INVERSE TRIGONOMETRY FUNCTIONS (XII, R. S. AGGARWAL)

EXERCISE 4A (Pg. no.: 89)

1. Find the principal value of :

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(\frac{1}{2}\right)$ (iii) $\cos^{-1}\left(\frac{1}{2}\right)$ (iv) $\tan^{-1}(1)$

(v) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (vi) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (vii) $\operatorname{cosec}^{-1}\left(\sqrt{2}\right)$

Sol. **(i)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$ **(ii)** $\sin^{-1}\left(\frac{1}{2}\right)=\sin^{-1}\left(\sin\frac{\pi}{6}\right)=\frac{\pi}{6}$

(iii) $\cos^{-1}\left(\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$ **(iv)** $\tan^{-1}(1)=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}$

(v) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$ **(vi)** $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\sec^{-1}\left(\sec\frac{\pi}{6}\right)=\frac{\pi}{6}$

(vii) Let $\operatorname{cosec}^{-1}\left(\sqrt{2}\right)=\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)=\frac{\pi}{4}$

2. Find the principal value of

(i) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ (iii) $\tan^{-1}\left(-\sqrt{3}\right)$

(iv) $\sec^{-1}(-2)$ (v) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ (vi) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Sol. **(i)** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)=-\sin^{-1}\left(\sin\frac{\pi}{4}\right)=-\frac{\pi}{4}$ **(ii)** $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\pi-\cos^{-1}\left(\cos\frac{\pi}{6}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

(iii) $\tan^{-1}\left(-\sqrt{3}\right)=-\tan^{-1}\left(\tan\frac{\pi}{3}\right)=-\frac{\pi}{3}$ **(iv)** $\sec^{-1}(-2)=\pi-\sec^{-1}\left(\sec\frac{\pi}{3}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

(v) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)=-\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)=-\frac{\pi}{4}$ **(vi)** $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\pi-\cot^{-1}\left(\cot\frac{\pi}{3}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

3. Evaluate $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}$

Sol. Let $\cos\left\{\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right\}=\cos\left\{\pi-\cos^{-1}\left(\cos\frac{\pi}{6}\right)+\frac{\pi}{6}\right\}=\cos\left\{\pi-\frac{\pi}{6}+\frac{\pi}{6}\right\}=\cos\pi=-1$

4. Evaluate $\sin\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right\}$

Sol. $\sin\left\{\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right\}=\sin\left\{\frac{\pi}{2}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)\right\}=\sin\left\{\frac{\pi}{2}+\frac{\pi}{3}\right\}$

$$= \sin\left(\frac{3\pi+2\pi}{6}\right) = \sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

EXERCISE 4B (Pg. no.: 126)

Find the principal value of each of the following

1. $\sin^{-1}\left(-\frac{1}{2}\right)$

Sol. $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\sin^{-1}\left(\sin\frac{\pi}{6}\right) = -\frac{\pi}{6}$

2. $\cos^{-1}\left(-\frac{1}{2}\right)$

Sol. Let $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

3. $\tan^{-1}(-1)$

Sol. $\tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}$

4. $\sec^{-1}(-2)$

Sol. $\sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

5. $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Sol. $\operatorname{cosec}^{-1}\left(\sqrt{2}\right) = -\frac{\pi}{4}$

6. $\cot^{-1}(-1)$

Sol. $\cot^{-1}(-1) = \pi - \cot^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

7. $\tan^{-1}\left(-\sqrt{3}\right)$

Sol. $\tan^{-1}\left(-\sqrt{3}\right) = -\tan^{-1}\left(\sqrt{3}\right) = \frac{-\pi}{3}$

8. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$

Sol. $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right) = \pi - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

9. $\operatorname{cosec}^{-1}(2)$

Sol. $\operatorname{cosec}^{-1}(2) = \frac{\pi}{6}$

Find the principle value of each of the following:

10. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Sol. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left\{\sin\left(\pi-\frac{\pi}{3}\right)\right\}=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$

11. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Sol. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{4}\right)\right\}=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=-\tan^{-1}\left(\tan\frac{\pi}{4}\right)=-\frac{\pi}{4}$

12. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Sol. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left\{\cos\left(\pi+\frac{\pi}{6}\right)\right\}=\cos^{-1}\left(-\cos\frac{\pi}{6}\right)=\pi-\cos^{-1}\cos\frac{\pi}{6}=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

13. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Sol. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=\cos^{-1}\left\{\cos\left(2\pi+\frac{\pi}{6}\right)\right\}=\cos^{-1}\left(\cos\frac{\pi}{6}\right)=\frac{\pi}{6}$

14. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Sol. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left\{\tan\left(\pi+\frac{\pi}{6}\right)\right\}=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$

15. $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$

Sol. $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)=\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)=\frac{\pi}{3}-\frac{5\pi}{6}=\frac{2\pi-5\pi}{6}=\frac{-3\pi}{6}=\frac{-\pi}{2}$

16. $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right\}$

Sol. $\sin\left\{\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right\}=\sin\left\{\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)\right\}=\sin\left\{\frac{\pi}{3}+\frac{\pi}{6}\right\}=\sin\left\{\frac{2\pi+\pi}{6}\right\}=\sin\frac{\pi}{2}=1$

17. $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$

Sol. $\cot\left(\tan^{-1}a+\cot^{-1}a\right)=\cot\left(\frac{\pi}{2}\right)=0$

18. $\operatorname{cosec}\left(\sin^{-1}a+\cos^{-1}a\right)$

Sol. $\operatorname{cosec}\left(\sin^{-1}a+\cos^{-1}a\right)=\operatorname{cosec}\left(\frac{\pi}{2}\right)=1$

19. $\sin\left(\sec^{-1}a+\operatorname{cosec}^{-1}a\right)$

Sol. $\sin\left(\sec^{-1}a+\operatorname{cosec}^{-1}a\right)=\sin\frac{\pi}{2}=1$

20. $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}$

Sol. $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)-=\frac{\pi}{3}+2.\frac{\pi}{6}=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

21. $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Sol. $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(\frac{-1}{2}\right)$

$$=\tan^{-1}+\pi-\cos^{-1}\frac{1}{2}-\sin^{-1}\frac{1}{2}=\frac{5\pi}{4}-\left(\frac{2\pi+\pi}{6}\right)=\frac{5\pi}{4}-\frac{\pi}{2}=\frac{5\pi-2\pi}{4}=\frac{3\pi}{4}$$

22. $\sin^{-1}\left\{\sin\frac{3\pi}{5}\right\}$

Sol. $\sin^{-1}\left\{\sin\frac{3\pi}{5}\right\}=\sin^{-1}\left\{\sin\left(\pi-\frac{2\pi}{5}\right)\right\}=\sin^{-1}\left\{\sin\frac{2\pi}{5}\right\}=\frac{2\pi}{5}$

EXERCISE 4C (Pg.no.: 129)

1. Prove that

(i) $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x, x<1$ (ii) $\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$

Sol. (i) $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x,\ x<1$

Let, $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

L.H.S. = $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

$\tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\}$

$=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\tan^{-1}x$

= R.H.S Proved

(ii) L.H.S = $\tan^{-1}x+\cot^{-1}(x+1)$

$=\tan^{-1}x+\tan^{-1}\frac{1}{x+1}=\tan^{-1}\left\{\frac{x+\frac{1}{x+1}}{1-x\cdot\frac{1}{x+1}}\right\}=\tan^{-1}\left\{\frac{x(x+1)+1}{(x+1)-x}\right\}$

$=\tan^{-1}\{x^2+x+1\}$ = R.H.S Proved.

2. prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)-2\sin^{-1}x, |x|\le\frac{1}{\sqrt{2}}$

Sol. Let, $x=\sin\theta, \Rightarrow \theta=\sin^{-1}x$

Now, L.H.S = $\text{Sin}^{-1}\left\{2x\sqrt{1-x^2}\right\}$

$\text{Sin}^{-1}\left\{2\sin\theta\cdot\sqrt{1-\cos^2\theta}\right\}$

$=\sin^{-1}\{2\sin\theta\cos\theta\}=\sin^{-1}(\sin2\theta)=2\theta=2\cdot\sin^{-1}x$ = R.H.S Proved

3. Prove that

(i) $\sin^{-1}\left(3x-4x^3\right)=3\sin^{-1}x, |x|\le\frac{1}{2}$ (ii) $\cos^{-1}\left(4x^3-3x\right)=3\cos^{-1}x, \frac{1}{2}\le x\le 1$

(iii) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ (iv) $\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), x^2<\frac{1}{3}$

Sol. (i) Let, x $\sin\theta \Rightarrow \theta = \sin^{-1}x$

Now, L.H.S = $\sin^{-1}(3x - 4x^3)$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin^{-1}(\sin 3\theta) = 3\theta = 3\cdot\sin^{-1}x$ = R.H.S Proved

(ii) Let, x $\cos\theta$

L.H.S = $\cos^{-1}(4x^3 - 3x)$

$= \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cdot\cos^{-1}x$ = R.H.S Proved

(iii) R.H.S = $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Let $x=\tan\theta \quad \therefore \theta=\tan^{-1}x=\tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)=\tan^{-1}\left(\tan 3\theta\right)=3\theta$

Putting the value of θ, we get, $3\tan^{-1}x=$ L.H.S. Proved

(iv) L.H.S= $\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x.\frac{2x}{1-x^2}}\right]$

$=\tan^{-1}\left[\frac{\frac{x-x^3+2x}{1-x^2}}{\frac{1-x^2-2x^2}{1-x^2}}\right]=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ = R.H.S. Proved

4. (i) $\cos^{-1}\left(1-2x^2\right)=2\sin^{-1}x$ (ii) $\cos^{-1}\left(2x^2-1\right)=2\cos^{-1}x$

(iii) $\sec^{-1}\left(\frac{1}{2x^2-1}\right)=2\cos^{-1}x$ (iv) $\cot^{-1}\left(\sqrt{1-x^2}-x\right)=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$

Sol. (i) L.H.S = $\cos^{-1}\left(1-2x^2\right)$

Let $x=\sin\theta \therefore \theta=\sin^{-1}x=\cos^{-1}\left(1-\sin^2\theta\right)=\cos^{-1}\left(\cos 2\theta\right)=2\theta$

Putting the value of θ we get, $2\sin^{-1}x$. Proved.

(ii) L.H.S = $\cos^{-1}\left(2x^2-1\right)$

Let $x=\cos\theta \;\therefore \theta=\cos^{-1}x=\cos^{-1}\left(2\cos^2\theta-1\right)=\cos^{-1}\left(\cos 2\theta\right)=2\theta$

Putting the value of θ, we get $2\cos^{-1}x=$ R.H.S. Proved

(iii) Let, $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

L.H.S = $\mathrm{Sec}^{-1}\left(\frac{1}{2x^2-1}\right)$

$=\mathrm{Sec}^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)=\mathrm{Sec}^{-1}\left(\frac{1}{\cos^2\theta}\right)=\mathrm{Sec}^{-1}(\sec^2\theta)=2\theta$

$=2\cdot\cos^{-1}x$ = R.H.S Proved

(iv) Let, $x = \cot\theta \Rightarrow \theta = \cos^{-1}x$

L.H.S. $= \text{Cot}^{-1}\left(\sqrt{1+x^2} - x\right)$

$= \text{Cot}^{-1}\left(\sqrt{1+\cot^2\theta} - \cot\theta\right) = \text{Cot}^{-1}(\text{Cosec}\theta - \cot\theta) = \text{Cot}^{-1}\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$

