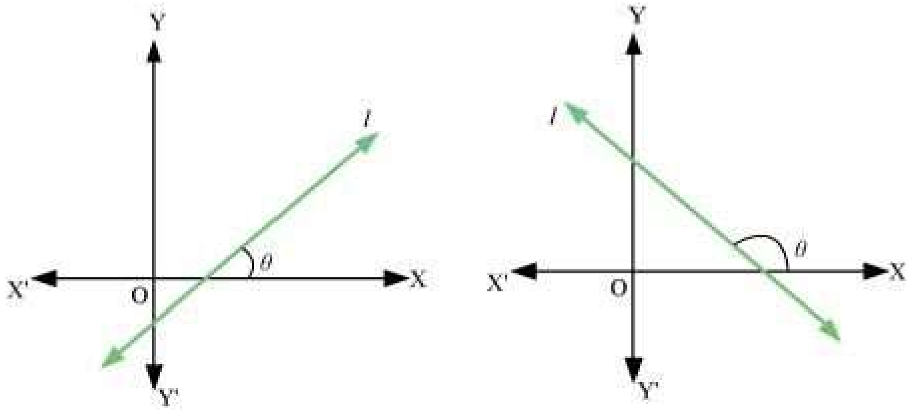
5. Straight Line

- **Slope of a line:** If θ is the inclination of a line *l* (the angle between positive *x*-axis and line *l*), then $m = \tan\theta$ is called the slope or gradient of line *l*.



  - The slope of a line whose inclination is 90º is not defined. Hence, the slope of the vertical line, *y*-axis is undefined.
  - The slope of the horizontal line, *x*-axis is zero.

  For example, the slope of a line making an angle of 135° with the positive direction of *x*-axis is $m = \tan 135° = \tan(180° - 45°) = -\tan 45° = -1$

- **Slope of line passing through two given points:**

  The slope (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2$.

  For example, the slope of the line joining the points (–1, 3) and (4, –2) is given by,
  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 3}{4 - (-1)} = -\frac{5}{5} = -1$

- **Conditions for parallelism and perpendicularity of lines:**

  Suppose l_1 and l_2 are non-vertical lines having slopes m_1 and m_2 respectively.

  - l_1 is parallel to l_2 if and only if $m_1 = m_2$ i.e., their slopes are equal.
  - l_1 is perpendicular to l_2 if and only if $m_1 m_2 = -1$ i.e., the product of their slopes is –1.

Example:
Find the slope of the line which makes an angle of 45° with a line of slope 3.
Solution:
Let *m* be the slope of the required line.

$$\therefore \tan 45° = \left|\frac{m-3}{1+3m}\right|$$
$$\Rightarrow \left|\frac{m-3}{1+3m}\right| = 1$$
$$\Rightarrow \left|\frac{m-3}{1+3m}\right| = \pm 1$$
$$\Rightarrow \frac{m-3}{1+3m} = 1 \quad \text{or} \quad \frac{m-3}{1+3m} = -1$$
$$\Rightarrow m-3 = 1+3m \quad \text{or} \quad m-3 = -1-3m$$
$$\Rightarrow -2m = 4 \quad \text{or} \quad 4m = 2$$
$$\Rightarrow m = -2 \quad \text{or} \quad m = \frac{1}{2}$$

- **Collinearity of three points:** Three points A, B and C are collinear if and only if slope of AB = slope of BC

- The equation of a horizontal line at distance a from the x-axis is either $y = a$ (above x-axis) or $y = -a$ (below x-axis).
- The equation of a vertical line at distance b from the y-axis is either $x = b$ (right of y-axis) or $x = -b$ (left of y-axis).

- **Point-slope form of the equation of a line**

The point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) if and only if its coordinates satisfy the equation. This means $y - y_0 = m(x - x_0)$.

