EXERCISE: 5.6 (NCERT)

DERIVATIVES OF FUNCTIONS IN PARAMETRIC FORM:

A relation expressed between two variables x and y in form of $x = f(t)$, $y = g(t)$ is said to be parametric form with t as parameter.

So that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ (Whenever $\frac{dx}{dt} \neq 0$)

Thus $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ provided $f'(t) \neq 0$

If x and y are connected parametrically by equations given in Exercises 1 to 10, without eliminating the parameter, find $\frac{dy}{dx}$

Q No 1 $x = 2at^2$; $y = at^4$.

$\therefore \frac{dx}{dt} = 2a \cdot 2t$; $\frac{dy}{dt} = a \cdot 4t^3$

$= 4at$ $\qquad = 4at^3$.

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$.

Q No 2 $x = a\cos\theta$; $y = b\cos\theta$

Sol. $\Rightarrow \frac{dx}{d\theta} = a(-\sin\theta) = -a\sin\theta$

and $\frac{dy}{d\theta} = b(-\sin\theta) = -b\sin\theta$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$

Q No. 3 $x = \sin t$; $y = \cos 2t$

$\therefore \frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -(\sin 2t).2$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cos t} = \frac{-2\times 2\sin t\cos t}{\cos t} = -4\sin t$

Q No 4 $x = 4t$; $y = \frac{4}{t}$

$\Rightarrow \frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 4\left(-\frac{1}{t^2}\right) = -\frac{4}{t^2}$.

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-4}{t^2}}{4} = -\frac{1}{t^2}$.

Q No 5 $x = \cos\theta - \cos 2\theta$; $y = \sin\theta - \sin 2\theta$.

$\Rightarrow \frac{dx}{d\theta} = (-\sin\theta) - (-\sin 2\theta \times 2) = -\sin\theta + 2\sin 2\theta$

and $\frac{dy}{d\theta} = \cos\theta - (\cos 2\theta \times 2) = \cos\theta - 2\cos 2\theta$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos\theta - 2\cos 2\theta}{-\sin\theta + 2\sin 2\theta}$.

Q No. 6 $x = a(\theta - \sin\theta)$; $y = a(1+\cos\theta)$

$\therefore \frac{dx}{d\theta} = a.\frac{d}{d\theta}\left[\theta - \sin\theta\right] = a(1-\cos\theta)$

and $\frac{dy}{d\theta} = a\frac{d}{dx}(1+\cos\theta) = a(0 - \sin\theta) = -a\sin\theta$.

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-a.2.\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a.\sin^2\frac{\theta}{2}} = -\cot\left(\frac{\theta}{2}\right)$

Q No. 7: $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

$\therefore \frac{dx}{dt} = \frac{d}{dt}\left(\frac{\sin^3 t}{\sqrt{\cos 2t}}\right) = \frac{\sqrt{\cos 2t}(3\sin^2 t\cos t) - \sin^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{(\sqrt{\cos 2t})^2}$

$$= \frac{3(\cos 2t)\sin^2 t\cos t + \sin 2t\,\sin^3 t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$= \frac{3(1-2\sin^2 t).\sin^2 t\cos t + (2\sin t\cos t)\sin^3 t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$= \frac{3\sin^2 t\cos t - 4\sin^4 t\cos t}{\cos 2t\sqrt{\cos 2t}}$$

and $\quad \frac{dy}{dt} = \frac{d}{dt}\left(\frac{\cos^3 t}{\sqrt{\cos 2t}}\right)$

$$= \frac{\sqrt{\cos 2t}\,(-3\cos^2 t\sin t) - \cos^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{(\sqrt{\cos 2t})^2}$$

$$= \frac{-3(\cos 2t)\cos^2 t\sin t + \cos^3 t\sin 2t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$= \frac{-3(2\cos^2 t-1).\cos^2 t.\sin t + \cos^3 t.2\sin t.\cos t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$= \frac{3\cos^2 t.\sin t - 4\cos^4 t\sin t}{(\cos 2t)\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos^2 t\sin t - 4\cos^4 t\sin t}{3\sin^2 t.\cos t - 4\sin^4 t.\cos t}$$

$$= \frac{\cos^2 t.\sin t(3-4\cos^2 t)}{\sin^2 t\cos t(3-4\sin^2 t)} = \frac{\cos t(3-4\cos^2 t)}{\sin t(3-4\sin^2 t)}$$

$$= \frac{3\cos t - 4\cos^3 t}{3\sin t - 4\sin^3 t} = \frac{-\cos 3t}{\sin 3t} = -\cot 3t.$$

Q No. 8 : $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$; $y = a\sin t$.

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{1}{2}\right\} = a\left\{-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right\}$$

$$= a\left\{-\sin t + \frac{1}{\sin t}\right\} = a\left\{\frac{1-\sin^2 t}{\sin t}\right\}$$

or $\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$

Also $y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \tan t.$$

Q No. 9: $x = a\sec\theta$, $y = b\tan\theta$.

$$\Rightarrow \frac{dx}{d\theta} = a\sec\theta\cdot\tan\theta. \quad \text{and} \quad \frac{dy}{d\theta} = b\sec^2\theta.$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\cdot\tan\theta} = \frac{b}{a}\left(\frac{\sec\theta}{\tan\theta}\right) = \frac{b}{a}\operatorname{cosec}\theta.$$

Q No 10: $x = a(\cos\theta + \theta\sin\theta)$ $y = a(\sin\theta - \theta\cos\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta + \theta\sin\theta) = a\left[-\sin\theta + \left(\theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right)\right]$$

$$= a\left[-\sin\theta + (\theta\cos\theta + \sin\theta)\right] = a\theta\cos\theta$$

Also $\frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta - \theta\cos\theta) = a\left[\cos\theta - \left(\theta\cdot\frac{d}{d\theta}\cos\theta + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right)\right]$

$$= a\left[\cos\theta - (\theta(-\sin\theta) + \cos\theta\times 1)\right] = a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\cdot\sin\theta}{a\theta\cdot\cos\theta} = \tan\theta.$$ [Note that $\frac{dy}{dx}$ does not exist when $\cos\theta = 0$]

Q No 11: If $x = \sqrt{a^{\sin^{-1}t}}$ $y = \sqrt{a^{\cos^{-1}t}}$

Show that $\frac{dy}{dx} = -\frac{y}{x}$.

Sol: $x = \sqrt{a^{\sin^{-1}t}} \Rightarrow x^2 = a^{\sin^{-1}t}$

Taking log on both sides we get

$$\log(x^2) = \log\left(a^{\sin^{-1}t}\right) = \sin^{-1}t\cdot\log a.$$

Similarly $\log y^2 = \cos^{-1}t \cdot \log a$.

$\Rightarrow \log x^2 + \log y^2 = \sin^{-1}t \log a + \cos^{-1}t \log a$ [Adding]

$\Rightarrow \log x^2 + \log y^2 = \log a\left(\sin^{-1}t + \cos^{-1}t\right)$

$\Rightarrow \log x^2 + \log y^2 = \log a\left(\pi/2\right)$

Differentiating both sides w.r.t x

$$\left(\frac{1}{x^2}\right)\left(2x\right) + \frac{1}{y^2}(2y) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{x}}{\frac{1}{y}} = -\frac{y}{x}$$

Hence the proof.

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