Chapter 5

Continuity and Differentiability

Exercise 5.1

Q. 1

Prove that the function f (x) = 5x – 3 is continuous at x = 0, at x = – 3 and at x = 5.

Answer:

The given function is f(x) = 5x -3

At x = 0, f (0) = 5 × 0 – 3 = -3

$\lim_{x \to 0} f(x) = \lim_{x \to 0}(5x - 3) =$ 5 × 0 – 3 = -3

Thus, $\lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at x = 0

At x = -3, f (-3) = 5 × (-3) – 3 = -18

$\lim_{x \to 3} f(x) = \lim_{x \to}(5x - 3)$ = 5 × (-3) – 3 = -18

Thus, $\lim_{x \to 3} f(x) = f(-3)$

Therefore, f is continuous at x = -3

At x = 5, f (5) = 5 × 5 – 3 = 22

$\lim_{x \to 5} f(x) = \lim_{x \to 5}(5x - 3)$ = 5 × 5 – 3 = 22

Thus,

Therefore, f is continuous at x = 5

Q. 2 Examine the continuity of the function f (x) = $2x^2$ – 1 at x = 3.

Answer:

The given function is $f(x) = 2x^2 - 1$

At $x = 3$, $f(x) = f(3) = 2 \times 3^2 - 1 = 17$

Left hand limit (LHL):

Right hand limit(RHL):

As, LHL= RHL = f (3)

Therefore, f is continuous at x = 3

Q. 3 A

Examine the following functions for continuity.

$f(x) = x - 5$

Answer:

a) The given function is $f(x) = x - 5$

We know that f is defined at every real number k and its value at k is k – 5.

We can see that $\lim_{x\to k} f(x) = \lim_{x\to k}(x-5) = k - 5 = f(k)$

Thus, $\lim_{x\to k} f(x) = f(k)$

Therefore, f is continuous at every real number and thus, it is continuous function.

Q. 3 B

Examine the following functions for continuity.

$f(x) = \frac{1}{x-5}$

Answer:

The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we get,

$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{x-5} = \lim_{(k-5)} \frac{1}{k-5}$

Also, f (k) = $\frac{1}{k-5}$ (Ask ≠ 5)

Thus, $\lim_{x \to k} f(x) = f(k)$

Therefore, f is continuous at point in the domain of f and thus, it is continuous function.

Q. 3 C

Examine the following functions for continuity.

f (x) = $\frac{x^2-25}{x+5}$

Answer:

The given function is f (x) = $\frac{x^2-25}{x+5}$, x ≠ 5

For any real number k ≠ 5, we get,

$\lim_{x \to k} f(x) = \lim_{x \to k} \frac{x^2-25}{x+5} = \lim_{x \to k} \frac{(x-5)(x+5)}{x+5} = \lim_{x \to k}(x-5)$ = (k – 5)

Also, f (k) = $\lim_{x \to k} \frac{(k-5)(k+5)}{k+5} = \lim_{x \to k}(k-5)$ = (k – 5)(ask ≠ 5)

Thus, $\lim_{x \to k} f(x)$ = f (k)

Therefore, f is continuous at point in the domain of f and thus, it is continuous function.

Q. 3 D

Examine the following functions for continuity.

f (x) = | x – 5|

Answer:

The given function is f (x) = |x – 5| = $\begin{cases} 5 - x, if\, x < 5 \\ x - 5, if\, x \geq 5 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.ee, k < 5, or k = 5 or k >5

Now, Case I: k<5

Then, f(k) = 5 – k

$\lim_{x \to k} f(x) = \lim_{x \to k}(5 - x)$ = 5 – k = f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number less than 5.

Case II: k = 5

Then, f(k) = f (5) = 5 – 5 = 0

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5}(5 - x)$ = 5 – 5 = 0

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5}(5 - x)$ = 5 – 5 = 0

= $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k)$

Hence, f is continuous at x = 5.

Case III: k > 5

Then, f(k) = k – 5

$\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5)$ = k – 5 = f(k)

Thus, $\lim_{x \to k} f(x) = f\ (k)$

Hence, f is continuous at all real number greater than 5.

Therefore, f is a continuous function.

Q. 4

Prove that the function f (x) = x^n is continuous at x = n, where n is a positive integer.

Answer:

It is given that function f (x) = x^n

We can see that f is defined at all positive integers, n and the value of f at n is n^n.

= $\lim_{x\to n} f(n) = \lim_{x\to n}(x^n)$ = n^n

Thus, $\lim_{x\to n} f\ (x) = f(n)$

Therefore, f is continuous at x =n, where n is a positive integer.

Q. 5

Is the function f defined by f (x) = $\begin{cases} x, i\ f x \leq 1 \\ 5, i\ f x > 1 \end{cases}$n Continuous at x = 0? At x = 1? At x = 2?

