Chapter 2

Inverse Trigonometric Functions

Exercise 2.2

Q. 1

Prove the following:

3 $\sin^{-1} x = \sin^{-1} (3x – 4x3)3$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Answer:

Let $x = \sin \theta$ then $\sin^{-1}x = \theta$

We have,

R.H.S = $\sin^{-1} (3x – 4x^3)^3 = \sin^{-1} (3 \sin \theta – 4 \sin^3 \theta)$

Now, we know that,

sin 3x = 3sin x - 4 sin3x

Therefore,

$= \sin^{-1}(\sin (3\theta))$

$= 3\ \theta$

$= 3\sin^{-1}x$

= L.H.S

Hence Proved

Q. 2

Prove the following:

$3 \cos^{-1} x = \cos^{-1} (4x^3 – 3x)$, $x \in \left[\frac{1}{2}, 1\right]$

Answer:

Let x = cos θ

Then, $\text{Cos}^{-1} = \theta$

Now, R.H.S. = $\cos^{-1}(4x^3 - 3x)$

$= \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$= \cos^{-1}(\cos^3\theta)$

$= 3\theta$

$= 3\cos^{-1}x$

= L.H.S.

Hence Proved

Q. 3

Prove the following:

$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Answer:

L.H.S. tan-1 $\frac{2}{11}$ + tan-1 $\frac{7}{24}$

$= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}}$ $\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$

$= \tan^{-1}\frac{\frac{48+77}{11\times24}}{\frac{11\times24-14}{11\times24}}$

$= \tan^{-1}\frac{48+77}{264-14}$

$= \tan^{-1}\frac{125}{250}$

$= \tan^{-1}\frac{1}{2}$

= R.H.S.

Hence Proved.

Q. 4

Prove the following:

$2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Answer:

L.H.S. = $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7}$

$= \tan^{-1} \frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1} \frac{1}{7}$

$= \tan^{-1} \frac{1}{\frac{3}{4}} + \tan^{-1} \frac{1}{7}$

$= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{7}$

$= \tan^{-1} \frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}$ $\left[since, \tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy}\right]$

$= \tan^{-1} \frac{\frac{28+3}{21}}{\frac{21-4}{21}}$

$= \tan^{-1} \frac{31}{17}$

= R.H.S.

Hence Proved.

Q. 5

Write the following functions in the simplest form:

$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$

Answer:

$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$

Now, Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

Therefore, $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+tan\theta^2}-1}{tan\theta}\right)$

$= \tan^{-1}\left(\frac{sec\theta-1}{tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-cos\theta}{sin\theta}\right)$

$= \tan^{-1}\left(\frac{2sin^2\frac{\theta}{2}}{2sin\frac{\theta}{2}cos\frac{\theta}{2}}\right)$

$= \tan^{-1}\left(tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

Therefore, $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \frac{1}{2}\tan^{-1}x$

Q. 6

Write the following functions in the simplest form:

$\tan^{-1} \frac{1}{\sqrt{x^2-1}}$, $|x| > 1$

Answer:

$\tan^{-1} \frac{1}{\sqrt{x^2-1}}$, $|x| > 1$

Let us take,

$x = \text{cosec}\,\theta = \theta = \text{cosec}^{-1}\,x$

[We have done this substitution on the bases of identity sec2θ – 1 = tan2θ]

Therefore, $\tan^{-1} \frac{1}{\sqrt{x^2-1}} = \tan^{-1} \frac{1}{\sqrt{cosec^2\theta-1}}$

Now we know that, cosec2θ – 1 = cot2θ

Therefore,

$= \tan^{-1} \frac{1}{cot\theta} = \tan^{-1} (\tan\theta) = \theta = cosec^{-1} x$

Q. 7

Write the following functions in the simplest form:

$\tan^{-1} = \left(\sqrt{\frac{1-cosx}{1+cosx}}\right) x < \pi$

Answer:

$\tan^{-1} = \left(\sqrt{\frac{1-cosx}{1+cosx}}\right) x < \pi$

$= \tan^{-1} \left(\sqrt{\frac{2sin^2\frac{x}{2}}{2cos^2\frac{x}{2}}}\right)$

$= \tan^{-1} \left(\frac{sin\frac{x}{2}}{cos\frac{x}{2}}\right)$

$= \tan^{-1} \left(tan\frac{x}{2}\right)$

$= \frac{x}{2}$

Hence, $\tan^{-1} \left(\sqrt{\frac{1-cosx}{1+cosx}}\right) = \frac{x}{2}$

Q. 8

Write the following functions in the simplest form:

$\tan^{-1}\left(\frac{cosx-sinx}{cosx+sinx}\right)$, $0 < x < \pi$

Answer:

