Chapter 5

State Space Analysis

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- System state
- State vector
- State equations
- State space representation
- Controllable canonical form
- Observable canonical form
- Diagonal canonical form
- Jordan canonical form
- Observability
- Kalman's test for observability
- Transfer function

Basic Definitions

System state: minimum information needed in order to completely determine the output of the system from a given moment, provided the input is known from that moment.

System variable: Any variable that responds to an input (or) initial conditions in a system.

State Variables: The smallest set of linearly, independent system variables such that the values of the set members at time t_0 along with known forcing function completely determine the value of all system variables for all $t \geq t_0$.

State Vector: Vector whose elements are the state variables.

State space: n-dimensional space whose axes represents the state variables.

State equations: A set of n simultaneous, first-order differential equations with 'n' variables, where 'n' variables to be solved are the state variables.

State Space Representation

A state space representation of an linear time invariant (LTI) system has the general for x.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) - Du(t) \\ x(t_0) &= X_0 \rightarrow \text{initial conditions} \end{aligned}$$

where

$x(t)$	:	State vector (n-dimensional)
$y(t)$	:	Output vector (P-dimensional)
$u(t)$	:	Input or control vector (m-dimensional)
A	:	Dynamic or system matrix ($n \times n$)
B	:	Input matrix ($n \times m$)
C	:	Output matrix ($p \times n$)
D	:	Feed forward (direct) matrix ($p \times m$)

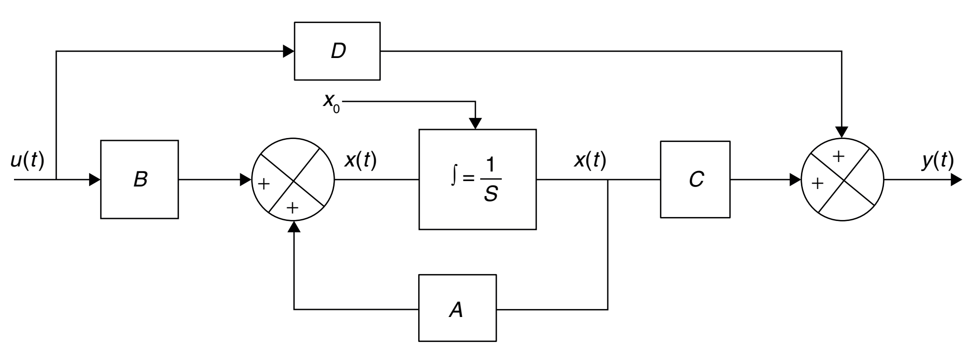


Figure 1 Block diagram representation of state space model

Advantages of State Space Analysis

1. It is applicable to multiple input, multiple output systems.
2. It is applicable to system with non-zero initial conditions.
3. It is applicable to both LTI and Non Linear time varying system.
4. All the internal states can be determined.
5. More accurate than transfer function.
6. It gives the information about the controllability and observability.

State Space Representation

Selecting the state variables must follow the following rules

- A minimum number of state variables must be selected.
- They must be linearly independent.

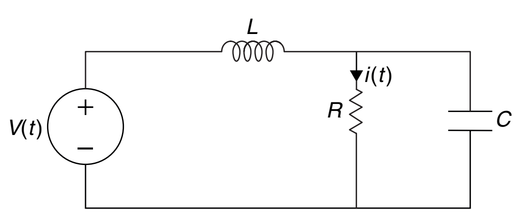
The minimum number of state variables required equals the order of the differential equation describing the system. From the transfer function point of view, the order of the differential equation is the order of the denominator of the T.F. after cancelling common factors in the numerator and denominator.

A practical way to determine the number of state variables is to count the number of independent energy storage elements in the system (Capacitor and inductors in electrical system and masses and springs in mechanical system). Procedure for state representation of electrical network

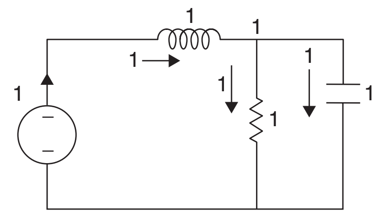
1. Write simple derivative equation for each energy storage element [Node equation for the node at which an inductor is connected and loop equation for the Loop in which capacitor is connected in electrical network].
2. Solve for the each derivate term as a linear combination of system variable and the input.
3. Each differentiated variable is selected as a state variable.
4. All other variables and output are written in terms of the state variables and the input.

Solved Examples

Example 1: Find the state space representation of the system shown in the figure if the output is the current through the resistor



Solution: Step 1: Label all the branch variables in the network



Step 2: Select the state variables: Write the derivative equations for all energy storage elements (L and C)

$$C.\frac{dv_c}{dt} = i_c;\ L.\frac{di_L}{dt} = V_L$$

Choose the differentiated quantities as the state variables (v_C and i_L).

Step 3: Write the each differentiated term as a linear, combination of system variables and the input.

$$C.\frac{dv_c}{dt} = i_C = f_1(v_C, i_C, V(t)) \quad (1)$$

$$L.\frac{di_L}{dt} = \vartheta_C = f_2(v_C, i_L, V(t)) \quad (2)$$

Applying node equation at V_1

$$i_C = -i_R + i_L = -\frac{1}{R}\vartheta_C + i_L \quad (3)$$

Applying loop equation in capacitor loop

$$V_L = -V_C + V(t) \quad (4)$$

From Equations (1), (2), (3) and (*4),

$$\frac{dv_c}{dt} = -\frac{1}{RC}\vartheta_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L}\vartheta_C + \frac{1}{L}V(t)$$

Output equation $i_R = \frac{1}{R}\vartheta_C$

Step 4: Obtain state space representation in vector matrix form

$$\begin{bmatrix} \dot{\vartheta}_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} \vartheta_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} v(t)$$

$$[i_R] = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix}$$

Transfer Function to State Space Model

This case corresponds to a linear system that can be represented as a n^{th}-order differential equation with constant co-efficient like:

$$\frac{d^n\, y(t)}{dt^{\,n}} + \frac{d^{n-1}\, y(t)}{dt^{n-1}} a_{n-1} + \frac{d^{n-2}\, y(t)}{dt^{\,n-2}} a_{n-2} + \dots.$$
$$+ a_1 \frac{dy(t)}{dt} + a_0$$
$$= b_0 U(t)$$

The classical transfer function (T.F.) representation of the system is obtained by applying the Laplace transform to the differential equation.

$$G(S) = \frac{Y(S)}{U(S)} = \frac{b_0}{s^n + a_{n-1}\, s^{n-1} + a_{n-2}\, s^{n-2} + \dots. + a_1 s + a_0}$$

If the state space representation has to be obtained, a convenient way to select state variables is to choose the output $y(t)$ and its 'n–1' derivatives as the state variables. This is called phase variables choice.

