Exercise 9.R

Answer 1CC.

(A). A differential equation is an equation that contains an unknown function and one or More derivatives.

(b). The order of a differential equation is the order of the highest derivative that occurs in the equation.

(c). The condition which used to find the particular solution that satisfies the differential equation. The condition is in the form y (t_0) =y_0.

Answer 1E.

(A) This is the direction field for the given differential equation

$$y' = y(y-2)(y-4)$$

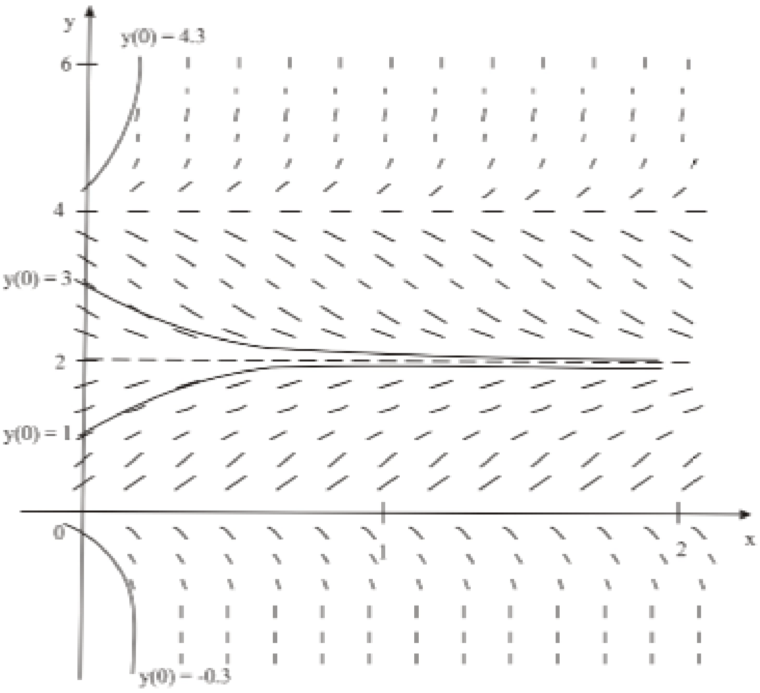


Fig. 1

(B) We see that directions of the slopes blow x-axis and above the line $y=4$ are downward and upward respectively so $\lim_{t\to\infty} y(t)$ is finite in the interval $0 \le C \le 4$

Equilibrium solutions are $\boxed{y=0, y=2, y=4}$

Answer 1P.

Given condition is $\left[f(x)\right]^2 = 100 + \int_0^x \{\left[f(t)\right]^2 + \left[f'(t)\right]^2\}dt$

Differentiating with respect to x

$$2f(x)f'(x) = \frac{d}{dx}\int_0^x \{\left[f(t)\right]^2 + \left[f'(t)\right]^2\}dt$$

$$= \left[f(x)\right]^2 + \left[f'(x)\right]^2 \qquad \text{[By fundamental theorem]}$$

$$\Rightarrow \left[f(x)\right]^2 + \left[f'(x)\right]^2 - 2f(x)f^1(x) = 0$$

$$\Rightarrow \left[f'(x) - f(x)\right]^2 = 0$$

$$\Rightarrow f'(x) - f(x) = 0$$

$$\Rightarrow f'(x) = f(x)$$

$$\Rightarrow \frac{d}{dx}f(x) = f(x)$$

Solution of this equation is $f(x) = f(0)e^x$

For $x=0$ $\left[f(0)\right]^2 = 100 + 0$

$$\Rightarrow f(0) = \pm 10$$

So $f(x) = \pm 10e^x$

So functions are $\boxed{f(x) = \pm 10e^x}$

Answer 1TFQ.

Given

$$y' = -1 - y^4$$

$$\Rightarrow y' = -(1+y^4)$$

$$\Rightarrow \frac{y'}{(1+y^4)} = -1$$

Hence all solutions of this differential equation are decreasing functions.

$\boxed{\text{The given statement is true.}}$

Answer 2CC.

Consider the differential equation $y' = x^2 + y^2$.

$y' = x^2 + y^2 \ge 0$ for all x and y

$y' = 0$ Only at the origin,

So there is a horizontal tangent at $(0,0)$ but nowhere else.

The graph of the solution is increasing on every interval.

Answer 2E.

(A) We sketch the direction field with calculating slopes on differential points and sketch the solutions curves that satisfy the initial conditions $y(0)=1, y(0)=-1$, $y(2)=1$, and $y(-2)=1$ [figure 1]

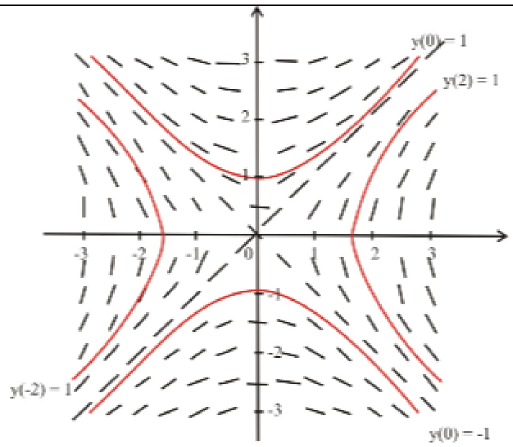


Fig. 1

(B) We have to solve $y'=x/y$

$$\Rightarrow y\,dy = x\,dx$$

Integrating both sides

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\Rightarrow y^2 = x^2 + 2C$$

$$\Rightarrow \boxed{y^2 - x^2 = 2C}$$

For initial condition $y(0)=1, \Rightarrow C = \frac{1}{2}$

The solutions is $\boxed{y^2 - x^2 = 1}$

For $y(0)=-1 \Rightarrow C = \frac{1}{2}$

The solutions is $\boxed{y^2 - x^2 = 1}$

For $y(2)=1$ $\boxed{y^2 - x^2 = -3}$ $C = -3/2$

For $y(-2)=1,$ $\boxed{y^2 - x^2 = -3}$ $C = -3/2$

All solution curves are hyperbolic

Answer 2P.

We have $f(x)=e^{x^2}$ then $f'(x)=2xe^{x^2}$

According to given situation we have

$$(fg)' = f'g'$$

By product rule

$$fg' + f'g = f'g'$$

$$e^{x^2}g' + 2xe^{x^2}g = 2xe^{x^2}g'$$

Dividing by e^{x^2}

$$g' + 2xg = 2xg'$$

$$\Rightarrow g'(1-2x) + 2xg = 0$$

$$\Rightarrow g'(1-2x) = -2xg$$

$$\Rightarrow \frac{1}{g(x)}dg(x) = \frac{-2x}{(1-2x)}dx$$

Integrating both sides

$$\int \frac{1}{g(x)} dg(x) = \int \frac{-2x}{(1-2x)} dx$$

$$\Rightarrow \ln|g(x)| = \int \frac{-2x}{(1-2x)} dx$$

Let 1-2x = t $\Rightarrow$ -2 dx = dt , so we have

$$\ln|g(x)| = \frac{1}{2}\int \frac{t-1}{t} dt$$

$$\Rightarrow \ln|g(x)| = -\frac{1}{2}\int\left(1-\frac{1}{t}\right)dt$$

$$\Rightarrow \ln|g(x)| = -\frac{1}{2}(t - \ln|t|) + c$$

$$\Rightarrow \ln|g(x)| = \frac{1}{2}\ln|t| - \frac{t}{2} + c$$

$$\Rightarrow \ln g(x) = \frac{1}{2}\ln|(1-2x)| - \frac{(1-2x)}{2} + c$$

$$\Rightarrow \ln g(x) - \ln\sqrt{2x-1} = \frac{2x-1}{2} + c$$

$\{$for the interval $(1/2, \infty)$, $g(x) > 0$ and $1-2x < 0$, so $\ln|1-2x| = \ln(2x-1)\}$

$$\Rightarrow \ln\frac{g(x)}{\sqrt{2x-1}} = x - \frac{1}{2} + c$$

Let $c - \frac{1}{2} = k$ (any constant)

$$\Rightarrow \ln\frac{g(x)}{\sqrt{2x-1}} = x + k$$

$$\Rightarrow \frac{g(x)}{\sqrt{2x-1}} = e^{x+k} = Ae^x \qquad A = e^k \text{ (any constant)}$$

So $\boxed{g(x) = A\left(\sqrt{2x-1}\right)e^x}$

Where A is any positive constant.

Answer 2TFQ.

Given

$$x^2y' + xy = 1$$

$$f(x) = \frac{\ln x}{x} = y$$

$$\Rightarrow y' = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

Consider

$$x^2y' + xy = x^2\left(\frac{1-\ln x}{x^2}\right) + x\left(\frac{\ln x}{x}\right)$$

$$= 1 - \ln x + \ln x$$

$$= 1$$

$\therefore \frac{\ln x}{x} = y$ is a solution of $x^2y' + xy = 1$

Hence the given statement is true.

Answer 3CC.

Consider $y' = F(x, y)$

Where F(x,y) is some expression in x and y. The differential equation says that the slope of the solution curve at a point (x,y) on the curve is F(x,y). If we draw short line segment with slope F(x,y) at several points (x,y), the result is called a direction field (or slope field). These line segments indicate the direction in which a solution cure is heading, so the direction field helps us visualize the general shape of these curves.

Answer 3E.

(A)

With help of the given direction field we sketch the curve of the solution of initial value problem. $y' = x^2 - y^2, \quad y(0) = 1$

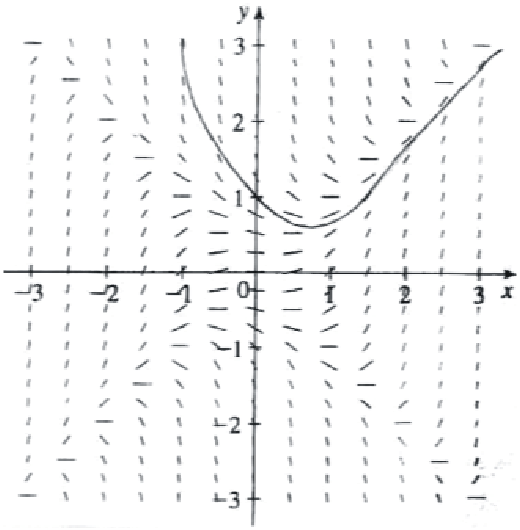


Fig. 1

Now from the graph we see that $\boxed{y(0.3) \approx 0.8}$

(B) We have $f(x, y) = x^2 - y^2, \; y(0) = 1$

If h = 0.1 then By Euler's method

$y(0.1) = y_1 = y_0 + h(f(x_0, y_0)) = 1 + 0.1(-1) = 0.9$

$y(0.2) = y_2 = y_1 + h(f(x_1, y_1)) = 0.9 + 0.1((0.1)^2 - (0.9)^2) = 0.82$

$y(0.3) = y_3 = y_2 + h\,f(x_2, y_2)$

$= 0.82 + 0.1((0.2)^2 - (0.82)^2)$

$= 0.75676$

$\Rightarrow \boxed{y(0.3) = 0.75676}$

(C) On the lines y = x and y = -x, the centers of the horizontal line segments of the directions field are located. When a solution curves crosses these lines, we get local maximum or minimum.

