Generating Functions.

Let $\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ be a sequence of real nos., then a function $f(x)$ defined by

$$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n + \cdots$$

is called "Generating function" of the sequence.

If the sequence contains infinitely many terms, then

$$f(x) = \sum_{n=0}^{\infty} (a_n \cdot x^n)$$

Q.1. For the sequence

$C_0, C_1, C_2, C_3, \ldots, C_n$ where $C_k = C(n,k)$, the generating func^n is ?

→ $f(x) = C_0 + C_1 \cdot x + C_2 \cdot x^2 + C_3 \cdot x^3 + \cdots + C_n \cdot x^n$

Generating function

= Binomial expansion of $(1+x)^n$

$$\boxed{f(x) = (1+x)^n}$$

Q.2. For the sequence

$C_0, -C_1, C_2, -C_3, \ldots, (-1)^n \cdot C_n$.

→ The Generating function is

$$f(x) = C_0 - C_1 \cdot x + C_2 \cdot x^2 - C_3 \cdot x^3 + \cdots + (-1)^n \cdot C_n \cdot x^n$$

= Binomial expansion of $(1-x)^{12}$

$\boxed{f(x) = (1-x)^{12}}$

Q.3. For the sequence,

$\{1, 1, 1, \cdots, 1\}$ (n terms)

Gen. funcⁿ = ?

→

The Generating Funcⁿ is

$f(x) = 1 + x + x^2 + \cdots + x^{n-1}$

$\boxed{f(x) = \left(\frac{1-x^n}{1-x}\right)}$

if $n = \infty$,

$f(x) = 1 + x + x^2 + \cdots \cdots$

= Binomial expansion of $(1-x)^{-1}$

$\boxed{f(x) = 1/(1-x)}$

Q.4. " — "

$\{1, -1, 1, -1, \cdots, (-1)^n, \cdots \infty\}$

→ The generating funcⁿ is

$f(x) = 1 - x + x^2 - x^3 + \cdots + \infty = (1+x)^{-1}$

$\boxed{f(x) = \frac{1}{1+x}}$

Note $\rightarrow (1-ax)^{-k} = \sum_{n=0}^{\infty} \{C(n-1+k, n).a^n.x^n\}$

where $(k = 1, 2, 3, \dots)$

and a is any constant.

1) $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \infty.$

$a = 1, \; k = 1$

2) $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n . x^n$

$a = -1, \; k = 1$

$= 1 - x + x^2 - x^3 + \dots \infty.$

3) $(1-ax)^{-1} = \sum_{n=0}^{\infty} (a^n.x^n)$

$k = -1$

$= 1 + ax + a^2.x^2 + a^3x^3 + \dots + a^n.x^n + \dots \infty$

4) $(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1).x^n$

$a = 1, \; k = 2$

$= 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots + \infty$

5) $(1-x)^{-3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{3} x^n$

$a = 1, \; k = -3$

$= 1+3x+6x^2+10x^3+\cdots\infty.$

6) $(1-x)^{-4} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} x^n$

Q.5. The generating funcn of the sequence $\{1,1,3,1,1,1,\ldots\infty\}$. is what?

$\rightarrow$ Gen. funcn $f(x) =$
$(1+x+3x^2+x^3+\cdots+\infty)$

$= (1+x+x^2+x^3+\cdots\infty) + 2x^2$

$= (1-x)^{-1} + 2x^2$

$$\boxed{f(x) = \frac{1+2x^2-2x^3}{1-x}}$$

Q.6. The generating funcn of the sequence $\{1,2,3,4,(n+1),\ldots\infty\}$.

$= 1+2x+3x^2+4x^3+\cdots\infty$

$= (1-x)^{-2}$

Q.7. The generating funcn of the sequence $\{0,1,2,3,\ldots,n,\ldots\infty\}$

$= 0+x+2x^2+3x^3+\cdots\infty$

$= x\,\underbrace{(1+2x+3x^2+4x^3+\cdots\infty)}$ $\leftarrow$ prev. Binomial expansion of $(1-x)^{-2}$.

$= x \cdot (1-x)^{-2}$

Q.8. The generating funcn of seq. of

$\{0^2, 1^2, 2^2, 3^2, 4^2, \ldots, n^2, \ldots \infty\}$ is

$= f_1(x) = \dfrac{x}{(1-x)^2} = 0 + 1.x + 2.x^2 + 3.x^3 + 4.x^4 \ldots \infty$

we know,

$$x \cdot \frac{d(f_1(x))}{dx} = \frac{x^2+x}{(1-x)^3} = x\{0 + 1 + 4x + 9x^2 + \ldots\}$$

$$= 0^2 + 1^2.x + 2^2.x^2 + 3^2.x^3 + \ldots \infty$$

$\therefore$ Required Generating Function $= \dfrac{x^2+x}{(1-x)^3}$.

Q.9. The Generating Functn of the sequence

$\{0^3, 1^3, 2^3, \ldots n^3, \ldots \infty\}$ is ?

$\rightarrow$ Required G.F. $= x \cdot \dfrac{d}{dx}\{f_2(x)\}$

$$= x \cdot \frac{d}{dx}\left\{\frac{x^2+x}{(1-x)^3}\right\}.$$

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Q.10. The generating funcn of the sequence

$\{a_0, a_1, a_2, \ldots\ldots, a_n, \ldots \infty\}$ where $a_n = (n+1)\cdot(n+2)$

is what?

$\rightarrow$ $f(x) = 2\cdot \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \cdot x^n$

$= 2\cdot(1-x)^{-3} = \boxed{\frac{2}{(1-x)^3}}$

Q.11. " — "

$\{a_0, a_1, a_2, \ldots, a_n, \ldots \infty\}$ where $a_n = n\cdot(n+1)$ is ?

