Chapter 13

Limits and Derivatives

Exercise 13.2

Question 1: Find the derivative of $x^2 - 2$ at $x = 10$.

Answer 1:

Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$

$$= \lim_{h\to 0}\frac{[(10+h)^2-2]-(10^2-2)}{h}$$

$$= \lim_{h\to 0}\frac{10^h+2.10.h+h^h-2-10^h+2}{h}$$

$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$

$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2: Find the derivative of 99x at x = 100.

Answer 2:

Let f(x) = 99x. Accordingly,

$$f'(100) = \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$

$$= \lim_{h\to 0}\frac{99(100+h)-99(100)}{h}$$

$$= \lim_{h\to 0}\frac{99\times100+99h-99\times100}{h}$$

$= \lim_{h \to 0} \frac{99h}{h}$

$= \lim_{h \to 0} (99) = 99$

Thus, the derivative of 99x at x = 100 is 99.

Question 3: Find the derivative of x at x = 1.

Answer 3:

Let f(x) = x. Accordingly,

$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0} \frac{(1+h) - 1}{h}$

$= \lim_{h \to 0} \frac{h}{h}$

$= \lim_{h \to 0} (1)$

= 1

Thus, the derivative of x at x = 1 is 1.

Question 4: Find the derivative of the following functions from first principle.

(I) $x^3 - 27$ (ii) (x – 1) (x – 2)

(iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Answer 4:

(I) Let f(x) = $x^3 - 27$. Accordingly, from the first principle,

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$

$= \lim_{h \to 0} \frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$

$= \lim_{h \to 0} \frac{h^3+3x^2h+3xh^2}{h}$

$= \lim_{h \to 0} (h^2 + 3x^2 + 3xh)$

$= 0 + 3x^2 + 0 = 3x^2$

(ii) Let f(x) = (x – 1) (x – 2). Accordingly, from the first principle,

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$

$= \lim_{h \to 0} \frac{(x^2+hx-2x+hx+hx^2-2h-x-h+2)-(x^2-2x-x+2)}{h}$

$= \lim_{h \to 0} \frac{(hx+hx+h^2-2h-h)}{h}$

$= \lim_{h \to 0} \frac{2hx+h^2-3h}{h}$

$= \lim_{h \to 0} (2x + h - 3)$

$= (2x + 0 - 3)$

$= 2x - 3$

(iii) Let $f(x) = \frac{1}{x^2}$

Accordingly, from the first principle,

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$

$= \lim_{h \to 0} \left[\frac{-h-2hx}{x^2(x+h)^2}\right]$

$= \frac{0-2x}{x^2(x+0)^2} = \frac{-2}{x^3}$

(iv) Let $f(x) = \frac{x+1}{x-1}$

Accordingly, from the first principle,

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{\left\{\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right\}}{h}$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h+1)}\right]$

$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)}\right]$

$= \lim_{h \to 0} \left[\frac{-2}{(x-1)(x+h-1)}\right]$

$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$

Question 5: For the function $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$

Prove that $f'(1) = 100f'(0)$

Answer 5:

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^{2}}{2}+x+1$$

$$=\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^{2}}{2}+x+1\right]$$

$$=\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\cdots+\frac{d}{dx}\left(\frac{x^{2}}{2}\right)+\frac{d}{d}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$=\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\cdots+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+\cdots+x+1$$

$$=f'(x)=x^{99}+x^{98}+\cdots+x+1$$

At $x=0$

= f' (0) =1

At $x=1$

$$=f'(1)=1^{99}+1^{98}+\cdots+1+1=(1+1+\cdots+1+1)_{100terms}=1\times100=100$$

Thus, $f'(1)=100\times f^{1}(0)$

Question 6: Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\cdots+a^{n-1}x+a^n$ for some fixed real number a.

Answer 6:

Let

$$f'(x)=\frac{d}{dx}(x^n+ax^{n-1}+a^nx^{n-2}+\cdots a^{n-1}x+a^n)$$

$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+\cdots+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

= $f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots + a^{n-1} + a^n(0)$

= $nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots + a^{n-1}$

Question 7: For some constants an a and b, find the derivative of

(i) (x – a) (x – b) (ii) $(ax^2 + b)^2$ (iii) $\frac{x-a}{x-b}$

Answer 7:

(i) Let f (x) = (x – a) (x – b)

$f(x) = x^2 - (a+b)x + ab$

$f'(x) = \frac{d}{dx}(x^2 - (a+b) + ab)$

= $\frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$

On using theorem $\frac{d}{dx}(x^n) = nx^{x-1}$, we obtain

$f'(x) = 2x - (a+b) + 0 = 2x - a - b$

(ii) Let $f(x) = (ax^2 + b)^2$

$f(x) = a^2x^4 + 2abx^{22} + b^2$

$f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2) = a^2\frac{d}{dx}(x^4) + 2ab\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$

= $4a^2x^3 + 4abx$

= $4ax(ax^2 + b)$

(iii) $f(x) = \frac{x-a}{x-b}$

$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$

By quotient rule,

$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$

$= \frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$

$= \frac{x-b-x+a}{(x-b)^2}$

$= \frac{a-b}{(x-b)^2}$

Question 8: Find the derivative of $\frac{x^n-a^n}{x-a}$ for some constant a.