Answer 3P.

By the definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We have been given $f(a+b) = f(a)f(b)$

So $$f'(x) = \lim_{h \to 0} \frac{f(x).f(h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x)[f(h) - 1]}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

We have f (0) =1

So $$f'(x) = f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow f'(x) = f(x) f'(0)$$

Since $f'(0) = 1$ so $\boxed{f'(x) = f(x)}$

Now we have to solve $f'(x) = f(x)$

$$\Rightarrow \frac{df(x)}{dx} = f(x)$$

$$\Rightarrow \frac{1}{f(x)} df(x) = dx$$

Integrating both sides we get

$$\ln[f(x)] = x + c$$

$$\Rightarrow f(x) = ke^x \qquad [\text{Let } k = e^c]$$

When x = 0, f (0) =1 so k= 1

Thus the function is $\boxed{f(x) = e^x}$

Answer 3TFQ.

Given $y' = x + y$

This equation is not separable.

Since we cannot express the given equation in the form $f(x)dx = g(y)dy$

$\boxed{\text{Therefore the statement is false.}}$

Answer 4CC.

Euler's method:

For the general first order initial value problem $y' = F(x, y)$, $y(x_0)=y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1=x_0+h$, $x_2=x_0+2h$,….. where h is the step size. The differential equation tells us that the slope at (x_0,y_0) is

$y' = F(x_0, y_0)$

Hence the approximate value of the solution when $x=x_1$ is $y_1 = y_0 + hF(x_0, y_0)$

Similarly

$$y_2 = y_1 + hF(x_1, y_1)$$

$$y_3 = y_2 + hF(x_2, y_2)$$

$$\vdots$$

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

Answer 4E.

(A) We have $y' = 2xy^2$, $y(0)=1$

So $f(x, y) = 2xy^2$ given $h = 0.2$

We have $x_0 = 0, y_0 = 1$

Then by Euler's method

$$\begin{aligned} y_1 &= y(0.2) \\ &= y(0) + hf(x_0, y_0) \\ &= 1 + 0.2(0) \\ &= 1 \end{aligned}$$

So $x_1 = 0.2, y_1 = 1.0$

Then $y_2 = y(0.4)$

$$\begin{aligned} &= y(0.2) + hf(x_1, y_1) \\ &= 1 + 0.2(2 \times 0.2 \times (1)^2) \\ &= 1 + 0.2(0.4) \end{aligned}$$

$$\boxed{y(0.4) \approx 1.08}$$

(B) Now $f(x,y)=2xy^2$ given $h=0.1$

We have $x_0=0, y_0=1$

Then by Euler's method

$$\Rightarrow y(0.1)=y_1$$
$$=y_0+h\,f(x_0,y_0)$$
$$=1+0.1\times(0)$$
$$=1$$

Now $x_1=0.1, y_1=1$

Then $y(0.2)\approx y_2=y_1+h\,f(x_1,y_1)$

$$=1+0.1\left(2\times(0.1)\times 1^2\right)$$
$$=1.02$$

$$y(0.3)\approx y_3=y_2+h\,f(x_2,y_2)$$
$$=1.02+0.1\left(2\times(0.2)(1.02)^2\right)$$
$$=1.06162$$

Then $y(0.4)\approx y_4=y_3+h\,f(x_3,y_3)$

$$=1.06162+0.1\left(2\times(0.3)\times(1.06162)^2\right)$$
$$\Rightarrow \boxed{y(0.4)\approx 1.1292}$$

(C) We have to solve $y'=2xy^2,\ y(0)=1$

$\Rightarrow y^{-2}dy=2x\,dx$ [Separating the variables]

Integrating both sides

$$\int y^{-2}dy=\int 2x\,dx$$
$$\Rightarrow -y^{-1}=x^2+C$$
$$\Rightarrow y^{-1}=-\left(x^2+C\right)$$
$$\Rightarrow \boxed{y=-1/\left(x^2+C\right)}$$

We have $y(0)=1$

So $1=-1/C \Rightarrow (C=-1)$

So $y=-1/\left(x^2-1\right)$

Or $\boxed{y=1/\left(1-x^2\right)}$

Now $y(0.4)=1/\left(1-(0.4)^2\right)$

$$=1/0.84$$

Or $y(0.4)=\dfrac{100}{84}$

$$=\frac{25}{21}$$

$$\boxed{y(0.4)=\frac{25}{21}}\approx\boxed{1.190476}$$

We see that estimation in part (a) is very low but estimation in part (b) is less than but close to the exact value.

Answer 4P.

We have $\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}dx\right)=-1$ ....(1)

Differentiate with respect to x by product rule

$$\left[\frac{d}{dx}\left(\int f(x)\,dx\right)\right]\left(\int \frac{1}{f(x)}dx\right)+\left[\frac{d}{dx}\left(\int \frac{1}{f(x)}dx\right)\right]\left(\int f(x)\,dx\right)=0$$

$$\Rightarrow f(x).\int \frac{1}{f(x)}dx+\frac{1}{f(x)}.\int f(x)\,dx=0$$

$$\Rightarrow [f(x)]^2\int \frac{1}{f(x)}dx+1\int f(x)\,dx=0$$

$$\Rightarrow [f(x)]^2\frac{-1}{\left(\int f(x)\,dx\right)}+\left(\int f(x)\,dx\right)=0 \qquad \left[\text{from (1)}, \int \frac{1}{f(x)}dx=-1/\int f(x)\,dx\right]$$

$$\Rightarrow -[f(x)]^2+\left(\int f(x)\,dx\right)^2=0$$

$$\Rightarrow \left[\left(\int f(x)\,dx\right)-f(x)\right]\left[\left(\int f(x)\,dx\right)+f(x)\right]=0$$

$$\Rightarrow \left(\int f(x)\,dx\right)-f(x)=0 \text{ or } \left(\int f(x)\,dx\right)+f(x)=0$$

Again differentiating both the equations

$$f(x)-f'(x)=0 \text{ or } f(x)+f'(x)=0$$

$$\Rightarrow f'(x)=f(x) \text{ or } f'(x)=-f(x)$$

Solutions of these equations are

$\boxed{f(x)=Ae^{x}}$ *or* $\boxed{f(x)=Ae^{-x}}$ Where A is any constant

These are two functions which satisfy the given condition.

Answer 4TFQ.

Given

$y'=3y-2x+6xy-1$

$=(3y-1)(2x+1)$

$\Rightarrow \frac{1}{3y-1}dy=(2x+1)dx$

Clearly the given equation is separable.

$\boxed{\text{Hence the given statement is true.}}$

Answer 5CC.

A separable equation is a first order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y.

In other words it can be written in the form $\frac{dy}{dx}$=g(x)f(y)

$\Rightarrow \frac{dy}{dx}=\frac{g(x)}{h(y)}$ *where* $h(y)=\frac{1}{f(y)},\ f(y)\neq 0$

$\Rightarrow h(y)dy=g(x)dx$

Integrating on both sides

$\Rightarrow \boxed{\int h(y)\,dy=\int g(x)\,dx}$ +c is the solution of the given differential equation.

Answer 5E.

We have to solve $y'=xe^{-\sin x}-y\cos x$

$$\Rightarrow \frac{dy}{dx}+(\cos x)y=xe^{-\sin x} \qquad ....(1)$$

Comparing with $\frac{dy}{dx}+P(x)y=Q(x)$, we get $P(x)=\cos x$

Integrating factor is $I=e^{\int P(x)dx}=e^{\int \cos x dx}$

$=e^{\sin x}$

Multiplying both sides of the equation (1) by $e^{\sin x}$

$$e^{\sin x}\frac{dy}{dx}+e^{\sin x}(\cos x)y=x$$

$$\Rightarrow \frac{d}{dx}\left(e^{\sin x}y\right)=\int xdx$$

Integrating both sides, we have

$$e^{\sin x}y=\frac{x^2}{2}+C$$

Thus $\boxed{y=e^{-\sin x}\left(\frac{x^2}{2}+C\right)}$

Answer 5P.

img src=”https://c1.staticflickr.com/1/673/32203209956_1cf5b34b22_o.png” width=”636″ height=”84″ alt=”stewart-calculus-7e-solutions-Chapter-9.R-Differential-Equations-5P”>

We take the derivative of both sides of the integral equation and simplify:

$$\frac{d}{dx}\left[\int_0^x f(x)\,dx\right]=\frac{d}{dx}\left[k\left[f(x)\right]^{n+1}\right]$$

$$\Rightarrow f(x)=k(n+1)\left[f(x)\right]^n\frac{df(x)}{dx}$$

$$\Rightarrow \frac{df(x)}{dx}=\frac{f(x)}{k(n+1)\left[f(x)\right]^n}$$

$$\Rightarrow \frac{df(x)}{dx}=\frac{\left[f(x)\right]^{n-1}}{k(n+1)}$$

Put $y=f(x)$

$$\frac{dy}{dx}=y^{1-n}\frac{1}{k(n+1)}$$

Separate variables and integrate:

$$\int y^{n-1}dy=\int\frac{1}{k(n+1)}dx$$

$$\Rightarrow \frac{1}{n}y^n=\frac{1}{k(n+1)}x+C$$

$$\Rightarrow \frac{1}{n}\left[f(x)\right]^n=\frac{1}{k(n+1)}x+C$$

To satisfy $f(0)=0$ it must hold $C=0$.

To satisfy $f(1)=1$ it must hold $\frac{1}{n}=\frac{1}{k(n+1)}\Rightarrow k=\frac{n}{n+1}$.

Then, since $f(x)\geq 0$,

$$\boxed{y=f(x)=x^{\frac{1}{n}}}.$$

Answer 5TFQ.

Given

$$e^x y'=y$$

$$\Rightarrow y'=\frac{y}{e^x}$$

$$\Rightarrow y'-e^{-x}y=0$$

This is a linear equation

$\boxed{\text{Hence the given statement is true.}}$

Answer 6CC.

A first order linear differential equation is one that can be put into the form $\frac{dy}{dx}+p(x)y=q(x)$, where p(x) and q(x) are continuous functions on a given interval.

To solve these type of equations first we find the integrating factor $IF=e^{\int p dx}$

The solution of the given equation is y(IF)= $\boxed{\int q(x)(IF)dx+c}$

Answer 6E.

img src=”https://c1.staticflickr.com/1/328/32122625541_c6584ae406_o.png” width=”633″ height=”143″ alt=”stewart-calculus-7e-solutions-Chapter-9.R-Differential-Equations-6E”>

Integrating both sides

We have $\int \frac{1}{1+x}dx=\int(1-t)dt$

$$\Rightarrow \ln|(1+x)|=t-\frac{t^2}{2}+C$$

$$\Rightarrow 1+x=e^{t-t^2/2+C}$$

$$\Rightarrow x=e^{t(1-t/2)+C}-1$$

$$\Rightarrow x=e^{C}e^{t(1-t/2)}-1$$

$$\Rightarrow \boxed{x=ke^{t(1-t/2)}-1} \qquad \text{Let } k=e^{C}$$

Answer 6TFQ.

