16

Probability and Probability Distribution

16.01 Introduction

We often make statements about probability. For example, a weather forecaster may predict that there is an 80% chance of rain tomorrow. A health news reporter may state that a smoker has a much greater chance of getting cancer than a non smoker does.

In earlier classes, we have studied the probability as a measure of uncertainty of an event in a random experiment. We have also established a relationship between the exiomatic theory and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship, we obtain probabilities of events associated with discrete sample space. In this chapter, we shall discuss the important concept of conditional probability, multiplication rule of probality and independence of events, the Baye's theorem, random variable and its probability distribution, the mean and variance of a probability distribution.

16.02 Conditional Probability

If we have two events form the same sample space, Does the information about the occurance of one of the events affect the probability of the other event ? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur. Consider the experiment of tossing two fair coins. The sample space of the experiment is

$$S=\{HH,\ HT,\ TH,\ TT\},\ \text{H = Head, T = Tail}$$

Since the coins are fair, we can assign the probability 1/4 to each sample point. Let A be the event at least one head appears and B be the event "first coin shows tail". Then

$$A=\{HT,\ TH,\ HH\},\quad B=\{TH,\ TT\}$$

$$\therefore \qquad P(A)=P(\{HT\})+P(\{TH\})+P(\{HH\})$$

$$=(1/4)+(1/4)+(1/4)=3/4$$

and $$P(B)=P(\{TH\})+P(\{TT\})$$

$$=(1/4)+(1/4)=1/2$$

also $$A\cap B=\{TH\}$$

$$\therefore \qquad P(A\cap B)=P(\{TH\})=1/4$$

Now, we have to find the probability of A, when event B has already occured with the information of occurence of B, we are sure that the case in which first coin does not result into a tail, should not be considered while finding the probability of A. This information reduces our sample space form the set S to its subset B for the event A

Thus sample point of event A which is favourable to event B is {TH}

Thus , Probability of A considering B as the sample space = 1/2

or, Probability of A given that the event B has occured = 1/2

This probability of the event A is called the *conditional probability of A given that B has already occured,* and is denoted by *P(A/B)*

i.e. $$P\left(\frac{A}{B}\right)=\frac{1}{2}$$

Thus, we can also write the conditional probability of A given that B has occured as $P(A/B)$

$$P\left(\frac{A}{B}\right)=\frac{\text{Number of elementary events favourable to }(A\cap B)}{\text{Number of elementary events which are favourable to }B}$$

$$=\frac{n(A\cap B)}{n(B)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that

$P(A/B)$ can also be written as

$$P\left(\frac{A}{B}\right)=\frac{\frac{n(A\cap B)}{n(S)}}{\frac{n(B)}{n(S)}}=\frac{P(A\cap B)}{P(B)}$$

note that it is valid only when $P(B)\neq 0$

Definition : If A and B are two events associated with the same sample space of a random experiment, the conditional probability of the event A given that B has occured is given by

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)};\ P(B)\neq 0$$

Similarly the conditional probability of the event B given that A has occured is given by

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)};\ P(A)\neq 0$$

16.03 Properties of conditional probability

Let A and B be events of a sample space S of an experiment, then we have

(i) $$P\left(\frac{S}{B}\right)=P\left(\frac{B}{B}\right)=1$$

We know that, $$P\left(\frac{S}{B}\right)=\frac{P(S\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$$

again $$P\left(\frac{B}{B}\right)=\frac{P(B\cap B)}{P(B)}=\frac{P(B)}{P(B)}=1$$

thus $$P\left(\frac{S}{B}\right) = P\left(\frac{B}{B}\right) = 1$$

(ii) $$P\left(\frac{\bar{A}}{B}\right) = 1 - P\left(\frac{A}{B}\right)$$

using property (i) $$P\left(\frac{S}{B}\right) = 1$$

$\Rightarrow$ $$P\left(\frac{A \cup \bar{A}}{B}\right) = 1 \qquad \left[\because S = A \cup \bar{A}\right]$$

$\Rightarrow$ $$P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{B}\right) = 1 \qquad [\because A \text{ and } \bar{A} \text{ are disjoint events}]$$

$\therefore$ $$P\left(\frac{\bar{A}}{B}\right) = 1 - P\left(\frac{A}{B}\right).$$

(iii) *If* A *and* B *are any two events of a sample space* S *and* F *is an event of* S *such that* $P(F) \neq 0$ then

(a) $$P\left(\frac{A \cup B}{F}\right) = P\left(\frac{A}{F}\right) + P\left(\frac{B}{F}\right) - P\left(\frac{A \cap B}{F}\right)$$

In particular, if A *and* B *are disjoint events, then*

(b) $$P\left(\frac{A \cup B}{F}\right) = P\left(\frac{A}{F}\right) + P\left(\frac{B}{F}\right)$$

We have

$$P\left(\frac{A \cup B}{F}\right) = \frac{P\left[(A \cup B) \cap F\right]}{P(F)}$$

$$= \frac{P\left[(A \cap F) \cup (B \cap F)\right]}{P(F)}$$

$$= \frac{P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)}{P(F)}$$

$$= \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P\left[(A \cap B) \cap F\right]}{P(F)}$$

$$= P\left(\frac{A}{F}\right) + P\left(\frac{B}{F}\right) - P\left(\frac{A \cap B}{F}\right).$$

Special Condition : When A and B are disjoint events, then $P\left(\frac{A\cap B}{F}\right)=0$

$$\therefore \qquad P\left(\frac{A\cup B}{F}\right)=P\left(\frac{A}{F}\right)+P\left(\frac{B}{F}\right).$$

Illustrative Examples

Example 1. If $P(A)=6/11,\ P(B)=5/11$ and $P(A\cup B)=7/11$ then find

(i) $P(A\cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$

Solution : (i) We know that $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow \qquad P(A\cap B)=P(A)+P(B)-P(A\cup B)$$

$$=\frac{6}{11}+\frac{5}{11}-\frac{7}{11}=\frac{4}{11}$$

(ii) $$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{4/11}{5/11}=\frac{4}{5}$$

(iii) $$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{4/11}{6/11}=\frac{2}{3}$$

Example 2. An instructor has a question bank consisting of 300 easy true / false questions, 200 difficult true/ false questions, 500 easy multiple choice questions. If a question is selected at random from the question bank, what is the probabiliy that it will be an easy question given that it is a multiple choice question ?

Solution : Let event A 'it is an easy question' and event B 'It is a multiple choice question' and we have to find $P(A/B)$

$$n(A)=300+500=800,\ n(B)=500+400=900$$

Here set $A\cap B$ denotes 'it is an easy multiple choice question'

$$\therefore \qquad n(A\cap B)=500$$

required probability $$=P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{n(A\cap B)}{n(B)}=\frac{500}{900}=\frac{5}{9}.$$

Example 3. Determine $P(A/B)$ in each case when a coin is tossed three times, where

(i)	A : head on third toss	B : heads on first two tosses
(ii)	A : at least two heads,	B : at most two heads
(iii)	A : at most two tails,	B : at least one tail

Solution : The sample space when a coin is tossed three times is as follows

$$S=\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(i) $A=\{HHH, HTH, THH, TTH\}$, $B=\{HHH, HHT\}$

then $\qquad A\cap B=\{HHH\}$

$\Rightarrow \qquad n(A)=4,\ n(B)=2,\ n(A\cap B)=1$

$\therefore \qquad P\left(\frac{A}{B}\right)=\frac{n(A\cap B)}{n(B)}=\frac{1}{2}.$

(ii) $A=\{HHH, HHT, HTH, THH\}$, $B=\{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\therefore \qquad A\cap B=\{HHT, HTH, THH\}$

$\Rightarrow \qquad n(A)=4,\ n(B)=7,\ n(A\cap B)=3$

$\therefore \qquad P\left(\frac{A}{B}\right)=\frac{n(A\cap B)}{n(B)}=\frac{3}{7}.$

(iii) $A=\{HHH, HHT, HTH, THH, HTT, THT, TTH\}$,

$B=\{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$\therefore \qquad A\cap B=\{HHT, HTH, THH, HTT, THT, TTH\}$

$\Rightarrow \qquad n(A)=7,\ n(B)=7,\ n(A\cap B)=6$

$\therefore \qquad P\left(\frac{A}{B}\right)=\frac{n(A\cap B)}{n(B)}=\frac{6}{7}.$

Example 4. A black and a red die are thrwon, then

(a) Find the conditional probability of obtaining a sum greater than 9, giving that the black die resulted in a 5.

(b) Find the conditional probabilility of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution : (*i*) Let event A denotes 'sum greater then 9' and event B denotes 'black dice resulted in a 5' now we have to find $P(A/B)$

$$A=\{(5,5),(6,4),(4,6),(6,5),(5,6),(6,6)\},\quad B=\{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$$

$$A\cap B=\{(5,5),(5,6)\}$$

$\Rightarrow \qquad n(A)=6,\ n(B)=6,\ n(A\cap B)=2$

thus required probability $\qquad =P\left(\frac{A}{B}\right)=\frac{n(A\cap B)}{n(B)}=\frac{2}{6}=\frac{1}{3}.$

(ii) Let event A denotes 'sum greater than 9' and event B denotes' red die resulted in a number less than 4' now we have to find $P(A/B)$

then $A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

$B = \{(6, 1), (6, 2), (6, 3), (5, 1), (5, 2), (5, 3), (4, 1), (4, 2), (4, 3),$
$(3, 1), (3, 2), (3, 3), (2, 1), (2, 2), (2, 3), (1, 1), (1, 2), (1, 3)\}$

$A \cap B = \{(6, 2), (5, 3)\}$

$\Rightarrow \quad n(A) = 5,\ n(B) = 18,\ n(A \cap B) = 2$

thus required probability $= P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{2}{18} = \frac{1}{9}.$

Example 5. A die is thrown three times, then event A and B defined as follows :
A : 4 appears on the third throw,
B : 6 and 5 appears respectively on first two tosses

determine $P(A/B)$.

Solution : When a coin is tossed three times the sample space S contains $= 6 \times 6 \times 6 = 216$ equally likely outcomes.

then $A = \{(1, 1, 4), (1, 2, 4), (1, 3, 4), (1, 4, 4), (1, 5, 4), (1, 6, 4)$
$(2, 1, 4), (2, 2, 4), (2, 3, 4), (2, 4, 4), (2, 5, 4), (2, 6, 4)$
$(3, 1, 4), (3, 2, 4), (3, 3, 4), (3, 4, 4), (3, 5, 4), (3, 6, 4)$
$(4, 1, 4), (4, 2, 4), (4, 3, 4), (4, 4, 4), (4, 5, 4), (4, 6, 4)$
$(5, 1, 4), (5, 2, 4), (5, 3, 4), (5, 4, 4), (5, 5, 4), (5, 6, 4)$
$(6, 1, 4), (6, 2, 4), (6, 3, 4), (6, 4, 4), (6, 5, 4), (6, 6, 4)\}$

$B = \{(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)\}$

$A \cap B = \{(6, 5, 4)\}$

$\Rightarrow \quad n(A) = 36,\ n(B) = 6,\ n(A \cap B) = 1$

Thus required probability $= P\left(\frac{A}{B}\right) = \frac{n(A \cap B)}{n(B)} = \frac{1}{6}.$

Example 6. Consider the experiment of throwing a die, if a multiple of 3 or 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event' the coin shows a tail', given that' at least one die shows a 3'.