The state space representation using the phase variable choice of the state variables is said to be in the 'Controllable canonical form' (CCF).

Controller canonical form of the given differential equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 1 & . & . & . & 0 \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & . & . & . & . & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ b_0 \end{bmatrix} U \quad Y = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Consider a transfer function with polynomial in numerator

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots\dots + a_{n-1}\, \dot{y} + a_n y = b_0 u^n + b_1 u^{n-1} + \dots..+ b_{n-1}\, \dot{u} + b_n u$$

where u is the input and y is the output. The transfer function can be written as

$$\frac{Y(S)}{U(S)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots.b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

State space representation of the system in controllable canonical form and observable canonical form are given by

Controllable Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & . & . & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & . & . & . & . & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} U \quad y = [b_n - a_n b_0 : b_{n-1} - a_{n-1}\, b_0 : \dots\dots : b_1 - a_1\, b_0] \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} + b_0 u.$$

The controllable canonical form is important in discussing the pole placement approach to the control system design.

Observable Canonical Form

The following state-space representation is called an observable canonical form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ . \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots\ldots & 0 & -a_n \\ 1 & 0 & \ldots\ldots & 0 & -a_{n-1} \\ . \\ . \\ . \\ . \\ . \\ 0 & 0 & \ldots\ldots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ .. \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ . \\ . \\ . \\ . \\ . \\ b_1 - a_1 b_0 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 & 0 & . & . & . & .0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

System matrix in observable canonical form is the transpose of system matrix in controllable canonical form.

Diagonal Canonical Form

Transfer function with numerator polynomial can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots b_{n-1} s + b_n}{(s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n)}$$

$$= b_0 + \frac{C_1}{s+p_1} + \frac{C_2}{s+P_2} + \ldots \frac{C_n}{s+P_n}$$

where $p_1, p_2, \ldots p_n$ are the location of poles.

The diagonal canonical form of the state space representation of this system is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} -p_1 & . & \ldots\ldots & 0 \\ 0 & -p_2 & \ldots\ldots & 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ . \\ .. \\ . \\ 1 \end{bmatrix} u \quad y = [C_1 \; C_2 \ldots\ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ .. \\ . \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form

When the system involves multiple roots, the diagonal canonical form must be modified into Jordon canonical form. For example, if there are three equal poles ($P_1 = P_2 = P_3$). Then the factored form $Y(s)/U(s)$ becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{(s+P_1)^3} + \frac{C_2}{(s+p_1)^2} + \frac{C_3}{(s+p_1)} + \frac{C_4}{s+P_4} + \ldots + \frac{C_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ . \\ . \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & -p_1 & 1 & 1. & 0 & . & . & . & \\ 0 & 0 & -p_1 & 1 & 0 & . & . & . & 0 \\ - & - & - & - & & - & - & - & . \\ 0 & 0 & 0 & . & -p_4 & . & . & . & 0 \\ . & & & . & & & & & \\ . & . & & . & & & & & \\ . & & & . & & & & & \\ . & . & & . & & & & & \\ 0 & 0 & 0 & . & 0 & . & . & . & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ . \\ . \\ .. \\ . \\ 1 \end{bmatrix} u \quad y = [C_1 \; C_2 \ldots\ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ .. \\ . \\ x_n \end{bmatrix} + b_0 u.$$

Eigen Values of an $n \times n$ Matrix A

Characteristic equation of a square matrix 'A' is given by

$$|\lambda I - A| = 0$$

The values of λ which satisfies the characteristic equation are called as eigen values of 'A' matrix.

Note:
The poles of the transfer function are given by the $|sI - A| = 0$. This function is same equation as the characteristic equation of 'A'. Therefore, we can conclude that the eigen values of the state model and the poles of the transfer function are the same.

Note:
Stability of SISO system depends on the eigen values of system matrix in its state space model.

Controllability

A state $x(t)$ is said to be controllable at $t = t_0$ if there exist a piecewise continuous input u(t) that will drive the state to any final state $x(t_f)$ for a finite time $(t_f - t_0) \geq 0$.

If every state $x(t_0)$ of the system is controllable in a finite time interval, the system is said to be completely controllable or simply, controllable.

Kalman's Test for Controllability

Consider a state space model

$$\dot{x} = Ax + Bu$$

Then controllability matrix

$$Q_C = [B{:}AB{:}A^2B{:}\ \ldots{:}\ A^{n-1}B]$$

Note:
A system is said to be controllable if the rank of Q_C is equal to the order of matrix 'A'.

i.e., $|Q_C| \neq 0.$

Note:
Number of uncontrollable states is computed as the difference between order of A matrix (n) and rank of Q_C matrix (P).

Observability

Linear time invariant system is said to be observable if given any input $u(t)$, there exist a finite time $t_f \geq t_0$ such that the knowledge of $u(t)$ for $t_0 \leq t, < f_f$, matrix A, B, C and D and the output $y(t)$ for $t_0 \leq t < t_f$ are sufficient to determined $x(t_0)$. If a every state of the system is observable for a finite t_f, we say that the system is completely observable or simply observable.

Kalman's Test for Observability

Consider a state space model

$$\dot{x} = Ax + Bu \text{ and } y = Cx + Du$$

Then observability matrix

$$Q_0 = [C\ CA\ CA^2\ \ldots\ldots CA^{n-1}]^T$$

Note:
The system is said to be observable if the rank of Q_0 is equal to the order of matrix A.

i.e., $|Q_c| \neq 0.$

Note:
Number of unobservable states is computed as the difference between order of A matrix (n) and rank of Q_0 matrix (q).

Solution of State Equations

The state equations of a linear system is

$$\dot{x} = Ax + Bu;\ x(t_0) = x(0)$$

Solution of above state equation is

$$x(t) = \underbrace{e^{At}\,x(0)}_{\substack{\text{Homogeneous}\\ \text{Solution}}} + \int_0^t \underbrace{e^{-A(t-\tau)}Bu(\tau)\,.d\tau}_{\substack{\text{Forced}\\ \text{Solution}}}$$

In the absence of the input to the system, response of the system or solution of state equations, with initial conditions alone is given by

$$X(t)\ e^{At}\ x(0) = e^{At}x_0$$

It is observed that the initial state x_0 at $t = 0$ is driven to a state $x(t)$ at time 't'. This transition in state is carried out by the matrix exponential e^{At}. Because of this property, e^{At} is known as 'State transition matrix' and is denoted by $\phi(t)$.