Given $y'+xy=e^y$

Which is not a linear equation

Since the right hand side function is not a function of x.

$\boxed{\text{Hence the given statement is false.}}$

Answer 7CC.

(a).

The differential equation that expresses the law of natural growth is $\frac{dp}{dt}=kp$

i.e., the population grows at a rate proportional to the size of the population.

(b).

If k is positive then the population increases

If k is negative then it decreases.

(c).

The solution of this equation is obtained from the procedure given below

$$\frac{dp}{dt}=kp$$

$$\Rightarrow \frac{dp}{p}=kdt$$

Integrating on both sides

$$\int\frac{dp}{p}=k\int 1dt$$

$$\Rightarrow \ln|p|=kt+c$$

$$\Rightarrow \boxed{p=e^{kt+c}=Ae^{kt}}$$

Answer 7P.

img src=”https://c1.staticflickr.com/1/422/31431283093_64c6d9565c_o.png” width=”626″ height=”181″ alt=”stewart-calculus-7e-solutions-Chapter-9.R-Differential-Equations-7P”>

So we have initial value problem

$$\frac{dx}{dt}=k(x), \quad x(0)=100-T_R$$

Solution of this problem is

$$x(t)=x(0)e^{kt}$$
$$\Rightarrow x(t)=(100-T_R)e^{kt}$$
$$\Rightarrow T(t)-T_R=(100-T_R)e^{kt}$$
$$\Rightarrow \boxed{T(t)=T_R+(100-T_R)e^{kt}}$$

By the given information's we have

$$T(10)=80°C \text{ and } T(20)=65°C$$

So $\quad 80=T_R+(100-T_R)e^{10k} \quad \text{.....(1)}$

And $\quad 65=T_R+(100-T_R)e^{20k} \quad \text{....(2)}$

Form (1) we have

$$e^{10k}=\frac{80-T_R}{100-T_R} \text{ then } e^{20k}=\frac{(80-T_R)^2}{(100-T_R)^2}$$
$$\Rightarrow e^{20k}=\frac{6400+T_R^2-160T_R}{(100-T_R)^2}$$

Putting this value of e^{20k} in equation (2)

We have $T_R+(100-T_R).\frac{(6400+T_R^2-160T_R)}{(100-T_R)^2}=65$

$$\Rightarrow T_R+\frac{(6400+T_R^2-160T_R)}{(100-T_R)}=65$$
$$\Rightarrow T_R(100-T_R)+(6400+T_R^2-160T_R)=65(100-T_R)$$
$$\Rightarrow 100T_R-T_R^2+6400+T_R^2-160T_R=6500-65T_R$$
$$\Rightarrow 65T_R-60T_R=6500-6400$$
$$\Rightarrow 5T_R=100$$
$$\Rightarrow T_R=\frac{100}{5}=20$$

Room temperature is $\boxed{T_R=20°C}$

Answer 7TFQ.

Given $\frac{dy}{dt}=2y\left(1-\frac{y}{5}\right), y(0)=1$

Compare this equation with $\frac{dy}{dt}=ky\left(1-\frac{y}{M}\right)$

$$\Rightarrow k=2, M=5, y_0=1$$
$$\Rightarrow y(t)=\frac{5}{1+Ae^{-2t}}, A=\frac{5-1}{1}=4$$
$$\Rightarrow y(t)=\frac{5}{1+4e^{-2t}}$$
$$\therefore y\rightarrow 5 \text{ as } t\rightarrow\infty$$

Hence the given statement is true.

Answer 8CC.

(a).

Logistic equation is $\boxed{\frac{dp}{dt}=kp\left(1-\frac{p}{M}\right)}$

(b).

The relative growth rate is almost constant when the population is small.
The relative growth decreases as the population p increases and becomes negative if p ever exceeds its carrying capacity M, the maximum population that the environment is capable of sustaining in the long run.

Answer 8E.

img src="https://c1.staticflickr.com/1/592/32122624531_195fffc2a9_o.png" width="703" height="143" alt="stewart-calculus-7e-solutions-Chapter-9.R-Differential-Equations-8E">

Integration factor is $I = e^{\int -x^{-2}dx}$

$$= e^{1/x}$$

Multiplying both sides of the equation (1) by $e^{1/x}$

$$\frac{dy}{dx}e^{1/x} - x^{-2}e^{1/x}y = 2x$$

$$\Rightarrow \frac{d}{dx}\left(e^{1/x}.y\right) = 2x$$

Integrating both sides

$$e^{1/x}y = \int 2x\,dx$$

$$\Rightarrow e^{1/x}y = x^2 + C$$

$$\Rightarrow \boxed{y = e^{-1/x}\left(x^2 + C\right)}$$

Answer 9CC.

(a). The equations $\frac{dR}{dt} = kR - aRW$, $\frac{dW}{dt} = -rW + bRW$ are known as Lotka-Volterra equations.

(b).

A solution of this system of equations is a pair of functions R(t) and W(t) that describe the populations of prey and predator as functions of time. Because the system is coupled, we cannot solve one equation and then the other, we have to solve them simultaneously.

Answer 9E.

img src="https://c1.staticflickr.com/1/546/32241331855_caf7b6040a_o.png" width="697" height="244" alt="stewart-calculus-7e-solutions-Chapter-9.R-Differential-Equations-9E">

The integrating factor is

$$I(t) = e^{\int(2t-1)dt}$$
$$= e^{t^2-t}$$

Multiply by the integrating factor to the equation (1) to get

$$e^{t^2-t}\frac{dr}{dt} + e^{t^2-t}(2t-1)r = e^{t^2-t}\cdot 0$$
$$\left(re^{t^2-t}\right)' = 0$$

Integrating on both sides with respect to t to get

$$\int\left(re^{t^2-t}\right)' dt = \int 0\,dt + C$$
$$re^{t^2-t} = 0 + C$$
$$re^{t^2-t} = C$$
$$r = Ce^{t-t^2}$$

Since $r(0) = 5$, we obtain that

$$r = Ce^{t-t^2}$$
$$5 = Ce^0$$
$$5 = C$$

Thus, the solution of the differential equation is $\boxed{r = 5e^{t-t^2}}$.

Answer 9P.

img src="https://c1.staticflickr.com/1/425/32092653092_f0a18266dd_o.png" width="692" height="455" alt="stewart-calculus-7e-solutions-Chapter-9.R-Differential-Equations-9P">

When dog is at (x, y)
Distance traveled by dog, S = Length of the curve form x to L

$$S = \int_x^L \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$S = -\int_L^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Then $\frac{dS}{dx} = -\frac{d}{dx}\int_L^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$\Rightarrow \frac{dS}{dx} = -\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{........(1)}$$

Since both are running with the same speed, so when dog reaches at (x, y), rabbit will be at (0, S) because rabbit is running along the Y-axis.

Slope of the line joining them is $\frac{dy}{dx} = \frac{S-y}{0-x}$

$$\Rightarrow \frac{dy}{dx} = \frac{y-S}{x}$$
$$\Rightarrow x\frac{dy}{dx} = y - S$$
$$\Rightarrow S = y - x\frac{dy}{dx}$$

Differentiating with respect to x

$$\frac{dS}{dx} = \frac{dy}{dx} - \left(x\frac{d^2x}{dx^2} + \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dS}{dx} = -x\frac{d^2x}{dx^2} \qquad ....(2)$$

Form (1) and (2)

$$-x\frac{d^2x}{dx^2} = -\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \boxed{x\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}} \qquad .....(3)$$

(B) Let $r = \frac{dy}{dx}$

Then equation becomes

$$x\frac{dr}{dx} = \sqrt{1+r^2}$$

$$\Rightarrow \frac{1}{\sqrt{1+r^2}}dr = \frac{1}{x}dx$$

Integrating both sides $\int \frac{1}{\sqrt{1+r^2}}dr = \int \frac{1}{x}dx$

$$\ln\left(r+\sqrt{1+r^2}\right) = \ln x + \ln c$$

We have taken ln c in place of c, and used $\int \frac{1}{\sqrt{a^2+u^2}}du = \ln\left(u+\sqrt{a^2+u^2+c}\right)$

$$\Rightarrow r + \sqrt{1+r^2} = cx$$

$$\Rightarrow \sqrt{1+r^2} = cx - r$$

Step 9 of 14

Squaring both sides

$$\Rightarrow 1+r^2 = c^2x^2 + r^2 - 2cxr$$

When $x = L, \frac{dy}{dx} = 0 \Rightarrow r = 0$

So $1 = c^2L^2 + 0 - 0 \qquad \Rightarrow c = \pm\frac{1}{L}$

We take only $c = \frac{1}{L}$

So $1+r^2 = \frac{1}{L^2}x^2 + r^2 - \frac{2}{L}xr$

$$\Rightarrow \frac{2}{L}xr = \frac{1}{L^2}x^2 - 1$$

$$\Rightarrow r = \frac{L}{2x}\left(\frac{1}{L^2}x^2 - 1\right)$$

$$\Rightarrow r = \frac{1}{2L}x - \frac{L}{2x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2L} - \frac{L}{2x}$$

$$\Rightarrow dy = \left(\frac{x}{2L} - \frac{L}{2x}\right)dx$$

Integrating both sides

$$\int dy = \int \left(\frac{x}{2L} - \frac{L}{2x} \right) dx$$

$$\Rightarrow y = \frac{x^2}{4L} - \frac{L}{2} \ln x + C_1$$

Since when x =L, y = 0.

So $\quad 0 = \frac{L^2}{4L} - \frac{L}{2} \ln L + C_1$

$$\Rightarrow 0 = \frac{L}{4} - \frac{L}{2} \ln L + C_1$$

$$\Rightarrow C_1 = \frac{L}{2} \ln L - \frac{L}{4} = \frac{L}{4}(2 \ln L - 1)$$

(C) When the point (x, y) is very close to origin

So $\quad \lim_{x \to 0^+} y = \lim_{x \to 0^+} \left(\frac{x^2 - L^2}{4L} \right) + \frac{L}{2} \lim_{x \to 0^+} \ln (L / x)$

Since $\frac{L}{x} \to \infty$ *as* $x \to 0^+$ and then $\lim_{x \to 0^+} \ln \left(\frac{L}{x} \right) \to \infty$

So when, $x \to 0^+, y \to \infty$ so dog $\boxed{\text{can not}}$ catch the rabbit.

Answer 10E.