Example: Find the equation of the line passing through (4, 5) and making an angle of 120° with the positive direction of x-axis?
Solution: Slope of the line, $m = \tan 120° = \tan\left(180° - 60°\right) = -\tan 60° = -\sqrt{3}$
Equation of the required line is,
$$y - 5 = -\sqrt{3}(x-4)$$
$$\Rightarrow y - 5 = -\sqrt{3}x + 4\sqrt{3}$$
$$\Rightarrow \sqrt{3}x + y - \left(5 + 4\sqrt{3}\right) = 0$$

- **Two-point form of the equation of a line**

The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

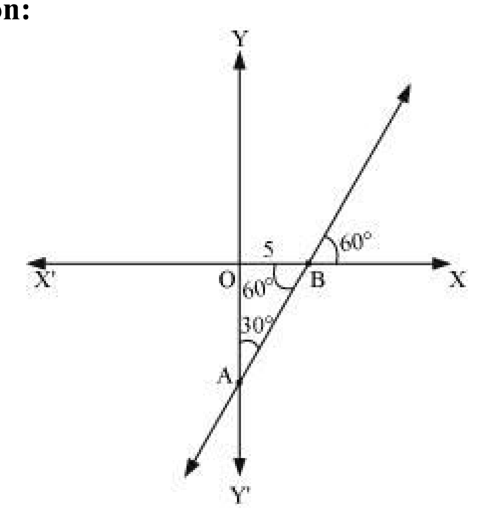
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Example: Find the equation of the line passing through the points (–5, 2) and (1, 6).
Solution: Equation of the line passing through points (–5, 2) and (1, 6) is

$$y-2=\frac{6-2}{1-(-5)}(x-(-5))$$
$$\Rightarrow y-2=\frac{4}{6}(x+5)$$
$$\Rightarrow y-2=\frac{2}{3}(x+5)$$
$$\Rightarrow 3y-6=2x+10$$
$$\Rightarrow 2x-3y+16=0$$

- **Slope-intercept form of a line**
  - The equation of the line, with slope m, which makes y-intercept c is given by $y = mx + c$.
  - The equation of the line, with slope m, which makes x-intercept d is given by $y = m(x - d)$.

Example:
Find the equation of the line which cuts off an intercept 5 on the x-axis and makes an angle of 30° with the y-axis.
Solution:



Slope of the line, $m = \tan 60° = \sqrt{3}$
OB = 5
Intercept on the x-axis, $c = -OB = -5$ and $\tan 60° = -5\sqrt{3}$
Equation of the required line is $y=\sqrt{3}x+\left(-5\sqrt{3}\right)$.

- **General equation of line**

Any equation of the form $Ax + By + C = 0$, where A and B are not zero simultaneously is called the general linear equation or general equation of line.

Slope of the line = $-\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = -\frac{A}{B}$
y- intercept = $-\frac{C}{B}$

Example:
Find the slope and the y-intercept of the line $2x - 3y = -16$.
Solution:
The equation of the given line can be rewritten as $2x - 3y + 16 = 0$.
Here, $A = 2$, $B = -3$ and $C = 16$.

Slope of the line $= -\frac{A}{B} = -\frac{2}{(-3)} = \frac{2}{3}$

Intercept on the y-axis $= -\frac{C}{B} = -\frac{16}{(-3)} = \frac{16}{3}$

- **Intercept form**

The equation of the line making intercepts a and b on x-axis and y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$

Example:
If a line passes through (3, 2) and cuts off intercepts on the axes in such a way that the product of the intercepts is 24, then find the equation of the line.

Solution:
The equation of a line in intercept form is

$\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots(1)$

Where, a and b are the intercepts on x and y axes respectively.

Since the line passes through (3, 2), we obtain

$\frac{3}{a} + \frac{2}{b} = 1$

$\Rightarrow 3b + 2a = ab$

$\Rightarrow 2a + 3b = 24 \quad \ldots(2)$ (Since product of intercepts is given as 24)

Now,

$(2a - 3b)^2 = 24ab$

$= (24)^2 - 24(24) \quad$ [From equation (2)]

$= 0$

$\therefore 2a - 3b = 0 \quad \ldots(3)$

On adding equations (2) and (3), we obtain

$4a = 24 \Rightarrow a = 6$

$\therefore 3b = 2a = 2\times 6 = 12$

$\Rightarrow b = 4$

Hence, from (1), the required equation of line is

$\frac{x}{6} + \frac{y}{4} = 1$

$\Rightarrow 4x + 6y = 24$

$\Rightarrow 2x + 3y = 12$

- **Normal form of the equation of a line**

The equation of the line at normal distance p from the origin and angle ω, which the normal makes with the positive direction of the x-axis is given by $x\cos\omega + y\sin\omega = p$

Example: Reduce the equation $x - \sqrt{3y} - 6 = 0$ to normal form and hence find the length of perpendicular to the line from the origin. Also find angle between the normal and positive direction of the x-axis.