Properties of State Transition Matrix $\phi(t)$

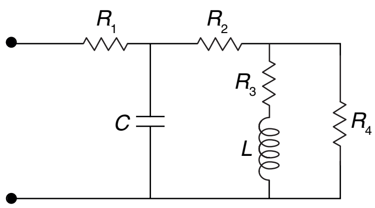
1. $\phi(0) = e^{A0} = I$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1}\ .\ e^{At_2}$
   $= \phi(t_1).\ \phi(t_2) = \phi(t_2).\ \phi(t_1)$
4. $[\phi(t)]^n = [e^{At}]^n = e^{An}+ = \phi(nt)$
5. $\phi(t_1 - t)\ .\ \phi(t - t_2) = e^{A(t1-t)}\ .\ e^{A(t-t2)}$
   $= e^{A(t1-t2)} = \phi(t_1 - t_2)$

Transfer Function

Given the state space model of SISO system as

$\dot{x} = Ax + Bu$
$Y = Cx + Du$
The transfer function of the system is
T. F. $= C[sI - A]^{-1}B + D$

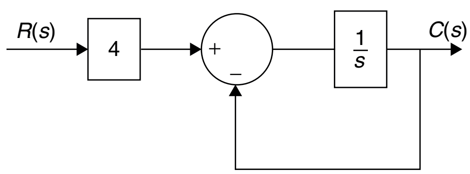
Example 2: The maximum number of states required to describe the network shown in figure is



(A) 1 (B) 2
(C) 3 (D) 4

Solution: (B)
No. of energy storage elements (L, C) = 2
No. of states required to analysis = 2

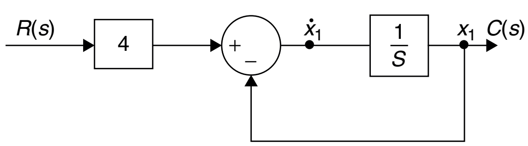
Example 3: The matrix of any state space equation for the transfer function $\frac{C(s)}{R(s)}$ of the system shown in figure is



(A) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (B) $[-1]$

(C) $[4]$ (D) $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

Solution: (B)
Output of integrator is considered as state and no. of integrators is equal to the no. of states.
No. of states = 1



$\dot{x}_1 = -x_1 + 4r(t)$
$[x^\circ{}_1] = [-1]\,[x_1] + [4]\,[r(t)]$
Matrix of state space equation is [–1]

Example 4: A system is described by the state equations. $\dot{x} = Ax + Bu$. The output is given by $y = Cx$

where $A = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

The transfer function $G(S)$ of the system is

(A) $\frac{s}{s^2+5s+7}$ (B) $\frac{s+7}{s^2+5s+7}$

(C) $\frac{1}{s^2+5s+7}$ (D) $\frac{s}{s^2+s+5}$

Solution: (B)
Transfer function of the state space model is

$$\text{T. F} = \frac{Y(s)}{U(s)} = C[s\text{I} - A]^{-1}B + D$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s+1)(s+4)}\begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}$$

$$\text{T. F} = -C\,[sI - A]^{-1}B = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{s^2+5s+7}$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+1-1 \\ 3+s+4 \end{bmatrix}}{s^2+5s+7}$$

$$= \frac{s+7}{s^2+5s+7}.$$

Example 5: Given that homogeneous state space equations $\dot{x} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}x$, the steady-state value of $x_{ss} = \underset{t\to\infty}{\text{Lt}}\, x(t)$, given the initial state value of $x[0] = [10 \; -10]^T$ is

(A) $x_{ss} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$ (B) $x_{ss} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$

(C) $x_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (D) $x_{ss} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$

Solution: (C)
Solution of homogenous equation $A\dot{x} = Ax$

$$x(t) = e^{At}\,x(0)$$

$$e^{At} = L\left[[sI - A]^{-1}\right]$$

$$sI - A = \begin{bmatrix} s+4 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{-1}{(s+4)(s+3)}\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+4)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$$

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right] = \begin{bmatrix} e^{-4t} & e^{-3t} - e^{-4t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 10e^{-4t} + 10e^{-3t} + 10e^{-4t} \\ -10e^{-3t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 20e^{-4t} - 10e^{-3t} \\ -10^{-3t} \end{bmatrix}$$

$$x_{ss} = \underset{t\to\infty}{\text{Lt}}\, x(t) = \begin{bmatrix} \underset{t\to\infty}{\text{Lt}} \left(20e^{-4t} - 10e^{-3t}\right) \\ \underset{t\to\infty}{\text{Lt}} - 10e^{-3t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 6: A second-order system starts with an initial condition of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ without any external input. The state transition matrix for the system is given by $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$

The state of the system at end of 2 seconds is given by

(A) $10^{-3} \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (B) $10^{-3} \times \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (D) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Solution: (B)

Solution of state equations without external input is

$$x(t) = e^{At} \times (0)$$

$$= \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 2e^{-3t} \\ 3e^{-4t} \end{bmatrix}$$

At $t = 2$ sec, $x(2) = \begin{bmatrix} 2e^{-3\times2} \\ 3e^{-4\times2} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.10^{-3}$

Example 7: For a system with transfer function $G(S) = \dfrac{2s+4}{s^3+4s^2+9s+4}$

The matrix 'A' in the state space form $\dot{x} = Ax + Bu$ is equal to

(A) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -9 & -4 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 & 0 \\ -2 & -4 & 1 \\ -4 & -9 & -4 \end{bmatrix}$

Solution: (A)

From the standard controllable canonical form of the transfer function

$$\text{T.F.} = \frac{2s+4}{s^3+4s^2+9s+4}$$

State space representation

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t).$$

Example 8: For the system $\dot{x} = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, which of the following statement about the system is true?

(A) Controllable and stable
(B) Uncontrollable and stable
(C) Controllable and unstable
(D) Uncontrollable and unstable

Solution: (D)

For stability analysis, location of poles are the roots of characteristic equation.

$$|sI - A| = \begin{vmatrix} s-3 & -4 \\ 0 & s-5 \end{vmatrix} = (s-3)(s-5) = 0$$

$$s = 3 \text{ and } 5$$

∴ Poles are located on RHS plane, system is unstable.