Solution : The results of the experiments can be shown as

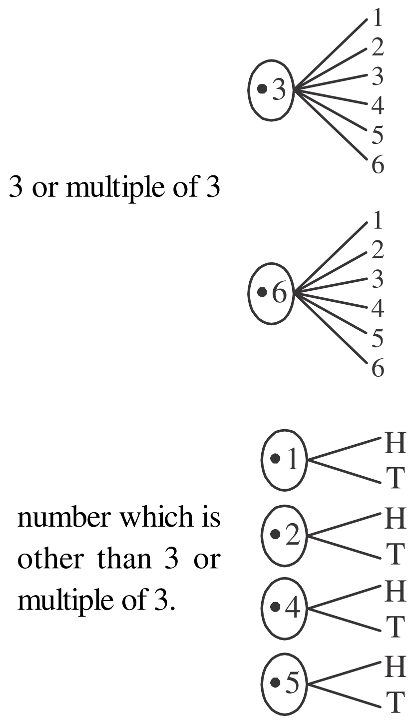

Fig. 16.01

The sample space is as follows

$$S=\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(6,1),(6,2),(6,3),(6,4)$$
$$(6,5),(6,6),(1,H),(1,T),(2,H),(2,T),(4,H),(4,T),(5,H),(5,T)\}$$

Let event A denotes ' tail on the coin' and event B denotes 'at least one die show a 3'.

then $\quad A=\{(1,T),(2,T),(4,T),(5,T)\}; \quad B=\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(6,3)\}$

$$A\cap B=\phi$$

$\Rightarrow \quad n(A)=4,\ n(B)=7,\ n(A\cap B)=\phi$

Required probability $=P\left(\dfrac{A}{B}\right)=\dfrac{n(A\cap B)}{n(B)}=\dfrac{0}{7}=0$

Exercise 16.1

1. If $P(A)=7/13,\ P(B)=9/13$ and $P(A\cap B)=4/13$ then find $P(A/B)$.

2. If $P(B)=0.5$ and $P(A\cap B)=0.32$ then find $P(A/B)$.

3. If $2P(A)=P(B)=5/13$ and $P\left(\dfrac{A}{B}\right)=\dfrac{2}{5}$ then find $P(A\cup B)$.

4. If $P(A)=0.6,\ P(B)=0.3$ and $P(A\cap B)=0.2$ then find $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$.

5. If $P(A)=0.8,\ P(B)=0.5$ and $P\left(\frac{B}{A}\right)=0.4$ then find that

(i) $P(A\cap B)$ (ii) $P\left(\frac{A}{B}\right)$ (iii) $P(A\cup B)$

6. Assune that each born child equally likely to be a boy or a girl. If a family has two children, it is given that if atleast one of them is a boy than find the probability that both the children to be a boy.

7. Two coins are tossed once then find $P(A/B)$

(i) A : tail appear on one coin, B : one coin shows head

(ii) A : no tail appears, B : no head appears

8. Mother, father and son line up at random for a family picture. If A and B are two event as follows then find $P(A/B)$

A : son on one end

B : father in middle

9. A fair die is rolled. Consider events $A=\{1,\ 3,\ 5\}$] $B=\{2,\ 3\}$ and $C=\{2,\ 3,\ 4,\ 5\}$ then find

(i) $P\left(\frac{A}{B}\right)$ and $P\left(\frac{B}{A}\right)$ (ii) $P\left(\frac{A}{C}\right)$ and $P\left(\frac{C}{A}\right)$ (iii) $P\left(\frac{A\cup B}{C}\right)$ and $P\left(\frac{A\cap B}{C}\right)$

10. Given that the two numbers apperaring on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

11. Ten cards numbered 1 to 10 are placed in box, mixed up throughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number ?

12. In a school, there are 1000 students, out ot which 430 are girls. It is known that 10% girls out of 430 study in class XII. What is the probability that a student chosen randomly studies in class XII if given that the chosen student is a girl ?

13. A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once ?

14. Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Find the conditional probability of the event that 'the die shows a number greater than 4' if given that 'there is at least one tail'.

16.04 Multiplication theorem on probability

Let A and B be two events associated with a sample space S. Clearly, the set A denotes the event that both A and B have occured. In other words $A\cap B$ denotes the simultaneous occurence of the events A and B. The event $A\cap B$ is also written as AB

We know that the conditional probability of event A given that B has occurred is

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)};\ P(B)\neq 0$$

$\Rightarrow$ $$P(A\cap B)=P(B)P\left(\frac{A}{B}\right) \qquad \text{(i)}$$

again $$P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)};\ P(A)\neq 0$$

or $$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)} \qquad [\because\ B\cap A=A\cap B]$$

$\therefore$ $$P(A\cap B)=P(A)P\left(\frac{B}{A}\right) \qquad \text{(ii)}$$

from (i) and (ii) $$P(A\cap B)=P(A)P\left(\frac{B}{A}\right)=P(B)P\left(\frac{A}{B}\right), \text{where } P(A)\neq 0 \ \text{ and } \ P(B)\neq 0$$

The above result is known as the *multiplication rule of probability.*

Note : Let A, B and C be any events of sample space then

$$P(A\cap B\cap C)=P(A)P\frac{B}{A}P\frac{C}{A\cap B}$$

$$=P(A)P\left(\frac{B}{A}\right)P\left(\frac{C}{AB}\right)$$

Thus the above expression denotes the multiplication rule of probability for more than two events

Illustrative Examples

Example 7. An urn contains 10 white and 15 black balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that first ball is white and second is black.

Solution : Let A and B denote respectively the events that ball drawn is white and second ball drawn is black then we have to find $(A\cap B)$

Now $P(A)=P$ (white ball in first draw) $=\frac{^{10}C_1}{^{25}C_1}=\frac{10}{25}$

Also given that the first ball drawn is white, i.e. event A has occured, now there are 9 white balls and fifteen black balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is white, is nothing but the conditional probability of B given that A has occured. i.e.

$\therefore$ $$P\left(\frac{B}{A}\right)=\frac{^{15}C_1}{^{24}C_1}=\frac{15}{24}$$

By multiplication rule of probability, we have

$$P(A\cap B)=P(A)P\left(\frac{B}{A}\right)=\frac{10}{25}\times\frac{15}{24}=\frac{1}{4}.$$

Example 8. Three cards are drawn successively, without replacement form a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is a queen ?

Solution : Let K denotes the event that the card drawn is king and Q be the event that the card drawn is a queen.

Clearly, we have to find $P(KKQ)$

Now $P(K) = P$ (card drawn is a king) $= 4/52$

Now there are three kings in (52-1) = 51 cards.

$\therefore$ $P\left(\frac{K}{K}\right) = P$ (the probability of second king with the condition that one king has already been drawn) $= \frac{3}{51}$

Now there are four queens left in 50 cards.

$\therefore$ $P\left(\frac{Q}{KK}\right) = P$ (the probability of third drawn card to be a queen, with the condition that two kings have already been drawn $= \frac{4}{50}$

By multiplication rule of probability, we have

$$P(KKQ) = P(K)P\left(\frac{K}{K}\right)P\left(\frac{Q}{KK}\right)$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}.$$

16.05 Independent Events

If A and B are two events such that the probability of occurence of one of them is not affected by occurence of the other. Such events are called *independent events*

Two events A and B are said to be independent, if

$$P\left(\frac{A}{B}\right) = P(A) \quad \text{when} \quad P(B) \neq 0$$

and

$$P\left(\frac{B}{A}\right) = P(B) \quad \text{when} \quad P(A) \neq 0$$

Now, by the multiplication rule of probability, we have

$$P(A \cap B) = P(A)P\left(\frac{B}{A}\right)$$

If A and B are independent, then

$$P(A \cap B) = P(A)P(B)$$

Note : Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A)P(B)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap C) = P(A)P(C)$$

and $$P(A\cap B\cap C)=P(A)P(B)P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent

Example : An unbaised die is thrown twice. Let the event A be 'odd number on the first throw' and B the event' odd number on the second throw'. Check the independence of the events A and B.

The sample space in tossing two coins is

$$S=\begin{Bmatrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{Bmatrix}$$

$\Rightarrow$ $$n(S)=36$$

Also getting an odd number on the first throw we have

$$n(A)=18$$

$$P(A)=\frac{18}{36}=\frac{1}{2}$$

similarly $$P(B)=\frac{18}{36}=\frac{1}{2}$$

and $$P(A\cap B)=P(\text{getting odd number on both the throws})=\frac{9}{36}=\frac{1}{4}$$

{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)} [be the sample points w.r. to event A and event B.]

clearly $$P(A\cap B)=1/4=1/2\times 1/2=P(A)P(B)$$

Thus A and B are independent events.

Illustrative Examples

Example 9. Events A and B are such that $P(A)=1/2,\ P(B)=7/12$ and P(A – not or B – not) $=1/4$ then are A and B independent events ?

Solution : Given $$P(A)=\frac{1}{2},\ P(B)=\frac{7}{12},\ P(\bar{A}\cup\bar{B})=\frac{1}{4}$$

$$P(\bar{A}\cup\bar{B})=\frac{1}{4}$$

$\Rightarrow$ $$P(\overline{A\cap B})=\frac{1}{4}\qquad\left[\because P(\bar{A}\cup\bar{B})=P(\overline{A\cap B})\right]$$

$\Rightarrow$ $$1-P(A\cap B)=\frac{1}{4}\qquad\left[\because P(\overline{A\cap B})=1-P(A\cap B)\right]$$

$\Rightarrow$ $$P(A \cap B) = 1 - 1/4 = 3/4$$

also $$P(A)P(B) = 1/2 \times 7/12 = 7/24$$

$\because$ $$P(A \cap B) \neq P(A)P(B)$$

therefore A and B are not independent events.

Example 10. A fair coin and an unbaised die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Solution : The sample space related to the experiment is -

$$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$$

and $A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\}$, $B = \{(H, 3), (T, 3)\}$

$\therefore$ $$A \cap B = \{(H, 3)\}$$

$$P(A) = \frac{6}{12} = \frac{1}{2}, \quad P(B) = \frac{2}{12} = \frac{1}{6}, \quad P(A \cap B) = \frac{1}{12}$$

clearly $$P(A \cap B) = P(A)P(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Therefore A and B are independent events.

Example 11. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the numeber is even,' and B be the event,'the number is red'. Are A and B independent ?

Solution : The sample space in rolling a die once is $= \{1, 2, 3, 4, 5, 6\}$

then $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ also $A \cap B = \{2\}$

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}$$

clearly $$P(A \cap B) = \frac{1}{6} \neq P(A).P(B).$$

Therefore A and B are not independent events.

Example 12. A die is thrown. If A is the event 'the number apperaring is a multiple of 3' and B be the event ' the number appearing is even' then find whether A and B are independent ?

Solution : The sample space in rolling a die once is $= \{1, 2, 3, 4, 5, 6\}$

then $A = \{3, 6\}$, $B = \{2, 4, 6\}$ and $A \cap B = \{6\}$

$$P(A) = \frac{2}{6} = \frac{1}{3}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}$$

clearly $$P(A \cap B) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = P(A)P(B)$$

Thus events A and B are independent events.