Solution: The given equation is $x - \sqrt{3y} - 6 = 0$.

$\Rightarrow x - \sqrt{3y} = 6 \quad \ldots(1)$

On dividing (1) by $\sqrt{\left(\sqrt{1}\right)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$, we obtain

$\frac{1}{2}x - \frac{\sqrt{2}}{2}y = 3$

$\Rightarrow x \cos 300° + y \sin 300° = 3$ … (2)

On comparing equation (2) with $x \cos\omega + y \sin \omega = p$, we obtain $\omega = 300°$ and $p = 3$

Therefore, the length of perpendicular to the line from the origin is 3 units and the angle between the normal and the positive x-axis is 300°.

Parametric form

The equation of a line in parametric form is x-x1cosθ=y-y1sinθ=r.

Also, from the above equation, we get:

x-x1=rcosθ and y-y1= rsinθ∴ x=x1+rcosθ and y=y1+ rsinθ

Thus, the coordinates of any point on the line at a distance r from the given point (x1,y1) are x1 + r cosθ, y1 + r sinθ.

- **Angle between two lines:** An acute angle, θ, between line l_1 and l_2 with slopes m_1 and m_2 respectively is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|,\ 1 + m_1 m_2 \neq 0$$.

Example 1: Two lines AB and CB, intersect at point B. The coordinates of end points are A(−4, −3), B(0, 5), and C(10, 5). Find the measures of angles between AB and CB.

Solution: Let the angle between the lines AB and BC be θ.

Slope of line $AB = \frac{5-(-3)}{0-(-4)} = \frac{8}{4} = 2$

Slope of line $BC = \frac{5-5}{10-0} = 0$

We know that the angle between two lines with slopes m_1 and m_2 is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$.

Therefore, $\tan\theta = \left|\frac{2-0}{1+2\times 0}\right| = 2$

$\Rightarrow \theta = \tan^{-1}(2)$.

Point of Intersection of Two Intersecting Lines

- If a1x+b1y=c1 and a2x+b2y=c2 are two intersecting lines, then the coordinates of their point of intersection are

  c1b2-c2b1a1b2-a2b1,a1c2-a2c1a1b2-a2b1, where a1b2-a2b1 ≠0.

- Three or more distinct lines are said to be concurrent if they pass through a common point, and the point of intersection is called the point of concurrence.

- If the three lines a1x+b1y+c1=0 ,a2x+b2y+c2=0 and a3x+b3y+c3=0 are concurrent, then a1b2c3-b3c2-b1a2c3-a3c2+c1a2b3-a3b2=0.

- **Distance of a Point From a Line**

The perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Example: Find the distance of point (1, –2) from the line $8x - 6y - 12 = 0$.
Solution: On comparing the equation of the given line i.e., $8x - 6y - 12 = 0$ with $Ax + By + C = 0$, we obtain $A = 8$, $B = -6$, $C = -12$
The distance (d) of point (1, –2) from line $8x - 6y - 12 = 0$ is

$$d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}} = \frac{|8\times1+(-6)(-2)+(-12)|}{\sqrt{8^2+6^2}} = \frac{|8+12-12|}{\sqrt{100}} = \frac{8}{10} = \frac{4}{5}$$

- **Distance between parallel lines**

The distance (d) between two parallel lines i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ is given by,

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2+B^2}}$$

Example: Find the distance between the lines $4x + 3y = 11$ and $4x + 3y = 8$.
Solution: The given lines are $4x + 3y - 11 = 0$ and $4x + 3y - 8 = 0$
Slope of the line $4x + 3y - 11 = 0$ is $-\frac{4}{3}$.
Slope of the line $4x + 3y - 8 = 0$ is $-\frac{4}{3}$.
Since the slopes of the given lines are equal, the lines are parallel.
Here, $A = 4$, $B = 3$, $C_1 = -11$ and $C2 = -8$

Distance between the lines $= \left|\frac{-11-(-8)}{\sqrt{4^2+3^2}}\right| = \left|\frac{-11+8}{\sqrt{16+9}}\right| = \left|\frac{-3}{\sqrt{25}}\right| = \frac{3}{5}$

Family of Lines

If u≡a1x+b1y+c1=0 and v≡a2x+b2y+c2=0 are two intersecting lines, then the equation $u + \lambda v = 0$ represents a family of lines, each passing through the point of intersection of the lines $u = 0$ and $v = 0$, where λ is a parameter.