$= \text{Cot}^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \text{Cot}^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2 \cdot \cos\theta/2}\right) = \text{Cot}^{-1}\left(\tan\theta/2\right) = \cot^{-1}\left\{\cot\left(\frac{\pi}{2} - \frac{1}{2}\theta\right)\right\}$

$= \frac{\pi}{2} - \frac{1}{2}\theta = \frac{\pi}{2} - \frac{1}{2}\cot^{-1}x$ = R.H.S Proved

5. (i) $\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}\right) = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{y}$ (ii) $\tan^{-1}\left(\frac{x+\sqrt{x}}{1-x^{3/2}}\right) = \tan^{-1}x + \tan^{-1}\sqrt{x}$

(iii) $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \frac{x}{2}$

Sol. (i) R.H.S $= \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{y} = \tan^{-1}\left[\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{x}.\sqrt{y}}\right] = \tan^{-1}\left[\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}\right]$ = L.H.S. Proved

(ii) Let, $x = \tan\alpha$ and $\sqrt{x} = \tan\beta \Rightarrow \alpha = \tan^{-1}x$ and $\beta = \tan^{-1}\sqrt{x}$

L.H.S. $= \tan^{-1}\left\{\frac{x+\sqrt{x}}{1-x^{3/2}}\right\}$

$= \tan^{-1}\left\{\frac{\tan\alpha + \tan\beta}{1-\tan\alpha \cdot \tan\beta}\right\} \begin{cases} x \cdot \sqrt{x} \\ = x^{3/2} \end{cases} = \tan^{-1}\{\tan(\alpha+\beta)\} = \alpha + \beta$

$= \tan^{-1}x + \tan^{-1}\sqrt{x}$ = R.H.S. Proved

(iii) L.H.S. $= \tan^{-1}\left\{\frac{\sin x}{1+\cos x}\right\}$

$= \tan^{-1}\left\{\frac{2 \cdot \sin\frac{x}{2} \cdot \cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right\} = \tan^{-1}\left\{\tan\frac{x}{2}\right\} = \frac{x}{2}$ = R.H.S Proved

6. Prove that

(i) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$ (ii) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

(iii) $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{3}$ (iv) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

(v) $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\frac{1}{3}$ (vi) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(vii) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$ (viii) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\tan^{-1}\frac{4}{3}$

Sol. (i) L.H.S. $= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11}$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\cdot\frac{2}{11}}\right) = \tan^{-1}\left(\frac{11+4}{22+2}\right) = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4}$ = R.H.S Proved

(ii) $\tan^{-1}\frac{7}{24}+\tan^{-1}\frac{2}{11}=\tan^{-1}\frac{1}{2}$

L.H.S = $\tan^{-1}\frac{7}{24}+\tan^{-1}\frac{2}{11}=\tan^{-1}\left[\frac{\frac{7}{24}+\frac{2}{11}}{1-\frac{7}{24}.\frac{2}{11}}\right]$

$$=\tan^{-1}\left[\frac{\frac{77+48}{26}}{\frac{264-14}{264}}\right]=\tan^{-1}\left(\frac{125}{250}\right)=\tan^{-1}\frac{1}{2}= \text{R.H.S. Proved}$$

(iii) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

L.H.S = $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}+\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)=\frac{\pi}{4}+\tan^{-1}1=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

(iv) L.H.S= $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\left[\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right]+\tan^{-1}\frac{1}{7}=\tan^{-1}\left[\frac{\frac{2}{3}}{1-\frac{1}{9}}\right]+\tan^{-1}\frac{1}{7}=\tan^{-1}\left[\frac{\frac{2}{3}}{\frac{9-1}{9}}\right]+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left[\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right]=\tan^{-1}\left[\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right]$$

$$=\tan^{-1}\frac{25}{25}=\tan^{-1}(1)=\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}\quad \text{R.H.S Proved}$$