Answer:

It is given that f (x) = $\begin{cases} x, i\ f x \leq 1 \\ 5, i\ f x > 1 \end{cases}$

Case I: x = 0

We can see that f is defined at 0 and its value at 0 is 0.

LHL

$\lim_{x\to 0^-} = \lim_{h\to 0} f(0-h)$

= $\lim_{h\to 0} -h = 0$

RHL

$\lim\limits_{x\to 0^+} = \lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0} h = 0$

LHL = RHL = f (0) Hence, f is continuous at x = 0.

Case II: x = 1

We can see that f is defined at 1 and its value at 1 is 1.

For x < 1

f(x) = x Hence, LHL:

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1$

For x > 1f(x) = 5therefore, RHL

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(5) = 5$

$= \lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$

Hence, f is not continuous at x = 1.

Case III: x = 2

As,

We can see that f is defined at 2 and its value at 2 is 5

LHL:

$\lim\limits_{x\to 2^-} = \lim\limits_{h\to 0} f(2-h)$

$= \lim\limits_{h\to 0} 5 = 5$

here f (2 - h) = 5, as h → 0 ⇒ 2 - h → 2RHL:

$\lim\limits_{x\to 2^+} = \lim\limits_{h\to 0}(2+h)$

$= \lim\limits_{h\to 0} 5 = 5$

LHL = RHL = f (2)

here f (2 + h) = 5, as h → 0 ⇒ 2 + h → 2

Hence, f is continuous at x = 2.

Q. 6

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} 2x + 3, if x \leq 2 \\ 2x - 3, i\ if x > 2 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} 2x + 3, if\ x \leq 2 \\ 2x - 3, i\ if x > 2 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 2, or k = 2 or k >2

Now, Case I: k < 2

Then, f(k) = 2k + 3

$\lim_{x \to k} f(x) = \lim_{x \to k}(2x + 3)$ = 2k + 3= f(k)

Thus, $\lim_{x \to k} f(x) = f\ (k)$

Hence, f is continuous at all real number less than 2.

Case II: k = 2

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x + 3)$ = 2×2 + 3 = 7

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(2x - 3)$ = 2×2 - 3 = 1

= $\lim_{x \to k^-} f(k) \neq \lim_{x \to k^+} f(k) = f(k)$

Hence, f is not continuous at x = 2.

Case III: k > 2

Then, f(k) = 2k - 3

$\lim_{x\to k} f(x) = \lim_{x\to k}(2x - 3)$ = 2k – 3 = f(k)

Thus, $\lim_{x\to k} f(x) = f(k)$

Hence, f is continuous at all real number greater than 2.

Therefore, x = 2 is the only point of discontinuity of f.

Q. 7

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} |x| + 3, i\ f x \leq -3 \\ -2x, i\ f - 3 < x < 3 \\ 6x + 2, i\ f x \geq 3 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} |x| + 3, i\ f x \leq -3 \\ -2x, i\ f - 3 < x < 3 \\ 6x + 2, i\ f x \geq 3 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 5 cases i.e., k < -3, k = -3, -3 < k < 3, k = 3 or k > 3

Now, Case I: k < -3

Then, f(k) = -k + 3

$\lim_{x\to k} f(x) = \lim_{x\to k}(-x + 3)$ = -k + 3= f(k)

Thus, $\lim_{x\to k} f(x) = f(k)$

Hence, f is continuous at all real number x < -3.

Case II: k = -3

f (-3) = -(-3) + 3 = 6

$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x + 3)$ =-(-3) + 3 = 6

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^-} (-2x)$ = -2 × (-3) = 6

= $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k)$

Hence, f is continuous at x = -3.

Case III: -3 < k < 3

Then, f(k) = -2k

$\lim_{x \to k} f(x) = \lim_{x \to k} (-2x)$ = -2k = f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous in (-3,3).

Case IV: k = 3

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x)$ = -2 × (3) = -6

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$ = 6 × 3 + 2 = 20

= $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$

Hence, f is not continuous at x = 3.

Case V: k > 3

Then, f(k) = 6k + 2

$\lim_{x \to k} f(x) = \lim_{x \to k} (6x + 2)$ = 6k + 2= f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number x < 3.

Therefore, x = 3 is the only point of discontinuity of f.

Q. 8

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} \frac{|x|}{x}, i\ f x \neq 0 \\ 0, i\ f x = 0 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} \frac{|x|}{x}, i\ f x \neq 0 \\ 0, i\ f x = 0 \end{cases}$

We know that if x > 0

⇒ |x| = -x and x > 0

⇒ |x| = x

So, we can rewrite the given function as:

f (x) = $\begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1, i\ f x < 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, i\ f x > 0 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 0, or k = 0 or k >0.