Controllability matrix $Q_C = \begin{bmatrix} B & AB \end{bmatrix}$

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$Q_C = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow |Q_C| = 0$$

∴ System is uncontrollable.

Exercises

Practice Problems I

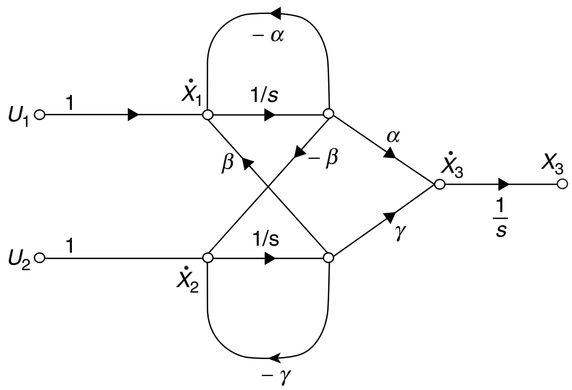
Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. Given that $\dot{X} = AX$ for the system described by the differential equation $\ddot{y} + 2\dot{y} + 3y = 0$. The matrix 'A' is

(A) $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ -1 & -3 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$

2. A signal flow graph of a system is given below



If the state equation that corresponds to the above signal flow graph is $\dot{X} = AX + BU$, the matrices 'A' and 'B' are

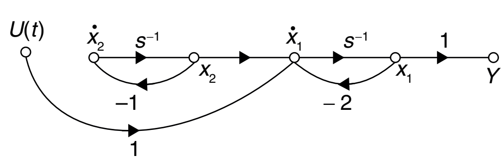
(A) $\begin{bmatrix} \beta & -\gamma & 0 \\ \gamma & \alpha & 0 \\ -\alpha & -\beta & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & \alpha & \gamma \\ 0 & -\alpha & -\gamma \\ 0 & \beta & -\beta \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} -\alpha & 0 & \beta \\ \gamma & 0 & \alpha \\ -\beta & 0 & -\alpha \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

3.



In the state diagram of a system shown in the above figure, which variables are controllable?

(A) $x_1(t)$
(B) $x_2(t)$
(C) Both $x_1(t)$ and $x_2(t)$
(D) Neither $x_1(t)$ nor $x_2(t)$

4. The state equations of a linear time-invariant system are represented by $\dfrac{dx(t)}{dt} = AX(t) + BU(t)$.

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state transition matrix $\phi(t)$ is

(A) $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t/3} & 0 \\ 0 & e^{t/3} \end{bmatrix}$ (D) $\begin{bmatrix} e^{t/3} & 0 \\ 0 & e^{-t/3} \end{bmatrix}$

5. A particular control system is described by the following state equations

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{U and}$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transfer function of this system is

(A) $\dfrac{Y(s)}{U(s)} = \dfrac{2}{s^2+2s+5}$ (B) $\dfrac{Y(s)}{U(s)} = \dfrac{4}{s^2+2s+5}$

(C) $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2+2s+5}$ (D) $\dfrac{Y(s)}{U(s)} = \dfrac{3}{s^2+2s+5}$

Common Data for Questions 6 and 7: A system is characterized by the following state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U, (t > 0)$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

6. The transfer function of the system is

(A) $\dfrac{s}{(s+2)(s+1)}$ (B) $\dfrac{1}{s(s+2)(s+1)}$

(C) $\dfrac{s}{(s-2)(s+1)}$ (D) $\dfrac{1}{(s+2)(s+1)}$

7. The state transition matrix of the system is

(A) $\begin{bmatrix} e^{-t}+2e^{-2t} & e^{-t}+e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}+e^{-2t} \end{bmatrix}$

(B) $\begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t}-2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$

(D) $\begin{bmatrix} e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}-2e^{-2t} & 2e^{-t}+e^{-2t} \end{bmatrix}$

8. Match List-I (Matrix) with List-II (dimensions) for the state equations:

$\dot{X}(t) = AX(t) + BU(t)$ and $Y(t) = CX(t) + DU(t)$ and select the correct answer using the codes given below the lists:

List-I	List-II
A	(1) $n \times p$
B	(2) $q \times n$
C	(3) $n \times n$
D	(4) $q \times p$

Codes:

	A	B	C	D
(A)	1	3	4	2
(B)	1	3	2	4
(C)	3	1	4	2
(D)	3	1	2	4

9. Consider the single-input, single-output system with its state variable representation:

$$\dot{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} U, \; Y = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} X$$

The system is

(A) Neither controllable nor observable.
(B) Controllable but not observable.
(C) Uncontrollable but observable.
(D) Both controllable and observable.

10. Consider the state transition matrix: $\phi(t) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

The eigen values of the matrix when $t = 0$ are

(A) 1, 2 (B) 2, 1
(C) 1, 1 (D) 1, 3

11. A linear system is described by the following state equation $-\dot{X} = AX(t) + BU(t)$.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The state transition matrix of the system is

(A) $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

(B) $\begin{bmatrix} -\cos t & \sin t \\ -\sin t & -\cos t \end{bmatrix}$

(C) $\begin{bmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{bmatrix}$

(D) $\begin{bmatrix} \cos t & -\sin t \\ \cos t & \sin t \end{bmatrix}$

12. Consider the system $\frac{dx}{dt} = AX + BU$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix}$, where p and q are arbitrary real numbers. Which of the following statements about the controllability of the system is true?

(A) The system is completely controllable for any non-zero values of p and q.
(B) Only $p = 0$ and $q = 0$ result in controllability.
(C) The system is uncontrollable for all values of p and q.
(D) We cannot conclude about controllability from the given data.

13. The state transition matrix of the system whose state equation are $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(A) $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \sin\sqrt{2}t \end{bmatrix}$

(B) $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}$

(C) $\begin{bmatrix} \sin t & -\cos t \\ +\cos t & \sin t \end{bmatrix}$

(D) $\begin{bmatrix} \sin\sqrt{2}t & \cos\sqrt{2}t \\ -\cos\sqrt{2}t & \sin\sqrt{2}t \end{bmatrix}$

14. If the eigen values of a 3×3 matrix A are 1, –3 and 4. The eigen values of $P^{-1}AP$ (Q P is a linear transformation) are

(A) 1, $\frac{-1}{3}$ and $\frac{1}{4}$
(B) 1, – 3 and 4
(C) 1, 9 and 16
(D) –1, 3 and – 4

15. The state equations of a system are $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$. The closed loop poles of the system are

(A) ± 2 (B) – 1, – 1
(C) + 1, – 2 (D) + 2, – 1

Practice Problems 2

Directions for questions 1 to 15: Select the correct alternative from the given choices.

