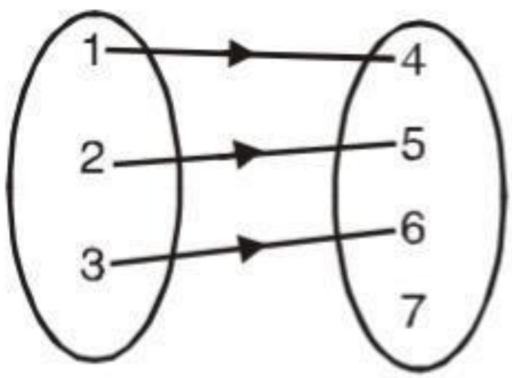
FUNCTIONS, LIMITS AND CONTINUITY

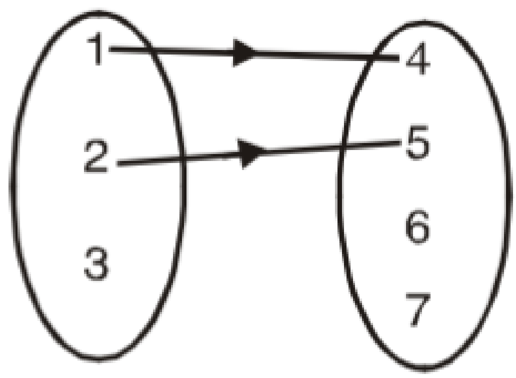
Function: Let X = {1, 2, 3} and Y = {4, 5, 6, 7}

Let us consider the following association between the sets X and Y. Let us associate 1 to 4; 2 to 5 and 3 to 6.

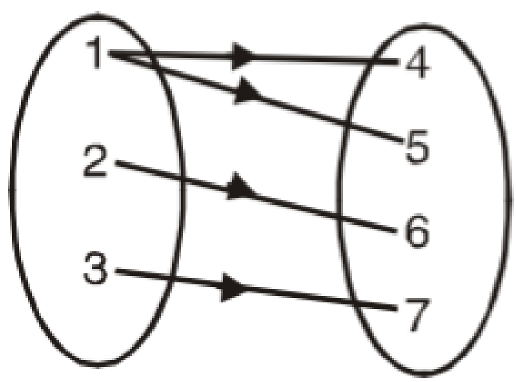
Def. of function: In this association each element of X has been associated with some element of Y. Moreover no element of X has been associated with more than one element of Y. Such an association between element X and Y is called a function.



Remark: 1. Any association between the elements of sets A and B need not be a function *e.g.* In above figure where X = {1, 2, 3} and Y = {4, 5, 6, 7} if we associate 1 to 4 and 2 to 5, then this association is not a function because 3 (an element of X) has been left unassociated.



If we associate 1 to 4 and 5; 2 to 6 and 3 to 7. Even this association is not a function because an element of X is associated to two elements 4 and 5 of Y.



2. For defining a function there are no restrictions on the set Y.
3. Functions can be defined from set X to X also.

Alternate def. of function. A function f from a non-empty set X to a set Y is a rule which associates to each element x in X a unique element y in Y.

Notations: The unique element of Y which f associates with x in X is defined by $f(x)$.

$$f: X \longrightarrow Y. \; X \xrightarrow{f} Y$$

usually denoted that f is a function from X to Y.

The set X is called the *domain* of f and the set Y is called *co-domain* of f. The set $\{y : y = f(x)$ for some $x \in X)$ is called the range of f, e.g., in above figure.

Domain of $f = X = \{1, 2, 3\}$

Co-domain of $f = Y = \{4, 5, 6, 7\}$

and Range of $f = \{4, 5, 6\} \subset Y$.

Range is always a subset of the co-domain.

The element $f(x)$ is called the image of x by (or under) the function f and x is called the *pre-image*.

Note:

(*i*) The word function is also replaced by mapping or correspondence or transformation.

(*ii*) The function : $f: X \rightarrow Y$ is denoted by $y = f(x)$

Here x ranges over X and is called independent variable and y ranges over a subset of Y and is called dependent variable.

(*iii*) If f is a function and x is an object in the domain of f then the image $f(x)$ of x under f is called the value of f at x.

Real Functions: Let $f : A \rightarrow B$ be a function from a non empty set A into non-empty set B. In general, the elements of set A and B need not to be numbers. In particular, if A and B are sub set of R (the set of real numbers) then the function f is called a real function.

Value of a function at a point: Let $y = f(x)$ be a function. For any real number 'a' in the domain of f, the value of y corresponding to $x = a$ is called the value of the function at a and is denoted by $f(a)$. For example if $f(x) = x^2 + 7$ then $f(4)$ = value of function at $x = 4$ is equal to $4^2 + 7 = 23$.

Classification of functions

(*i*) **Algebraic functions:** A function which is obtained by a finite number of algebraic operations (e.g., addition, subtraction, multiplication, division, raising to powers, extracting roots etc.) on identify, and constant functions, is called an algebraic function.

Thus, $4x^2 + 7x - 8, \sqrt{x} + \frac{6}{x}$, etc. are all algebraic functions.

Transcendental function: A function which is not algebraic, is called transcendental function. These functions may be

(a) *Trigonometric,* such as $\sin x$, $\tan x$, etc.

(b) *Inverse-Trigonometric,* such as $\sin^{-1}x$, $\cos^{1} x$, etc.

(c) *Logarithmic,* such as $\log_{10}x \log_{a} x$, etc.

(d) *Exponential,* such as 3^x, e^x, $(\cos x)^{\sin x}$, etc.

Explicit and Implicit functions

(a) *Explicit function.* A variable y is said to be an explicit function of another variable x when the value of y is directly expressed in terms of x.

If $y = 4x^2 + 7x + 8$ then y is an explicit function of x.

(b) *Implicit function.* When the variables x and y are expressed in a functional relation, then either variable is an implicit function of the other.

If $3x^2 + 7y^2 + 3axy + 4bx = 0$, then x is an implicit function of y and vice-versa.

Even and odd functions

(a) *Even function.* A function 'f' is said to be even function of x if it remains unaltered in magnitude as well as in sign when x is replaced by $-x$.

f is even if $f(-x) = f(x)$.