$\rightarrow$ $f(x) = \sum_{n=0}^{\infty} n\cdot(n+1)\, x^n$

$= x\cdot \sum_{n=\cancel{0}0} n(n+1)\cdot x^{n-1}$

Replace 'n' with 'n+1',

$\therefore$ $f(x) = 2\cdot x\cdot \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2}\cdot x^n$

$= 2x\cdot(1-x)^{3} = \boxed{\frac{2x}{(1-x)^3}}$

Q.12. Generating funcn of the sequence

$\{a_0, a_1, a_2, \ldots, a_n, \ldots \infty\}$

where, $a_n = (n+1)(n+2)(n+3)$ is what?

$\rightarrow$ $f(x) = 6 \cdot \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} \cdot x^n$

$= 6 \cdot (1-x)^{-4} = \frac{6}{(1-x)^4}$

$\boxed{f(x) = \frac{6}{(1-x)^4}}$

Q.13. Generating funcn of the sequence

$\{1, -2, 4, -8, \ldots, (-2)^n, \ldots, \infty\}$ is ?

$\rightarrow$ $f(x) = \sum_{n=0}^{\infty} (-2)^n \cdot x^n$

$= (1+2x)^{-1}$

$\boxed{f(x) = \frac{1}{1+2x}}$

Q.14. The coefficient of x^{20} in the expansion of

$(x^3 + x^4 + x^5 + \ldots \infty)^5$ is ?

⇒ we know,

$$(1-ax)^{-k} = \sum_{n=0}^{\infty} C(n-1+k, n) \cdot x^n$$

Given,

$$\Rightarrow x^{15} \cdot (1 + x^2 + x^3 + \cdots \infty)^5$$

$$\Rightarrow x^{15} \cdot (1-x)^{-5}$$

$$\Rightarrow x^{15} \cdot \sum_{n=0}^{\infty} C(n+4, n) \cdot x^n$$

The coefficient of x^{20} = $C(n+4, n)$.

taking $n = 5$

$= C(9, 5)$

$= \boxed{126}$.

\#. To find no. of nonnegative integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_n = k$$

let the generating soln be

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdots f_n(x)$$

where $f_i(x) = 1 + x + x^2 + \cdots \infty \quad (i = 1, 2, \cdots, n)$

$= (1-x)^{-1}$

The coefficient of 'x^k' in the expansion of $f(x)$ is the answer to our problem

$$\boxed{(1-x)^{-n} = \sum_{r=0}^{\infty} \frac{C(n-1+k,\,k)\,.\,x^k}{V(n,k)}}$$

$\therefore$ coefficient of x^k in the expansion of $f(x) =$

$\boxed{V(n,k)}$

In the above example, if we have constraints on the variables like $2 \le x_i \le 6$, then we have to choose

$f_i(x) = x^2 + x^3 + x^4 + x^5 + x^6$

if suppose, $1 \le x_1 \le 5$

then $f_1(x) = x^1 + x^2 + x^3 + x^4 + x^5$

Q.15. No. of non-negative integer solutions to the equation

$x_1 + x_2 + x_3 + \dots + x_5 = 15$

where $1 \le x_1 \le 5$, $2 \le x_2 \le 5$, $x_3 \ge 2$,

$x_4 \ge 2$, $x_5 \ge 2$.

$\rightarrow$ The generating func for the problem is

$f(x) = (x + x^2 + x^3 + x^4 + x^5).(x^2 + x^3 + x^4 + x^5).$
$(x^2 + x^3 + x^4 + \dots \infty)^3$

$= x\cdot(1+x+x^2+x^3+x^4)\cdot x^2(1+x+x^2+x^3+x^4)\cdot$

$(x^2)^3\cdot(1+x+x^2+x^3+\dots\infty)^3$

$= x^9\cdot\left(\frac{1-x^5}{1-x}\right)^2\cdot(1-x)^{-3}$

$= x^9\cdot(1-2x^5+x^{10})(1-x)^{-5}$

$=(x^9-2\cdot x^{14}+x^{19})\cdot\sum_{n=0}^{\infty} C(n+4,n)\cdot x^n$

$\left(\because (1-x)^{-k}=\sum_{K=0}^{\infty} C(n-1+k,n)\cdot x^n \quad V(n,k)\right)$

Required no. of nonneg integer solutions to the given problem i.e. coeff of ~~...~~ x^{15} in the expansion of $f(x)$.

$= C(10,6) - 2C(5,1)$

$= \boxed{200}$

Q.16. How many ways we can choose a commitee of 9 members from 3 political parties so that no party has absolute majority in the comittee

→ Required no. of ways =

no. of non negative integer sol^ns to the eq^n

$x_1+x_2+x_3=9$.

where $1 \le x_i \le 4$

where $x_i \to$ no. of representatives from ith party P_i

The generating funcn of this problem is

$f(x) = f_1(x) * f_2(x) * f_3(x)$

where $f_i(x) = x + x^2 + x^3 + x^4$

$\therefore f(x) = (x + x^2 + x^3 + x^4)^3$

$= x^3 (1 + x + x^2 + x^3)^3$

$= x^3 \cdot \left(\frac{1-x^4}{1-x}\right)^3$

$= x^3 \cdot (1 - 3x^4 + 3x^8 - x^{12})\ (1-x)^{-3}$

$= (x^3 - 3 \cdot x^7 + 3 \cdot x^{11} - x^{15}) \cdot \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$

when x^3, $n = 6$
x^7, $n = 2$

Coeff. of x^9 = The required no. of ways

$= \frac{7 \times 8^4}{2} - \frac{3 \times 3 \times 4^2}{2}$

$= 28 - 6\cancel{8} = 28 - 18$

$= \boxed{10}$