We must solve the given initial-value problem:

$(1 + \cos x) y' = (1 + e^{-y}) \sin x$, $y(0) = 0$

$$(1 + \cos x) y' = (1 + e^{-y}) \sin x$$

$$\Rightarrow (1 + \cos x) \frac{dy}{dx} = (1 + e^{-y}) \sin x$$

Separating the variables we get

$$(1 + \cos x) \frac{dy}{dx} = (1 + e^{-y}) \sin x$$

$$(1 + \cos x) dy = (1 + e^{-y}) \sin x dx$$

$$\Rightarrow \frac{dy}{(1 + e^{-y})} = \frac{\sin x dx}{(1 + \cos x)} \qquad \text{...... (1)}$$

Integrate:

$$\int \frac{dy}{(1 + e^{-y})} = \int \frac{\sin x dx}{(1 + \cos x)}$$

For the left side of the equation (1)

Let

$u = 1 + e^{-y}$

Then

$du = -e^{-y} dy$

That means

$$dy = -\frac{1}{e^{-y}} du$$

$$= -\frac{1}{(u-1)} du$$

So

$$\int \frac{dy}{\left(1+e^{-y}\right)} = -\int \frac{1}{u\,(u-1)}du$$

$$= -\int \left(\frac{1}{u-1} - \frac{1}{u}\right)du$$

$$= \int \frac{1}{u}du - \int \frac{1}{u-1}du$$

$$= \ln u - \ln (u-1)$$

$$= \ln\left(\frac{u}{u-1}\right)$$

$$= \ln\left(\frac{1+e^{-y}}{e^{-y}}\right)$$

$$= \ln\left(1+e^{y}\right)$$

For the right side of the equation (1)
Let

$u = 1+\cos x$

$\Rightarrow du = -\sin x dx$

$$\int \frac{\sin x dx}{(1+\cos x)} = -\int \frac{du}{u}$$

$$= -\ln u + C$$

$$= -\ln(1+\cos x) + C \qquad (\text{since } u = 1+\cos x)$$

So substituting in (1) we get

$\ln\left(1+e^{y}\right) = -\ln(1+\cos x) + C$

Using the initial condition $y(0) = 0$ we get

$\ln(1+1) = -\ln(1+1) + C$

$\Rightarrow \ln 2 = -\ln 2 + C$

$\Rightarrow C = 2\ln 2$

Substituting the value of C we get

$$\ln\left(1+e^{y}\right) = -\ln(1+\cos x) + 2\ln 2$$

$$= -\ln(1+\cos x) + \ln 4$$

$\Rightarrow \ln 4 = \ln\left(1+e^{y}\right) + \ln(1+\cos x)$

$\Rightarrow \ln 4 = \ln\left(1+e^{y}\right)(1+\cos x)$

$\Rightarrow 4 = \left(1+e^{y}\right)(1+\cos x)$

$\Rightarrow 4 = 1+\cos x + e^{y}(1+\cos x)$

$\Rightarrow e^{y}(1+\cos x) = 3-\cos x$

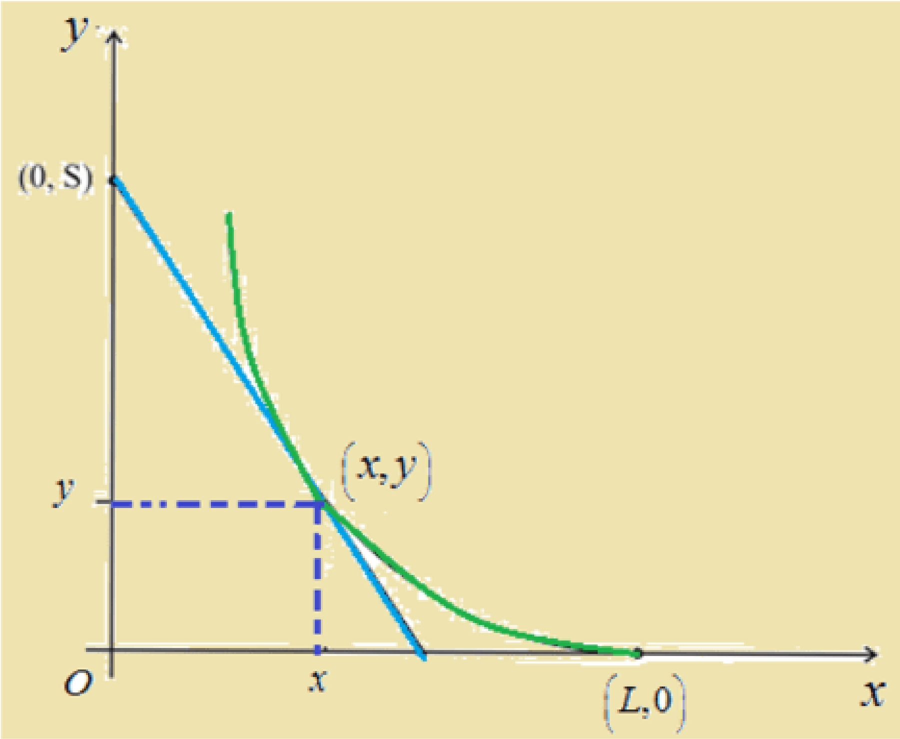
$$\Rightarrow e^{y} = \frac{(3-\cos x)}{(1+\cos x)}$$

$$\Rightarrow y = \ln\left(\frac{3-\cos x}{1+\cos x}\right)$$

Thus solution of the given initial value problem is $\boxed{y = \ln\left(\frac{3-\cos x}{1+\cos x}\right)}$

Answer 10P.

Consider the data: The rabbit is at the origin, and also the dog is at the point, $(L,0)$ at the instant the dog first sees the rabbit, the rabbit runs up the $y-$axis and the dog always runs straight for the rabbit, and the dog runs at the same speed as the rabbit.



(a)

When the dog is at the point, (x,y)

Take the distance traveled by the dog is

S = Length of the curve from x to L

$$S=\int_x^L \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S=-\int_L^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Differentiate this with respect to x, we get

Then $\frac{dS}{dx}=-\frac{d}{dx}\int_L^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$\frac{dS}{dx}=-\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{...... (1)}$$

When dog reaches at (x,y), rabbit will be at $(0,S)$ because the rabbit is running along the $Y-$axis.

Slope of the line joining them is $\frac{dy}{dx}=\frac{S-y}{0-x}$

$$\frac{dy}{dx}=\frac{y-S}{x}$$

$$x\frac{dy}{dx}=y-S$$

$$S=y-x\frac{dy}{dx}$$

Differentiate with respect to x, we get

$$\frac{dS}{dx}=\frac{dy}{dx}-\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)$$

$$\frac{dS}{dx}=-x\frac{d^2y}{dx^2} \quad \ldots\ldots (2)$$

From the data, dog runs twice as fast as the rabbit.

So, from (1) and (2),

$$2\left(-x\frac{d^2y}{dx^2}\right)=-\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$-2x\frac{d^2y}{dx^2}=-\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$2x\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Hence, the differential equation is $\boxed{2x\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$.

Take $r=\frac{dy}{dx}$, then equation becomes

$$2x\frac{dr}{dx}=\sqrt{1+r^2}$$

$$2\left(\frac{1}{\sqrt{1+r^2}}\right)dr=\frac{1}{x}dx$$

Integrate on both sides $2\int\frac{1}{\sqrt{1+r^2}}dr=\int\frac{1}{x}dx$

$$2\ln\left(r+\sqrt{1+r^2}\right)=\ln x+\ln c$$

Here, we used the formula,

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left(u+\sqrt{a^2+u^2+c}\right).$$

$$\left(r+\sqrt{1+r^2}\right)^2 = cx$$

$$r+\sqrt{1+r^2} = \sqrt{cx}$$

$$\sqrt{1+r^2} = \sqrt{cx}-r$$

$$1+r^2 = \left(\sqrt{cx}-r\right)^2$$

Squaring on both sides

$$1+r^2 = cx+r^2-2r\sqrt{cx}$$

$$1 = cx-2r\sqrt{cx}$$

When $x = L, \frac{dy}{dx} = 0 \Rightarrow r = 0$

So

$$1+(0)^2 = cL+(0)^2-2(0)\sqrt{cL}$$

$$1 = cL$$

$$c = \frac{1}{L}$$

Substitute, $c = \frac{1}{L}$, then

$$1 = cx-2r\sqrt{cx}$$

$$1 = \frac{1}{L}x-2r\sqrt{\left(\frac{1}{L}\right)x}$$

$$1 = \frac{1}{L}x-2r\sqrt{\frac{x}{L}}$$

$$2r\sqrt{\frac{x}{L}} = \frac{1}{L}x-1$$

$$2r\sqrt{\frac{x}{L}} = \frac{x}{L}-1$$

$$2r = \sqrt{\frac{L}{x}}\left(\frac{x}{L}-1\right)$$

$$r = \frac{1}{2}\sqrt{\frac{L}{x}}\left(\frac{x}{L}-1\right)$$

$$r = \frac{1}{2}\left(\sqrt{\frac{x}{L}}-\sqrt{\frac{L}{x}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\sqrt{\frac{x}{L}}-\sqrt{\frac{L}{x}}\right)$$

$$dy = \frac{1}{2}\left(\sqrt{\frac{x}{L}}-\sqrt{\frac{L}{x}}\right)dx$$

$$dy = \frac{1}{2}\left(\frac{1}{\sqrt{L}}\sqrt{x}-\sqrt{L}\sqrt{\frac{1}{x}}\right)dx$$

$$dy = \left(\frac{1}{2\sqrt{L}}\sqrt{x}-\frac{\sqrt{L}}{2}\sqrt{\frac{1}{x}}\right)dx$$

Integrate both sides

$$\int dy = \int\left(\frac{1}{2\sqrt{L}}\sqrt{x} - \frac{\sqrt{L}}{2}\sqrt{\frac{1}{x}}\right)dx$$

$$\int dy = \frac{1}{2\sqrt{L}}\int\left(\sqrt{x}\right)dx - \frac{\sqrt{L}}{2}\int\left(\frac{1}{\sqrt{x}}\right)dx$$

$$y = \frac{1}{2\sqrt{L}}\left(\frac{2x^{\frac{3}{2}}}{3}\right) - \frac{\sqrt{L}}{2}\left(2\sqrt{x}\right) + C$$

$$y = \frac{1}{\sqrt{L}}\left(\frac{x^{\frac{3}{2}}}{3}\right) - \sqrt{L}\left(\sqrt{x}\right) + C$$

$$y = \frac{x^{\frac{3}{2}}}{3\sqrt{L}} - \sqrt{Lx} + C$$

When, $x = L, y = 0$.

$$y = \frac{x^{\frac{3}{2}}}{3\sqrt{L}} - \sqrt{Lx} + C$$

$$0 = \frac{L^{\frac{3}{2}}}{3\sqrt{L}} - \sqrt{L(L)} + C$$

$$\frac{L^{\frac{3}{2}}}{3\sqrt{L}} - \sqrt{L(L)} + C = 0$$

$$\frac{L}{3} - L + C = 0$$

$$C = \frac{2L}{3}$$

Hence, the solution is,

$$\boxed{y = \frac{x^{\frac{3}{2}}}{3\sqrt{L}} - \sqrt{Lx} + \frac{2L}{3}}$$

(b)

From the data, dog runs half as fast as the rabbit.