(b) *Odd function*. A function 'f' is said to be an odd function of x if it remains unaltered in magnitude but changes in sign when x is replaced by $-x$.

$$f \text{ is odd if } f(-x) = -f(x).$$

(i) Polynomial function. A function 'f' defined by

$$f(x) = a_0x^n + a_1\, x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$$

where n is a positive integer and $a_0, a_1 ... a_{n-1}, a_n$ are constants and $a_0 \neq 0$ is called a polynomial function of nth degree. Thus $5x^3 - 6\,x^2 + 7x + 1$, $4x^4 - 3x^2 + 1$ are polynomial functions of degree 3 and 4 respectively.

when $n = 0$, it is a constant function

when $n = 1$, it is a linear function,

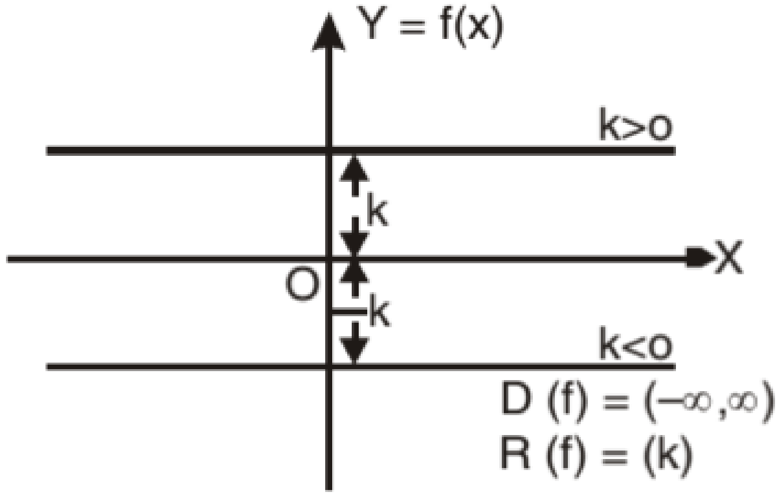
when $n = 2$, it is a quadratic function,

when $n = 3$, it is a cubic function

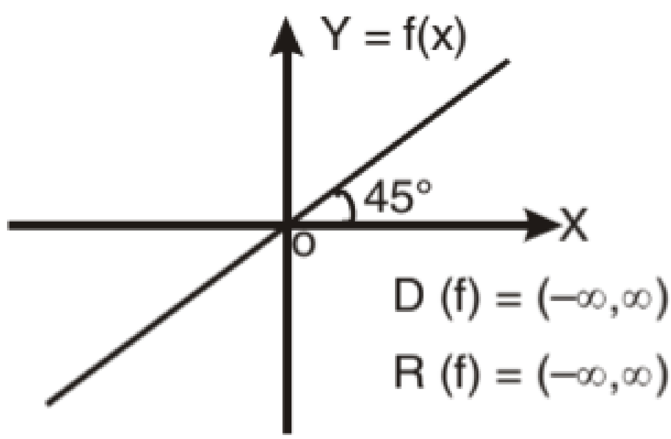
when $n = 4$, it is a biquadratic function etc.

Some standard functions and their graphs

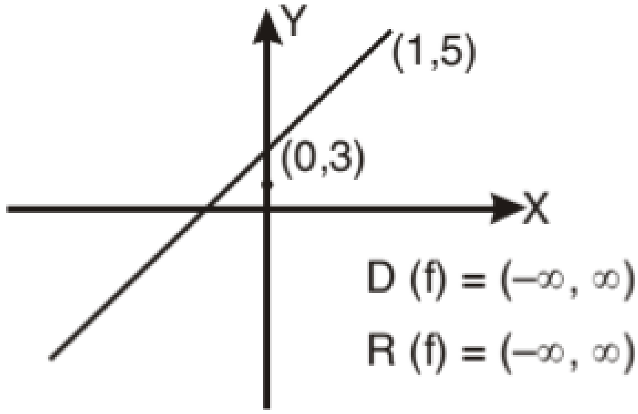
(1) Constant function: A function of the type $y = f(x) = k$ is called a constant function where k is a fixed real number. The graph of this function is a straight line parallel to x-axis.



(2) Identity function: The function $y = f(x) = x$ is called the identity function. The graph of $y = f(x) = x$ can also be drawn by finding the values of y for any two different values of x.



(3) Linear function: A function of the type $y = f(x) = ax + b$, $a \neq 0$ is called a linear function. Thus, graph of $y = f(x) = 2x + 3$ is a straight line.



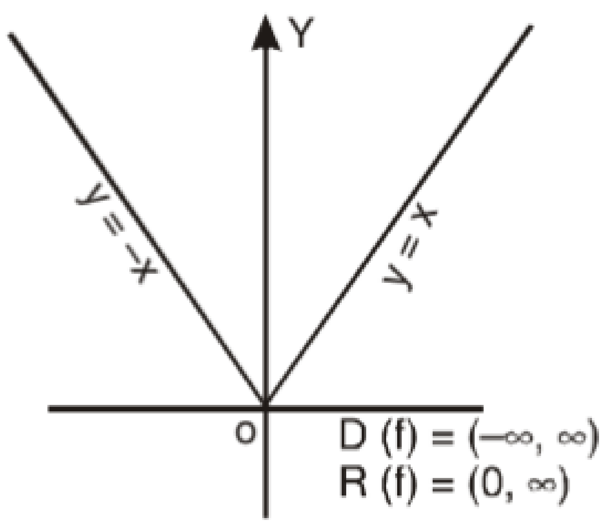
(4) Absolute value function. The function $y = f(x) = |x|$ is called the absolute value function. For every real value of x, the value of $y = |x|$ is unique.

The graph of the function cosists of

(*i*) line $y = -x, x \in (-\infty, 0)$

(*ii*) point (0, 0)

(*iii*) line $y = x, x \in (0, \infty)$

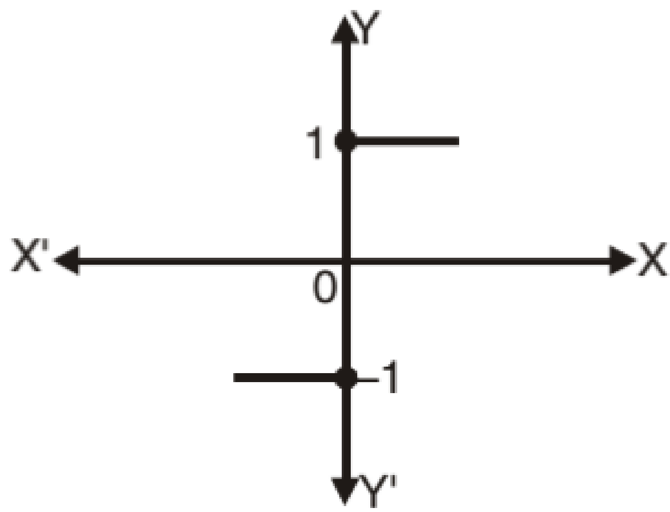


(5) Signum function: The function $f = f(x)$

$$= \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ is called the signum function.