Answer 8:

Let, $f(x) = \frac{x^n-a^n}{x-a}$

$f'(x) = \frac{d}{dx}\left(\frac{x^n-a^n}{x-a}\right)$

By quotient rule,

$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$

$= \frac{(x-a)\left(nx^{n-1}-0\right)-(x^n-a^n)}{(x-a)^2}$

$= \frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$

Question 9: Find the derivative of

(i) $2x - \frac{3}{4}$

(ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) x–3 (5 + 3x)

(iv) $x^5 (3 – 6x–9)$

(v) x–4 (3 – 4x–5)

(vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Answer 9:

(i) Let, $f(x) = 2x - \frac{3}{4}$

$= f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$

$= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$

= 2 – 0

= 2

(ii) Let f (x) = $(5x^3 + 3x – 1) (x – 1)$

By Leibnitz product rule,

$f'(x) = (5x^3 + 3x - 1)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(5x^3 + 3x - 1)$

$= (5x^3 + 3x - 1)(1) + (x-1)(5.3x^2 + 3 - 0)$

$= (5x^3 + 3x - 1) + (x-1)(15x^2 + 3)$

$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$

$= 20x^3 15x^2 + 6x - 4$

(iii) Let f (x) = x– 3 (5 + 3x)

By Leibnitz product rule,

$f'(x) = x^{-3}\frac{d}{dx}(5 + 3x) + (5 + 3x)\frac{d}{dx}(x^{-3})$

$= x^{-3}(0 + 3) + (5 + 3x)(-3x^{-3-1})$

$= x^{-3}(3) + (5 + 3x)(-3x^{-4})$

$= 3x^{-3} - 15x^{-4} - 9x^{-3}$

$= -6x^{-3} - 15x^{-4}$

(iv) Let f (x) = x^5 (3 – 6x–9)

By Leibnitz product rule,

$f'(x) = x^5 \frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9})\frac{d}{dx}(x^5)$

$= x^5\{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4)$

$= x^5(54x^{-10}) + 15x^4 - 30x^{-5}$

$= 54x^{-5} + 15x^4 - 30x^{-5}$

$= 24x^{-5} + 15x^4$

$= 15x^4 + \frac{24}{x^5}$

(v) Let f (x) = x–4 (3 – 4x–5)

By Leibnitz product rule,

$f'(x) = x^{-4} \frac{d}{dx}(3 - 4x^{-5}) + (3 - 4x^{-5})\frac{d}{dx}(x^{-4})$

$= x^{-4}\{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1}$

$= x^{-4}(20x^{-6}) + (3 - 4x^{-5})(-4x^{-5})$

$= 20x^{-10} - 12x^{-5} + 16x^{-10}$

$= 36x^{-10} - 12x^{-5}$

= - $\frac{12}{x^5} + \frac{36}{x^{10}}$

(vi) Let f (x) = $\frac{2}{x+1} - \frac{x^2}{3x-1}$

$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$

By quotient rule,

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left(\frac{(x+1)(0)-2(1)}{(x+1)^2}\right) - \left(\frac{(3x-1)(2x)-(x^2)(3)}{(3x+1)^2}\right)$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Question 10: Find the derivative of cos x from first principle.

Answer 10:

Let f (x) = cos x. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x+h)-cosx}{h}$$

$$= \lim_{h\to 0}\left[\frac{cosxcosh-sinxsinh-cosx}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-cosx(1-\cos h)-sinxsinh}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-cosx(1-\cos h)}{h} - \frac{sinxsinh}{h}\right]$$

$$= -\cos x\left(\lim_{h\to 0}\frac{1-cosh}{h}\right) - sinx\lim_{h\to 0}\left(\frac{sinh}{h}\right)$$

= - $cosx(0) - sinx(1)$ $\left[\lim_{h\to 0}\frac{1-cosh}{h} = 0 and \lim_{h\to 0}\frac{sinh}{h} = 1\right]$

= - sin x

= $f'(x)$ = - sin x

Question 11: Find the derivative of the following functions:

(i) sin x cos x

(ii) sec x

(iii) 5 sec x + 4 cos x

(iv) cosec x

Answer 11:

(i) Let f (x) = sin x cos x.