1. The state equation of a linear system is given by $\dot{X} = AX + BU$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The state transition matrix of the system is
   (A) $\begin{bmatrix} e^{-t} & e^{-t}+e^{-2t} \\ 0 & e^{2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-t} & e^{-t}-e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$
   (C) $\begin{bmatrix} 0 & e^{-2t} \\ e^{-t} & e^{-t}-e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} 0 & e^{-t}-e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix}$

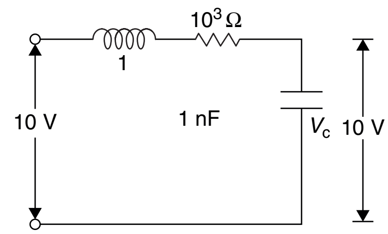
2. The eigen value and eigen vector pairs (λ_i, V_i) for the system are
   (A) $\left[-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[-2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$
   (B) $\left[-2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[-1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$
   (C) $\left[-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$
   (D) $\left[2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right]$ and $\left[1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$

3. The system matrix A is:
   (A) $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$
   (C) $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

4. Let $X = \dot{X} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$, $Y = \begin{bmatrix} b & 0 \end{bmatrix} X$
   where 'b' is an unknown constant. This system is
   (A) Observable for all values of b.
   (B) Unobservable for all values of b.
   (C) Observable for all non-zero values of b.
   (D) Unobservable for all non-zero values of b.

5. A state variable representation of a system is given by the expression
   $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t),\ Y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   The transfer function of the system is
   (A) $\dfrac{2}{s+1}$ (B) $\dfrac{2s}{(s-1)(s+1)}$
   (C) $\dfrac{2}{(s-1)(s+1)}$ (D) $\dfrac{1}{(s+1)}$

6. The state space model for an electrical network shown in fig where the current I and voltage across the capacitor V_c are the state variables.

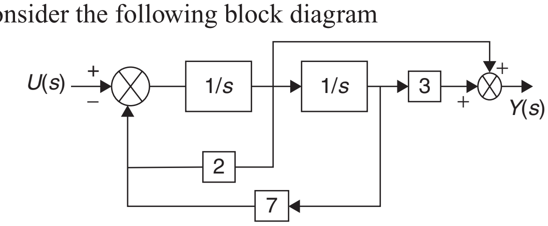


   (A) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 10^6 & 10^3 \\ 10^9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10^3 \end{bmatrix} \times 10$
   (B) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 10^9 & 0 \\ 10^6 & 10^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10^3 \end{bmatrix} \times 10$
   (C) $\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -10^6 & -10^3 \\ 10^9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10^3 \\ 0 \end{bmatrix} \times 10$
   (D) None of these

7. The poles and zero of the following system:
   $$\dot{X} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
   $y = [-17\ -5]\,x + [1]\,u$ are
   (A) Poles: – 4, – 5 and zeros: – 1, – 3
   (B) Poles: – 3, – 4, and zeros: – 3, – 2
   (C) Poles: – 2, – 3 and zeros: – 2, – 4
   (D) None of these

8. The second-order system $\dot{x} = Ax$ has $A = \begin{bmatrix} -1 & -2 \\ +1 & -1 \end{bmatrix}$.
   The values of its damping factor and natural frequency ω_n are, respectively,
   (A) 1.732, 0.577 (B) 1.414, 0.6
   (C) 0.577, 1.732 (D) 0.6, 1.414

Common Data for Questions 9 and 10:
Consider the following block diagram



9. In the above figure, the state space representation in vector matrix from is
   (A) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$Y = [3\ 1]\,x$

(B) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$Y = [3\ 1]\,x$

(C) $\dot{x} = \begin{bmatrix} 1 & 0 \\ -2 & -7 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$Y = [1\ 3]\,x$

(D) None of these

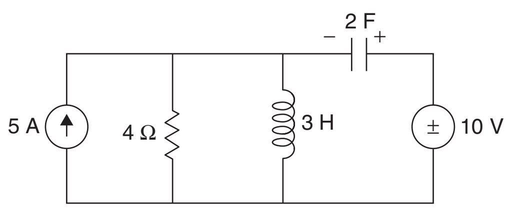
10. From the question number (9), the transfer function of the system is

(A) $\dfrac{s}{s^2 + 2s + 7}$ (B) $\dfrac{s+3}{s^2 + 2s + 7}$

(C) $\dfrac{3}{s^2 + 2s + 7}$ (D) $\dfrac{2s+3}{s^2 + 2s + 7}$

Common Data for Questions 11 and 12:

The circuit diagram is shown in the following figure.



11. The state equations of the above circuit are

(A) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1}{3} \\ \dfrac{-1}{2} & \dfrac{-1}{8} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} \dfrac{1}{3} & 0 \\ \dfrac{1}{8} & \dfrac{-1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(B) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1}{3} \\ \dfrac{1}{2} & \dfrac{-1}{8} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} \dfrac{1}{3} & 0 \\ \dfrac{1}{8} & \dfrac{-1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(C) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{3} & 0 \\ \dfrac{1}{8} & \dfrac{-1}{2} \end{bmatrix} \begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 0 & \dfrac{1}{3} \\ \dfrac{1}{8} & \dfrac{-1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(D) None of these

12. From the Q. No. (11), the eigen values are

(A) $-0.06 \pm j0.06$
(B) $-0.06 \pm j0.403$
(C) $-0.403 \pm j0.06$
(D) $-0.12 \pm j0.12$

13. A continuous time, linear time invariant system is described by $\dfrac{d^2y}{dt} + 4\dfrac{dy}{dt} + 3y(t) = 2\dfrac{dx}{dt} + 4x$.

Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}\,u(t)$ is given by

(A) $(e^{-t} - e^{-3t})\,u(t)$
(B) $(e^{-t} + e^{-3t})\,u(t)$
(C) $(e^{t} - e^{+3t})\,u(t)$
(D) $(e^{t} + e^{3t})\,u(t)$

14. The system with the state equation

$\dot{x} = Ax + Bu$ and $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(A) System is controllable.
(B) System is uncontrollable.
(C) System is controllable and observable.
(D) None of these

15. Consider the state transition matrix:

$$\phi(t) = L^{-1}\,[\phi(s)] = L^{-1} \begin{bmatrix} \dfrac{s+1}{s^2+3s+2} & \dfrac{-1}{s^2+3s+2} \\ 0 & \dfrac{s+2}{s^2+3s+2} \end{bmatrix}$$

The eigen values of the system are

(A) 0 and – 2
(B) – 1 and 2
(C) 1 and 2
(D) – 1 and – 2

Previous Years' Questions

Data for Question 1:

A state variable system

$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$, with the initial condition $X(0) = [-1\ 3]^{\mathrm{T}}$ and the unit step input $u(t)$ has

1. And the state transition equation **[2005]**

(A) $X(t) = \begin{bmatrix} t - e^{-t} \\ e^{-t} \end{bmatrix}$ (B) $X(t) = \begin{bmatrix} t - e^{-t} \\ 3e^{-3t} \end{bmatrix}$

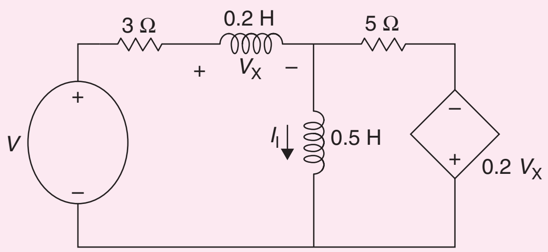
(C) $X(t) = \begin{bmatrix} t - e^{-3t} \\ 3e^{-3t} \end{bmatrix}$ (D) $X(t) = \begin{bmatrix} t - e^{-3t} \\ e^{-t} \end{bmatrix}$

2. For a system with the transfer function $H(s) = \dfrac{3(s-2)}{4s^2-2s+1}$, the matrix A in the state space form $x = Ax + Bu$ is equal to **[2006]**

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -4 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 1 \\ 1 & -2 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -4 \end{bmatrix}$

3. The state equation for the current I_1 shown in the network shown below in terms of the voltage V_x and the independent source V, is given by **[2007]**



(A) $\dfrac{dI_1}{dt} = -1.4V_x - 3.75I_1 + \dfrac{5}{4}V$

(B) $\dfrac{dI_1}{dt} = -1.4V_x - 3.75I_1 - \dfrac{5}{4}V$

(C) $\dfrac{dI_1}{dt} = -1.4V_x + 3.75I_1 + \dfrac{5}{4}V$

(D) $\dfrac{dI_1}{dt} = -1.4V_x + 3.75I_1 - \dfrac{5}{4}V$

Common Data for Questions 4 and 5:

A system is described by the following state and output equations.

$$\frac{dx_1(t)}{dt} = -3x_1(t) + x_2(t) + 2u(t)$$

$$\frac{dx_2(t)}{dt} = -2x_2 + u(t)$$

$$y(t) = x_1(t)$$

where $u(t)$ is the input and $y(t)$ is the output

4. The system transfer function is **[2009]**

(A) $\dfrac{s+2}{s^2+5s-6}$ (B) $\dfrac{s+3}{s^2+5s+6}$

(C) $\dfrac{2s+5}{s^2+5s+6}$ (D) $\dfrac{2s-5}{s^2+5s-6}$

5. The state transition matrix of the above system is **[2009]**

(A) $\begin{bmatrix} e^{-3t} & 0 \\ e^{-2t}+e^{-3t} & e^{-2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-3t} & e^{-2t}-e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-3t} & e^{-2t}+e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} e^{3t} & e^{-2t}-e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$

6. The second-order dynamic system

$$\frac{dX}{dt} = PX + Qu$$

$$y = RX$$

has the matrices P, Q and R as follows:

$$P = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix} R = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The system has the following controllability and observability properties: **[2014]**

(A) Controllable and observable
(B) Not controllable but observable
(C) Controllable but not observable
(D) Not controllable and not observable

7. Consider the system described by the following state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u; y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If u is unit step input, then the steady-state error of the system is **[2014]**

(A) 0 (B) 1/2 (C) 2/3 (D) 1

8. The state variable description of an LTI system is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where y is the output and u is the input. The system is controllable for **[2012]**

(A) $a_1 \neq 0, a_2 = 0, a_3 \neq 0$
(B) $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$
(C) $a_1 = 0, a_2 = 0, a_3 = 0$
(D) $a_1 \neq 0, a_2 \neq 0, a_3 = 0$

Common Data for Questions 9 and 10:
The state variable formulation of a system is given as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \; x_1(0) = 0, \; x_2(0) = 0$$

and $\quad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

9. The system is **[2013]**
(A) Controllable but not observable
(B) Not controllable but observable
(C) Both controllable and observable
(D) Both not controllable and observable

10. The response $y(t)$ to a unit step input is **[2013]**
(A) $\frac{1}{2} - \frac{1}{2}e^{-2t}$
(B) $1 - \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}$
(C) $e^{-2t} - e^{-t}$
(D) $1 - e^{-t}$

11. For the system governed by the set of equations:
$dx_1/dt = 2x_1 + x_2 + u$
$dx_2/dt = -2x_1 + u$
$y = 3x_1$
the transfer function $Y(s)/U(s)$ is given by **[2015]**
(A) $3(s + 1)/(s^2 - 2s + 2)$
(B) $3(2s + 1)/(s^2 - 2s + 1)$
(C) $(s + 1)/(s^2 - 2s + 1)$
(D) $3(2s + 1)/(s^2 - 2s + 2)$

12. Consider the following state-space representation of a linear time-invariant system. **[2016]**

$x(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x(t), \; y(t) = c^T x(t), \; c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the value of $y(t)$ for $t = \log_e 2$ is ______.

Answer Keys

Exercises

Practice Problems 1

1. D **2.** C **3.** A **4.** B **5.** C **6.** D **7.** B **8.** C **9.** A **10.** C
11. A **12.** C **13.** B **14.** B **15.** B

Practice Problems 2

1. B **2.** A **3.** D **4.** C **5.** D **6.** C **7.** A **8.** C **9.** B **10.** B
11. B **12.** B **13.** A **14.** B **15.** D

Previous Years' Questions

1. C **2.** B **3.** A **4.** C **5.** B **6.** C **7.** A **8.** D **9.** A **10.** A
11. A **12.** 6

Test
Control Systems

Time: 60 min.

