = \angle EDC$ $\quad$ (Vertically opposite angles)

$\Rightarrow$ By Side Angle Side rule, $\Delta ADB \cong \Delta EDC$

$$AB - CE$$

Now in, Δ ACE

$$AC - CE > AE$$

$\Rightarrow$ $AC \quad AB > AE$ $[\because CE - AB]$

$\Rightarrow$ $AC - AB > 2\,AD$ $[\because AE - 2AD]$ **Hence Proved**

Example 16. If *ABCD* is a quadrilateral, then prove that

(i) $AB + BC + CD + DA > 2\,AC$

(ii) $AB + BC + CD + DA > AC + BD$

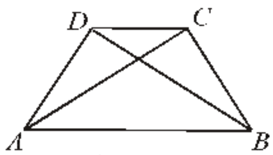

Fig. 7.42

Solution : Given : According to figure 7.42, *ABCD* is *a* quadrilateral.

To Prove : (i) $AB \quad BC + CD + DA > 2\,AC$

(ii) $AB - BC + CD - DA > AC - BD$

Construction : Join diagonals *AC* and *BD*.

Proof : We know that the sum of two sides of a triangle is more than the third side. So

In Δ ABC, AB + BC > AC ...(1)

In Δ *ADC*, $AD - DC > AC$ ...(2)

In Δ ABD, $AB \quad AD > BD$ ...(3)

In Δ BCD, $BC + CD > BD$ ...(4)

Adding (1) and (2), we get

AB + BC + AD + CD > 2AC ...(i)

Again, adding (1), (2), (3) and (4), we get

$$2(AB + BC - AD - DC) > 2\,(AC + BD)$$

$\Rightarrow$ AB + BC + AD + DC > AC + BD ...(ii) **Hence Proved**

Example 17 : In Fig. 7.43, *O* is any point in the interior of Δ ABC, then prove that AB + AC > *OB* + *OC*.

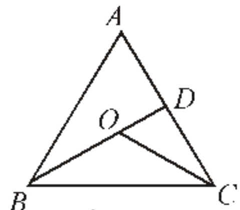

Fig. 7.43

Solution : Given : *O* is an interior point in Δ ABC.

To Prove : AB + AC > OB + *OC*

Construction : Produce *BO*, which meets AC at *D*.

Proof : We know that in a triangle, sum of two sides is more than the third side.

$\therefore$ *In* Δ *ABD*, $AB \quad AD > BD$

$\Rightarrow$ $AB + AD > OB \quad OD$ ...(1)

Similarly, in $\Delta OCD, OD + DC > OC$ ...(2)

Adding (1) and (2), we get

$$AB + AD + OD - DC > OB - OD + OC$$

$\Rightarrow$ $AB + (AD + DC) > OB + OC$

$\Rightarrow$ $AB + AC > OB + OC$ **Hence Proved**

Important points

In this chapter, you have studied the following points.

1. Two figures are congruent, if they have same size and same measure.
2. Two circles of same radius are congruent.
3. Two squares of same sides are congruent.
4. If ΔABC and ΔPQR are congruent under the correspondence $A \leftrightarrow P, B \leftrightarrow Q$ and $C \leftrightarrow R$, then in notation form, they are written *as* $\Delta ABC \cong \Delta PQR$.
5. If two sides and one included angle of one triangle are equal to the corresponding two sides and included angle of other triangle, then two triangles are congruent. (SAS rule of congruence)
6. If two angles and included side of a triangle are equal to the corresponding two angles and included side of other triangle, then the two triangles are congruent. (ASA rule of congruence)
7. If two angles and one side of a triangle are equal to the corresponding two angles and one side of other triangle, then two triangles are congruent. (AAS rule of congruence)
8. Angles opposite to equal sides of a triangle are equal.
9. Side opposite to equal angles of a triangle are equal.
10. Each angle of an equilateral triangle is 60°.
11. If the three sides of a triangle are equal to corresponding three sides of other triangle, then two triangles are congruent. (SSS rule of congruence)
12. 'If in two right triangles, hypotenuse and one side of a triangle are equal to hypotenuse and one side of other triangle, then two triangles are congruent. (RHS rule of congruence)
13. In a triangle, angle opposite to longer side is greater.
14. In a triangle, side oposite to greater angle is longer.
15. In a triangle, sum of two sides is greater than the third side.