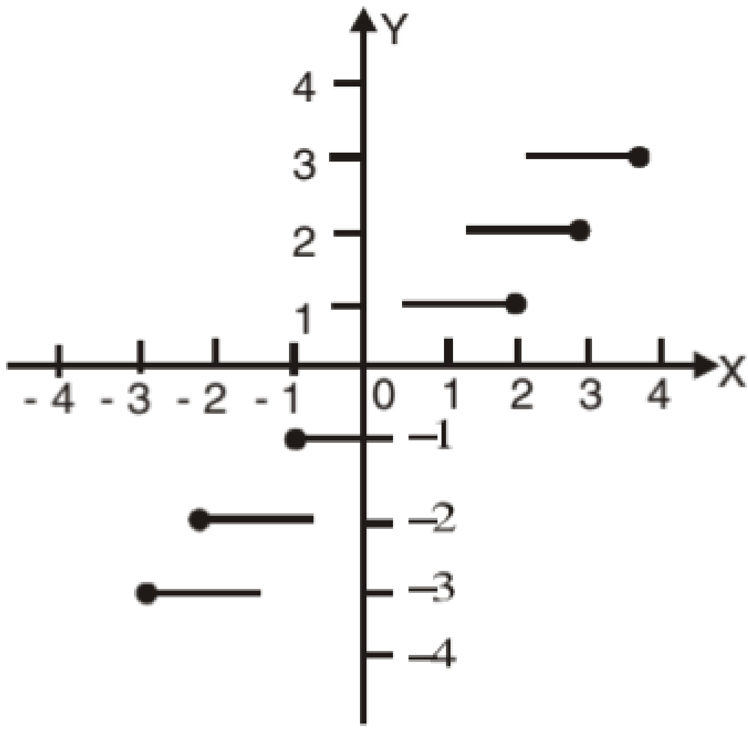
$\therefore$ The graph of the function consists of

(*i*) line $y = -1, x \in (-\infty, 0)$
(*ii*) point (0, 0)
(*iii*) line $y = 1, x \in (0, \infty)$

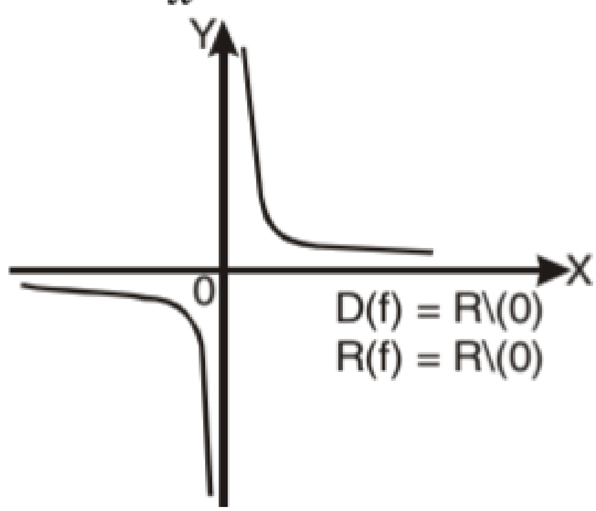


(6) Greatest Integer Function: The function $y = f(x) = [x]$. where x is the greatest integer, $\leq x$, is called the greatest integer function,

For example [4,6] = 4, because 4 is the greatest integer which is not greater than 4, 6.



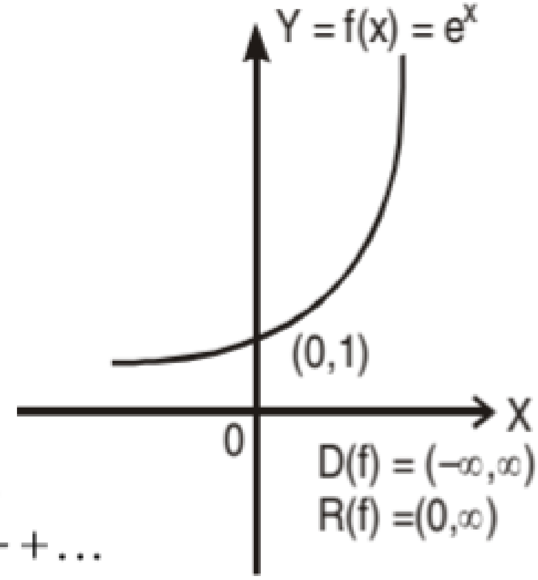
(7) Reciprocal function: The function $y = f(x) = \dfrac{1}{x}$, $x \neq 0$ is called the reciprocal function. For every non-zero value of x, the value of $y = \dfrac{1}{x}$ *is unique.*



(8) Exponential function: The function $y = f(x) = e^x$ is called the exponential func-tion.

We know for any value of x.

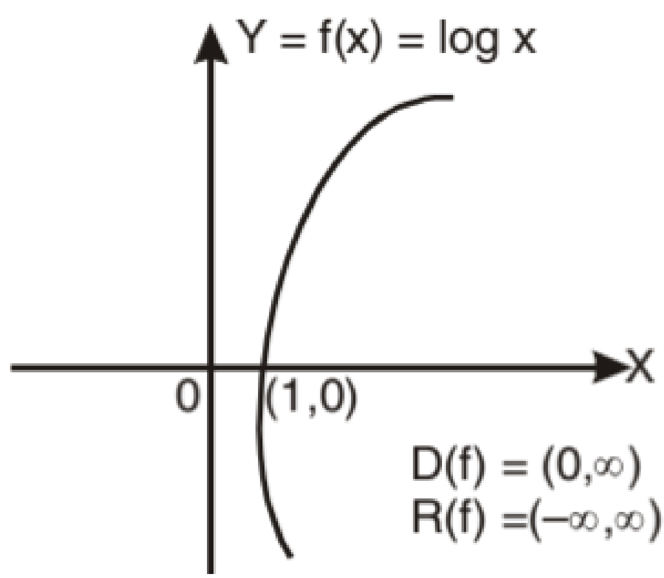
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$



(9) Logarithmic function: The function $y = f(x) = \log x$, $x \in (0, \infty)$ is called logarithmic function

For $0 < x < \infty$

$y = \log x$ iff $x = e^y$



Trigonometric function: The trigonometric functions with their respective domains and ranges are enlisted below :

	Circular Function	Domain	Range
(*i*)	$\sin x$	R	$[-1, 1]$
(*ii*)	$\cos x$	R	$[-1, 1]$
(*iii*)	$\tan x$	$R-(2n+1)\dfrac{\pi}{2}$	R
(*iv*)	$\cot x$	$R-n\pi$	R
(*v*)	$\sec x$	$R-(2n+1)\dfrac{\pi}{2}$	$(-\infty, -1] \cup [1, \infty)$
(*vi*)	$\text{cosec } x$	$R-n\pi$	$(-\infty, -1] \cup [1, \infty)$