So, from (1) and (2), we get

$$\frac{1}{2}\left(-x\frac{d^2y}{dx^2}\right) = -\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$-x\frac{d^2y}{dx^2} = -2\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$x\frac{d^2y}{dx^2} = 2\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Hence, the differential equation is $\boxed{x\frac{d^2y}{dx^2} = 2\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$.

Take $r = \frac{dy}{dx}$, then equation becomes

$$x\frac{dr}{dx} = 2\sqrt{1+r^2}$$

$$\left(\frac{1}{\sqrt{1+r^2}}\right)dr = \frac{2}{x}dx$$

Integrate on both sides $\int\frac{1}{\sqrt{1+r^2}}dr = 2\int\frac{1}{x}dx$

$$\ln\left(r+\sqrt{1+r^2}\right) = \ln x^2 + \ln c$$

$$r+\sqrt{1+r^2} = cx^2$$

Here, we used the formula, $\int\frac{1}{\sqrt{a^2+u^2}}du = \ln\left(u+\sqrt{a^2+u^2+c}\right)$.

$$\sqrt{1+r^2} = cx^2 - r$$

Squaring on both sides

$$1+r^2 = c^2x^4 + r^2 - 2cx^2r$$

$$1 = c^2x^4 - 2cx^2r$$

When $x = L, \frac{dy}{dx} = 0 \Rightarrow r = 0$

So

$$1 = c^2L^4 - 2cL^2(0)$$

$$1 = c^2L^4$$

$$c^2 = \frac{1}{L^4}$$

$$c = \frac{1}{L^2}$$

Substitute, $c = \frac{1}{L^2}$, then

$$1 = c^2x^4 - 2cx^2r$$
$$1 = \frac{1}{L^4}x^4 - 2\left(\frac{1}{L^2}\right)x^2r$$
$$1 = \frac{x^4}{L^4} - \frac{2x^2r}{L^2}$$
$$r = \frac{x^2}{2L^2} - \frac{L^2}{2x^2}$$
$$\frac{dy}{dx} = \frac{x^2}{2L^2} - \frac{L^2}{2x^2}$$
$$dy = \frac{1}{2}\left(\frac{x^2}{L^2} - L^2x^{-2}\right)dx$$

Integrate both sides

$$\int dy = \frac{1}{2}\int\left(\frac{x^2}{L^2} - L^2x^{-2}\right)dx$$
$$y = \frac{1}{2}\left(\frac{x^3}{3L^2}\right) - L^2\left(-\frac{1}{x}\right) + C$$
$$y = \frac{x^3}{6L^2} + \frac{L^2}{x} + C$$

When, $x = L, y = 0$.

$$y = \frac{x^3}{6L^2} + \frac{L^2}{x} + C$$
$$0 = \frac{L^3}{6L^2} + \frac{L^2}{L} + C$$
$$0 = \frac{L}{6} + L + C$$
$$C = \frac{7L}{6}$$

Hence, the solution is,

$$y = \frac{x^3}{6L^2} + \frac{L^2}{x} + C$$
$$\boxed{y = \frac{x^3}{6L^2} + \frac{L^2}{x} + \frac{7L}{6}}$$

When the point (x, y) is very close to origin

Answer 11E.

Consider the initial-value problem

$$xy' - y = x\ln x, \quad y(1) = 2$$

Need to solve the differential equation.

We must first divide both sides by the coefficient of y' to put the differential equation into standard form.

$$y' - \frac{1}{x}y = \ln x \quad \cdots\cdots(1)$$

The differential equation (1) is in the standard form for a linear equation.

The integrating factor is

$$\begin{aligned} I(t) &= e^{\int -\frac{1}{x}dx} \\ &= e^{-\ln x} \\ &= e^{\ln x^{-1}} \\ &= x^{-1} \\ &= \frac{1}{x} \end{aligned}$$

Multiply by the integrating factor to the equation (1) to get

$$\begin{aligned} \frac{1}{x}\cdot y' - \frac{1}{x}\cdot\frac{1}{x}y &= \frac{1}{x}\cdot\ln x \\ \frac{1}{x}y' - \frac{1}{x^2}y &= \frac{1}{x}\ln x \\ \left(\frac{1}{x}y\right)' &= \frac{1}{x}\ln x \end{aligned}$$

Integrating on both sides with respect to x to get

$$\begin{aligned} \int\left(\frac{1}{x}y\right)'dx &= \int\frac{1}{x}\ln x\,dx + C \\ \frac{1}{x}y &= \frac{1}{2}(\ln x)^2 + C \\ y &= \frac{x}{2}(\ln x)^2 + Cx \end{aligned}$$

Since $y(1) = 2$, we obtain that

$$\begin{aligned} y &= \frac{x}{2}(\ln x)^2 + Cx \\ 2 &= \frac{1}{2}(\ln 1)^2 + C(1) \\ 2 &= 0 + C \\ C &= 2 \end{aligned}$$

Thus, the solution of the differential equation is $\boxed{y = \frac{x}{2}(\ln x)^2 + 2x}$.

Answer 11P.

(A) We have $\frac{dv}{dt} = 60000\pi \quad \text{ft}^3/\text{h}$

Radius of the cone, r = 1.5 × height of the cone

$$\Rightarrow r = 1.5h = \frac{3}{2}h$$

Volume of the cone is $v=\frac{1}{3}\pi r^2 h$

$$=\frac{1}{3}\pi.\left(\frac{3}{2}\right)^2 h^3$$

$$=\frac{9}{12}\pi h^3$$

$$\Rightarrow v=\frac{3}{4}\pi h^3$$

Differentiating with respect to t

$$\frac{dv}{dt}=\frac{3}{4}\pi.3h^2\frac{dh}{dt}$$

$$\Rightarrow\frac{dv}{dt}=\frac{9}{4}\pi h^2\frac{dh}{dt}$$

So $$\frac{dh}{dt}=\frac{(dv/dt)}{\frac{9}{4}\pi h^2}$$

$$\Rightarrow 9\pi h^2\frac{dh}{dt}=4\frac{dv}{dt}$$

$$\Rightarrow 9\pi h^2\frac{dh}{dt}=4\times 60000\pi$$

$$\Rightarrow 9\pi h^2\,dh=240000\pi\,dt$$

Integrating both sides $\int 9\pi h^2\,dh=\int 240000\pi\,dt$

$$\Rightarrow 9\pi\frac{h^3}{3}=240000\pi t+c$$

$$\Rightarrow 3\pi h^3=240000\pi t+c$$

$$\Rightarrow h^3=80000t+\frac{c}{3\pi}$$

Let k = $c/3\pi$

Then $h^3=80000t+k$

When t = 0, h = 60

So $60^3=k=216000$

Then $h^3=80000t+216000$

At the top, h = 100 ft

So we have to find t such that h = 100

$$\Rightarrow(100)^3=80000t+216000$$

$$\Rightarrow 1000000=80000t+216000$$

$$\Rightarrow t=\frac{784000}{80000}\approx 9.8\,hours$$

$$\boxed{t=9.8\,\text{hours}}$$

(B) Area of the floor of the silo is $=\pi(200)^2\text{ ft}^2=40000\pi\text{ ft}^2$

When radius of the silo = 200ft

Area of the base of pile is $A=\pi r^2$, where r is radius

But $r=\frac{3}{2}h$

So $A=\pi\left(\frac{3}{2}h\right)^2$

When h = 60ft

Then A= $\frac{9}{4}\pi(60)^2=8100\pi\text{ ft}^2$

So remaining area of the floor of silo $=40000\pi-8100\pi=\boxed{31900\pi}\text{ ft}^2$

Now $A=\frac{9}{4}\pi h^2$

Differencing with respect to t

$$\frac{dA}{dt}=\frac{9}{2}\pi h.\frac{dh}{dt} \quad ft^2/h$$

For h = 60 ft

$$\frac{dh}{dt}=\frac{240000\pi}{9\pi\times60\times60}=\frac{80000}{10800}=\frac{800}{108} \qquad \text{[From part (a)]}$$

So $\frac{dA}{dt}=\frac{9}{2}\pi(60)\times\frac{800}{108} \quad ft^2/h$

$$=2000\pi\approx\boxed{6283\ ft^2/h}$$

(C) When h = 90 ft

$$\frac{dv}{dt}=60000\pi-20000\pi=40000\pi \quad ft^3/h$$

Then from part (a) $\frac{dh}{dt}=\frac{4\left(\frac{dv}{dt}\right)}{9\pi h^2}$

$$\Rightarrow\frac{dh}{dt}=\frac{4\times40000\pi}{9\pi h^2}$$

$$\Rightarrow 9\pi h^2 dh=160000\pi\, dt$$

Integrating both sides $\int 9\pi h^2\, dh=\int 16000\,\pi\, dt$

$$\Rightarrow 9\pi\frac{h^3}{3}=160000\pi t+c$$

$$\Rightarrow 3\pi h^3=16000\pi t+c$$

$$\Rightarrow h^3=\frac{160000}{3}t+\frac{c}{3\pi}$$

Let $\frac{c}{3\pi}=k$

Then $h^3=\frac{160000}{3}t+k$

When t = 0 h = 90 ft so $k=90^3$

Then $h^3=\frac{160000}{3}t+90^3$

We have to find t such that h = 100 ft

$$\Rightarrow(100)^3-(90)^3=\frac{160000}{3}t$$

$$\Rightarrow 271000=\frac{160000}{3}t$$

$$\Rightarrow t=\frac{813000}{160000}\approx 5.1 \text{ h}$$

$$\Rightarrow\boxed{t=5.1\text{ h}}$$

Answer 12E.