Example 13. Events A and B are such that $P(A)=1/2,\ P(A\cup B)=3/5$ and $P(B)=r$ then find r if ,

(i) the events are mutually exculsively

(ii) the events are independent

Solution : (i) If events A and B are mutually exclusively then

$$P(A\cup B)=P(A)+P(B)$$

$$3/5=1/2+r \quad\Rightarrow\quad r=1/10$$

(ii) If events A and B are independent events

$$P(A\cap B)=P(A)P(B)=(1/2)r$$

Given $$P(A\cup B)=3/5$$

$$\Rightarrow \quad P(A)+P(B)-P(A\cap B)=3/5$$

$$\Rightarrow \quad 1/2+r-P(A\cap B)=3/5$$

$$\Rightarrow \quad 1/2+r-(1/2)r=3/5$$

$$\Rightarrow \quad 1/2+(1/2)r=3/5$$

$$\Rightarrow \quad r/2=3/5-1/2$$

$$\Rightarrow \quad r=1/5$$

Example 14. Three coins are tossed simultaneously. Consider the event A 'three heads or three tails', B' at least two heads' and C' at most two heads'. Of the pairs (A,B), (A,C) and (B,C), which are independent ? which are dependent ?

Solution : The sample space of tossing three coins is

$$S=\{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$$

then $$A=\{HHH,\ TTT\},\quad B=\{HHT,\ HTH,\ THH,\ HHH\}$$

and $$C=\{TTT,\ TTH,\ THT,\ HTT,\ THH,\ HTH,\ HHT\}$$

Also

$$A\cap B=\{HHH\},\quad A\cap C=\{TTT\}\ \text{ and }\ B\cap C=\{HHT,\ HTH,\ THH\}$$

$$P(A)=2/8=1/4,\qquad P(B)=4/8=1/2,\qquad P(C)=7/8$$

$$P(A\cap B)=1/8,\qquad P(A\cap C)=1/8,\quad P(B\cap C)=3/8$$

Clearly $P(A\cap B)=P(A)P(B)=1/4\times 1/2=1/8$

similarly $$P(A\cap C)\neq P(A)P(C)$$

and $$P(B\cap C)\neq P(B)P(C)$$

Thus A and B are independent events and A and C, and B and C are dependent.

Example 15. If in a random experiment A and B are independent events then prove that

(i) $\overline{A}$ and B are dependent events.

(ii) A and $\overline{B}$ are independent events

(iii) $\overline{A}$ and $\overline{B}$ are also independent events

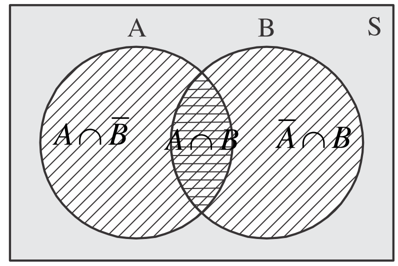

Fig. 16.02

Solution : Given A and B are independent events

$$P(A\cap B)=P(A)P(B)$$

It is clear form the venn diagram that $A\cap B$ and $\overline{A}\cap B$ are mutually exclusive such that

$$(A\cap B)\cup(\overline{A}\cap B)=B$$

By addition theroem of Probability

$$P(B)=(A\cap B)+P(\overline{A}\cap B)$$

$$\Rightarrow \quad P(\overline{A}\cap B)=P(B)-P(A\cap B)$$

$$=P(B)-P(A)P(B) \qquad [\because P(A\cap B)=P(A)P(B)]$$

$$=P(B)[1-P(A)]$$

$$=P(B)P(\overline{A})$$

$$=P(\overline{A})P(B)$$

Therefore $\overline{A}$ and B are independent events.

(ii) It is clear form the venn diagram that $A\cap B$ and $A\cap\overline{B}$ are mutually exclusive events such that

$$(A\cap B)\cup(A\cap\overline{B})=A$$

By addition theroem of probability

$$P(A)=P(A\cap B)+P(A\cap\overline{B})$$

$$\Rightarrow \quad P(A\cap\overline{B})=P(A)-P(A\cap B)$$

$$=P(A)-P(A)P(B)$$

$$=P(A)[1-P(B)]$$

$$=P(A)P(\overline{B})$$

Therefore A and $\overline{B}$ are independent events

(iii) $$P(\overline{A}\cap\overline{B})=P(\overline{A\cup B})$$

$$=1-P(A\cup B)$$

$$P(\overline{A}\cap\overline{B})=1-[P(A)+P(B)-P(A\cap B)]$$
$$=1-[P(A)+P(B)-P(A)P(B)]$$
$$=1-P(A)-P(B)+P(A)P(B)$$
$$=[1-P(A)]-P(B)[1-P(A)]$$
$$=[1-P(A)][1-P(B)]$$
$$=P(\overline{A})P(\overline{B})$$

Therefoere $\overline{A}$ and $\overline{B}$ are independent events

Example 16. If A and B are two independent events, then find the probability of occurence of at least one of A and B.

Solution : P (at least one of A and B) $=P(A\cup B)$

$$=P(A)+P(B)-P(A\cap B)$$
$$=P(A)+P(B)-P(A)P(B)$$ [$\because$ Events A and B are indepndent]
$$=P(A)+P(B)[1-P(A)]$$
$$=P(A)+P(B)P(\overline{A}) \qquad [\because P(A)+P(\overline{A})=1]$$
$$=1-P(\overline{A})+P(B)P(\overline{A})$$
$$=1-P(\overline{A})[1-P(B)]$$
$$=1-P(\overline{A})P(\overline{B})$$

Exercise 16.2

1. If A and B are two events such that $P(A)=1/4$, $P(B)=1/2$ and $P(A\cap B)=1/8$, then find $P(\overline{A}\cap\overline{B})$
2. If $P(A)=0.4$, $P(B)=p$ and $P(A\cup B)=0.6$ and A and B are independent events then find the value of p.
3. If A and B are independent events and $P(A)=0.3$ and $P(B)=0.4$ then find

(i) $P(A\cap B)$ (ii) $P(A\cup B)$ (iii) $P\left(\frac{A}{B}\right)$ (iv) $P\left(\frac{B}{A}\right)$

4. If A and B are independent events and $P(A)=0.3,\ P(B)=0.6$ then find

(i) $P(A\cap B)$ (ii) $P(A\cap \bar{B})$ (iii) $P(A\cup B)$ (iv) $P(\bar{A}\cap \bar{B})$

5. A bag contains 5 white, 7 Red and 8 black balls. If four balls are drawn without replacement then find the probability that all are white.
6. If a coin is tossed thrice then find the probability of getting an odd number atleast once.
7. Two cards are drawn without replacement form a well shuffled pack of 52 cards Find the probability that both are black.
8. Two coins are tossed. Find the probability of getting two heads when it is known that one Head has already occured.
9. In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.
   (i) Find the probability that she reads neither Hindi nor English newspapers.
   (ii) If she reads Hindi newspaper, find the probability that she reads English newspaper.
   (iii) If she reads English newspaper, find the probability that she reads Hindi newspaper.
10. A, solves 90% of the problems of the book and B, solves 70 % of the problems of the same book. If a question is taken at random then find the probability that at least one of them solve the question.
11. Three students are given the mathematical question to solve. Probability of solving the problem by the three are 1/2, 1/3 and 1/4. What is the probability that the question will solved ?
12. A bag contains 5 white and 3 black balls. Four balls are drawn one by one without replacements. Find the probability that the balls are of different colors.
13. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that
   (i) the problem is solved
   (ii) exacty one of them solves the problem

16.06 Partition of a Sample Space

A set of events $E_1, E_2,..., E_n$ is said to represent a partition of the sample space S if

(i) $E_i \cap E_j = \phi,\ i = j,\ i,\ j = 1, 2, 3, ..., n$

(ii) $E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = S$ and

(iii) $P(E_i) > 0,$ for all $i = 1, 2, ..., n$

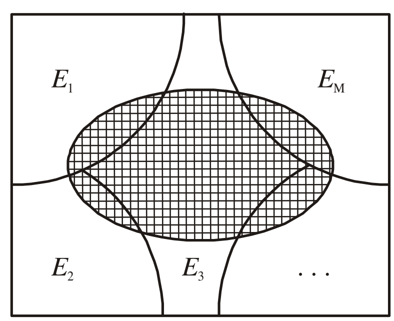

Fig. 16.03

In other words , the events $E_1, E_2,..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have non zero probabilities.

Example : As an example, we see that any non empty event E and its complement E form a partition of the sample space S since they satisfy

$$E \cap E' = \phi \quad \text{and} \quad E \cup E' = S.$$

16.07 Theorem on Total Probability

Statement : Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, ..., E_n$ has non zero probability of occurence. Let A be any event associated with S, then

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + ... + P(E_n)P\left(\frac{A}{E_n}\right)$$

$$= \sum_{j=1}^{n} P(E_j)P\left(\frac{A}{E_j}\right)$$

Statement : Given that $E_1, E_2, ..., E_n$ is a partition of the sample space S. Therefore

$\therefore \quad S = E_1 \cup E_2 \cup ... \cup E_n \quad (1)$

and $\quad E_i \cap E_j = \phi \ \ \forall \ i \neq j, \ i, \ j = 1, 2, ..., n$

for any event A

$$A = A \cap S$$

$$= A \cap (E_1 \cup E_2 \cup ... \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)$$

$\because \quad A \cap E_i$ and $A \cap E_j$ are the subsets of set E_i and E_j which are also disjoin for $i \neq j$

$\therefore \quad$ for $i \neq j, \ i, \ j = 1, 2, ..., n$, $A \cap E_i$ and $A \cap E_j$ are also disjoint.

$\therefore \quad P(A) = P\left[(A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)\right]$

$$= P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_n)$$

now $\quad P(A \cap E_i) = P(E_i)P\left(\frac{A}{E_i}\right), \qquad \left[\because \ P(E_i) \neq 0 \ \ \forall \ \ i = 1, 2, ..., n\right]$

Now, by multiplication rule of probability, we have

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + ... + P(E_n)P\left(\frac{A}{E_n}\right)$$

$$\Rightarrow \quad P(A) = \sum_{j=1}^{n} P(E_j)P\left(\frac{A}{E_j}\right).$$

Illustrative Examples

Example 17. In a class two- third of the students are boys and remainig are girls. Probability of a girl securing first division is 0.25 whereas probability of a boy securing first division is 0.28. A student is selected at random, find the probability that he or she gets first division.

Solution : Let event E_1 denotes ' a boy is selected' and event E_2 deontes ' a girl is selected' and let event A represent ' a student gets first divisions '.

then $$P(E_1)=2/3, P(E_2)=1/3$$

and $$P\left(\frac{A}{E_1}\right)=0.28, \quad P\left(\frac{A}{E_2}\right)=0.25$$

using theroem of total probability

$$P(A)=P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)=\frac{2}{3}\times 0.28+\frac{1}{3}\times 0.25=0.27$$

16.08 Baye's Theorem

Famous mathematician, John Baye's solved the problem of finding inverse probability by using conditional probability. The formula developed by him is known as '*Baye's theroem'* which was published posthumously in 1763

Statement : If $E_1, E_2, ..., E_n$ are n non empty events which constitute a partition of sample space S, ie., $E_1, E_2, ..., E_n$, are pairwise disjoint and $E_1 \cup E_2 \cup, ..., \cup E_n = S$ and A is any event of non zero probability, then

$$P\left(\frac{E_i}{A}\right)=\frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^{n}P(E_j)P\left(\frac{A}{E_j}\right)}, \quad i=1, 2, 3, ..., n$$

Proof : By formula of conditional probability, we know that

$$P\left(\frac{E_i}{A}\right)=\frac{P(A\cap E_j)}{P(A)}=\frac{P(E_i)P\left(\frac{A}{E_i}\right)}{P(A)} \qquad \text{(by multiplication rule of probability)}$$

$$=\frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{j=1}^{n}P(E_j)P\left(\frac{A}{E_j}\right)} \qquad \text{(by the result of theorem of total probability)}$$

Illustrative Examples

Example 18. In a factory which manufactures bolts, machines A, B and C maufacture respectively 25%, 35%, and 40% of the bolts. Of their outputs 5, 4, and 2 percent are respectively defective bolts. A bolt is drawn at random form the product and is found to be defective. What is the probability that it is manufactured by the machine B ?