Miscellaneous Exercise - 7

Multiple Choice Questions (1 to 16)

1. Which one of the following is not the condition of congruence of triangles.

   (a) SAS (b) ASA (c) SSA (d) SSS

2. If $AB = QR, BC = PR$ and $CA = PQ$, then :

   (a) $\Delta ABC \cong \Delta PQR$ (b) $\Delta CBA \cong \Delta PRQ$

   (c) $\Delta BAC \cong \Delta RPQ$ (d) $\Delta PQR \cong \Delta BCA$

3. In ΔABC, $AB = AC$ and $\angle B = 50°$, then $\angle C$ is equal to :

   (a) 40° (b) 50° (c) 80° (d) 130°

4. In ΔABC, $BC = AB$ and $\angle B = 80°$, then $\angle A$ is equal to :

   (a) 80° (b) 40° (c) 50° (d) 100°

5. In ΔPQR, $\angle R = \angle$P and QR = 4cm and $PR = 5$ cm, then lenght of PQ is :

   (a) 4 cm (b) 5 cm (c) 2 cm (d) 2.5 cm

6. D is a point situated on side BC of Δ ABC, such that AD bisects $\angle BAC$, then.

   (a) $BD = CD$ (b) $BA > BD$ (c) $BD > BA$ (d) $CD > CA$

7. It is given that $\Delta ABC = \Delta FDE$ and $AB = 5$ cm, $\angle B = 40°$ and $\angle A = 80°$. Which one of the following is true?

   (a) DF = 5 cm, $\angle$ F = 60° (b) $DF = 5cm, \angle E = 60°$

   (c) DE = 5cm, $\angle$ E = 60° (d) $DE = 5cm, \angle D = 40°$

8. Lenghts of two sides of a triangle are 5 cm and 1.5 cm. Following cannot be the length of its third side.

   (a) 3.6 cm (b) 4.1 cm (c) 3.8 cm (d) 3.4 cm

9. In Δ PQP if $\angle$ R $\angle$Q then

   (a) $QR > PR$ (b) $PQ > PR$ (c) $PQ < PR$ (d) $QR < PR$

10. In triangles ABC and PQR, $AB = AC$, $\angle C = \angle P$ and $\angle B = \angle Q$. These triangles are :

    (a) Isosceles but not congruent (b) Isosceles and congruent

    (c) Congruent but not isosceles (d) Neither congruent nor isosceles

11. In triangles ABC and DEF, $AB = FD$ *and* $\angle A = \angle D$. Two triangles are congruent by SAS rule if :

    (a) $BC = EF$ (b) AC = DE (c) $AC = EF$ (d) $BC = FE$

12. In right triangle ABC, if $\angle C$ is a right angle then longest side will be.

    (a) AB (b) 5C

(c) CA (d) None of theite

13. Difference of two sides of a triangle to its third side is :

(a) More (b) same (c) less (d) half

14. If two sides of a triangle are unequal, then angle opposite to longest side is :

(a) Greater (b) smaller (c) equal (d) half

15. The perimeter of triangle is then sum of its medians.

(a) More (b) Less (c) Equal (d) Half

16. The sum of three altitudes of triangles is its perimeter :

(a) more (b) equal (c) half (d) less

17. If in ΔABC, $AB - AC$ and $\angle A < 60°$ then write the relation between BC and AC.

18. In Fig 7.44, write the relation between ABand AC.

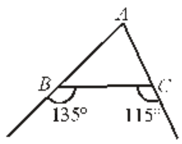

Fig. 7.44

19. In any triangle ABC, $\angle A > \angle B$ and $\angle B > \angle C$, then what will be the smallest side?

20. Find all the angles of an equilateral triangle.

21. P is any point lie on the bisector of $\angle$ABC. If through P, a line is drawn parallel to BA meets line BC at Q then prove that ΔBPQ is an isosceles triangle.