We need to solve the given initial-value problem

$y'=3x^2e^y$, $y(0)=1$

That means

$$\frac{dy}{dx}=3x^2e^y$$

Separating the variables we get

$$\frac{dy}{e^y} = 3x^2 dx$$

On integrating we get

$$\int e^{-y} dy = 3\int x^2 dx$$

$$\Rightarrow -e^{-y} = x^3 + C$$

Using the initial condition $y(0) = 1$ we get

$$-e^{-1} = 0 + C$$

$$\Rightarrow C = -\frac{1}{e}$$

Substituting the value of C we get

$$-e^{-y} = x^3 - \frac{1}{e}$$

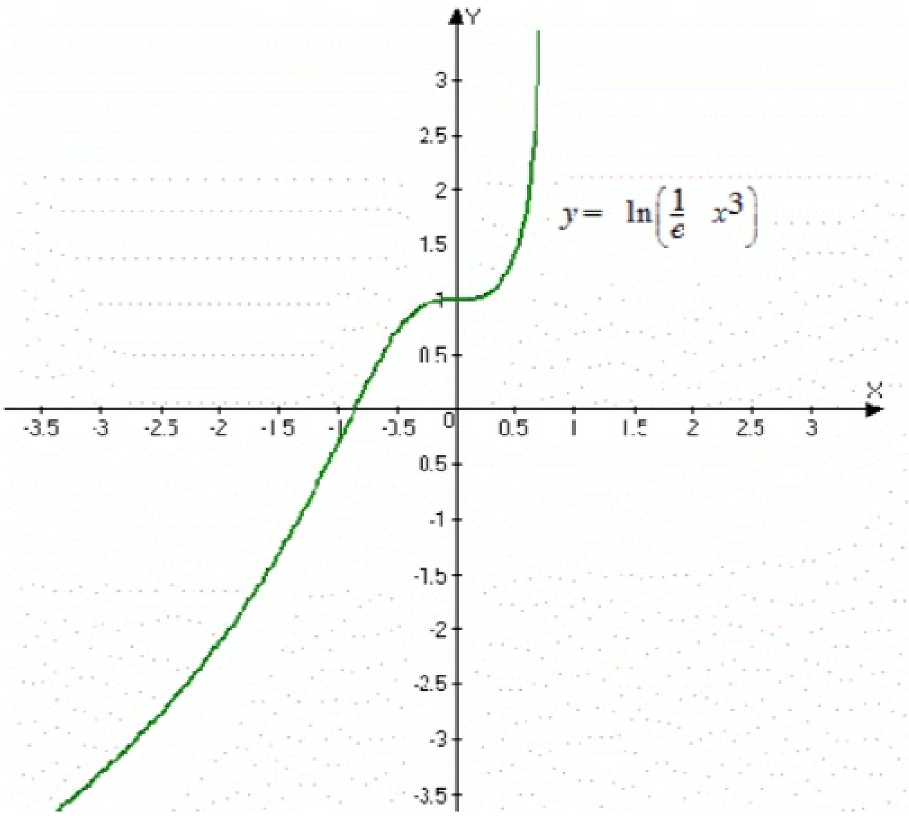
$$\Rightarrow e^{-y} = \frac{1}{e} - x^3$$

$$\Rightarrow \quad y = -\ln\left(\frac{1}{e} - x^3\right)$$

Thus the solution of the given initial value problem is

$$\boxed{y = -\ln\left(\frac{1}{e} - x^3\right)}$$

Graph of the solution of the given initial value problem:



Answer 13E.

Consider the following family of curves:

$$y = ke^x \quad \cdots\cdots(1)$$

The objective is to find the orthogonal trajectories of the family of curves.

The first step is to find a single differential equation that is satisfied by all members of the family.

Differentiate $y = ke^x$.

$$\frac{dy}{dx} = ke^x \quad \cdots\cdots(2)$$

This differential equation depends on *k*, but the task is to find an equation that is valid for all values of *k* simultaneously.

Hence, eliminate *k*.

From equation (1) obtain the following:

$$k = \frac{y}{e^x}$$

Substitute the value of *k* in equation (2).

$$\frac{dy}{dx} = \frac{y}{e^x} \cdot e^x$$
$$\frac{dy}{dx} = y$$

This implies that the slope of the tangent line at any point (x, y), on one of the parabolas is $\frac{dy}{dx} = y$.

On an orthogonal trajectory, the slope of the tangent line must be the negative reciprocal of this slope.

Therefore, the orthogonal trajectory must satisfy the following differential equation:

$$\frac{dy}{dx} = -\frac{1}{y}$$

Now, solve the differential equation.

Observe that this differential equation is separable and hence it can be solved it as follows:

$$y\,dy = -dx$$
$$\int y\,dy = -\int dx$$
$$\frac{y^2}{2} = -x + C, \quad \text{Here } C \text{ is a parameter.}$$
$$x + \frac{y^2}{2} = C$$

Thus, the orthogonal trajectory of the family of curves $y = ke^x$ is $\boxed{x + \frac{y^2}{2} = C}$.

Answer 13P.

Let $P(a,b)$ be any point on the required curve and m be the slope of the tangent line to the curve at the point $P(a,b)$.

Then the equation of the normal to the curve at the point $P(a,b)$ and perpendicular to the tangent line at *P* is as follows:

$$y-b=\frac{-1}{m}(x-a)$$
$$y=-\frac{1}{m}x+\left(b+\frac{a}{m}\right)$$

Here, the *y*-intercept of the line is $b+\frac{a}{m}$ which is equal to 6.

So,

$$b+\frac{a}{m}=6$$
$$6-b=\frac{a}{m}$$
$$m=\frac{a}{6-b}$$

Then, the slope of the line at any point (x,y) is as follows:

$$m=\frac{dy}{dx}=\frac{x}{6-y}.$$

Rewrite the equation as follows:

$$(6-y)dy=xdx$$

Then, the slope of the line at any point (x,y) is as follows:

$$m=\frac{dy}{dx}=\frac{x}{6-y}.$$

Rewrite the equation as follows:

$$(6-y)dy=xdx$$

Integrate on both sides to get the following:

$$\int(6-y)dy=\int xdx$$
$$6y-\frac{y^2}{2}=\frac{x^2}{2}+c$$

Since the curve passes through the point $(3,2)$, this point satisfies the curve equation.

Substitute the point $(3,2)$ in the equation, $6y-\frac{y^2}{2}=\frac{x^2}{2}+c$ and solve for the constant c.

$$6(2)-\frac{2^2}{2}=\frac{3^2}{2}+c$$
$$c=\frac{11}{2}$$

Therefore, the equation of the curve is as follows:

$$6y-\frac{y^2}{2}=\frac{x^2}{2}+\frac{11}{2}$$
$$12y-y^2=x^2+11$$
$$x^2+y^2-12y+11=0$$
$$x^2+(y-6)^2=25$$

Hence, the required equation of the curve is $\boxed{x^2+(y-6)^2=25}$.

Answer 14E.

Consider the family of curves

$$y=e^{kx} \quad \cdots\cdots(1)$$

Need to find the orthogonal trajectories of the family of curves.

The first step is to find a single differential equation that is satisfied by all members of the family.

Differentiate $y=e^{kx}$ to get

$$\frac{dy}{dx}=ke^{kx} \quad \cdots\cdots(2)$$

This differential equation depends on *k*, but we need an equation that is valid for all values of *k* simultaneously.

So, need to eliminate *k*.

From equation (1) to get

$$y=e^{kx}$$
$$\ln y=kx$$
$$k=\frac{\ln y}{x}$$

Substitute the value of *k* in equation (2).

$$\begin{aligned}\frac{dy}{dx} &= \frac{\ln y}{x}\cdot e^{\frac{\ln y}{x}\cdot x}\\ &= \frac{\ln y}{x}\cdot e^{\ln y}\\ &= \frac{\ln y}{x}\cdot y\\ &= \frac{y\ln y}{x}\end{aligned}$$

This means that the slope of the tangent line at any point (x, y) on one of the parabola is $\frac{dy}{dx} = \frac{y\ln y}{x}$.

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope.

Therefore, the orthogonal trajectory must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{x}{y\ln y}$$

Now solve the differential equation.

Observe that this differential equation separable and we solve it as follows:

$$\begin{aligned} y\ln y\,dy &= -x\,dx\\ \int y\ln y\,dy &= -\int x\,dx\\ \ln y\cdot\frac{y^2}{2} - \int\frac{1}{y}\cdot\frac{y^2}{2}dy &= -\frac{x^2}{2} + C_1, \quad \text{where } C_1 \text{ is a parameter.}\\ \frac{y^2\ln y}{2} - \frac{y^2}{4} &= -\frac{x^2}{2} + C_1\\ \frac{x^2}{2} - \frac{y^2}{4} + \frac{y^2\ln y}{2} &= C_1\\ 2x^2 - y^2 + 2y^2\ln y &= C\end{aligned}$$

Thus the orthogonal trajectory of the family of curves $y = e^{kx}$ is $\boxed{2x^2 - y^2 + 2y^2\ln y = C}$.

Answer 15E.

Given the initial value problem

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{2000}\right),\ P(0) = 100$$

(a) We need to find the solution of the given initial value problem and to find the Population when $t = 20$

Comparing with the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \quad \text{...... (1)}$$

We get

$k = 0.1,\ M = 2000,\ P_0 = 100$

We have the solution of logistic equation given by (1) is

$$P(t) = \frac{M}{1+Ae^{-kt}}, \text{ where } A = \frac{M-P_0}{P_0}$$

Here we have

$$P(t) = \frac{2000}{1+Ae^{-0.1t}}, \text{ and } A = \frac{2000-100}{100} = 19$$

Thus the solution of the initial value problem is $\boxed{P(t) = \frac{2000}{1+19e^{-0.1t}}}$

Now the population when $t = 20$ is given by

$$\begin{aligned} P(20) &= \frac{2000}{1+19e^{-0.1(20)}} \\ &= \frac{2000}{1+19e^{-2}} \\ &\approx 560 \end{aligned}$$

Thus population when $t = 20$ is $= \boxed{560}$

(b) Now for the population to reach 1200 we set

$$1200 = \frac{2000}{1+19e^{-0.1t}}$$

$$\Rightarrow 1+19e^{-0.1t} = \frac{5}{3}$$

$$\Rightarrow \quad 19e^{-0.1t} = \frac{2}{3}$$

$$\Rightarrow \quad e^{-0.1t} = \frac{2}{57}$$

That means

$$\begin{aligned} t &= -10\ln\left(\frac{2}{57}\right) \\ &\approx 33.5 \end{aligned}$$

Thus when $\boxed{t = 33.5}$ population reaches 1200

Answer 15P.

(a)

Consider the initial value problem

$$\frac{dP}{dt}=0.1P\left(1-\frac{P}{2000}\right), P(0)=100$$

Solve the differential equation for *P* using variable separable method.

$$\frac{dP}{dt}=0.1P-\frac{(0.1P)P}{2000}$$
$$=\frac{200P-0.1P^2}{2000}$$

$$\frac{dP}{200P-0.1P^2}=\frac{dt}{2000}$$ Separate the variables

$$\left[\frac{0.005}{P}-\frac{0.005}{P-2000}\right]dP=\frac{dt}{2000}$$

Write the left hand side into partial fractions

$$\int\left[\frac{0.005}{P}-\frac{0.005}{P-2000}\right]dP=\int\frac{dt}{2000}$$ Integrate on both sides

$$\int\frac{0.005}{P}dP-\int\frac{0.005}{P-2000}dP=\int\frac{dt}{2000}$$

$$\int\frac{1}{P}dP-\int\frac{1}{P-2000}dP=\frac{1}{0.005}\cdot\frac{1}{2000}\int dt$$

$$\int\frac{1}{P}dP-\int\frac{1}{P-2000}dP=\frac{1}{10}\int dt$$

$\ln P-\ln(P-2000)=\frac{1}{10}t+\ln C$ Use integration formula $\ln\left(\frac{P}{P-2000}\right)=\frac{1}{10}t+\ln C$ Use

$\ln a-\ln b=\ln\left(\frac{a}{b}\right)$ $\frac{P}{P-2000}=Ce^{\frac{1}{10}t}$

$$P=PCe^{\frac{1}{10}t}-2000Ce^{\frac{1}{10}t}$$

$$P\left(1-Ce^{\frac{1}{10}t}\right)=-2000Ce^{\frac{1}{10}t}$$

$$P=\frac{-2000Ce^{\frac{1}{10}t}}{1-Ce^{\frac{1}{10}t}}$$

Find the value of *C* by using the initial condition $P(0)=100$

$$100=\frac{-2000Ce^{\frac{1}{10}(0)}}{1-Ce^{\frac{1}{10}(0)}}$$

$$100=\frac{-2000C}{1-C}$$

$$00(1-C)=-2000C$$
$$100-100C=-2000C$$
$$100=-1900C$$
$$C=-19$$

Then,

$$P=\frac{-2000(-19)e^{\frac{1}{10}t}}{1-(-19)e^{\frac{1}{10}t}}$$

Answer 16E.