(v) $\tan^{-1}2-\tan^{-1}1=\tan^{-1}\frac{1}{3}$

L.H.S $\tan^{-1}2-\tan^{-1}1=\tan^{-1}\left[\frac{2-1}{1+2(1)}\right]=\tan^{-1}\left[\frac{1}{1+2}\right]=\tan^{-1}\frac{1}{3}=$ R.H.S. Proved

(vi) Let, $\tan^{-1}1 = A$, $\tan^{-1}2 = B$ and $\tan^{-1}3 = C$

$\Rightarrow \tan A = 1$, $\tan B = 2$ and $\tan C = 3$

Now, $\tan(A + B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

$\Rightarrow \tan(A + B) = \frac{1+2}{1-1\times 2} \Rightarrow \tan(A + B) = -3 \Rightarrow \tan(A + B) = -\tan C$

$\Rightarrow \tan(A + B) = \tan(-C)$ or, $\tan(\pi - C)$

$\Rightarrow A + B = -C$ or, $A + B = \pi - C$

$\because$ A, B & C > O $\quad\therefore A + B \neq -C \quad\therefore A + B = \pi - C$

$\Rightarrow A + B + C = \pi \quad\Rightarrow \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ Proved

(vii) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

L.H.S. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1/5+1/8}{1-1/5.1/8}$

$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{8+5}{39}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{13}{39}=\tan^{-1}\frac{1}{2}+\frac{1}{3}$

$=\tan^{-1}\frac{1/2+1/3}{1-1/3.1/3}=\tan^{-1}\frac{3+2}{6-1}=\tan^{-1}\frac{5}{5}=\tan^{-1}=\frac{\pi}{4}$

(viii) $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3}=2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right)=\tan^{-1}\frac{4}{3}$

L.H.S $=2\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\right)=2\left[\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}.\frac{2}{9}}\right)\right]=\left[\tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)\right]$

$=\left[\tan^{-1}\left(\frac{17}{34}\right)\right]=2\tan^{-1}\frac{1}{2}=\tan^{-1}\left[\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}\right]=\tan^{-1}\left(\frac{1}{1-\frac{1}{4}}\right)=\tan^{-1}\frac{4}{3}$ R.H.S. Proved.

7. **Prove that**

(i) $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$

(ii) $\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{2}{\sqrt{5}}=\frac{\pi}{2}$

(iii) $\cos^{-1}\frac{3}{5}+\sin^{-1}\frac{12}{13}=\sin^{-1}\frac{56}{65}$

(iv) $\cos^{-1}\frac{4}{5}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{27}{11}$

(v) $\tan^{-1}\frac{1}{3}+\sec^{-1}\frac{\sqrt{5}}{2}=\frac{\pi}{4}$

(vi) $\sin^{-1}\frac{1}{\sqrt{17}}+\cos^{-1}\frac{9}{\sqrt{85}}=\tan^{-1}\frac{1}{2}$

(vii) $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

Sol. (i) L.H.S. $=\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\left[\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right]$

$=\cos^{-1}\left\{\frac{4}{5}.\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right\}=\cos^{-1}\left\{\frac{48}{65}-\frac{3}{5}.\frac{5}{13}\right\}=\cos^{-1}\left(\frac{45-15}{65}\right)=\cos^{-1}\frac{33}{65}$

(ii) L.H.S.

$=\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{2}{\sqrt{5}}\quad==\sin^{-1}\frac{1}{\sqrt{5}}+\cos^{-1}\sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2}$

$=\sin^{-1}\frac{1}{\sqrt{5}}+\cos^{-1}\sqrt{1-\frac{4}{5}}\quad=\sin^{-1}\frac{1}{\sqrt{5}}+\cos^{-1}\frac{1}{\sqrt{5}}\quad=\frac{\pi}{2}=$ R.H.S. Proved