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. The time response of a first-order control system subjected to unit impulse input is

(A) $C_{(t)} = \frac{1}{T}e^{-t/T}$ (B) $C_{(t)} = e^{-t/T}$

(C) $C_{(t)} = 1 - e^{-t/T}$ (D) $C_{(t)} = T - e^{-t/T}$

2. For a second-order system with the closed-loop transfer function $T(s) = \frac{16}{s^2+4s+16}$. The damped frequency of oscillations is

(A) 4 rad/sec (B) $4\sqrt{3}$ rad/sec

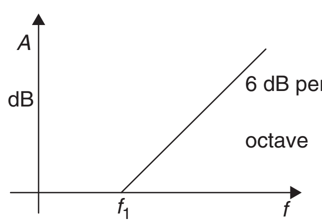
(C) $2\sqrt{3}$ rad/sec (D) 2 rad/sec

3. For a unity feedback closed loop system $G(s) = \frac{K}{s(s+4)(s+3)}$. The range of K for which the system is stable will be given by

(A) $K > 84$ (B) $0 < K < 84$

(C) $0 > K > 84$ (D) $0 < K > 84$

4. The function corresponding to the Bode plot of the figure is



(A) $A = \frac{jf}{f_1}$ (B) $A = \frac{1}{(1 - jf_1/f_1)}$

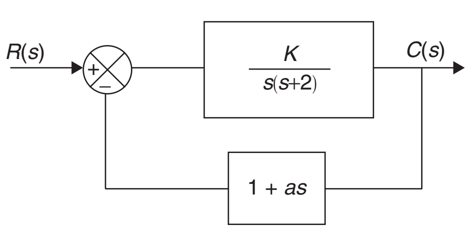
(C) $A = \frac{1}{(1 + jf_1/f)}$ (D) $A = 1 + jf/f_1$

5. Given a unity feedback system with $G(s) = \frac{K}{s(s+4)}$, the value of K for damping ratio of 0.5 is

(A) 1 (B) 16

(C) 4 (D) 2

6. For the system shown in the figure, with a damping ratio ξ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec, the values of K and 'a' are



(A) $K = 4$, $a = 0.35$ (B) $K = 8$, $a = 0.455$

(C) $K = 16$, $a = 0.225$ (D) $K = 64$, $a = 0.9$

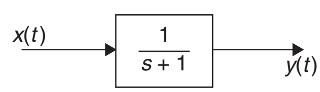
7. For an RLC series circuit, match list I with list II and select the correct answer using the codes given below the lists:

List – I (condition)	**List – II (Transient current response)**
(P) $R = 0$	(1) Undamped oscillations
(Q) $R < 2\sqrt{\frac{L}{C}}$	(2) Damped oscillations
(R) $R = 2\sqrt{\frac{L}{C}}$	(3) Critically damped response
(S) $R > 2\sqrt{\frac{L}{C}}$	(4) Non – oscillatory response

Codes:

	P	Q	R	S
(A)	1	2	3	4
(B)	1	4	3	2
(C)	3	2	1	4
(D)	3	4	1	2

8. In the system shown below, $x(t) = \sin t\, u(t)$. In steady-state, the response $y(t)$ will be



(A) $\frac{1}{\sqrt{2}}\sin\left[t - \frac{\pi}{4}\right]$ (B) $\frac{1}{\sqrt{2}}\sin\left[t + \frac{\pi}{4}\right]$

(C) $\frac{1}{\sqrt{2}}e^{-t}\sin t$ (D) $\sin t - \cos t$

9. A differentiator has a transfer function whose
(A) Phase increases linearly with frequency
(B) Amplitude remains constant
(C) Amplitude increases linearly with frequency
(D) Amplitude decreases linearly with frequency

10. If $f(s) = L[f(t)] = \frac{3(s+2)}{s^2+6s+12}$, then the initial and final values of $f(t)$ are respectively

(A) 0, 3 (B) 3, 0

(C) 0, 1/2 (D) $\frac{1}{2}$, 0

11. The transfer function of a system is given by

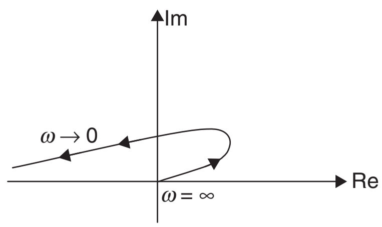
$$\frac{Y_{(s)}}{X_{(s)}} = \frac{1}{s^3+8s^2+19s+12}$$

Obtain the pole – zero form representation of matrix A

(A) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -19 & -8 \end{bmatrix}$ (B) $A = \begin{bmatrix} -8 & -19 & -12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

12. The Nyquist plot of a control system is shown below. For this system, $G(s)H(s)$ is equal to



(A) $\dfrac{K}{(l+sT_1)}$ (B) $\dfrac{K}{s^2(l+sT_1)}$

(C) $\dfrac{K}{s^3(l+sT_1)}$ (D) $\dfrac{K}{s^2(l+sT_1)\,(l+sT_2)}$

13. A system described by the transfer function $H(S) = \dfrac{1}{s^3+\alpha s^2+Ks+4}$ is stable. The constraints on a and K are

(A) $\alpha > 0$, $\alpha K < 4$ (B) $\alpha > 0$, $\alpha K > 4$
(C) $\alpha > 0$, $\alpha K > 0$ (D) $\alpha < 0$, $\alpha K < 0$

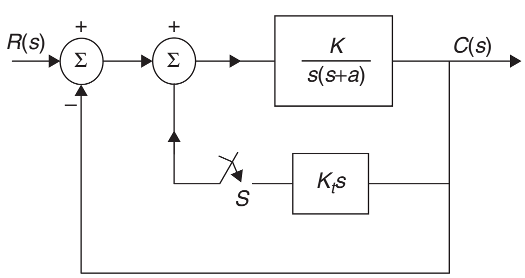
14. A system has an open-loop transfer function of $\dfrac{10}{s(1+0.5s)(1+0.2s)}$ and is has a feedback whose transfer function is $H(s) = 1 + Ts$. What should be the value of T so that the closed-loop system operates in a stable manner?