(A) Let $t = 0$ as year 1990, so $P(0) = 5.28$ billion

Rate of growth of population is proportional to it's size

So $\frac{dP}{dt} = kP$

Solution of this problem is

$$P(t) = P(0)e^{kt}$$

$$\Rightarrow P(t) = 5.28e^{kt} \qquad \text{(in billion)}$$

We have been given that, in year 2000, population was 6.07 billion

So $P(10) = 6.07$

Then $6.07 = 5.28e^{10k}$

$$\Rightarrow e^{10k} = \frac{6.07}{5.28}$$

$$\Rightarrow 10k = \ln\left(\frac{6.07}{5.28}\right)$$

$$\Rightarrow k = \frac{1}{10}\ln\left(\frac{6.07}{5.28}\right) \approx 0.01394325$$

So $\boxed{P(t) = 5.28e^{0.01394325t}}$ in billion

Population in year 2020 will be

$$P(30) = 5.28e^{0.01394325\times 30}$$

$$\boxed{P(30) \approx 8.02 \text{ billion}}$$

(B) We have to find t such that P (t) = 10 (in billion)

$$\Rightarrow 10 = 5.28e^{0.01394325t}$$

$$\Rightarrow e^{0.01394325t} = \frac{10}{5.28}$$

$$\Rightarrow 0.01394325t = \ln\left(\frac{10}{5.28}\right)$$

$$\Rightarrow t = \frac{1}{0.01394325}\times\ln\left(\frac{10}{5.28}\right) \approx 45.8 \text{ years}$$

So population will be 10 billion after 45.8 years, or in $\boxed{\text{year 2035.}}$

(C) Now carrying capacity is $K = 100$ billion

And from part (A), $k \approx 0.01394325$

Then logistic model of the population is

$$P(t) = \frac{K}{1+Ae^{-kt}}, \qquad A = \frac{k - P_0}{P_0}$$

Here $A = \frac{100-5.28}{5.28} = \frac{94.72}{5.28} \approx 17.94, \qquad P(0) = 5.28$

$$\boxed{P(t) = \frac{100}{1+17.94e^{-0.01394325t}}}$$

Then population in 2020

$$P(30) = \frac{100}{\left(1+17.94e^{-0.01394325\times 30}\right)} \approx \boxed{7.81 \text{ billion}}$$

It is slightly less than the estimation by exponential model.

(D) We have to find t such that $P(t)=10$ (in billion)

$$\Rightarrow \frac{100}{1+17.94e^{-0.01394325t}}=10$$

$$\Rightarrow 1+17.94e^{-0.01394325t}=10$$

$$\Rightarrow 17.94e^{-0.01394325t}=9$$

$$\Rightarrow e^{-0.01394325t}=9/17.94$$

$$\Rightarrow -0.01394325t=\ln(9/17.94)$$

$$\Rightarrow t=\frac{\ln(17.94/9)}{0.01394325}\approx 49.47 \text{ years}$$

So population will be 10 billion in year 2039. that is later than the prediction of 2035.

Answer 17E.

(A) We have been given rate of growth in length is proportional to $L_\infty - L$,

So $\frac{dL}{dt}=k(L_\infty - L)$, Since $[L(0)$, at time t=0

Here L_∞ is maximum length, which is constant

Solving this equation $\frac{dL}{dt}=k(L_\infty - L)$

$$\Rightarrow \frac{1}{L_\infty - L}dL=k\,dt$$

$$\Rightarrow \int \frac{1}{(L_\infty - L)}dl=\int k\,dt \quad \text{[Integrating both sides]}$$

$$\Rightarrow -\ln|L_\infty - L|=kt+c$$

$$\Rightarrow \ln(L_\infty - L)=-kt-c$$

$$\Rightarrow L_m - L=e^{-kt-c}$$

$$\Rightarrow L=L_\infty - e^{-kt}.e^{-c}$$

$$\Rightarrow L=L_\infty - e^{-c}.e^{-kt}$$

At time t = 0, Length is L (0)

So $L(0)=L_\infty - e^{-c}.e^{\infty}$

$$\Rightarrow e^{-c}=L_\infty - L(0)$$

Then expression for L (t) becomes $\boxed{L(t)=L_\infty - [L_\infty - L(0)]e^{-kt}}$

(B) We have $L_\infty = 53cm$, $L(0)=10cm$ and $k=-0.2$

Then expression for the length at time t is

$$L(t)=53-(53-10)e^{-0.2t}$$

$$\Rightarrow \boxed{L(t)=53-43e^{-0.2t}}$$

Answer 18E.

Volume of pure water in the tank = 100 L

Let x (t) be the amount of salt after time t in minutes then x (0) =0 because tank contains only pure water. The rate of change of the amount of salt is.

$$\frac{dx}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\text{Rate in } = 0.1\frac{\text{kg}}{\text{L}} \times \frac{10\text{L}}{\text{min}}$$

$$= 1\text{kg/min}$$

And since tank always contains 100 L of liquid.

So concentration at time t is $\frac{x(t)}{100}$ kg/min

$$\text{So rate out} = \frac{x(t)}{100} \times 10 \quad \text{kg/min}$$

$$= \frac{1}{10}x(t) \quad \text{kg/min}$$

Then $$\frac{dx}{dt} = 1 - \frac{1}{10}x(t)$$

$$\Rightarrow 10\frac{dx}{dt} = (10 - x)$$

$$\Rightarrow \frac{10}{(10-x)}dx = dt$$

Integrating both sides

$$\int \frac{10}{(10-x)}dx = \int 1dt$$

$$\Rightarrow -10\ln|(10-x)| = t + c$$

Since $x < 10$, because amount of salt cannot more than 0.1% of the amount of solution in the tank. So $10 - x > 0$ for all x in the interval (0, 10)

So $$-10\ln(10-x) = t + c$$

$$\Rightarrow \ln(10-x) = \frac{-1}{10}(t+c)$$

$$\Rightarrow 10 - x = e^{-(t+c)/10}$$

$$\Rightarrow x = 10 - e^{-c/10}.e^{-t/10}$$

When $t = 0$, $x(t) = 0$

$$\text{So } 0 = 10 - e^{-c/10}.e^{0}$$

$$\text{Or } e^{-c/10} = 10$$

$$\text{So } x(t) = 10 - 10e^{-t/10}$$

$$\Rightarrow \boxed{x(t) = 10\left(1 - e^{-t/10}\right)}$$

The amount of salt after 6 minutes

$$x(6) = 10\left(1 - e^{-6/10}\right) \approx 4.512 \text{ kg}$$

Amount of salt after 6 minutes $\approx \boxed{4.512}$ kg

Answer 19E.

Let N is the total population and x be the number of infected people

Then according to given condition, rate of spread is

$$\Rightarrow \frac{dx}{dt} = kx(N - x)$$

$$\Rightarrow \frac{dx}{dt} = kNx\left(1 - \frac{x}{N}\right)$$

This is a logistic equation, solution of equation is

$$x(t) = \frac{N}{1 + Ae^{-kNt}} \qquad A = \frac{N - x_0}{x_0}$$

$$\Rightarrow x(t) = \frac{N}{1 + \frac{(N - x_0)}{x_0}e^{-kNt}}$$

$$\Rightarrow x(t) = \frac{x_0 N}{\left(x_0 + (N - x_0)e^{-kNt}\right)}$$

We have $N = 5000$,

And $x_0 = 160$ = [Number of infected people at the beginning of the week]

$$\text{Then } x(t) = \frac{160 \times 5000}{160 + (5000 - 160)e^{-5000kt}}$$

$$\Rightarrow x(t) = \frac{800000}{160 + 4840e^{-5000kt}}$$

We have been given

x (7) = 1200 = [number of infected people at the end of the week]

Then $$\Rightarrow 1200 = \frac{800000}{160 + 4840e^{-5000k\times 7}}$$

$$\Rightarrow 160 + 4840e^{-35000k} = \frac{800000}{1200}$$

$$\Rightarrow 4840e^{-35000k} = \frac{8000}{12} - 160$$

$$\Rightarrow 4840e^{-35000k} = \frac{6080}{12}$$

$$\Rightarrow e^{-35000k} = \frac{6080}{12\times 4840}$$

$$\Rightarrow -35000k = \ln\left(\frac{6080}{12\times 4840}\right)$$

$$\Rightarrow k = \frac{-1}{35000}\ln\left(\frac{6080}{12\times 4840}\right) \approx 6.45\times 10^{-5}$$

Then $$x(t) = \frac{800000}{160 + 4840e^{-5000\times 6.45\times 10^{-5}\times t}}$$

Now we have to find t such that $x(t) = 80\%$ of $5000 = \frac{80\times 5000}{100} = 4000$

Then $$4000 = \frac{800000}{160 + 4840e^{-5000\times 6.45\times 10^{-5}\times t}}$$

$$\Rightarrow 160 + 4840e^{-5000\times 6.45\times 10^{-5}\times t} = 200$$

$$\Rightarrow 4840e^{-5000\times 6.45\times 10^{-5}\times t} = 40$$

$$\Rightarrow -5000\times 6.45\times 10^{-5}\times t = \ln\left(\frac{4}{484}\right)$$

$$\Rightarrow t = -\frac{1}{5000\times 6.45\times 10^{-5}}\times \ln\left(\frac{1}{121}\right) \approx 15\,days$$

$$\boxed{t \approx 15 \text{ days}}$$

Answer 20E.

We have the relation between R and S as

$$\frac{1}{R}\frac{dR}{dt}=\frac{k}{S}\frac{dS}{dt}$$

By the chain rule, we have $\frac{dR}{dt}=\frac{dR}{dS}\cdot\frac{dS}{dt}$

Then $$\frac{1}{R}\frac{dR}{dS}\frac{dS}{dt}=\frac{k}{S}\frac{dS}{dt}$$

$$\Rightarrow\frac{1}{R}\frac{dR}{dS}=\frac{k}{S}$$

$$\Rightarrow\frac{1}{R}dR=\frac{k}{S}dS$$

Integrating both sides, we have

$$\int\frac{1}{R}dR=\int\frac{k}{S}dS$$

$$\Rightarrow\ln|R|=k\ln|S|+C$$

$$\Rightarrow\ln|R|=\ln\left[|S|\right]^k+C$$

$$\Rightarrow\ln R-\ln\left[S^k\right]=C \qquad [\text{since } R>0, S>0$$

$$\Rightarrow\ln\frac{R}{S^k}=C$$

$$\Rightarrow\frac{R}{S^k}=e^C$$

$$\Rightarrow R=e^C S^k \qquad \text{Let } e^C=A \quad \text{is a constant}$$

Then $\boxed{R=AS^k}$

Answer 21E.