(iii) L.H.S= $\cos^{-1}\frac{3}{5}+\sin^{-1}\frac{12}{13}$

$==\sin^{-1}\sqrt{1-\left(\frac{3}{5}\right)^2}+\sin^{-1}\frac{12}{13}=\sin^{-1}\sqrt{1-\frac{9}{25}}+\sin^{-1}\frac{12}{13}=\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{12}{13}$

$$=\sin^{-1}\left\{\frac{4}{5}\cdot\sqrt{1-\left(\frac{12}{13}\right)^2}+\frac{12}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right\}=\sin^{-1}\left\{\frac{4}{5}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{3}{5}\right\}$$

$$=\sin^{-1}\left(\frac{56}{65}\right)=\text{R.H.S Proved}$$

(iv) L.H.S. $\cos^{-1}\frac{4}{5}+\sin^{-1}\frac{3}{5}$

$$=\cos^{-1}\frac{4}{5}+\cos^{-1}\sqrt{1-\left(\frac{3}{5}\right)^2}=\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{4}{5}=2\cos^{-1}\frac{4}{5}=\cos^{-1}\left\{2\left(\frac{4}{5}\right)^2-1\right\}$$

$$=\cos^{-1}\left\{\frac{32}{25}-1\right\}=\cos^{-1}\frac{7}{25}=\text{R.H.S. Proved}$$

(v) L.H.S. $=\tan^{-1}\frac{1}{3}+\sec^{-1}\frac{\sqrt{5}}{2}$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-1}\quad\left\{\begin{array}{l}\because \sec^{-1}x\\ =\tan^{-1}\sqrt{x^2-1}\end{array}\right.$$

$$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}\right)=\tan^{-1}1=\frac{\pi}{4}\ \text{R.H.S Proved}$$

(vi) Let, $\sin^{-1}\frac{1}{\sqrt{17}}=\theta$

$$\Rightarrow\sin\theta=\frac{1}{\sqrt{17}}\Rightarrow\cos\theta=\sqrt{1-\left(\frac{1}{\sqrt{17}}\right)^2}\Rightarrow\cos\theta=\sqrt{1-\frac{1}{17}}\Rightarrow\cos\theta=\sqrt{\frac{16}{17}}=\frac{4}{\sqrt{17}}\Rightarrow\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow\tan\theta=\frac{1/\sqrt{17}}{4/\sqrt{17}}=\frac{1}{4}\Rightarrow\theta=\tan^{-1}\frac{1}{4}$$

$$\therefore\sin^{-1}\frac{1}{\sqrt{17}}=\tan^{-1}\frac{1}{4}$$

Let, $\cos^{-1}\frac{9}{\sqrt{85}}=\alpha\Rightarrow\cos\alpha=\frac{9}{\sqrt{85}}\Rightarrow\sin\alpha=\sqrt{1-\frac{81}{85}}$

$$\Rightarrow\sin\alpha=\frac{2}{\sqrt{85}}$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}\Rightarrow\tan\alpha=\frac{2/\sqrt{85}}{9/\sqrt{85}}\Rightarrow\tan\alpha=\frac{2}{9}\Rightarrow\alpha=\tan^{-1}\frac{2}{9}\ \therefore\cos^{-1}\frac{9}{\sqrt{85}}=\tan^{-1}\frac{2}{9}$$

L.H.S. $=\sin^{-1}\frac{1}{\sqrt{17}}+\cos^{-1}\frac{9}{\sqrt{85}}$

$$\Rightarrow\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}\Rightarrow\tan^{-1}\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\cdot\frac{2}{9}}\Rightarrow\tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)$$

$$\Rightarrow\tan^{-1}\frac{17}{34}\Rightarrow\tan^{-1}\frac{1}{2}=\text{R.H.S. Proved}$$

(vii) Let, $\sin^{-1}\frac{3}{5}=\theta\Rightarrow\sin\theta=\frac{3}{5}$

Now, $\cos\theta=\sqrt{1-\sin^2\theta}$

$$\Rightarrow \cos\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} \Rightarrow \cos\theta = \sqrt{1-\frac{9}{25}} \Rightarrow \cos\theta = \sqrt{\frac{25-9}{25}} \Rightarrow \cos\theta = \frac{4}{5}$$

$$\because \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \tan\theta = \frac{3/5}{4/5} \Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4} \Rightarrow \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$$