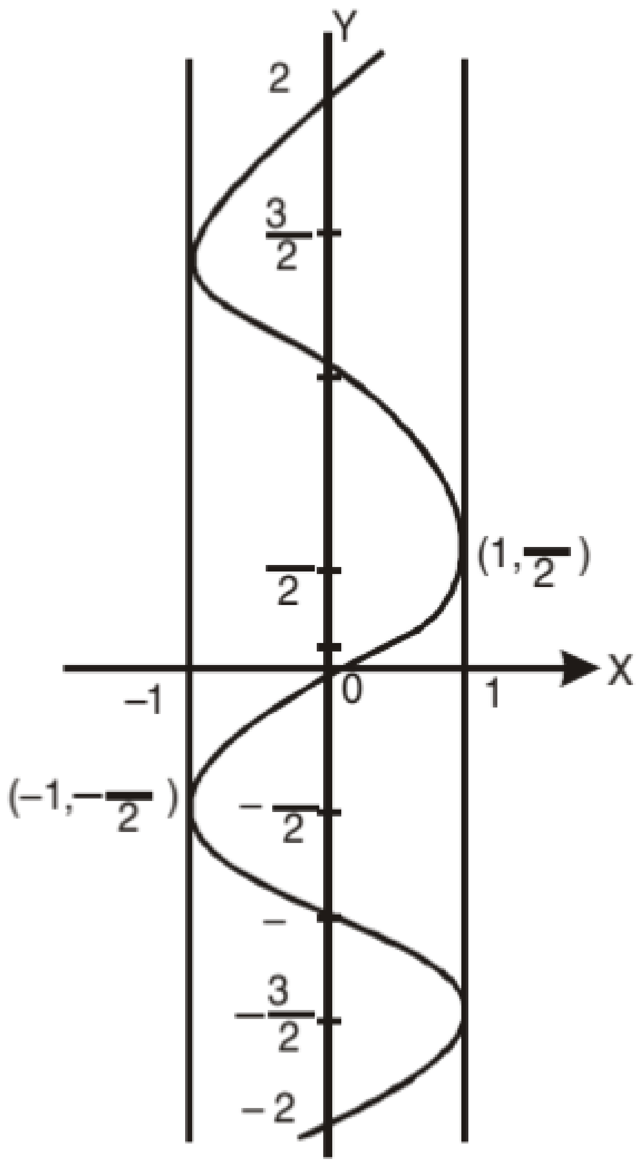
where R is the set of all real numbers and n is any integer.

Inverse Trigonometric function: As $\sin\dfrac{\pi}{6} = \dfrac{1}{2} = \sin\dfrac{5\pi}{6}$, it follows that $\sin x$ is not one-one. In fact, each of the trigonometric function assumes the same value at infinitely many points. We shall restric the domain suitably so that their inverse functions may exists.

(*i*) Graph of $y = \sin^{-1}x$

Domain $= -1 \le x \le 1$

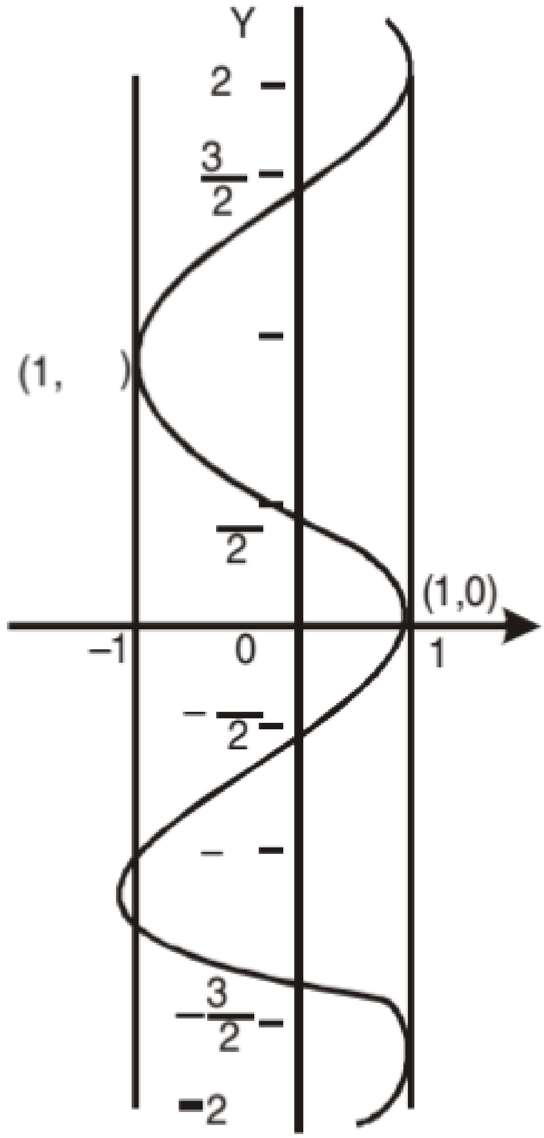
Range $= -\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$