Solution : Let events B_1, B_2 and B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space. Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that

$$P(B_1) = 25\% = \frac{25}{100} = 0 \cdot 25$$

$$P(B_2) = 35\% = \frac{35}{100} = 0 \cdot 35$$

and $$P(B_3) = 4\% = \frac{40}{100} = 0 \cdot 40$$

Again $P\left(\frac{E}{B_1}\right) =$ Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 0.05

Similarly,

$$P\left(\frac{E}{B_2}\right) = 0 \cdot 04, \ P\left(\frac{E}{B_3}\right) = 0 \cdot 02$$

Hence, by Baye's Theorem, we have

$$P\left(\frac{B_2}{E}\right) = \frac{P(B_2) P\left(\frac{E}{B_2}\right)}{P(B_1) P\left(\frac{E}{B_1}\right) + P(B_2) P\left(\frac{E}{B_2}\right) + P(E_3) P\left(\frac{E}{B_3}\right)}$$

$$= \frac{0 \cdot 35 \times 0 \cdot 04}{0 \cdot 25 \times 0 \cdot 05 + 0 \cdot 35 \times 0 \cdot 04 + 0 \cdot 40 \times 0 \cdot 02}$$

$$= \frac{0 \cdot 0140}{0 \cdot 0345} = \frac{28}{69}$$

Example 19. Given three identical boxex I, II, and III each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold.

Solution : Let E_1, E_2, E_3 be the events that boxes I, II, and III are chosen, respectively

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Also, let A be the event that ' the coin drawn is of gold'

P (a gold coin form bag I) = $P\left(\frac{A}{E_1}\right) = \frac{2}{2} = 1$

P (a gold coin form bag II) $P\left(\frac{A}{E_2}\right) = 0$

P(a gold coin from bag III) $P\left(\frac{A}{E_3}\right) = \frac{1}{2}$

Now, the probability that the other coin in the box is of gold = the probability that gold coin is drawn form the box I.

$$P\left(\frac{E_1}{A}\right)$$

By Bayes' theroem, we know that

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$= \frac{1/3 \times 1}{1/3 \times 1 + 1/3 \times 1/2}$$

$$= \frac{1/3}{1/3 + 1/6} = \frac{1/3}{2+1/6} = \frac{1/3}{3/6} = \frac{1}{3} \times \frac{6}{3} = \frac{2}{3}$$

Example 20. A man is know to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Solution : Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur Then

Probability that six occurs $= P(S_1) = \frac{1}{6}$

Probability that six does not occur $= P(S_2) = \frac{5}{6}$

Probability that the man reports that six occurs when six has actually occured on the die = Probability that the man speaks the truth =

$$= P\left(\frac{E}{S_1}\right) = \frac{3}{4}$$

Probability that the man reports that six occurs when six has not actually occured on the die = Probability that the man does not speak the truth

$$= P\left(\frac{E}{S_2}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

Thus, by Baye's theroem, we get

Probability that the report of the man that six has occured is actually a six

$$= P\left(\frac{S_1}{E}\right) = \frac{P(S_1)P\left(\frac{E}{S_1}\right)}{P(S_1)P\left(\frac{E}{S_1}\right) + P(S_2)P\left(\frac{E}{S_2}\right)}$$

$$= \frac{1/6 \times 3/4}{1/6 \times 3/4 + 5/6 \times 1/4} = \frac{3/24}{3/24 + 5/24} = \frac{3/24}{8/24}$$

$$= \frac{3}{24} \times \frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is 3/8

Example 21. Suppose that the reliability of a HIV test is specified as follows: Of perople having HIV, 90 % of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV -ive but 1% are diagnosed as showing HIV +ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ive. What is the probability that the person actually has HIV ?

Solution : Let E denotes the event that the person selected is actually having HIV and A the event that the person's HIV test is diagnosed as +ive. We need to find $P\left(\frac{E}{A}\right)$

Also E' denotes the event that the person selected is actually not having HIV. Clearly , $\{E, E'\}$ is a partition of the sample space of all people in the population. We are given that

$$P(E) = 0 \cdot 1\% = \frac{0 \cdot 1}{100} = 0 \cdot 001$$

$$P(E') = 1 - P(E) = 1 - 0 \cdot 001 = 0 \cdot 999$$

P(Person tested as HIV +ive given that he/she is actually having HIV)

$$P\left(\frac{A}{E}\right) = 90\% = \frac{9}{10} = 0 \cdot 9$$

P(Person tested as HIV+ive given that he/she is actually not having HIV)

$$P\left(\frac{A}{E'}\right)=1\%=\frac{1}{100}=0\cdot 01$$

Now, by Baye's theorem

$$P\left(\frac{E}{A}\right)=\frac{P(E)P\left(\frac{A}{E}\right)}{P(E)P\left(\frac{A}{E}\right)+P(E')P\left(\frac{A}{E'}\right)}$$

$\therefore$
$$P\left(\frac{E}{A}\right)=\frac{0\cdot 001\times 0\cdot 9}{0\cdot 001\times 0\cdot 9+0\cdot 999\times 0\cdot 01}$$

$$=\frac{90}{1089}=0\cdot 083 \text{ approx.}$$

Thus, the probability that a person selected at random is actually having HIV given that he/she is tested HIV +ive is 0.083.

Exercise 16.3

1. Bag I contain 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random form one of the bags and it is found to be red. Find the probability that it was drawn form Bag II.
2. A doctor is to visit a patient. From the past experience, it is known that the probabilites that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$ The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{12}$, if he comes by train, bus and scooter respectively, but if he comes by he comes by train ?
3. Bag I contains 3 Red and 4 black balls while Bag II contains 4 Red and 5 black balls. One ball is transfered from Bag I to Bag II and then a ball is drawn form Bag II and it was found to be Red. Find the probability that the transfered ball is black.
4. A bag contains 3 Red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.
5. There are three coins. One is a two headed coin (having head on both faces), another is a baised coin that comes up heads 75% of the time and third is an unbaised coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?
6. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

7. Students in a college, it is known that 60% reside in hostel and 40% are day scholar (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% if day scholar attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probabilioty that the student is a hostlier ?
8. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilioty of accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meet with an accident, What is the probability that he is a scooter driver ?
9. In answering a question on a multiple choice test, a student either knows the answer or gusses. Let 3/4 be the probability that he knows the answer and 1/4 be the probability that he gusses. Assuming that a student who guesses at the answer will be correct with probability 1/4 What is the probability that the student know the answer given that he answered it correctly ?
10. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male ? Assume that there are equal number of males and females.
11. Two groups are competing for the position on the Board of directors of corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins . Find the probability that the new product introduced was by the second group.
12. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin three times and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3, or 4 with the die ?
13. A card form a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.
14. A bag contains 3 Red and 7 Black balls. Two balls are selected at random without replacement. If the second drawn ball is Red then what is the probability that the first ball drawn is also Red ?

16.09 Random variable and its Probability Distribution

We have already learnt about random experiments and formation of sample spaces. Sample spaces are set of all possible results of any random experiment. The result of any random experiments may be numerical or non-numerical. In most of these experiments, we were not only interested in the particular outcome that occurs but rather in some number associated with that outcomes as shown in following example / experiments.

(*i*) In tossing two dice, we may be interested in the sum of the number on the two dice.

(*ii*) In tossing a coin 50 times, we may be interested in the sum of the number of heads obtained.

(*iii*) In the experiment of taking out four articles (one after the other) at random from a lot of 20 articles in which 6 are defective, we want to know the number of defective in the sample of four and not in the particular sequence of defective and non defective articles. In all of the above experiments, we have a rule which assigns to each outcome of the experiment a single real number. This single real number may vary with different outcome of a random experiment and hence, is called random variable. A random variable is usually denoted by X. If you recall the defination of a function, you will realise that the random variable X is really speaking a function whose domain is the set of outcomes(or sample space) of a random experiment. A random variable can take any real value, therefore, its co-domain is the set of real numbers. Hence, a random variable can be defined as follows :

Definition : A random variable is a real valued function whose domain is the sample space of a random experiment

Random variables are generally expresses as X, Y, Z

For example, let us consider the experiment of tossing a coin three times in succession.

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below :

$$X(HHH) = 3,\ X(HHT) = 2 = X(HTH) = X(THH),$$

$$X(HTT) = 1 = X(THT) = X(TTH),\ X(TTT) = 0$$

NOTE : more than one random variables can be defined on the same sample space

Random variables are of two types :

(i) Discrete Random variable

(ii) Continuous Random variable

(i) Discrete Random variable : If a random variable takes a finite or infinite value then that variable is called as discrete random variable. For example -

(a) number of students in a class.

(b) the number of printed errors in a book

(c) the number of complaints received in an office

(ii) Continuous Random variable : If a random variable takes all the values in a fixed interval then it is called as continuous random variable for example -

(a) height of a person

(b) $X = \{x \in R : 0 < x < 1\}$ etc.

NOTE : In this chapter random variable means discrete random variable only.

16.10 Probability distribution of a Random Variable

Probability distribution of a random variable is discription of collection of all possible results and probability related to them. The probability distribution of a random variable X is the system of numbers

$$X = x : x_1 \quad x_2 \quad x_3 \ldots x_n$$

$$P(X) : p_1 \quad p_2 \quad p_3 \ldots p_n$$

$$\text{where } p_i > 0 \text{ and } \sum_{i=1}^{n} p_i = 1; \quad i = 1, 2, \ldots, n$$

The real numbers $x_1, x_2, x_3, \ldots, x_n$ are the possible values of the random variable X with possible probabilities $p_1, p_2, p_3, \ldots, p_n$ etc.

For example, let us consider the experiment of tossing a coin two times in succession. The sample space of the experiment is

$$S = \{HH, HT, TH, TT\}$$

If X denotes the number of heads obtained, then X is a random variable and for each outcome, its value is as given below :

$$X(HH) = 2,\ X(HT) = 1 = X(TH),\ X(TT) = 0$$

Here X tales the values 0, 1 and 2 whose corresponding probabilities are $1/4$, $2/4$ and $1/4$, thus the proabbility distribution is

$X = x$	:	0	1	2
$P(X)$	:	$1/4$	$2/4$	$1/4$

where $p_i > 0$ and $\sum p_i = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1,$

Illustrative Examples

Example 22. The probability distribution of a random variable X is given below :

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find

(i) k (ii) $P(X < 6)$ (iii) $P(X \geq 6)$ (iv) $P(0 < X < 5)$

Solution : (i) The sum of probabilities in a probability distribution is always 1. Therefore

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)+P(X=7)=1$$

$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

or $\quad (10k - 1)(k + 1) = 0$

or $\quad 10k - 1 = 0 \qquad [\because k \geq 0]$

$\Rightarrow \quad k = \frac{1}{10}$

(ii) $\quad P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2$

$\Rightarrow \quad k^2 + 8k$

$\Rightarrow \quad (1/10)^2 + 8(1/10) = \frac{81}{100}$

(iii) $\quad P(X \geq 6) = P(X=6) + P(X=7)$

$\Rightarrow \quad 2k^2 + 7k^2 + k$

$\Rightarrow \quad 9k^2 + k$

$\Rightarrow \quad 9(1/10)^2 + 1/10 = \frac{19}{100}$

(iv) $\quad P(0 < X < 5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$\Rightarrow \quad k + 2k + 2k + 3k = 8k$

$\Rightarrow \quad 8/10 = 4/5$

Example 23. Three cards are drawn succesively with replacement form a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Solution : The number of aces is a random variable. Let it be denoted by X. Clearly, X can take the values 0, 1, 2 or 3.

$$P(X=0)=\text{P (non-ace and non-ace)}=\frac{^{48}c_3}{^{52}c_3}=\frac{4324}{5525}$$

$$P(X=1)=\text{P (ace and two non-ace)}=\frac{^{4}c_1\times{}^{48}c_2}{^{52}c_3}=\frac{1128}{5525}$$

$$P(X=2)=\text{P (two ace and one non-ace)}=\frac{^{4}c_2\times{}^{48}c_1}{^{52}c_3}=\frac{72}{5525}$$

$$P(X=3)=\text{P (ace and ace and ace)}=\frac{^{4}c_3}{^{52}c_3}=\frac{1}{5525}$$

Thus, the required probability distribution is

X	0	1	2	3
$P(X)$	$\frac{4324}{5525}$	$\frac{1128}{5525}$	$\frac{72}{5525}$	$\frac{1}{5525}$

Example 24. Let X denote the number of hours you study during a randonly selected school day. The probability that X can take the values *x*, has the following form, where *k* is some unkonwn constant.

$$P(X=x)=\begin{cases} 0\cdot1 & ; \quad \text{If } x=0 \\ kx & ; \quad \text{If } x=1 \text{ or } 2 \\ k(5-x) & ; \quad \text{If } x=3 \text{ or } 4 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(i) Find the value of *k*.