L.H.S. $= 2\cdot\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31}$

$$\Rightarrow 2\cdot\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} \Rightarrow \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) - \tan^{-1}\frac{17}{31} \Rightarrow \tan^{-1}\left(\frac{\frac{6}{4}}{1-\frac{9}{16}}\right) - \tan^{-1}\frac{17}{31}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) - \tan^{-1}\frac{17}{31} \Rightarrow \tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right) \Rightarrow \tan^{-1}\frac{\frac{24}{7}-\frac{17}{3}}{1+\frac{24}{7}\cdot\frac{17}{31}}$$

$$\Rightarrow \tan^{-1}\frac{\frac{24\times31-17\times7}{7\times31}}{\frac{7\times31+24\times17}{7\times31}} \Rightarrow \tan^{-1}\frac{744-119}{217+408} \Rightarrow \tan^{-1}\frac{625}{625} \Rightarrow \tan^{-1}1 \Rightarrow \frac{\pi}{4} = \text{R.H.S.}$$

8. Solve for x

(i) $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$ (ii) $\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\frac{2}{3}$

(iii) $\cos\left(\sin^{-1}x\right)=\frac{1}{9}$ (iv) $\cos\left(2\sin^{-1}x\right)=\frac{1}{9}$

(v) $\sin^{-1}\frac{8}{x}+\sin^{-1}\frac{15}{x}=\frac{\pi}{2}$

Sol. (i) $\tan^{-1}(X+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$

$$\Rightarrow \tan^{-1}\frac{x+1+x-1}{1-(x+1)(x-1)} = \tan^{-1}\frac{8}{31} \Rightarrow \tan^{-1}\frac{2x}{1-(x^2-1)} = \tan^{-1}\frac{8}{31}$$

$$\Rightarrow \frac{2x}{1-x^2+1} = \frac{8}{31} \Rightarrow 62x = 8(2-x^2) \Rightarrow 31x = 4(2-x^2) \Rightarrow 31x = 8-4x^2$$

$$\Rightarrow 4x^2+31x-8=0 \Rightarrow 4x^2+32x-x-8=0 \Rightarrow 4x(x+8)-(x+8)=0$$

$$\Rightarrow (x+8)(4x-1)=0 \quad \Rightarrow x=-8 \text{ or } x=\frac{1}{4}$$

for x = -8

L.H.S. $= \tan^{-1}(-8+1)+\tan^{-1}(-8-1)$

$= \tan^{-1}(-8)+\tan^{-1}(-9)$

Which is –ve.

But, R.H.S is true

Therefore. x = -8 is let possible

Hence, $x = \frac{1}{4}$ is the required solution.

(ii) $\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\frac{2}{3}$

$\Rightarrow \tan^{-1}\dfrac{(2+x)+(2-x)}{1-(2+x)(2-x)} = \tan^{-1}\dfrac{2}{3} \Rightarrow \dfrac{4}{1-(4-x^2)} = \dfrac{2}{3} \Rightarrow \dfrac{4}{-3+x^2} = \dfrac{2}{3}$

$\Rightarrow 12 = -6 + 2x^2 \Rightarrow -2x^2 + 12 + 6 = 0 \Rightarrow -x^2 + 9 = 0 \quad \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$ Ans.

(iii). Let, $\sin^{-1}x = \theta$

$\Rightarrow x = \sin\theta$

Now, $\cos(\sin^{-1}x) = \dfrac{1}{9}$

$\Rightarrow \cos\theta = \dfrac{1}{9} \Rightarrow \cos^2\theta = \dfrac{1}{81} \Rightarrow \sin^2\theta = 1 - \dfrac{1}{81} \Rightarrow \sin\theta = \pm\sqrt{\dfrac{81-1}{81}} \Rightarrow \sin\theta = \pm\sqrt{\dfrac{80}{81}}$

$\Rightarrow x = \pm\dfrac{4\sqrt{5}}{9}$ Ans.