(*ii*) Graph of $y = \cos^{-1} x$

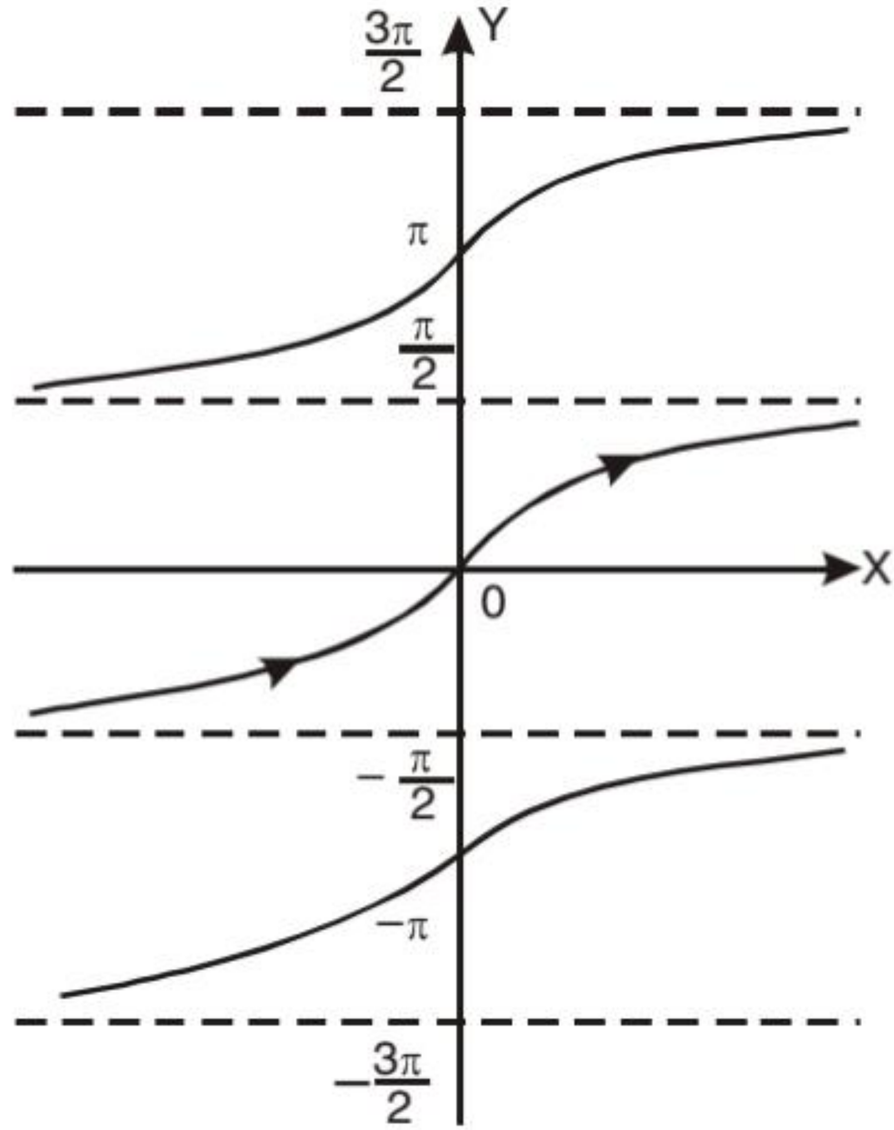
Domain = $-1 \le x \le 1$

Range = $0 \le y \le \pi$



(*iii*) Graph of $y = \tan^{-1} x$

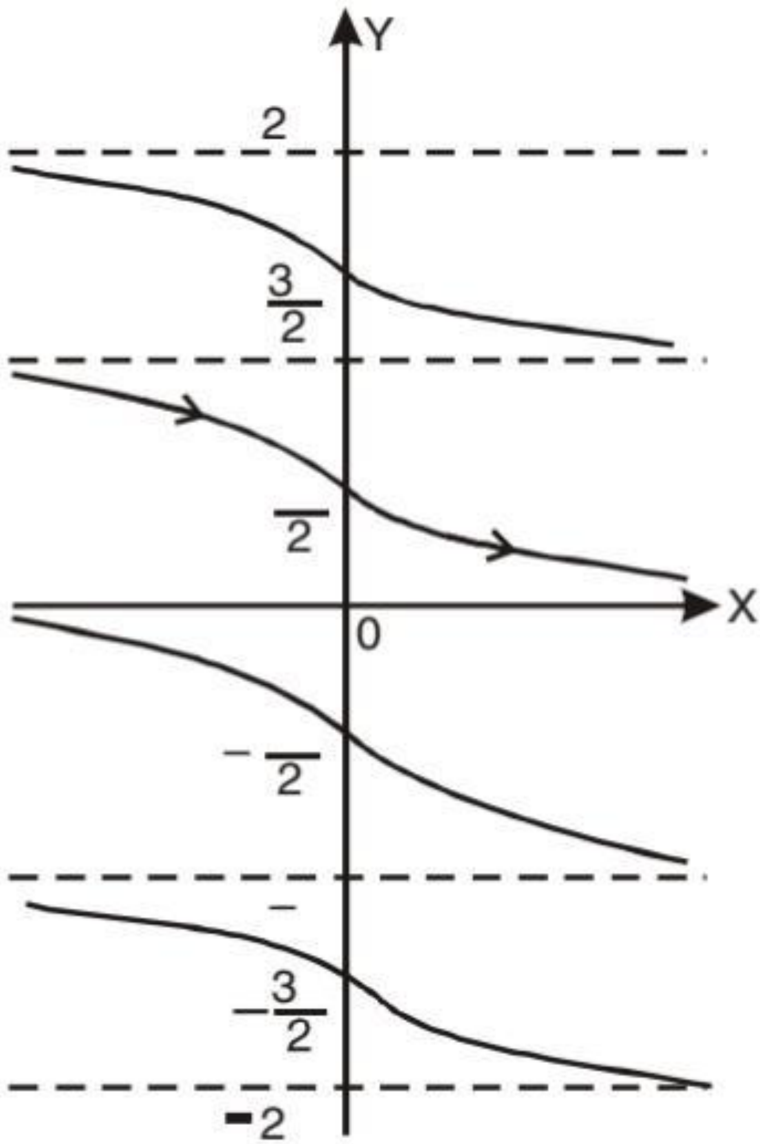
Domain = R, Range = $\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$