$\tan^{-1}\left(\frac{cosx-sinx}{cosx+sinx}\right)$

Dividing by cos x,

$= \tan^{-1}\left(\frac{1-\frac{sinx}{cosx}}{1+\frac{sinx}{cosx}}\right)$

$= \tan^{-1}\left(\frac{1-tanx}{1+tanx}\right), \left[\therefore \frac{sinx}{cosx} = tanx\right]$

$= \tan^{-1} - \tan^{-1}$ (tan x) $\left[\therefore tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$

As we know $\tan(\pi/4) = 1$

$= \tan^{-1}\left(tan\left(\frac{\pi}{4}\right)\right) + \tan^{-1}$ (tan x)

$= \frac{\pi}{4} - x$

Hence, $\tan^{-1}\left(\frac{cosx-sinx}{cosx+sinx}\right) = \frac{\pi}{4} - x.$

Q. 9

Write the following functions in the simplest form:

$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$, $|x| < a$

Answer:

$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$

We will solve this problem on the bases of the identity 1 - sin2θ = cos2θ

So, for $a^2 - x^2$, we can substitute x = a sin θ or x = a cos θ

Now, let us put x = a sin θ

$= \frac{x}{a} = \sin\theta$

$= \theta = \sin^{-1}\left(\frac{x}{a}\right)$

Therefore,

$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$

$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$

Hence, $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

Q. 10

Write the following functions in the simplest form:

$\tan^{-1}\left\{\frac{3a^3x-x^3}{a^3-3ax^2}\right\}$, $a > 0; \frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Answer:

$\tan^{-1}\left\{\frac{3a^3x-x^3}{a^3-3ax^2}\right\}$

Put $x = a\tan\theta \Rightarrow \frac{x}{a} = \tan\theta = \tan^{-1}\frac{x}{a}$

Now,

$\tan^{-1}\left(\frac{3a^3x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^3.a\tan\theta-a^3\tan^3\theta}{a^3-3a.a^2\tan^2\theta}\right)$

$= \tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)\left[\therefore \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta} = \tan 3\theta\right]$

$= 3\theta$

$= 3\tan^{-1}\frac{x}{a}$

Q. 11

Find the values of each of the following:

$\tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right]$

Answer:

$\tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right]$

We will solve the inner bracket first.

So, we will first find the principal value of $\sin^{-1}\frac{1}{2}$

We know that, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

Therefore,

$\tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right] = \tan^{-1}\left[2cos\left(2 \times \frac{\pi}{6}\right)\right]$

$= \tan^{-1}\left[2\cos\left(\frac{\pi}{3}\right)\right]$

$= \tan^{-1}\left[2 \times \frac{1}{2}\right]\left[since, \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}\right]$

$= \tan^{-1}1$

$= \pi/4$

Hence,

The value of $\tan^{-1}\left[2cos\left(2sin^{-1}\frac{1}{2}\right)\right] = \frac{\pi}{4}$

Q. 12

Find the values of each of the following:

$\cot(\tan^{-1}a + \cot^{-1}a)$

Answer:

$\cot(\tan^{-1}a + \cot^{-1}a)$

= cot $\left(\frac{\pi}{2}\right)$, $\left[\because tan^{-1}x + cot^{\ -1}y = \frac{\pi}{2}\right]$

= 0

Hence, the value of cot ($\tan^{-1}$a + $\cot^{-1}$a) = 0

Q. 13

Find the values of each of the following:

tan $\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right]$, |x| < 1, y = 0

Answer:

tan $\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right]$

We will solve this problem by expressing sin2θ and cos2θ in terms of tan θ

Now let us put, x = tan θ. Then we will have,

θ = $\tan^{-1}$x

∴ $\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+tan^2\theta} = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

Now again, Let’s put, y = tan∅. Then we will have,

∅ = tan-1y

∴ $\cos^{-1}\frac{1-y^2}{1+y^2} = \sin^{-1}\left(\frac{1-tan^2\emptyset}{1+tan^2\emptyset}\right) = \cos^{-1}(\cos 2\emptyset) = 2\emptyset = 2\tan^{-1}y$

Now,

tan $\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right]$

= tan $\frac{1}{2}$ [2$\tan^{-1}$ x + 2$\tan^{-1}$ y]

= tan [$\tan^{-1}$ x + $\tan^{-1}$ y]

$= \tan\left[tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \frac{x+y}{1-xy}$

Hence, the value of

$\tan\frac{1}{2}\left[sin^{-1}\frac{2x}{1+x^2} + cos^{-1}\frac{1-y^2}{1+y^2}\right] = \frac{x+y}{1-xy}$

Q. 14

Find the values of each of the following:

If $\sin\left(sin^{-1}\frac{1}{5} + cos^{-1}x\right)$ = 1, then find the value of x

Answer:

Sin $\left(sin^{-1}\frac{1}{5} + cos^{-1}x\right)$ = 1

= $\sin^{-1}\frac{1}{5} + \cos^{-1}$ x = sin-1 1

= $\sin^{-1}\frac{1}{5} + \cos^{-1}$ x = $\frac{\pi}{2}$, $\left[since, sin^{-1}1 = \frac{\pi}{2}\right]$

