Application of Derivatives

Rate of change : If a quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Differentials : Let $y = f(x)$ be any function of x which is differentiable in (a, b). The derivatives of this function at some point x of (a, b) is given by the relation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) \quad \Rightarrow dy = f'(x)\,dx$$

differential of the function

Increasing and decreasing functions A function f is said to be,

(a) increasing on an interval (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in (a, b)$

(b) decreasing on an interval (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in (a, b)$

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a, b). Then

(a) f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Tangent to a curve The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$\boxed{y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)}$$

$\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ OR $f'(x_0) = m =$ slope of tangent at (x_0, y_0)

If $\frac{dy}{dx}$ does not exist at the point (x_0, y_0), then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$

If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right]_{x = x_0} = 0$

Normal to the curve

Equation of the normal to the curve $y = f(x)$ at a point (x_0, y_0) is given by,

$$\boxed{y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)}$$

$\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ OR $f'(x_0) = m =$ slope of tangent at (x_0, y_0)

If $\frac{dy}{dx}$ at the point (x_0, y_0) is zero. then equation of the normal is $x = x_0$

If $\frac{dy}{dx}$ at the point (x_0, y_0) does not exist, then the normal is parallel to x-axis and its eq. $y = y_0$

$$\boxed{\text{Slope of the normal} = \frac{-1}{\text{slope of the tangent}}}$$

Approximation Let $y = f(x)$, Δx be a small increament in x and Δy be the increment in y corresponding to the increament in x, i.e. $\Delta y = f(x + \Delta x) - f(x)$. Then approximate value of $\Delta y = \left(\frac{dy}{dx}\right)\Delta x$.

✓ Maximum or Minimum value of a function (Absolute Maxima or Absolute Minima)

A function f is said to attain maximum value at a point $a \in D_f$, if $f(a) \geq f(x)$ $\forall$ $x \in D_f$ then $f(a)$ is called absolute maximum value of f.

A function f is said to attain minimum value at a point $b \in D_f$, if $f(b) \leq f(x)$ $\forall$ $x \in D_f$ then $f(b)$ is called absolute maximum value of f.

✓ Local Maxima and local Minima (Relative Extrema)

Local Maxima A function $f(x)$ is said to attain a local maxima at $x = a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ of 'a' such that $f(x) < f(a)$ $\forall$ $x \in (a-\delta, a+\delta)$, $x \neq a$, then $f(a)$ is the local maximum value of $f(x)$ at $x = a$.

Local Minima A function $f(x)$ is said to attain a local minima at $x = a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ of 'a' such that $f(x) > f(a)$ $\forall$ $x \in (a-\delta\ a+\delta)$, $x \neq a$, then $f(a)$ is the local minimum value of $f(x)$ at $x = a$.

(a) First derivative test :

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, then 'c' is a point of local maxima and $f(c)$ is local maximum value

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, then 'c' is a point of local minima and $f(c)$ is local minimum value

(iii) If $f'(x)$ dosen't changes sign as x increases through c, then c is neither a point of local minima nor a point of local maxima. Such a point is called point of inflection.

(b) Second Derivative Test : Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. In this case $f(c)$ is called local maximum value.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. In this case $f(c)$ is called local minimum value.

(iii) The test fails of $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to first derivative test.

✓ Working Rule for finding absolute maximum or absolute minimum values

Step I : Find all the critical points of f in the given interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step II : Take the end points of the interval.

Step III : At all these points, Calculate the value of f.

Step IV : Identify the maximum and minimum value of f out of the values calculated in Step III. The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

✓ Critical Point A Point C in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.