(ii) What is the probability that

(a) you study at least two hours ?

(b) Exactly two hours ?

(c) At most two hours ?

Solution : The probability disrtibution of X is

X	:	0	1	2	3	4
$P(X)$	:	$0\cdot1$	k	$2k$	$2k$	k

(*i*) We know that

$$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$$

$$0\cdot1+k+2k+2k+k=1$$

$$\Rightarrow \quad 6k=0\cdot9$$

$$\text{or} \quad k=0\cdot15$$

(*ii*) (a) required probability

when $P(X \geq 2) = P(X=2) + P(X=3) + P(X=4)$

$$= 2k + 2k + k = 5k = 5 \times 0{\cdot}15 = 0{\cdot}75.$$

(b) required probability

when $P(X=2) = 2k = 2 \times 0{\cdot}15 = 0{\cdot}30.$

(c) required probability

when $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= 0{\cdot}1 + k + 2k = 3k + 0{\cdot}1$$

$$= 0{\cdot}1 + 3 \times 0{\cdot}15 = 0{\cdot}55$$

Example 25. A coin is baised so that the head is 3 times as likely to occur as tail. If the coin is tosses twice, find the probability distribution of number of tails.

Solution : Let p denotes the probability of getting tail in tossing a coin once then probability of getting head will $3p$

Thus getting " number of head " and " number of tails " are mutually exclussive and exhaustive events

$$P(H) + P(T) = 1$$

$\Rightarrow$ $3p + p = 1$

or $p = 1/4$

$\therefore$ $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$

Let X denote the number of tails in tossing a coin twice then X will take values 0, 1 and 2

$P(X=0) =$ P (not getting Tail)

= P (getting both Heads)

= P(HH)

= P(H) P(H) {$\because$ both are independent}

$$= \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

$P(X=1) =$ P (getting one Tail and one Head) = P(HT) + P(TH)

= P(H) P(T) + P(T) P(H)

$$= \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{3}{8}$$

$P(X=2) =$ P(getting both Tails)

$$= P(TT) = P(T)\,P(T) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Now the probability distribution of X

X :	0	1	2
$P(X)$:	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

16.11 Mean of a Random Variable

In many problems, it is desirable to describe some feature of the random variable by means of a single number that can be computed form its probability distribution. Few such numbers are mean, median and mode. In this section, we shall discuss mean only. Mean is a measure of location or central tendency in the sense that it roughly locates a *middle* or *average value* of the random varible

Let X be a random variable whose possible values $x_1, x_2, x_3, ..., x_n$ occur with probabilities $p_1, p_2, ..., p_n$ respectively.

The mean of a random variable X is also called the expection of X, denoted by E(X).

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + ... + x_n p_n$$

The mean of X, denoted by μ is the number $\sum_{i=1}^{n} x_i p_i$ i.e. the mean of X is the weighted average of the possible values of X, each being weighted by its probability with which it occurs.

Let a dice be thrown and the random variable X be the number that appears on the dice. Find the mean or expectation of X.

The sample space is $= \{1, 2, 3, 4, 5, 6\}$

Now the probability distribution with random variable X–

$X = x$	:	1	2	3	4	5	6
$P(x)$	:	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\therefore \quad \mu = E(X) = \sum x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 + x_5 p_5 + x_6 p_6$$

$$= 1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6$$

$$= 21/6 = 7/2$$

NOTE : This does not mean at all that in the experiment of tossing a coin we get the number 7/2. This number indicates that if the coin is tossed for longer period then the number we get in average tossing is 7/2

Illustrative Examples

Example 26. Three coins are tossed, If X denotes the number of Heads then find the mean or expectations of X

Solution : Here X takes the values 0, 1, 2, and 3

$$P(X = 0) = P(TTT) = \frac{1}{8}$$

$$P(X=1)=P\text{ (HTT या TTH या THT)}=\frac{3}{8}$$

$$P(X=2)=P\text{(HHT या THH या HTH)}=\frac{3}{8}$$

and $$P(X=3)=P(HHH)=\frac{1}{8}$$

Probability distribution of variable X is–

$X=x$	0	1	2	3
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Mean of $$\text{X } = \overline{X} = E(X) = \sum x_i p_i$$

$$= 0\times 1/8 + 1\times 3/8 + 2\times 3/8 + 3\times 1/8 = 12/8 = 3/2$$

Example 27. Two cards are drawn simultaneously (or successively with replacement) from a well shuffled pack of 52 cards. Find the mean and probability of the number of aces.

Solution : Let X denote the number of aces.

Variable X take the values 0, 1 and 2

P(X = 0) = P (not getting an ace)

= P (no ace and no ace) =P (no ace) . P (no ace)

$= 48/52 \times 48/52 = 144/169$

P(X = 1) = P (getting an ace)

= P(ace and no ace or no ace and ace)

= P(ace) P(no ace) + P(no ace) P(ace)

$= 4/52 \times 48/52 + 48/52 \times 4/52 = 24/169$

P(X = 2) = P(gettting both the aces ace)

=P(ace and ace)

=P(ace) P(ace)

$= 4/52 \times 4/52 = 1/169$

Probability distribution of variable X-

X	:	0	1	2
$P(X)$	:	$144/169$	$24/169$	$1/169$

Mean $$= \overline{X} = E(X) = \sum x_i p_i$$

$$= 0\times 144/169 + 1\times 24/169 + 2\times 1/169 = 26/169.$$

16.12 Variance of a random variable

Let X be a random whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p_1, p_2, ..., p_n$ respectively then variance of X is given by var(X) or σ_X^2

$$\sigma_X^2 = \text{var}(X) = E(X-\mu)^2 = \sum_{i=1}^{n}(x_i-\mu)^2 p_i$$

The positive square root of variance as "$+\sqrt{\operatorname{var}(X)}$" is called as standard deviation

$$\sigma_X = +\sqrt{\operatorname{var}(X)} = +\sqrt{\sum_{i=1}^{n}(x_i-\mu)^2 p_i}$$

Alternative Formula to find Variance.

$$\operatorname{var}(X) = \sum_{i=1}^{n}(x_i-\mu)^2 p_i$$

$$= \sum_{i=1}^{n}\left(x_i^2+\mu^2-2\mu x_i\right)p_i$$

$$= \sum_{i=1}^{n}x_i^2 p_i + \sum_{i=1}^{n}\mu^2 p_i - 2\sum_{i=1}^{n}\mu x_i p_i$$

$$= \sum_{i=1}^{n}x_i^2 p_i + \mu^2\sum_{i=1}^{n}p_i - 2\mu\sum_{i=1}^{n}x_i p_i$$

$$= \sum_{i=1}^{n}x_i^2 p_i + \mu^2(1) - 2\mu(\mu)$$

$$= \sum_{i=1}^{n}x_i^2 p_i - \mu^2$$

$$= \sum_{i=1}^{n}x_i^2 p_i - \left(\sum_{i=1}^{n}x_i p_i\right)^2$$

$$\operatorname{var}(X) = E(X^2) - \{E(X)\}^2 \qquad \text{Where } E(X^2) = \sum_{i=1}^{n}x_i^2 p_i$$

For Example : Find the Variance of head in three tosses of a fair coin.

Solution : We have to find the variance of head in three tosses of a fair coin

The sample space S={HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Here X can take the values 0, 1, 2 and 3 whose probabilities are $1/8, 3/8, 3/8 \; ; 1/8$

The probability distribution of X is -

$X = x$	:	0	1	2	3
$P(X)$	:	1/8	3/8	3/8	1/8

Variance of X var (X) $= E(X^2) - [E(X)]^2$

where $$E(X) = \sum_{i=1}^{n}x_i p_i = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4$$

$$= 0\times1/8 + 1\times3/8 + 2\times3/8 + 3\times1/8 = 3/2$$

and $$E(X^2)=\sum_{i=1}^{n} x_i^2 p_i = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + x_4^2 p_4$$

$$=(0)^2\times 1/8+(1)^2\times 3/8+(2)^2\times 3/8+(3)^2\times 1/8$$

$$=0+3/8+12/8+9/8=3$$

$\therefore$ $$\text{var}(X)=E(X^2)-[E(X)]^2$$

$$=3-(3/2)^2=3-9/4=3/4.$$

Example 28. Two dice are thrown simultaneously. If X denotes the number of sixes, find the variance of X,

Solution : The Sample Space is tossing two coins is -

$$X=\begin{Bmatrix} (1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6) \\ (2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6) \\ (3,1),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6) \\ (4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6) \\ (5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6) \\ (6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6) \end{Bmatrix}$$

X can take the values 0, 1 and 2

$$P(X=0)=\text{P(not getting six)}=25/36$$

$$P(X=1)=\text{P(getting six on one dice)}=10/36$$

$$P(X=2)=\text{P(getting six on both the die)}=1/36$$

The probability distribution of variable X -

X	:	0	1	2
$P(X)$	:	25/36	10/36	1/36

$$E(X)=\sum_{i=1}^{n} x_i p_i = 0\times 25/36+1\times 10/36+2\times 1/36=12/36=1/3$$

$$E(X^2)=\sum_{i=1}^{n} x_i^2 p_i=(0)^2\times 25/36+(1)^2\times 10/36+(2)^2\times 1/36=14/36=7/18$$

$\therefore$ $$\text{var}(X)=E(X^2)-\{E(X)\}^2=\frac{7}{18}-\left(\frac{1}{3}\right)^2=\frac{5}{18}.$$

Example 29. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the variance

Solution : X takes the value 2, 3, 4, 5, 6

$P(\mathrm{X}=2)=$ P (getting a number greater than 2)

$= P$ (getting 1 and then 2) or (getting 2 and then 1)

$=1/6\times 1/5+1/6\times 1/5=2/30=1/15$

$P(\mathrm{X}=3)=$ P (getting a number greater than 3)

$= P$ (getting a number lesser than 3 and than 3) (getting 3 or a number lesser than 3)

$$=2/6\times 1/5+1/6\times 2/5=4/30=2/15$$

similarly $P(X=4)=3/6\times 1/5+1/6\times 3/5=6/30=1/5$

and $P(X=5)=4/6\times 1/5+1/6\times 4/5=8/30=4/15$

also $P(X=6)=5/6\times 1/5+1/6\times 5/5=10/30=1/3$

Thus the probability distribution of X -

X	:	2	3	4	5	6
$P(X)$	:	1/15	2/15	1/5	4/15	1/3

$$E(X)=\sum x_i p_i = 2\times 1/15+3\times 2/15+4\times 1/5+5\times 4/15+6\times 1/3=70/15=14/3$$

$$E(X^2)=\sum x_i^2 p_i = (2)^2\times 1/15+(3)^2\times 2/15+(4)^2\times 1/5+(5)^2\times 4/15+(6)^2\times 1/3$$

$$=4/15+18/15+16/5+100/15+36/3=350/15=70/3$$

$$\mathrm{var}(X)=E(X^2)-\{E(X)\}^2$$

$$=70/3-(14/3)^2=70/3-196/9=14/9.$$

Example 30. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the proabability distribution of the random variable X? Find mean, variance and standard deviation of X.