(A) $T \geq 23.33$ (B) $T \geq 2.333$
(C) $T \geq 0.42$ (D) $T \geq 0.042$

15. The number of roots of the equation $2s^4 + s^3 + 3s^2 + 5s + 7 = 0$ that lie in the right half of S plane is

(A) Zero (B) One
(C) Two (D) Three

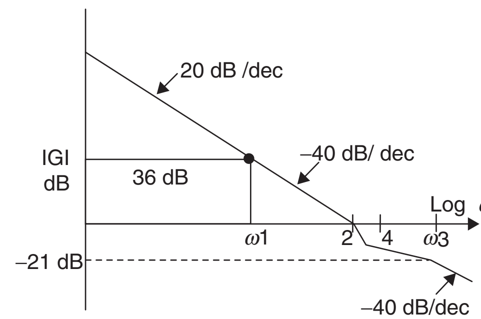
16. The system shown in the given figure, is subjected to a unit ramp input



By closing the switch 'S'

(A) Steady-state error will increase and damping coefficient 'ζ' will decrease
(B) Both steady-state error and damping coefficient 'ζ' will increase
(C) Both steady-state error and damping coefficient 'ζ' will decrease
(D) Steady-state error will decrease and damping coefficient 'ζ' will increase

17. The asymptotic magnitude Bode plot of a system is given in the figure. The system is a minimal phase system. The transfer function of the system is



(A) $\dfrac{K\left(1+\dfrac{s}{\omega_2}\right)}{\left(1+\dfrac{s}{\omega_1}\right)\left(1+\dfrac{s}{\omega_3}\right)}$ (B) $\dfrac{K\left(1+\dfrac{s}{\omega_1}\right)}{s\left(1+\dfrac{s}{\omega_2}\right)\left(1+\dfrac{s}{\omega_3}\right)}$

(C) $\dfrac{K\left(1+\dfrac{s}{\omega_2}\right)}{s\left(1+\dfrac{s}{\omega_1}\right)\left(1+\dfrac{s}{\omega_3}\right)}$ (D) $\dfrac{K\left(1+\dfrac{s}{\omega_1}\right)}{\left(1+\dfrac{s}{\omega_2}\right)\left(1+\dfrac{s}{\omega_3}\right)}$

18. The open loop transfer function of a feed back control system is given by

$$G(s)\,H(s) = \frac{K(s+5)}{s\,(s+6)\,(s^2-6s+10)}$$

In the root locus diagram, the asymptotes of the root loci for large values of K meet at

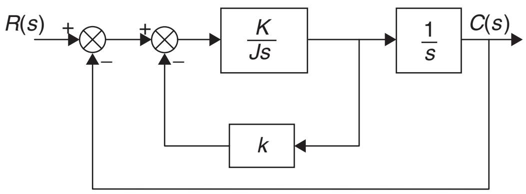
(A) $\left(\dfrac{4}{3}, 0\right)$ (B) $\left(\dfrac{-5}{3}, 0\right)$

(C) $\left(\dfrac{5}{3}, 0\right)$ (D) $\left(\dfrac{-10}{3}, 0\right)$

19. A servo mechanism represented by the equation $\dfrac{d^2\theta}{dt^2} + 10\dfrac{d\theta}{dt} = 150\,E$, where $E = (r - \theta)$ is the actuating signal. The value of damped frequency of oscillation is

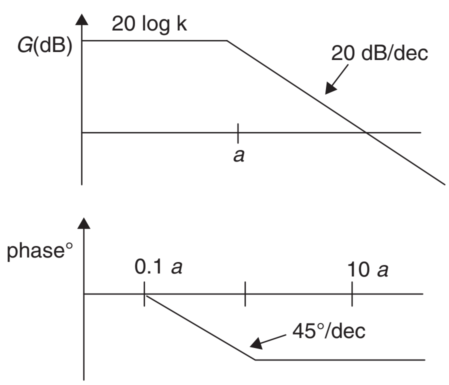
(A) 11.17 rad/sec (B) 12.25 rad/sec
(C) 13.62 rad/sec (D) 9.81 rad/sec

20. A closed loop control system is shown in the figure. The maximum overshoot in the unit step response is to be limited to 25% and the peak time is 2 sec. Assuming the inertia constant $J = 1$ kg–m^2, the value of 'K' and 'k' are respectively



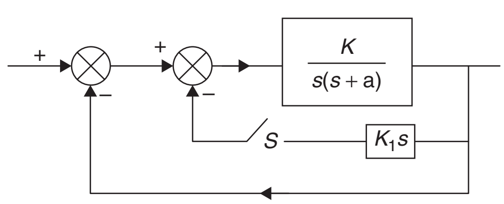
(A) 4.62, 0.471 (B) 2.95, 0.471
(C) 4.62, 0.961 (D) 2.95, 0.961

21. The asymptotic Bode plot of the transfer function $\dfrac{K}{1+\dfrac{s}{a}}$ is given in the figure. The error in the phase angle and dB gain at a frequency of $\omega = 0.5a$ are respectively



(A) 4.06°, 0.97 dB (B) 5.7°, 3 dB
(C) 4.9°, 3dB (D) 5.7°, 0.97 dB

22. The system shown in the given figure, is subjected to a unit ramp input on closing the switch 's'



(A) Steady-state error will increase and damping co-efficient 'ζ' will decrease
(B) Both steady-state error and damping coefficient 'ζ' will increase
(C) Both steady-state error and damping coefficient 'ζ' will decrease
(D) Steady-state error will decrease and damping co-efficient 'ζ' will increase

23. A system is described by

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U,\ Y = \begin{bmatrix} 1 & 0 \end{bmatrix} X$$

The transfer functions is

(A) $\dfrac{1}{s^2+2s+3}$ (B) $\dfrac{1}{2s^2+3s+1}$

(C) $\dfrac{s+3}{s^2+3s+2}$ (D) $\dfrac{1}{3s^2+2s+1}$

Common Data for Questions 24 and 25:

A system is described by the following state and output equations

$$\frac{dx_1(t)}{dt} = -3x_1(t) + x_2(t) + 2u(t)$$

$$\frac{dx_2(t)}{dt} = -2x_2(t) + u(t)$$

$$y(t) = x_1(t)$$

where $u(t)$ is the input and $y(t)$ is the output.

24. The system transfer function is ________.

(A) $\dfrac{s+2}{s^2+5s-6}$ (B) $\dfrac{s+3}{s^2+5s+6}$

(C) $\dfrac{2s+5}{s^2+5s+6}$ (D) $\dfrac{2s-5}{s^2+5s-6}$

25. The state-transition matrix of the system is _______.

(A) $\begin{bmatrix} e^{-3t} & 0 \\ e^{-2t}+e^{-3t} & e^{-2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{3t} & e^{-2t}-e^{3t} \\ 0 & e^{2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-3t} & e^{-2t}+e^{3t} \\ 0 & e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} e^{-3t} & e^{-2t}-e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$

Answer Keys

1. A	**2.** C	**3.** B	**4.** D	**5.** B	**6.** C	**7.** A	**8.** A	**9.** C	**10.** B
11. C	**12.** D	**13.** B	**14.** D	**15.** C	**16.** B	**17.** C	**18.** C	**19.** A	**20.** B
21. A	**22.** B	**23.** C	**24.** C	**25.** D					