(iv) $\cos(2\sin^{-1}x) = \dfrac{1}{9}$

$\Rightarrow \cos(\cos^{-1}(1-2x^2)) = \dfrac{1}{9} \quad \Rightarrow 1 - 2x^2 = \dfrac{1}{9} \Rightarrow 2x^2 = 1 - \dfrac{1}{9} \Rightarrow 2x^2 = \dfrac{8}{9} \Rightarrow x^2 = \dfrac{4}{9}$

$\Rightarrow x = \pm\dfrac{2}{3}$ Ans.

(v) $\sin^{-1}\dfrac{8}{x} + \sin^{-1}\dfrac{15}{x} = \dfrac{\pi}{2} \Rightarrow \sin^{-1}\dfrac{15}{x} = \dfrac{\pi}{2} - \sin^{-1}\dfrac{8}{x} \Rightarrow \sin^{-1}\dfrac{15}{x} = \cos^{-1}\dfrac{8}{x} \qquad \Rightarrow \sin^{-1}\dfrac{15}{x} = \sin^{-1}\sqrt{1-\left(\dfrac{8}{x}\right)^2}$

$\Rightarrow \dfrac{15}{x} = \sqrt{1-\left(\dfrac{8}{x}\right)^2} \Rightarrow \left(\dfrac{15}{x}\right)^2 = 1 - \left(\dfrac{8}{x}\right)^2 \Rightarrow \left(\dfrac{15}{x}\right)^2 + \left(\dfrac{8}{x}\right)^2 = 1 \Rightarrow 225 + 64 = x^2 \quad \Rightarrow x^2 = 289$

$\Rightarrow x = 17$ Ans.

9. Solve for x

(i) $\cos\left(\sin^{-1}x\right) = \dfrac{1}{2}$ (ii) $\tan^{-1}x = \sin^{-1}\dfrac{1}{\sqrt{2}}$ (iii) $\sin^{-1}x - \cos^{-1}x = \dfrac{\pi}{6}$

Sol. (i) $\cos(\sin^{-1}x) = \dfrac{1}{2} \Rightarrow \cos\left(\cos^{-1}\sqrt{1-x^2}\right) = \dfrac{1}{2} \Rightarrow \sqrt{1-x^2} = \dfrac{1}{2} \Rightarrow 1 - x^2 = \dfrac{1}{4}$

$\Rightarrow x^2 = 1 - \dfrac{1}{4} \Rightarrow x^2 = \dfrac{3}{4} \Rightarrow x = \pm\dfrac{\sqrt{3}}{2}$ Ans.

(ii) $\tan^{-1}x = \sin^{-1}\dfrac{1}{\sqrt{2}} \qquad \Rightarrow \tan^{-1}x = \dfrac{\pi}{4} \Rightarrow x = \tan\left(\dfrac{\pi}{4}\right) \Rightarrow x = 1$

(iii) $\sin^{-1}x - \cos^{-1}x = \dfrac{\pi}{6}$

$\Rightarrow \sin^{-1}x - \left(\dfrac{\pi}{2} - \sin^{-1}x\right) = \dfrac{\pi}{6} \Rightarrow 2\sin^{-1}x - \dfrac{\pi}{2} = \dfrac{\pi}{6} \Rightarrow 2\sin^{-1}x = \dfrac{\pi}{2} + \dfrac{\pi}{6} \Rightarrow 2\sin^{-1}x = \dfrac{3\pi+\pi}{6}$

$\Rightarrow \sin^{-1}x = \dfrac{4\pi}{6\times 2} \Rightarrow \sin^{-1}x = \dfrac{\pi}{3} \Rightarrow x = \sin\dfrac{\pi}{3} \Rightarrow x = \dfrac{\sqrt{3}}{2}$ Ans.