Solution : X can take values 14, 15, 16, 17,18, 19, 20 and 21

$\therefore\ P(X=14)=2/15,\quad P(X=15)=1/15,\quad P(X=16)=2/15,\quad P(X=17)=3/15,$

$P(X=18)=1/15,\quad P(X=19)=2/15,\quad P(X=20)=3/15,\quad P(X=21)=1/15$

The probability distribution of X -

X	:	14	15	16	17	18	19	20	21
$P(X)$	:	2/15	1/15	2/15	3/15	1/15	2/15	3/15	1/15

$Mean\ of\ X = E(X)=\sum x_i p_i$

$$=14\times 2/15+15\times 1/15+16\times 2/15+17\times 3/15+18\times 1/15+19\times 2/15+20\times 3/15+21\times 1/15$$

$$=263/15=17{\cdot}53$$

$$=E(X^2)=\sum x_i^2 p_i$$

$$=(14)^2\times(2/15)+(15)^2\times(1/15)+(16)^2\times(2/15)+(17)^2\times(3/15)+(18)^2\times(1/15)+(19)^2\times(2/15)+(20)^2\times(3/15)+(21)^2\times(1/15)$$

$$=\frac{392}{15}+\frac{225}{15}+\frac{512}{15}+\frac{867}{15}+\frac{324}{15}+\frac{722}{15}+\frac{1200}{15}+\frac{441}{15}=\frac{4683}{15}$$

$$\text{var}(X)=E(X^2)-\{\text{E}(X)\}^2$$

$$=\frac{4683}{15}-\left(\frac{263}{15}\right)^2=\frac{70245-69169}{225}=\frac{1076}{225}$$

$$\text{Standard Deviation }=\sqrt{\frac{1076}{225}}=2{\cdot}186$$

Exercise 16.4

1. State which of the following are not the probability distribution of a random variable. Give reasons for your answer.

(i)

X	:	0	1	2
$P(X)$	:	0.4	0.4	0.2

(ii)

X	:	0	1	2
$P(X)$	:	0.6	0.1	0.2

(iii)

X	:	0	1	2	3	4
$P(X)$	:	0.1	0.5	0.2	–.01	0.3

2. Find the probability distribution of number of heads in two tosses of a coin.
3. Four rotten oranges by mistake are mixed with 16 good oranges. Two oranges are drawn and found to be rotten, find the probability distribution.
4. An urn contains 4 white and 3 red balls. Three balls are drawn at random and found to be red, find the probability distribution.
5. From a lot of 10 object which includes 6 defective, a sample of 4 objects is drawn at random. If the random variable of defective objects is denoted as X, then find-

   (i) Probability distribution of X (ii) $P(X \le 1)$ (iii) $P(X < 1)$

   (iv) $P(0 < X < 2)$.
6. A die is rolled so that getting an even number is twice as likely to occur odd number. If a die is rolled twice then considering the random variable X as the square of the number , find the probability distribution.
7. An urn contains 4 white and 6 red balls. Four balls are drawn at random, find the probability distribution of number of white balls.
8. Find the probability distribution of getting a doublet in rolling two dice three times.
9. A pair of dice is rolled. Let X, the sum of the numbers on the dice. Find the mean of X.
10. Find the variance of the number obtained on a throw of an unbiased die.
11. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).
12. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings.

16.13 Bernoulli Trials

Each time we toss a coin or roll a die or perform any other experiments, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or faliure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred as 'success' or 'failure' are called *Bernoulli trials*

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes : success or failure.

(iv) The probability of success remains the same in each trial.

For example, throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results is success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent experiments.

16.14 Binomial Distribution

Let an experiments is repeated n times. Therefore it is an experiments of n-Bernoulli trials where every experiments is independent and let S and F denote respectively success and failure in each trial.

Let the probability of getting a success in an experiments is (p) and failure be $(q = 1 - p)]$ let in n- Bernoulli's trials experiment, the probability for r successes and (n - r) failure

$$P(X = r) = P(r\,\text{success}).P\left[(n-r)\,failure\right]$$

$$= P(\underbrace{SSS...S}_{r\,times}\ \underbrace{FFF...F}_{(n-r)\,times})$$

$$= P(S)P(S)P(S)...P(S)\quad P(F)P(F)P(F)...P(F)$$

$$= ppp....p\ \ qqq...q$$

$$P(X = r) = p^r q^{n-r}$$

This result shows r success and $(n-r)$ failure in an experiments but in n experiments r success can be found through ${}^nc_r$ procedures and probabilities of every procedure remains $p^r q^{n-r}$

Thus Probability of r *success in* n-Bernoulli's experiments is

$$P(X = r) = {}^nc_r p^r q^{n-r};\quad r = 0,1,2,....,n \text{ and } q = 1-p$$

The distribution of number of successes X *in* n-Bernouli's experiments is given by-

X	0	1	2	...	r	...	n
$P(X)$	${}^nC_0 p^0 q^{n-0} = {}^nC_0 q^n$	${}^nc_1 p^1 q^{n-1}$	${}^nc_2 p^2 q^{n-2}$	...	${}^nc_r p^r q^{n-r}$	...	${}^nc_n p^n q^{n-n} = {}^nc_n p^n$

The above probability distribution is known as *binomial distribution with parameters* n and p, because for given values of n and p, we can find the complete probability distribution.

A binomial distribution with n-Bernoulli trial and probability of success in each trial as p, is denote by B(n,p).

NOTE : $$\sum_{r=0}^{n} P(X = r) = \sum_{r=o}^{n} {}^nC_r p^r q^{n-r}$$

$$= {}^nC_0 p^o q^n + {}^nC_1 p^1 q^{n-1} + ... + {}^nC_n p^n q^{n-n} = (q+p)^n = 1.$$

Illustrative Examples

Example 31. A die is thrown 7 times. If 'getting a sum 7' is a success, what is the probability of (*i*) no success ? (*ii*) 6 successes? (*iii*) at least 6 successes? (*iv*) at most 6 successes?

Solution : Let the probability of getting a sum 7 be p then $p = 6/36 = 1/6$

[$\because$ there are six ways of getting a sum 7 on the dice]

$$(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$$

$$q = 1 - p = 1 - 1/6 = 5/6$$

Let the number of successes be X then

Let the number of successes be X then

$$P(X = r) = {}^7c_r (1/6)^r (5/6)^{7-r}; \quad r = 0,1,2,3,4,5,6,7$$

(i) P (no success) $= P(X = 0)$

$$= {}^7C_o \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{7-0} = \left(\frac{5}{6}\right)^7$$

(ii) P (6 successes) $= P(X = 6)$

$$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6} = \frac{35}{6^7}$$

(iii) P (atleast 6 successes) $= P(X \geq 6)$

$$= P(X = 6) + P(X = 7)$$

$$= {}^7C_6 \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{7-6} + {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^{7-7}$$

$$= 7\left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^7 = \frac{1}{6^5}$$

(iv) P (at most 6 successes) $= P(X \leq 6)$

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$= 1 - P(X > 6)$$

$$= 1 - P(X = 7)$$

$$= 1 - {}^7C_7 \left(\frac{1}{6}\right)^7 \left(\frac{5}{7}\right)^{7-7} = 1 - \left(\frac{1}{6}\right)^7.$$

Example 32. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

Solution : Let the probability of getting a number 6 is p then $p = \frac{1}{6}, \quad q = 1 - \frac{1}{6} = \frac{5}{6}$

Required probability = P (getting two 6 in first 5 throws). P (getting 6 in the sixth throw)

$$= \left({}^5C_2 p^2 q^{5-2}\right)(p) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \times \frac{1}{6}$$

$$= \frac{10 \times 125}{6^6} = \frac{625}{23328}.$$

Example 33. A fair coin is tossed 5 times. Find the probability of getting atleast 3 Heads.

Solution : Let the probability be p then $p = 1/2,\ q = 1/2$ Let X denote getting a number 5 then $n = 5$ and $p = 1/2$ such that

$$P(X = r) = {}^5C_r \left(\frac{1}{2}\right)^{5-r} \left(\frac{1}{2}\right)^r = {}^5C_r \left(\frac{1}{2}\right)^5 ; \text{ where } r = 0, 1, 2, 3, 4, 5$$

Required probability $= P$ (atleast 3 Heads)

$$= P(X \geq 3)$$

$$= P(X = 3) + (X = 4) + (X = 5)$$

$$= {}^5C_3 \left(\frac{1}{2}\right)^5 + {}^5C_4 \left(\frac{1}{2}\right)^5 + {}^5C_5 \left(\frac{1}{2}\right)^5$$

$$= \left({}^5C_3 + {}^5C_4 + {}^5C_5\right)\left(\frac{1}{2}\right)^5 = \left(\frac{10+5+1}{32}\right) = \frac{1}{2}$$

Example 34. A die is rolled 6 times. If getting an even number is a success than find the following probabilities -

(i) exactly 5 successes (ii) atleast 5 successes (iii) almost 5 successes

Solution : Let the probability be p then $p = 3/6 = 1/2$ and $q = 1 - p = 1 - 1/2 = 1/2$

and let $n = 6$ and $p = 1/2$ then -

$$P(X = r) = {}^6C_r \left(\frac{1}{2}\right)^{6-r} \left(\frac{1}{2}\right)^r = {}^6C_r \left(\frac{1}{2}\right)^6 ; \text{ where } r = 0, 1, 2, 3, 4, 5, 6$$

(i) P (exactly 5 successes) $= P(X = 5) = {}^6C_5 \left(\frac{1}{2}\right)^6 = \frac{3}{32}.$

(ii) P (atleast 5 successes) $= P(X \geq 5) = P(X = 5) + P(X = 6)$

$$= {}^6C_5 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6$$

$$= \frac{6}{64} + \frac{1}{64} = \frac{7}{64}.$$

(iii) P (atmost 5 successes) $= P(X \le 5)$

$$= P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$$

$$= 1 - P(X > 5)$$

$$= 1 - P(X = 6)$$

$$= 1 - {}^6C_6\left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}.$$

Example 35. The probability of a shooter hitting a target is 1/4 How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 2/3 ?