(*iv*) Graph of $y = \cot^{-1} x$

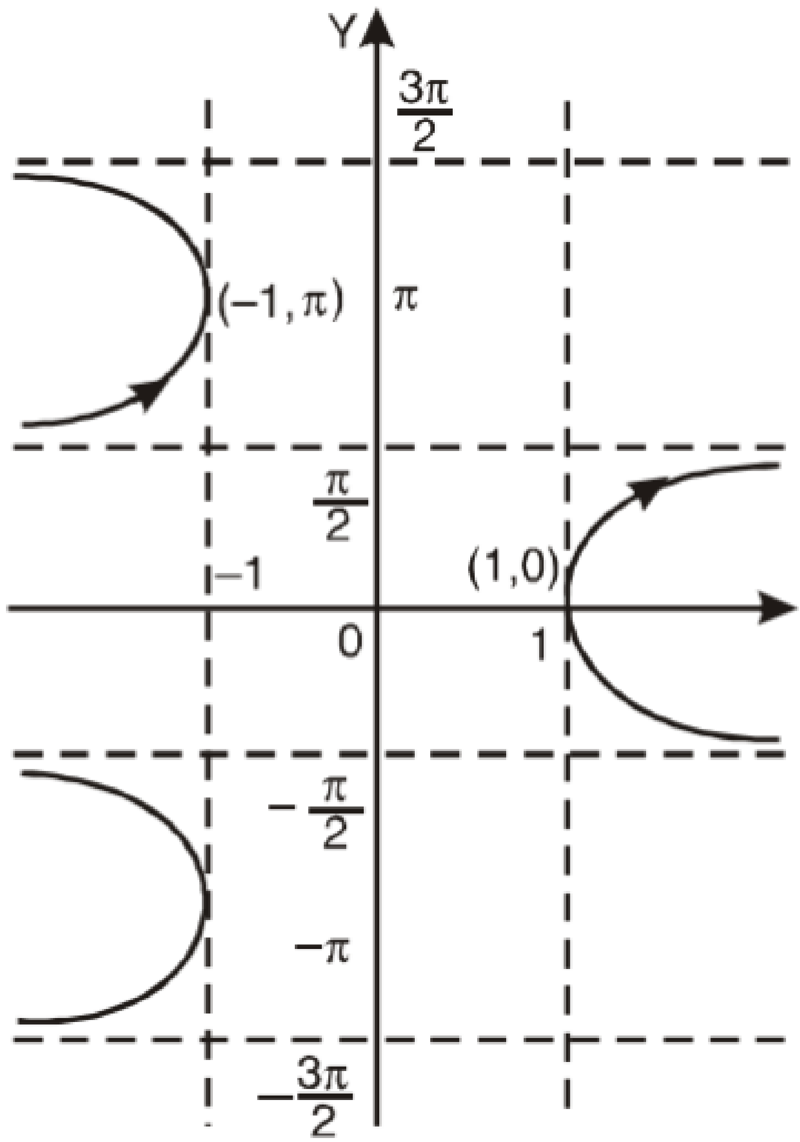
Domain = R

Range = $0 < y < \pi$



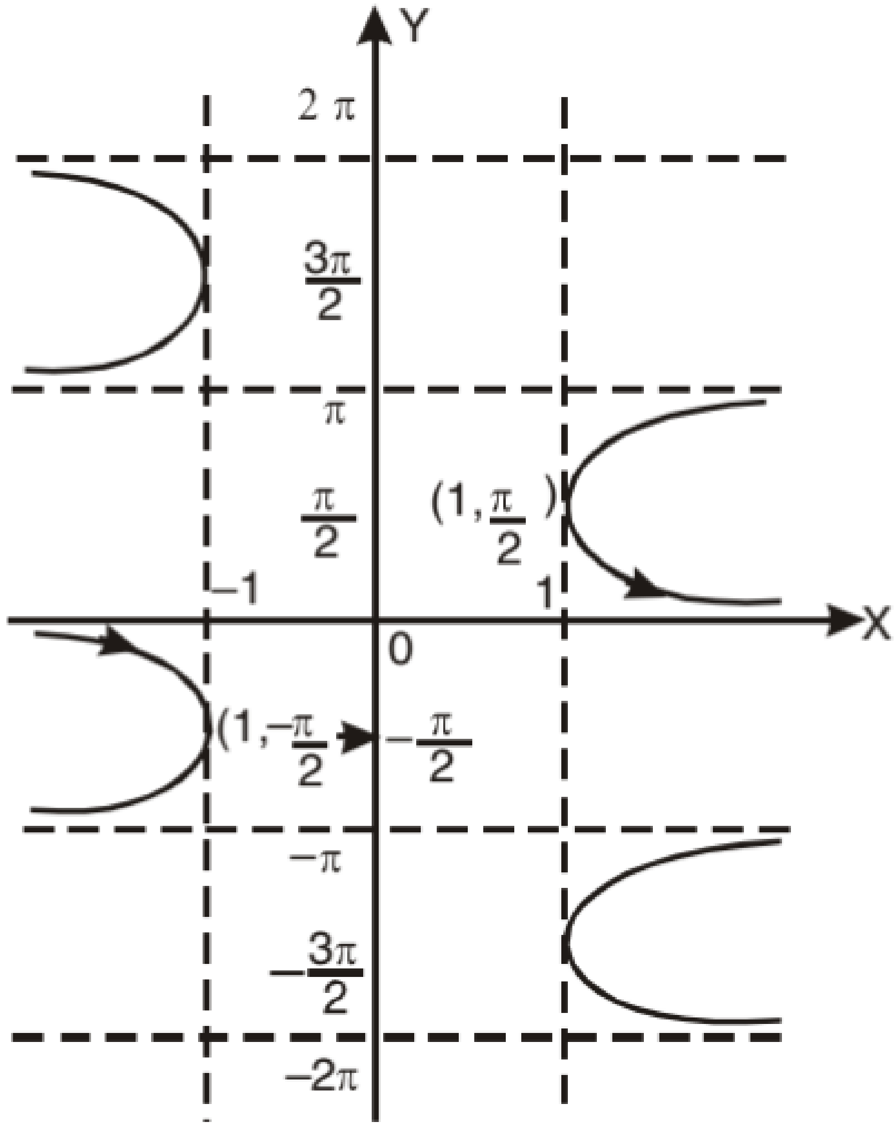
(v) Graph of $y = \sec^{-1}x$

Domain	Range
$x \le -1$	$\frac{\pi}{2} < y \le \pi$
$x \ge 1$	$0 \le y < \frac{\pi}{2}$



(*vi*) Graph of $y = \text{cosec}^{-1}x$

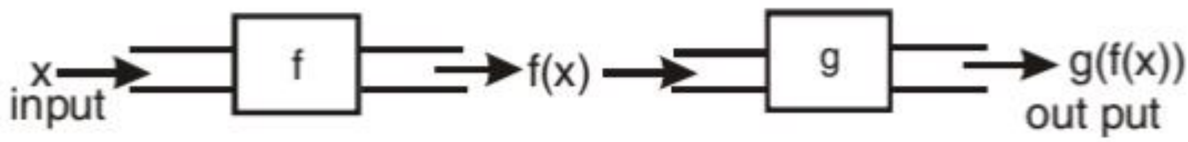
Domain	Range
$x \leq -1$	$-\frac{\pi}{2} \leq y < 0$
$x \geq 1$	$0 < y \leq \frac{\pi}{2}$



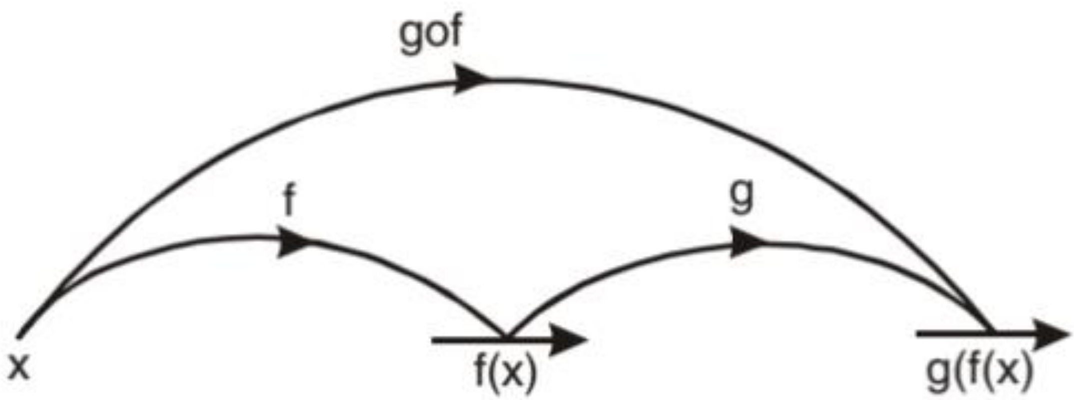
Composite Function: Let $f : A \to A$ and $g : B \to C$ then the composite of the functions f and g denoted by gof or gf is mapping $gof : A \to C$ such that