We have the differential equation

$$\frac{dh}{dt}=\frac{-R}{V}\left(\frac{h}{k+h}\right) \qquad \text{where } R, V \text{ and } k \text{ are constant}$$

Solving this equation

$$\left(\frac{k+h}{h}\right)\frac{dh}{dt}=\frac{-R}{V}$$

$$\Rightarrow\left(\frac{k}{h}+1\right)dh=-\frac{R}{V}dt$$

Integrating both sides

$$\int\left(\frac{k}{h}+1\right)dh=-\frac{R}{V}\int dt$$

$$\Rightarrow k\ln|h|+h=-\frac{R}{V}t+C$$

Since $h>0$, so $\boxed{k\ln h+h=\left(\frac{-R}{V}\right)t+C}$

This is the relation between h and t.

Answer 22E.

(A) Given equations are

$$\frac{dx}{dt}=0.4x-0.002xy$$

$$\frac{dy}{dt}=-0.2y+0.000008xy$$

Comparing with standard predator –prey equations

$$\frac{dR}{dt}-kR-aRW,\quad \frac{dW}{dt}=-rW+bRW$$

Where R denotes number of preys, and W denotes number of predators

In our problem variable $\boxed{x \text{ denotes the number of insects}}$

And variable $\boxed{y \text{ denotes the number of birds}}$ because bird is predator and insect is prey.

(B) For equilibrium solutions, x and y both, should be constant, and then

$$\frac{dx}{dt}=0,\frac{dy}{dt}=0$$

$$\Rightarrow\frac{dx}{dt}=x(0.4-0.002y)=0$$

$$\Rightarrow\frac{dy}{dt}=y(-0.2+0.000008x)=0$$

One solutions is $\boxed{y=0 \text{ and } x=0}$

Means if there are no insects or birds, populations are not going to increase.

For other solution, we must have

$$0.4-0.002y=0 \quad \text{and} \quad -0.2+0.000008x=0$$

$$\Rightarrow y=\frac{0.4}{0.002} \quad \text{and} \quad x=\frac{0.2}{0.000008}$$

$$\boxed{y=200} \quad \text{and} \quad \boxed{x=25000}$$

Thus there are 200 birds and 25000 insects in the equilibrium population. It means that 25000 insects are enough for 200 birds.

(C) Since $\frac{dy}{dt}=\frac{dy}{dx}\,\frac{dy}{dt}$ by chain rule

So $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-0.2+0.000008xy}{0.4x-0.002xy}$

$$\Rightarrow\boxed{\frac{dy}{dx}=\frac{-0.2y+0.000008xy}{0.4x-0.002xy}}$$

(D) We sketch the solution curve through the point P_0 (40000, 100)

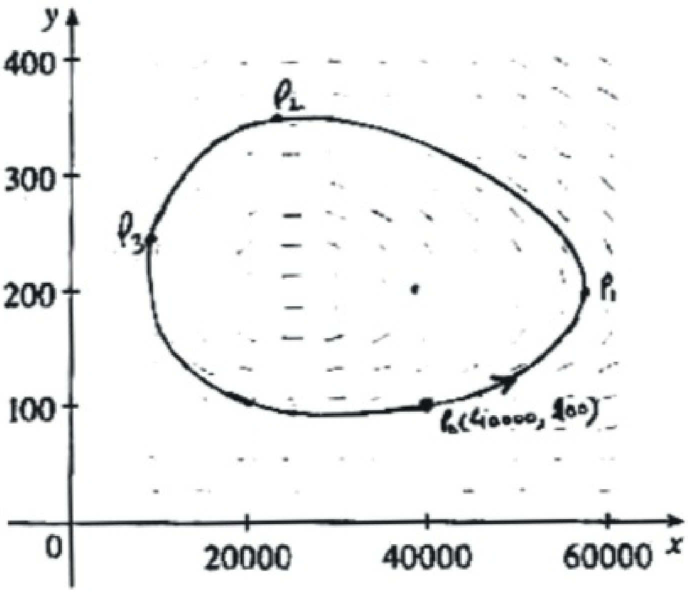


Fig. 1

If we put $x = 40000$ and $y = 100$ in equation of $\frac{dx}{dt}$

$$\frac{dx}{dt} = 0.4 \times 40000 - 0.002 \times 100 \times 40000 = 8000$$

So $\frac{dx}{dt} > 0$ this means that x is increasing at P_0 , so we have to move anti clock wise. We see that number of birds is not enough to maintain a balance between the populations so insect's populations increase. Up to its maximum capacity and number of birds also increase. After the point P_1 the number of insects starts to decrease and number of birds reaches its maximum. Then after both the populations start to decrease up to P_3 .due to decrease in population of birds, number of insects again starts to increase and this happens when the populations return to their initial value of $x = 40000$ and $y = 100$.

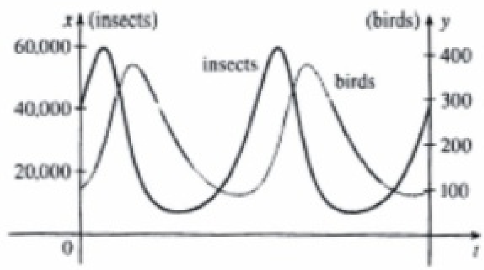


Fig. 2

Both the graphs have same period of time

Answer 23E.

(A) Given equations are

$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.002xy$$

and $$\frac{dy}{dt} = -0.2y + 0.000008\,xy$$

When birds are absent; y= 0, and $\frac{dx}{dt} = 0$

So $\frac{dx}{dt} = 0.4x(1 - 0.000005x) = 0$

$\Rightarrow x = 0 \text{ or } x = 1/(0.000005)$

$\Rightarrow \boxed{x = 200000}$

So number of insects will be stable at $\boxed{200000.}$

(B) For equilibrium solutions, we must have

$$\frac{dx}{dt} = 0 \text{ and } \frac{dy}{dt} = 0$$

So $0.4x(1 - 0.000005x) - 0.002xy = 0$

And $-0.2y + 0.000008xy = 0$

$\Rightarrow 0.4x\left[(1 - 0.000005x) - 0.005y\right] = 0$

$and \ -0.2y(1 - 0.000004x) = 0$

For the second equation

$$y = 0 \text{ or } x = \frac{1}{0.000004} = 25000$$

When y = 0 in the first equation

$$x = 0 \text{ or } x = \frac{1}{0.000005} = 200000$$

And if x = 25000 then

$(0.4) \times 25000(1 - 0.000005 \times 25000) - 0.002 \times 25000 \times y = 0$

$\Rightarrow 10000(1 - 0.125) - 50y = 0$

$\Rightarrow 8750 - 50y = 0$

$\Rightarrow y = \frac{8750}{50} = 175$

For the second equation

$$y = 0 \text{ or } x = \frac{1}{0.000004} = 25000$$

When y = 0 in the first equation

$$x = 0 \text{ or } x = \frac{1}{0.000005} = 200000$$

And if x = 25000 then

$(0.4) \times 25000(1 - 0.000005 \times 25000) - 0.002 \times 25000 \times y = 0$

$\Rightarrow 10000(1 - 0.125) - 50y = 0$

$\Rightarrow 8750 - 50y = 0$

$\Rightarrow y = \frac{8750}{50} = 175$

(C) In the given figure we see that bird population fluctuate around 175, and insect population fluctuate around 25000. At last these populations will be stabilizing at these values.

Answer 25E.

(A) We have $\frac{d^2y}{dx^2} = k\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Let $\frac{dy}{dx} = z$ then $\frac{dz}{dx} = k\sqrt{1+z^2}$

$$\Rightarrow \frac{1}{\sqrt{1+z^2}}dz = kdx$$

$$\Rightarrow \int \frac{1}{\sqrt{1+z^2}}dz = \int k\,dx$$

$$\Rightarrow \ln\left(z+\sqrt{1+z^2}\right) = kx + c \qquad \left[\int \frac{1}{\sqrt{a^2+u^2}}du = \ln\left(u+\sqrt{a^2+u^2}\right)+C\right]$$

$$\Rightarrow z+\sqrt{1+z^2} = Ce^{kx} \qquad \left(Let\, e^c = C\right)$$

$$\Rightarrow \sqrt{1+z^2} = Ce^{kx} - z$$

From figure we see that at $x = 0$, slope of the wire is 0, so $z = 0$ at $x = 0$

Then $\sqrt{1} = C \quad \Rightarrow C = 1$ or $C = -1$

We take $C = 1$,

$$\Rightarrow \sqrt{1+z^2} = e^{kx} - z$$

Squaring both sides

$$\Rightarrow 1+z^2 = e^{2kx} + z^2 - 2ze^{kx}$$

$$\Rightarrow 2ze^{kx} = e^{2kx} - 1$$

$$\Rightarrow z = \frac{e^{kx}-1}{2e^{kx}}$$

Squaring both sides

$$\Rightarrow 1+z^2 = e^{2kx} + z^2 - 2ze^{kx}$$

$$\Rightarrow 2ze^{kx} = e^{2kx} - 1$$

$$\Rightarrow z = \frac{e^{kx} - 1}{2e^{kx}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{2kx} - 1}{2e^{kx}}$$

$$\Rightarrow dy = \frac{e^{2kx} - 1}{2e^{kx}} dx$$

Integrating both sides, we have

$$\int dy = \int \frac{e^{2kx} - 1}{2e^{kx}} dx$$

$$\Rightarrow y = \frac{1}{2} \int \left(e^{kx} - e^{-kx}\right) dx$$

$$\Rightarrow y = \frac{1}{2}\left(\frac{e^{kx}}{k} + \frac{e^{-xy}}{k}\right) + c$$

$$\Rightarrow y = \frac{1}{2k}\left(e^{kx} + e^{-kx}\right) + c$$ This is the equation of wire

Since this wire passes though (o, a)

Then $a = \frac{1}{2k}\left(e^{0} + e^{0}\right) + c \Rightarrow c = a - \frac{1}{k}$

so $y = \frac{1}{2k}\left(e^{kx} + e^{-kx}\right) + a - \frac{1}{k}$

Now, since $\cosh kx = \dfrac{e^{kx} + e^{-kx}}{2}$

Therefore $\boxed{y = (1/k)\cosh kx + (a - 1/k)}$

(B) Now we have $y = \frac{1}{k}\cosh kx + \left(a - \frac{1}{k}\right)$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{k}\sinh kx\left(k\right)$$

$$\Rightarrow \frac{dy}{dx} = \sinh kx$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \sinh^2 kx$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\sinh^2 kx$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \cosh^2 kx \qquad \left(\cosh^2 x - \sinh^2 x = 1\right)$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \cosh kx$$

$$\frac{dy}{dx} = \frac{1}{k}\sinh kx\left(k\right)$$

$$\Rightarrow \frac{dy}{dx} = \sinh kx$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \sinh^2 kx$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\sinh^2 kx$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \cosh^2 kx \qquad \left(\cosh^2 x - \sinh^2 x = 1\right)$$

$$\Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \cosh kx$$