Solution : Let a person hit the target n times

as per the question $p = 1/4$ and $q = 1 - p = 1 - 1/4 = 3/4$ then

$$P(X = r) = {}^nC_r\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}; \text{ where } r = 0, 1, 2, ..., n$$

Given P (hitting the target atleast once) $> 2/3$

$$P(X \ge 1) > 2/3$$

$$\Rightarrow \quad 1 - P(X = 0) > 2/3$$

$$\Rightarrow \quad 1 - {}^nC_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{n-0} > \frac{2}{3}$$

$$\Rightarrow \quad 1 - \left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$\Rightarrow \quad \left(\frac{3}{4}\right)^n < \frac{1}{3}$$

$$\Rightarrow \quad n = 4, 5, 6, ... \left[\because \left(\frac{3}{4}\right)^1 > \frac{1}{3}, \left(\frac{3}{4}\right)^2 > \frac{1}{3}, \left(\frac{3}{4}\right)^3 > \frac{1}{3}\, but \left(\frac{3}{4}\right)^4 < \frac{1}{3}, \left(\frac{3}{4}\right)^5 < \frac{1}{3}, ...\right]$$

The person should hit the target atleast 4 times.

Example 36. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is just one step away from the starting point.

Solution : Let p denote the probability that the man takes a step forward. Then $p = 0.4$,

$$\therefore \quad q = 1 - p = 1 - 0.4 = 0.6$$

Let X denote the number of steps taken in the forward direction. Since the steps are independent of each other, therefore X is a binomial variate with parameters $n = 11$ and $p = 0.4$ such that

$$P(X = r) = {}^{11}C_r(0.4)^r(0.6)^{11-r}; r = 0, 1, 2, ...,11$$

Since the man is one step away from the initial point, he is either one step forward or one step backward from the initial point at the end of eleven steps. If he is one step forward, then he must have taken six steps forward and five steps backward and if he is one step backward, then he must have taken five steps forward and six steps backward. Thus, either $X=6$ or $X=5$

$\therefore$ required probability $=P[(X=5)$ or $(X=6)]$

$\Rightarrow$ required probability $=P(X=5)+P(X=6)$ (both the event are mutually exclusive)

$\Rightarrow$ required probability $={}^{11}c_5\,(0{\cdot}4)^5\,(0{\cdot}6)^{11-5}+{}^{11}c_6\,(0{\cdot}4)^6\,(0{\cdot}6)^{11-6}$

$\Rightarrow$ required probability $={}^{11}c_5\,(0{\cdot}4)^5\,(0{\cdot}6)^6+{}^{11}c_6\,(0{\cdot}4)^6\,(0{\cdot}6)^5=462(0{\cdot}24)^5$.

Exercise 16.5

1. If a fair coin is tossed 10 times, find the probability of
   (*i*) exactly six heads (*ii*) at least six heads (*iii*) at most six heads
2. An urn contains 5 white, 7 red and 8 black balls. If four balls are drawn with replacement then what is the probability that
   (*i*) all balls are white (*ii*) only three balls are white
   (*iii*) none of the balls is white (*iv*) alleast three balls are white
3. In a hurdle race, a player has to cross 10 hurdless. The probability that he will clear each hurdle is 5 / 6. What is the probability that he will knock down fewer that 2 hurdle.
4. Five dice are thrown at once. If getting an even number is a success then find the probability of three successes.
5. Ten eggs are drawn successively with replacement form a lot conatining 10% defective eggs. Find the probability that there is at least one defective egg.
6. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1 / 100. What is the probability that he will win a prize.
   (i) at least once (ii) exactly once (iii) at least twice?
7. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05. Find the probability that out of 5 such bulbs
   (i) none (ii) not more than one (iii) more than one (iv) at least one
   will fuse after 150 days of use.
8. In a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?
9. In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it falls tails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.
10. A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?
11. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards,. What is the probability that
    (i) all the five cards are spades ? (ii) only 3 cards are spades ?
    (iii) none is a spade ?
12. Suppose X has a binomial $B(6,1/2)$ Show that X = 3 is the most likely outcome.
13. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

Miscellaneous Examples

Example 37. A and B throw two dice alternatively. If A throws 6 before B throws 7 then A wins and if B throws 7 before A throws 6 then B wins. If A starts playing then find the probability that A wins.

Solution : We can get 6 in five ways

$\{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\} = 5/36$

and Probability of not getting $6 = 1 - 5/36 = 31/36$

similarly we can get 7 in six different ways

$\{(1,6)\ (2,5)\ (3,4)\ (4,3)\ (5,2)(6,1)\}$

$\therefore$ probability of getting $7 = 6/36 = 1/6$

and probability of not getting $7 = 1 - 1/6 = 5/6$

Let two events A and B are defined such that

A 'getting 6 in one throw'

B 'getting 7 in one throw'

then $P(A) = \frac{5}{36}, \quad P(\overline{A}) = \frac{31}{36}$

and $P(B) = \frac{1}{6}$ o $P(\overline{B}) = \frac{5}{6}$

A	A_W	$A_L B_L A_W$	$A_L B_L A_L B_L A_W$	...
B	$A_L B_W$	$A_L B_L A_L B_W$	$A_L B_L A_L B_L A_L B_W$	...

where A_W and A_L are of winning and losing of events A, Similarly B_W and B_L are winning are losing of events B

If A starts playing then the probability of winning A

$$P(A_W) + P(A_L B_L A_W) + (A_L B_L A_L B_L A_W) + \ldots$$

$$= P(A) + P(\overline{A})P(\overline{B})P(A) + \ldots$$

$$= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \ldots$$

$$= \frac{5}{36}\left[1 + \left(\frac{31}{36} \times \frac{5}{6}\right) + \ldots\right]$$

$$= \frac{5}{36} \frac{1}{\left[1 - \left(\frac{31}{36} \times \frac{5}{6}\right)\right]} \qquad [S_\infty = \frac{a}{1-r}]$$

$$= \frac{5}{36} \frac{36 \times 6}{216 - 155} = \frac{30}{61}.$$

Example 38. If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assuming that the individual entries of the determinant are chosen independently, each value being assumed with probability 1/2).

Solution : Let the given determinant be $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,

where $a_{ij} = 0$ or 1; $i, j = 1, 2$

It is clear that $\Delta \le 0$ if $a_{11} = 0$ or $a_{22} = 0$ neither $a_{11} = 0$ nor $a_{22} = 0 \Rightarrow a_{11} = 1 = a_{22}$ when $a_{11} = a_{22} = 1$ then $\Delta = 0$ if $a_{12} = a_{21} = 1$ so $a_{12} \ne 1$, $a_{21} \ne 1$ following are three posibility of values of Δ.

$$a_{11} = a_{22} = 1, a_{12} = 1, a_{21} = 0$$

$$a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 1$$

$$a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 0$$

Retired probability $= P(a_{11} = a_{22} = 1, a_{12} = 1, a_{21} = 0) + P(a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 1)$
$+P(a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 0)$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

$$= \frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{3}{16}.$$

Example 39. Find the mean of binomial distribution $B(4, 1/3)$.

Solution : Let X be a random variable whose probability distribution is $B(4, 1/3)$

here $n = 4,\ p = 1/3,\ q = 1 - p = 1 - 1/3 = 2/3$

and $P(X = x) = {}^4C_x \left(\frac{2}{3}\right)^{4-x}\left(\frac{1}{3}\right)^x$; $x = 0, 1, 2, 3, 4$

thus the probability distribution is

$X:$	0	1	2	3	4
$P(x_i):$	${}^4C_0\left(\frac{2}{3}\right)^{4-0}\left(\frac{1}{3}\right)^0 = \frac{16}{81}$	${}^4C_1\left(\frac{2}{3}\right)^{4-1}\left(\frac{1}{3}\right)^1 = \frac{32}{81}$	${}^4C_2\left(\frac{2}{3}\right)^{4-2}\left(\frac{1}{3}\right)^2 = \frac{24}{81}$	${}^4C_3\left(\frac{2}{3}\right)^{4-3}\left(\frac{1}{3}\right)^3 = \frac{8}{81}$	${}^4C_4\left(\frac{2}{3}\right)^{4-4}\left(\frac{1}{3}\right)^4 = \frac{1}{81}$

Mean $\mu = E(X) = \sum x_i p_i$

$$= 0\times\frac{16}{81}+1\times\frac{32}{81}+2\times\frac{24}{81}+3\times\frac{8}{81}+4\times\frac{1}{81}$$

$$= \frac{32+48+24+4}{81} = \frac{108}{81} = \frac{4}{3}$$

Miscellaneous Exercise 16

1. Two events A and B are mutually independent if -

   (A) $P(A)=P(B)$ (B) $P(A)+P(B)=1$

   (C) $P(\bar{A}\bar{B})=[1-P(A)][1-P(B)]$ (D) A and B are mutually exclusive

2. What is the probability of getting even prime number on both the dice if pair of dice is rolled together ?

   (A) 1 / 3 (B) 0 (C) 1 / 36 (D) 1 / 12

3. If A and B ae events so that $A \subset B$ and $P(B) \neq 0$, then which of the following statement is true ?

   (A) $P\left(\frac{A}{B}\right) < P(A)$ (B) $P\left(\frac{A}{B}\right) \geq P(A)$ (C) $P\left(\frac{A}{B}\right) = P\frac{(B)}{(A)}$ (D) None of these

4. Two cards are drawn from the well shuffled pack of 52 cards Let X denote the number of aces, then find X -

   (A) 5 / 13 (B) 1 / 13 (C) 37 / 221 (D) 2 / 13

5. Let X takes the value 0, 1, 2, 3. The mean of X is 1.3. If $P(X=3)=2P(X=1)$ and $P(X=2)=0{\cdot}3$ then find $P(X=0)$.

   (A) 0.2 (B) 0.4 (C) 0.3 (D) 0.1

6. The probability of a girl being a racer is 4 / 5. Find the probability of 4 girls being a racer out of 5 girls.

   (A) $\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)$ (B) ${}^5C_1\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)$ (C) ${}^5C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^4$ (D) None of these

7. A box contains 100 objects out of which 10 are defective. The probability of the given 5 objects , find the probability that none of them are defective-

   (A) $\left(\frac{1}{2}\right)^5$ (B) 10^{-1} (C) $\frac{9}{10}$ (D) $\left(\frac{9}{10}\right)^5$

8. A couple has two children, find the probability -

   (*i*) that both are males if it is known that the elder one is a male

   (*ii*) that both children are female, if it is known that the elder child is a female

   (*iii*) that both children are males, if it is known that at least one of the children is male

9. Two integers are chosen form the numbers 1 to 11. Find the probability that both the numbers are odd if it is known that the sum of both is an even number.