22. ABC is a right angled triangle in which $AB - AC$. Bisector of $\angle$ A meets BC at D. Prove that $BC - 2AD$.

23. ΔABC and ΔDEC are lie on the same base BC such that points A and D are on the opposite side of BC, and $AB = AC$ and $DB = DC$. Show that AD is the perpendicular bisector of BC.

24. ABC'is an isosceles triangle, in which $AC - BC$. AD and BE are altitudes on BC and AC respectively. Prove that $AE - BD$.

25. Prove that sum of any two sides of a triangle is more than twice the corrosponding median of third side.

26. In a ΔABC, D is the mid-point of side AC such that $BD - \frac{1}{2}AC$. Show that $\angle ABC$ is a right angle.

27. In a right triangle prove that line segment joining the mid point of hypotenuse to its opposite vertex is half of the hypotenuse.

28. In Fig. 7.45, if $AB - AC$, then write the relation between ABand AD.

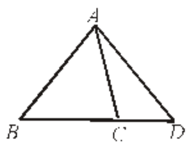

Fig. 7.45

29. AD is a median of ΔABC. Is it true that AB + BC + CA >$2AD$. Give reason for you answer.

30. M is any point on side BC of ΔABC such that AM is the bisector of $\angle BAC$. Is it true that perimeter of triangle is more than 2 AM? Give reason for your answer.

31. Q is any point situated on side SR of ΔPSR such that $PQ - PR$. Prove that : PS > PQ.

32. S is any point situated on side QR of ΔPQR. Show that $PQ - QR + RP > 2PS$

33. D is any point situated on side AC of Δ ABC, with AB = AC. Show that $CD < BD$.

34. In Fig. 7.46. $\angle B > \angle A$ and $\angle D > \angle E$, then prove that $AE > BD$.

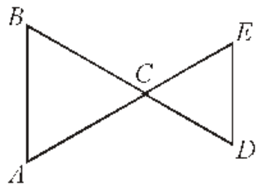

Fig. 7.46

35. In ΔABC, $AB > AC$ and D is any point on side BC prove that $AB > AD$.

36. Prove that sum of three sides of a triangle is more than the sum of its three medians. [Hint: use example 1]

37. In Fig. 7.47. O is the interior point in a triangle, then prove that :
$(BC + AB - AC) < 2\ (OA - OB -$ OC)

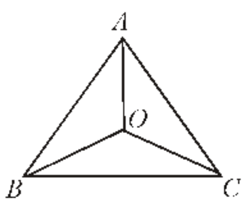

Fig. 7.47

38. Prove that sum of three altitudes of triangle is less then perimeter of triangle.

39. Prove that difference of any two sides of any triangle is less than its third side.

40. Bisectors of $\angle B$ and $\angle C$ of an isosecles triangle with AB = AC, intersect each other at point O. show that adjacent $\angle ABC$ is equal to exterior $\angle BOC$.

41. In Fig. 7.48 AD is the bisector of $\angle BAC$. Prove that AB > BD.

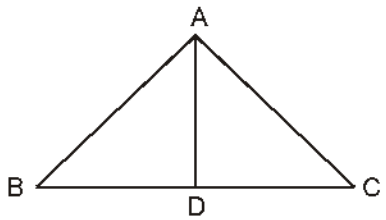

Fig. 7.48

Answer

Exercise 7.1

1. QR: These will be congruent by ASA

2. RP: These will be congruent by AAS

3. No, angle should be between the two sides.

4. No, sides should be corresponding.

5. No, BC should be equal to PQ

6. Yes, these are corresponding sides.

Exercise 7.2

1. 64° **3.** 90°

Miscellaneous Exercise -7

1. C	**2.** B	**3.** B	**4.** C	**5.** A	**6.** B
7. B	**8.** D	**9.** B	**10.** A	**11.** B	**12.** A
13. C	**14.** A	**15.** A	**16.** D		

17. BC < AC

18. AB > AC

19. AB

20. 60, 60, 60

28. AD > AB

29. Yes AB + BD > AD and AC + CD > AD

30. Yes AB + BM >AM and AC + CM > AM