$(gof)(x) = g\ [f(x)],\ \forall\ x \in A$

$(gof)(x)$ is defined whenever both $f(x)$ and $g(x)$ are defined. We can denote it by a machine or arrow diagram.



gof machine is composed of the *f*(first) and then *g* machine.



Operation on Real functions: Let *f* and *g* be two functions whose domains are D_1 and D_2. Then the sum of function is denoted by $f+g$ is defined by the rule

$(f+g)x = f(x)+g(x)$ for all x in D

where $D = D_1 \cap D_2$ is the common part of the two domains D_1 and D_2 and $D_1 \cap D_2 \neq \phi$

Difference and product functions denoted by $f–g$ and fg one defined by the rules

$(f–g)x = f(x)–g(x)$ for all x in $D = D_1 \cap D_2$

$(fg)(x) = f(x).\ g(x)$ for all x in $D = D_1 \cap D_2$.

The quotient function f by g denoted by f/g is defined by the rule.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for all } x \text{ in D.}$$

where D is common part of two domains D_1 and D_2 excluding those points at which $g(x) = 0$.

The modulus $|f|$ of the function f can also be denoted by the rule $|f|(x) = |f(x)|$

Limit of a function: Meaning of *x* approaches *a* or "*x* tends to *a*". The expression x tends to a, means, the set of infinite (real) values of x which are very close to a and are slightly greater than a and slightly less than a but $x \neq a$.

Limit of a function. Let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ when } x \neq 1$$

$$= \frac{(x+1)(x-1)}{x-1} = x + 1$$

Let x approach 1 through values less than 1,

Putting $x = .9, .99, .999$

$$f(x) = 1.9, 1.99, 1.999.... \qquad ...(i)$$

Let x approach 1 through values greater than 1,

Putting $x = 1.1, 1.01, 1.001,....$

$f(x) = 2.1, 2.01, 2.001,...$ ..(*ii*)

Here, we note that as x takes values nearer and nearer to 1, remaining always less than 1 [as in (*i*) or greater than 1 [as in (*ii*), $f(x)$ takes values which nearer and near to 2. The difference between $f(x)$ and 2 i.e..., $|f(x)-2|$ can be made as small as we please by giving to x values close to 1. We say $f(x)$ tends to the limit 2 as x tends to 1.

A function $f(x)$ is said to have a limiting values (approx. value) l as x tends to a if the numerical difference between $f(x)$ and l *i.e.*,, $|f(x)-l|$ can be made as small as we please by giving to x values very close to a but not equal to a.

(*i*) *Left handed and right handed limit.* The limiting value of $f(x)$ as $x \to a^+$ is called the right hand limit of $f(x)$ and is written as

$$\underset{x \to a^+}{\text{Lt}} f(x).$$

The limiting value of $f(x)$ as $x \to a^-$ is called the left hand limit of $f(x)$ and is writen as

$$\underset{x \to a^-}{\text{Lt}} f(x).$$

(*ii*) *Existence of the limit of a function* $\underset{x \to a^+}{\text{Lt}} f(x)$ is said to exist if the left-handed and right-handed limits both exists and are equal.

Thus if $\underset{x \to a^-}{\text{Lt}} f(x) = \underset{x \to a^+}{\text{Lt}} f(x) = l$, only then we say that $\underset{x \to a}{\text{Lt}} f(x)$ exists and is $= l$

(*iii*) *Distinction between the value and the limit of a function.* The value of a function $f(x)$ at $x = a$ is obtained by putting $x = a$. The limit of the function $f(x)$ as $x \to a$ is obtained by considering the values of x very close to a.

Thus $\underset{x \to a}{\text{Lt}} f(x)$ may exist even if the function is not defined at $x = a$.

Theorems on Limits

Theorem 1. The limit of a constant quantity is the constant itself.

i.e., $\underset{x \to a}{\text{Lt}}\, c = c$

where c is a constant.

Theorem 2. The limit of the sum of two (or more) functions is equal to the sum of their limits.

$$i.e., \quad \underset{x \to a}{\text{Lt}} [\psi(x) + \varphi(x)] = \underset{x \to a}{\text{Lt}}\, \psi(x) + \underset{x \to a}{\text{Lt}}\, \varphi(x)$$

Theorem 3. The limit of the difference of two functions is equal to the difference of their limits.

i.e., $\underset{x \to a}{\text{Lt}} [\psi(x) - \varphi(x)] = \underset{x \to a}{\text{Lt}} \psi(x) - \underset{x \to a}{\text{Lt}} \varphi(x)$

Theorem 4. The limit of the product of two functions is equal to the product of their limits.

i.e., $\underset{x \to a}{\text{Lt}} [\psi(x)\, \varphi(x)] = [\underset{x \to a}{\text{Lt}} \psi(x)]\, [\underset{x \to a}{\text{Lt}} \varphi(x)]$.

Theorem 5. The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the denominator is not zero.

$$\underset{x \to a}{\text{Lt.}} \frac{\varphi(x)}{\psi(x)} = \frac{\underset{x \to a}{\text{Lt.}} \phi(x)}{\underset{x \to a}{\text{Lt.}} \psi(x)}$$

provided $\underset{x \to a}{\text{Lt.}} \psi(x) \neq 0.$

Theorem 6. The limit of the product of a constant and a function is equal to the product of the constant and limit of the function.

i.e., $\underset{x \to a}{\text{Lt.}} [c.\, \psi(x)] = c\, [\underset{x \to a}{\text{Lt.}} \psi(x)]$,

where c is a constant and $\psi(x)$ is any function of x.

Methods for finding the limits of a function
Ist Method

(*By factorization*)

When $f(x)$ is of the form $\dfrac{g(x)}{h(x)}$

(*i*) Factorise $g(x)$ and $h(x)$ and cancel the common factors.

(*ii*) Put the value of x.