10. An electronic assembly consists of two sub system, say, A and B. Form previous testing procedures, the following probabilities are assumed to be known :

    P (A fails) = 0.2

    P (B fails alone) = 0.15

    P (A and B fail) = 0.15

    Evaluate the following probabilities

    (i) P(A fails | B has failed)

    (ii) P(A fails alone)

11. Let A and B be two independent events. The probability that both occur together is 1 / 8 and probability that both do not occur is 3 / 8. Determine $P(A)$ and $P(B)$.
12. Anil speaks truth in 60 % of the cases and Anand speaks truth in 90 % of the cases. Find the probability that both of them contradicts on a statement.
13. Three people A, B and C toss a coin one by one. A person wins if he gets Heads first. Assuming that the game continues , if A starts the game, find the probability that A wins.
14. The probability of a person remains alive for the next 25 years is 4 / 5 and the probability that his wife remain alive for the same 25 yeas is 3 / 4, Find the probabilities that -
    (*i*) both are alive for the 25 years
    (*ii*) atleast one of them remain alive for the next 25 years.
    (*iii*) Only wife remain alive for the next 25 years.
15. In a group of children there are 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and 3 boys. One child is selected at random form each group. Find the probability that the out of the three children selected there is 1 girl and 2 boys.
16. Bag 1 contains 3 black and 4 white balls and Bag II contains 4 black and 3 white balls. A die is thrown. If it shows 1 or 3 then a ball is drawn form Bag I and if some other number appear a ball is drawn from Bag II. Find the probability the drawn ball is black.
17. A person has under taken a construction job. The probabilities are 0.65 that there will be strike , 0.80 that the construction job will be completed on time if there is no strike and 0.32 that the construction job will be completed on time.
18. Bag I contains 8 white and 4 black balls and Bag II contains 5 white and 4 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. Find the probability that the drawn ball is white in colour.
19. On a multiple choice examination with four choices a student either guesses or knowns or cheat the answer. Find the probability of guessing or cheating the answer if it is known that he answers the question correctly.
20. A letter comes form the two cities *TATANAGAR* or *CALCUTTA*. Only alphabets TA is visible on the envelope. Find the probability that the letter comes from the city.
    (*i*) *CALCUTTA*
    (*ii*) *TATANAGAR*,
21. A manufacturer has three machine operators A, B and C. The first operator A produces 1 %, whereas the other two opeartors B and C produce 5% and 7% defective items resp. A is on the job for 50 % of the time, B is on the job for 30 % of the time and C is on the job for 20 % of the time. A defective item is produced, what is the probability that it was produced by A ?
22. A random variable X has a probability distribution $P(X)$ of the following form where K is some number:

$$P(X=x)=\begin{cases} k & \text{if } x=0 \\ 2k & \text{if } x=1 \\ 3k & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

    (i) Find the value of K
    (ii) Find $P(X<2)$, $P(X\le 2)$ and $P(X\ge 2)$
23. A random variable takes all negative integral values and the value of X is 'r' whose probability is directly

proportional to α^r where $0 < \alpha < 1$ then find $P(X = 0)$

24. Let X be the random variable with values x_1, x_2, x_3, x_4 such that

$$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$$

Find the probability distribution of X

25. A fair coin is tossed to get one head or five tails. if X denotes the number of tosses then find the mean of X.

26. Three cards are drawn from the well shuffled deck of 52 cards. Find the probability distribution of number of red cards drawn. Also find the mean of the distribution.

IMPORTANT POINTS

1. If any random experiment if A and B are two events related to sample space then the conditional probability of event A, given the occuence of the event B is given by

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}; \qquad P(B) \neq 0.$$

similarly $$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}; \qquad P(A) \neq 0$$

2. $0 \leq P\left(\frac{A}{B}\right) \leq 1,$ $\qquad P\left(\frac{\overline{A}}{B}\right) = 1 - P\left(\frac{A}{B}\right)$

3. If S is a sample space and A and B are two events then event F is such that $P(F) \neq 0$ then

$$P\left(\frac{A \cup B}{F}\right) = P\left(\frac{A}{F}\right) + P\left(\frac{B}{F}\right) - P\left(\frac{A \cap B}{F}\right)$$

4. Multiplication Rule of Probability

$$P(A \cap B) = P(A)P\left(\frac{B}{A}\right); \quad P(A) \neq 0 \text{ ;k } P(A \cap B) = P(B)P\left(\frac{A}{B}\right); \quad P(B) \neq 0$$

5. If A and B are independent, then

$$P\left(\frac{A}{B}\right) = P(A),\ P(B) \neq 0; \qquad P\left(\frac{B}{A}\right) = P(B),\ P(A) \neq 0$$

and $P(A \cap B) = P(A)P(B)$

6. Therorem of total probability

Let $A_1, A_2, A_3, ..., A_n$ be 'n' partition of a sample spaces let A be any event associated with S, i.e. then

$P(A_j) \neq 0;\ j = 1, 2, ..., n$

$$P(E) = P(A_1)P\left(\frac{E}{A_1}\right) + P(A_2)P\left(\frac{E}{A_2}\right) + ... + P(A_n)P\left(\frac{E}{A_n}\right) = \sum_{j=1}^{n} P(A_j)P\left(\frac{E}{A_j}\right)$$

7. Baye's Theorem -

$$P\left(\frac{A_i}{E}\right) = \frac{P(A_i)P\left(\frac{E}{A_i}\right)}{\sum_{j=1}^{n} P(A_j)P\left(\frac{E}{A_j}\right)}$$

8. A random variable is a real valued function whose domain is the sample space of a random experiment.

9. The probability distribution of a random variable X is the system of numbers

$$\begin{array}{llllll} X = x & : & x_1 & x_2 & x_3 \ldots x_n \\ P(x) & : & p_1 & p_2 & p_3 \ldots p_n \end{array} \; ; \text{ where } p_i > 0, \quad \sum_{i=1}^{n} p_i = 1; \quad i = 1, 2, \ldots, n$$

10. Let X be a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p_1, p_2, \ldots, p_n$ respectively. The mean of X, denoted by μ is the number $\sum_{i=1}^{n} x_i p_i$

(a) The mean of a random variable X is also called the expectation of X, denoted by E (X).

(b) Varience of X

$$= \text{var}(X) = \sigma_x^2 = E(X-\mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

(c) $\text{var}(X) = E(X^2) - \{E(X)\}^2$

(d) Standard Deviation

$$\sigma_x = +\sqrt{\text{var}(x)} = +\sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p_i}$$

11. Trials of a random experiments are called Bernoulli trials, if they satisfy the following conditions :

(i) There should be a finite number of trials.

(ii) The trials should be independent.

(iii) Each trial has exactly two outcomes : success or failure.

(iv) The probability of success remains the same in each trial.

12. Probability of r successes in binomial distribution is $B(n, p)$

$$P(X = r) = {}^nC_r p^r q^{n-r}; \quad r = 0, 1, 2, \ldots, n \text{ where } q = 1 - p.$$

ANSWERS

Exercise 16.1

1. 4 / 9 2. 16 / 25 3. 11 / 26 4. $P\left(\frac{A}{B}\right) = \frac{2}{3}$, $P\left(\frac{B}{A}\right) = \frac{1}{3}$ 5. (i) 0.32 ; (ii) 0.64 ; (iii) 0.98

6. 1/ 3 7. (i) $P\left(\frac{A}{B}\right) = 1$ (ii) $P\left(\frac{A}{B}\right) = 0$ 8. $P\left(\frac{A}{B}\right) = 1$

9. (i) $P\left(\frac{A}{B}\right) = \frac{1}{2}$, $P\left(\frac{B}{A}\right) = \frac{1}{3}$; (ii) $P\left(\frac{A}{C}\right) = \frac{1}{2}$, $P\left(\frac{C}{A}\right) = \frac{2}{3}$; (iii) $P\left(\frac{A \cup B}{C}\right) = \frac{3}{4}$, $P\left(\frac{A \cap B}{C}\right) = \frac{1}{4}$

10. 1 / 15 11. 4 / 7 12. 0.1 13. 2 / 5 14. 2 / 9

Exercise 16.2

1. 3 / 8 2. 1 / 3 3. (i) 0.12 ; (ii) 0.58 ; (iii) 0.3; (iv) 0.4 4. (i) 0.18 ; (ii) 0.12 ; (iii) 0.72 ; (iv) 0.28
5. 1 / 969 6. 7 / 8 7. 25 / 102 8. 1 / 3 9. (i) 1 / 5; (ii) 1 / 3 ; (iii) 1 / 2
10. 0.97 11. 3 / 4 12. 1 / 7 13. (i) 2 / 3 ; (ii) 1 / 2

Exercise 16.3

1. 35 / 68 2. 1 / 2 3. 16 / 31 4. 2 / 3 5. 4 / 9 6. 22 / 133 7. 9 / 13
8. 1 / 52 9. 12 / 13 10. 20 / 21 11. 2 / 9 12. 8 / 11 13. 11 / 50 14. 2 / 9

Exercise 16.4

1. (i)

2.

$X = x$	0	1	2
$P(x)$	1/4	1/2	1/4

3.

$X = x$	0	1	2
$P(x)$	12/19	32/95	3/95

4.

$X = x$	0	1	2	3
$P(x)$	4/35	18/35	12/35	1/35

5. (i)

$X = x$	0	1	2	3
$P(x)$	1/6	1/2	3/10	1/30

(ii) 2 / 3 (iii) 1 / 6 (iv) 1 /2

6.

$X = x$	0	1	2
$P(x)$	4/9	4/9	1/9

7.

$X = x$	0	1	2	3	4
$P(x)$	1/14	8/21	6/14	4/35	1/210

8.

$X = x$	0	1	2	3
$P(x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

9. 7 10. 35 / 12 11. 7 / 10, 21 / 100

12. $\frac{34}{221}, \frac{6800}{(221)^2}, 0{\cdot}37$

Exercise 16.5

1. (i) 105 / 512 ; (ii) 193 / 512 ; (iii) 53 / 64 2. (i) $\left(\frac{1}{4}\right)^4$ (ii) $3\left(\frac{1}{4}\right)^3$ (iii) $\left(\frac{3}{4}\right)^4$ (iv) $\frac{13}{4^4}$

3. $\frac{5^{10}}{2\times 6^9}$ 4. $\frac{13}{16}$ 5. $1-\frac{9^{10}}{10^{10}}$ 6. (i) $1-\left(\frac{99}{100}\right)^{50}$ (ii) $\frac{1}{2}\left(\frac{99}{100}\right)^{49}$ (iiii) $1-\frac{149}{100}\left(\frac{99}{100}\right)^{49}$

7. (i) $\left(\frac{19}{20}\right)^5$ (ii) $\frac{6}{5}\left(\frac{19}{20}\right)^4$ (iii) $1-\frac{6}{5}\left(\frac{19}{20}\right)^4$ (iv) $1-\left(\frac{19}{20}\right)^5$ 8. $\frac{11}{243}$

9. $\dfrac{{}^{20}C_{12} + {}^{20}C_{13} + \ldots + {}^{20}C_{20}}{2^{20}}$ 10. $\left(\dfrac{9}{10}\right)^4$ 11. (i) $\dfrac{1}{1024}$; (ii) $\dfrac{45}{512}$; (iii) $\dfrac{243}{1024}$

13. $\dfrac{25}{216}$

Miscellaneous Exercise - 16

1. (C) 2. (C) 3. (B) 4. (D) 5. (B) 6. (C) 7. (D)

8. (i) 1 / 2 ; (ii) 1 / 2; (iii) 1 / 3 9. 3 / 5 10. (i) 1 / 2 ; (ii) 0.05

11. $P(A)=\dfrac{1}{2},\ P(B)=\dfrac{1}{4}$;k $P(A)=\dfrac{1}{4},\ P(B)=\dfrac{1}{2}$ 12. 0.42 13. 4 / 7, 2 / 7, 1 / 7

14. (i) $\dfrac{3}{5}$; (ii) $\dfrac{19}{20}$; (iii) $\dfrac{3}{20}$ 15. $\dfrac{13}{32}$ 16. $\dfrac{11}{21}$ 17. 0.488 18. $\dfrac{83}{150}$ 19. $\dfrac{24}{29}$

20. (i) $\dfrac{4}{11}$; (ii) $\dfrac{7}{11}$ 21. $\dfrac{5}{34}$ 22.(i) $\dfrac{1}{6}$; (ii) $P(X<2)=\dfrac{1}{2},\ P(X\le 2)=1,\ P(X\ge 2)=\dfrac{1}{2}$

23. $(1-\alpha)$

24.

X :	x_1	x_2	x_3	x_4
$P(X)$:	$\dfrac{15}{61}$	$\dfrac{10}{61}$	$\dfrac{30}{61}$	$\dfrac{6}{61}$

25. 1.9