= $\sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}$ x

= $\sin^{-1}\frac{1}{5} = \sin^{-1}$ x, $\left[since, sin^{-1}x + cos^{-1}x = \frac{\pi}{2}\right]$

On comparing the co-efficient on both sides we get,

= x = $\frac{1}{5}$

Q. 15

Find the values of each of the expression

$\text{Sin}^{-1}\left(sin\frac{2\pi}{3}\right)$

Answer:

Sin^{-1} $\left(sin\frac{2\pi}{3}\right)$

(For sin^{-1} (sin x) type of problem we have to always check whether the angle is in the principal range or not. This angle must be in the principal range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

So here, $\frac{2\pi}{2} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, sin^{-1} $\left(sin\frac{2\pi}{3}\right)$ can be written as,

Sin^{-1} $\left(sin\frac{2\pi}{3}\right)$

= sin^{-1} $\left(\text{sin}\,\pi-\frac{\pi}{3}\right)$

= sin^{-1} $\left(\text{sin}\frac{\pi}{3}\right)$ where $\frac{\pi}{3} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

= $\frac{\pi}{3}$

Hence, sin^{-1} $\left(sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$.

Q. 16

Find the values of each of the expression

tan^{-1} $\left(\tan\frac{3\pi}{4}\right)$

Answer:

tan^{-1}$\left(\tan\frac{3\pi}{4}\right)$

(For tan^{-1} (tan x) type of problem we have to always check whether the angle is in the principal range or not. This angle must be in the principal range.$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$)

So here, $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ can be written as,

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$= \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$ where $-\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, [since, $\tan(\pi - x) = -\tan x$

$= -\frac{\pi}{4}$

Hence, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$

Q. 17

Find the values of each of the expression

$\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

Answer:

Let $\sin^{-1}\left(\frac{3}{5}\right) = y$ so $\sin y = \frac{3}{5}$ and $y \in \left(0, \frac{\pi}{2}\right)$, so all ratio of y are positive and

Hence, $\cos y = \frac{4}{5}$ and $\tan y = \frac{3}{4}$, so $\tan^{-1}\left(\frac{3}{4}\right) = y$

Also,

$\cot^{-1}\left(\frac{3}{2}\right) = \tan^{-1}\frac{2}{3}$ as $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$

So, $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$

$= \tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)$

$= \tan\left(tan^{-1}\ \frac{17}{6}\right) = \frac{17}{6}$

Hence, $\tan\left(sin^{-1}\frac{3}{5} + cot^{\ -1}\frac{3}{2}\right) = \frac{17}{6}$

Q. 18

Find the values of each of the expression

$Cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

A. $\frac{7\pi}{6}$

B. $\frac{5\pi}{6}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{3}$

Answer:

$Cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

(For $\cos^{-1}(\cos x)$ type of problem we have to always check whether the angle is in the principal range or not. This angle must be in the principal range. $\left[-\frac{(\pi}{2}, \frac{(\pi}{2}\right]$)

So here,

$\frac{7\pi}{6} \notin [0, \pi]$

Now, $\cos^{-1}\left(\cos\frac{7\pi}{67}\right)$ can be written as,

$Cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

$= \cos^{-1}\left[\cos\left(\pi + \frac{\pi}{6}\right)\right]$

= - $\cos^{-1}\left(\cos\frac{\pi}{6}\right)$ where - $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, [since, cos (π + x) = - cos x]

= π – $\cos^{-1}\left(\cos\frac{\pi}{6}\right)$ as cos-1(-x) = π – cos-1

= π - $\frac{\pi}{6} = \frac{5\pi}{6}$

Hence, cos $\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$.

Q. 19

Find the values of each of the expression

Sin $\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. 1

Answer:

$\text{Sin}^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right)$ as $\sin^{-1}$ (-x) = $\sin^{-1}$ x

= $-\frac{\pi}{6}$ as sin $\left(\frac{\pi}{6}\right) = \frac{1}{2}$

We all know that the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Therefore, sin $\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$ = sin $\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$ = sin $\left(\frac{3\pi}{6}\right)$ = sin $\left(\frac{\pi}{2}\right)$ = 1

Hence, the value of sin $\left(\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right)$

Q. 20

Find the values of each of the expression

$\tan^{-1}\sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right)$

is equal to

A. π

B. $-\frac{\pi}{2}$

C. 0

D. $2\sqrt{3}$

Answer:

$\tan^{-1}\sqrt{3 -} \cot^{-1}\left(-\sqrt{3}\right)$

$= \tan^{-1}\sqrt{3} - \left(\pi - cot^{-1}\sqrt{3}\right)$

$= \tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} - \pi$

$= \frac{\pi}{2} - \pi$

$= -\frac{\pi}{2}$