Accordingly, from the first principle,

$$f'(x) \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{sin(x+h)cos(x+h)-2sinxcosx}{h}$$

$$= \lim_{h\to 0}\frac{1}{2h}[2sin(x+h)cos(x+h) - sin2x]$$

$$= \lim_{h\to 0}\frac{1}{2h}[sin2(x+h) - sin2x]$$

$$= \lim_{h\to 0}\frac{1}{2h}\left[2cos\frac{2x+2h+2x}{2}.sin\frac{2x+2h-2x}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[cos\frac{4x+2h}{2}sin\frac{2h}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}[cos(2x+h)sinh]$$

$$= \lim_{h\to 0}cos(2x+h). \lim_{h\to 0}\frac{sinh}{h}$$

$= \cos(2x + 0) . 1$

$= cos2\ x$

(ii) Let f (x) = sec x.

Accordingly, from the first principle,

$$f"(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sec(x+h)-sec\ x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{cosx}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{cosx-\cos(x+h)}{cosxcos(x+h)}\right]$$

$$= \frac{1}{cosx} . \lim_{h\to 0} \frac{1}{h}\left[\frac{-2sin\left(\frac{x+x+h}{2}\right)sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{cosx} . \lim_{h\to 0} \frac{1}{h}\left[\frac{-2sin\left(\frac{2s+h}{2}\right)sin\left(\frac{h}{2}\right)}{\cos(x+h)}\right]$$

$$= \frac{1}{cosx} . \lim_{h\to 0} \frac{\left[sin\left(\frac{2x+h}{2}\right)\frac{sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$

$$= \frac{1}{cosx} . \lim_{\frac{h}{2}\to 0} \frac{sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} . \lim_{h\to 0} \frac{sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

$$= \frac{1}{cosx} . 1 . \frac{sinx}{cosx}$$

$= sec\ x\ tan\ x$

(iii) Let f (x) = 5 sec x + 4 cos x.

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5sec(x+h)+4cos(x+h)-[5secx+4cosx]}{h}$$

$$= 5\lim_{h\to 0} \frac{[\sec(x+h)-secx]}{h} + 4\lim_{h\to 0} \frac{[\cos(x+h)-cosx]}{h}$$

$$= 5\lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{cosx}\right] + 4\lim_{h\to 0} \frac{1}{h}[\cos(x+h) - cosx]$$

$$= 5\lim_{h\to 0} \frac{1}{h}\left[\frac{cosx-\cos(x+h)}{cosxcos(x+h)}\right] + 4\lim_{h\to 0} \frac{1}{h}[cosxcosh - sinxsinh - cosx]$$

$$= \frac{5}{cosx}\lim_{h\to 0} \frac{1}{h}\left[\frac{-2sin\left(\frac{x+x+h}{2}\right)sin\left(\frac{x-x-h}{2}\right)}{cos(x+h)}\right] + 4\lim_{h\to 0} \frac{1}{h}[-cosx(1-cosh) - sinxsinh]$$

$$= \frac{5}{cosx}.\lim_{h\to 0} \frac{1}{h}\left[\frac{-2sin\left(\frac{2x+h}{2}\right)sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right] + 4\left[-cosx\lim_{h\to 0} \frac{(1-cosh)}{h} - sinx\lim_{h\to 0} \frac{sinh}{h}\right]$$

$$= \frac{5}{cosx}\lim_{h\to 0}\left[\frac{sin\left(\frac{2x+h}{2}\right).\frac{sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)}\right] + 4[(-cosx).(0) - (sinx).1]$$

$$= \frac{5}{cosx}.\left[\lim_{h\to 0} \frac{sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0} \frac{sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] - 4sinx$$

$$= \frac{5}{cosx}.\frac{sinx}{cosx}.1 - 4sinx$$

$$= 5sec\ x\ tan\ x - 4sin\ x$$

(iv) Let f (x) = cosec x.

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} [cosec(x+h) - cosecx]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{sinx}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{sinx - \sin(x+h)}{\sin(x+h) sinx}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{2cos\left(\frac{x+x+h}{2}\right).sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h) sinx}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{2cos\left(\frac{2x+h}{2}\right)sin\left(-\frac{h}{2}\right)}{\sin(x+h) sinx}\right]$$

$$= \lim_{h\to 0} \frac{-cos\left(\frac{2x+h}{2}\right).\frac{sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h) sinx}$$

$$= \lim_{h\to 0} \left(\frac{-cos\frac{2x+h}{2}}{\sin(x+h) sinx}\right). \lim_{\frac{h}{2}\to 0} \frac{sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-cosx}{sinxsinx}\right). 1$$

$$= - cosec\ x\ cot\ x$$