Example. *Evaluate* $\underset{x \to a}{\text{Lt.}} \dfrac{\sqrt{x}-\sqrt{a}}{x-a}$

Sol.
$$\underset{x \to a}{\text{Lt.}} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \underset{x \to a}{\text{Lt.}} \frac{\sqrt{x}-\sqrt{a}}{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)}$$

[Factorising]

$$= \underset{x \to a}{\text{Lt.}} \frac{1}{\sqrt{x}+\sqrt{a}}$$

$[\because x \to a \therefore x-a \neq 0]$

$$= \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$$

2 nd Method (*By substitution*)

To evaluate $\underset{x \to a}{\text{Lt.}} \dfrac{g(x)}{h(x)}$

(*i*) Put $x = a + h$, where $(h \neq 0)$ is very small.
As $x \to a$, then $h \to 0$

(*ii*) Simplify the numerator and denominator so as to cancel h throughout as $h \neq 0$.

(*iii*) Put $h = 0$ and get the required limit.

Example: *Evaluate* $\underset{x \to 1}{\text{Lt.}} \dfrac{x^n - 1}{x - 1}$

Sol. Put $x = 1 + h$, where h is very small.

Since $x \to 1 \therefore 1 + h \to 1$ *i.e.*, $h \to 0$

$$\therefore \underset{x \to 1}{\text{Lt.}} \frac{x^n - 1}{x - 1} = \underset{h \to 0}{\text{Lt.}} \frac{(1+h)^n - 1}{(1+h) - 1}$$

$$= \underset{h \to 0}{\text{Lt.}} \frac{\left[\left(1 + nh + \dfrac{n(n-1)}{2!} h^2 + \ldots\right) - 1\right]}{h}$$

[Binomial theorem)

$$= \underset{h \to 0}{\text{Lt.}} \frac{nh + \frac{n(n-1)}{2!} h^2 + \ldots}{h}$$

$$= \underset{h \to 0}{\text{Lt.}} \left[n + \frac{n(n-1)}{2!} h + \ldots \right] \quad \text{[Cancelling } h \text{ as } h \neq 0.$$

$= n + 0 + 0 \ldots.$

$= n.$

3rd Method. In this case we rationalise the factor which contains radical sign. Then simplfy.

Example: *Evaluate* $\underset{x \to 0}{\text{Lt.}} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Sol. $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$

[Rationalising the numerator]

$$= \underset{x \to 0}{\text{Lt}} \frac{(1+x) - (1-x)}{x\left[\sqrt{1+x} + \sqrt{1-x}\right]}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2x}{x\left[\sqrt{1+x} + \sqrt{1-x}\right]}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = \frac{2}{1+1} = \frac{2}{2} = 1.$$

4th Method. $\underset{x \to \infty}{\text{Lt}} f(x) = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$

(*i*) Divide the numerator and denominator by the highest power of *x* of occuring in *f*(*x*).

(*ii*) Use the idea of $\dfrac{1}{x}, \dfrac{1}{x^2} \ldots \to 0 \; as \; x \to \infty.$

Example: *Evaluate* $\underset{x \to \infty}{\text{Lt}} \dfrac{4x^2 + 5x + 6}{3x^2 + 4x + 5}$

Sol. $\underset{x \to 0}{\text{Lt}} \dfrac{4x^2 + 5x + 6}{3x^2 + 4x + 5}$

$$= \underset{x \to \infty}{\text{Lt}} \frac{4 + \frac{5}{x} + \frac{6}{x^2}}{3 + \frac{4}{x} + \frac{5}{x^2}}$$

[Dividing the numerator and denominator by x^2]

$$= \frac{4+0+0}{3+0+0} \quad \left[\because \frac{5}{x}, \frac{6}{x^2}, \frac{4}{x}, \frac{5}{x^2} \text{ all} \to 0 \text{ at } x \to \infty\right]$$

$$= \frac{4}{3}$$

Some important limits:

(*i*) $\underset{\theta \to 0}{\text{Lt}} \cos\theta = 1$

(*ii*) $\underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1$

(*iii*) $\underset{\theta \to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1$

Cor 1. $\underset{\theta \to 0}{\text{Lt}} \sec\theta = 1$

Cor 2. $\underset{\theta \to 0}{\text{Lt}} \sin\theta = 0$

Cor 3. $\underset{\theta \to 0}{\text{Lt}} \frac{\sin^{-1}\theta}{\theta} = 1$

Cor 4. $\underset{\theta \to 0}{\text{Lt}} \frac{\tan^{-1}\theta}{\theta} = 1$

(*iv*) $\underset{n \to 0}{\text{Lt}} \left(1 + \frac{1}{n}\right)^n = e$

(*v*) $\underset{n\to\infty}{\text{Lt}}(1+n)^{1/n} = e$

(*vi*) $\underset{x\to 0}{\text{Lt}}\dfrac{a^x - 1}{x} = \log_e a$

(*vii*) $\underset{x\to 0}{\text{Lt}}\log\dfrac{(1+x)}{x} = 1$

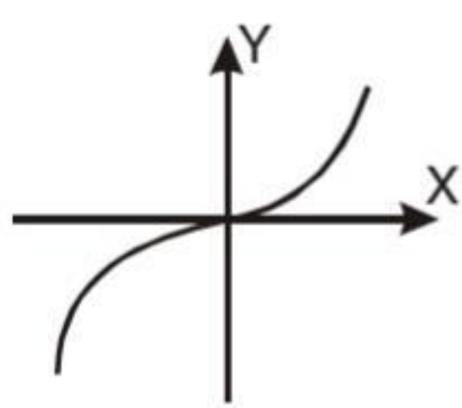
(*viii*) $\underset{x\to 0}{\text{Lt}}\dfrac{x^n - a^n}{x-a} = na^{n-1}$

where n is an integer, $n \neq 0$.

(*ix*) $\underset{x\to 0}{\text{Lt}}\dfrac{e^x - 1}{x} = 1$

Continuity of Functions

(*i*) A function $y = f(x)$ is said to be continuous in an interval if its graph is obtained by moving the pen over the various points without lifting it from the paper. On the other hand, if we have to lift the pen in drawing the



graph, then the function is said to be discontinuous. The graph of a discontinuous function is broken. It has gaps or jumps at points of discontinuity.

(*ii*) A function $f(x)$ is said to be continuous at $x = a$, if

(a) $\underset{x \to a}{\text{Lt}} f(x)$ exists *i.e.* both

$\underset{x \to a^-}{\text{Lt}} f(x)$ and $\underset{x \to a^+}{\text{Lt}} f(x)$ exists and are equal

(b) $f(a)$ exists.

(c) $\underset{x \to a}{\text{Lt}} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$ it is said to be discontinuous at $x = a$.