Now, Case I: k < 0

Then, f(k) = -1

$\lim_{x \to k} f(x) = \lim_{x \to k} (-1)$ = -1= f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number less than 0.

Case II: k = 0

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1)$ = -1

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1)$ = 1

= $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x) = f(k)$

Hence, f is not continuous at x = 0.

Case III: k > 0

Then, f(k) = 1

$\lim_{x \to k} f(x) = \lim_{x \to k} (1)$ = 1 = f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number greater than 1.

Therefore, x = 0 is the only point of discontinuity of f.

Q. 9

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} \frac{x}{|x|}, i\ f x < 0 \\ -1, i\ f x \geq 0 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} \frac{x}{|x|}, i\ f x < 0 \\ -1, i\ f x \geq 0 \end{cases}$

We know that if x < 0

⇒ |x| = -x

So, we can rewrite the given function as:

f (x) = $\begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, if\, x < 0 \\ -1, if\, x \geq 0 \end{cases}$

⇒ f(x) = -1 for all x ϵ R

Let k be the point on a real line.

Then, f(k) = -1

$\lim_{x \to k} f(k) = \lim_{x \to k} (-1)$ = -1= f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Therefore, the given function is a continuous function.

Q. 10

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} x + 1, if\, x \geq 0 \\ x^2 + 1, if\, x < 0 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} x + 1, if\, x \geq 0 \\ x^2 + 1, if\, x < 0 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 1, or k = 1 or k >1

Now, Case I: k < 1

Then, f(k) = $k^2 + 1$

$\lim_{x \to k} f(x) = \lim_{x \to k} (x^2 + 1)$ = $k^2 + 1$= f(k)

Thus, $\lim_{x \to k} f(x) = f\ (k)$

Hence, f is continuous at all real number less than 1.

Case II: k = 1

Then, f(k) = f (1) = 1 + 1 = 2

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2 + 1)$ = 1² + 1 = 2

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x + 1)$ = 1 + 1 = 2

= $\lim_{x\to k^-} f(x) = \lim_{x\to k^+} f(x) = f(k)$

Hence, f is continuous at x = 1.

Case III: k > 1

Then, f(k) = k + 1

$\lim_{x\to k} f\ (x) = \lim_{x\to k}(x + 1)$ = k + 1 = f(k)

Thus, $\lim_{x\to k} f(x) = f\ (k)$

Hence, f is continuous at all real number greater than 1.

Q. 11

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} x^3 - 3, i\ f x \leq 2 \\ x^2 + 1, i\ f x > 2 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} x^3 - 3, i\ f x \leq 2 \\ x^2 + 1, i\ f x > 2 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 2, or k = 2 or k > 2

Now, Case I: k < 2

Then, f(k) = k^3 - 3

$\lim_{x\to k} f(x) = \lim_{x\to k}(x^3 - 3)$ = k^3 - 3= f(k)

Thus, $\lim_{x\to k} f(x) = f(k)$

Hence, f is continuous at all real number less than 2.

Case II: k = 2

Then, f(k) = f (2) = 2^3 - 3 = 5

$\lim_{x\to 2^-} f\ (x) = \lim_{x\to 2^-}(x^3 - 3)$ = 2^3 - 3 = 5

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^2 + 1)$ = 2^2 + 1 = 5

= $\lim_{x\to k^-} f(x) = \lim_{x\to k^+} f(x) = f(k)$

Hence, f is continuous at x = 2.

Case III: k > 2

Then, f(k) = 2^2 + 1 = 5

$\lim_{x\to k} f(x) = \lim_{x\to k}(x^2 + 1)$ = 2^2 + 1 = 5 = f(k)

Thus, $\lim_{x\to k} f\ (x) = f\ (k)$

Hence, f is continuous at all real number greater than 2.

Q. 12

Find all points of discontinuity of f, where f is defined by

f (x) = $\begin{cases} x^{10} - 1, i\ f x \le 1 \\ x^2, i\ f x > 1 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} x^{10} - 1, i\ f x \le 1 \\ x^2, i\ f x > 1 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 1, or k = 1 or k > 1

Now,

Case I: k < 1

Then, $f(k) = k^{10} - 1$

$\lim_{x \to k} f(x) = \lim_{x \to k}(x^{10} - 1) = k^{10} - 1 = f(k)$

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number less than 1.

Case II: k = 1

Then, $f(k) = f(1) = 1^{10} - 1 = 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(x^{10} - 1) = 1^{10} - 1 = 0$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(x^2) = 1^2 = 1$

$= \lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$

Hence, f is not continuous at x = 1.

Case III: k > 1

Then, $f(k) = 1^2 = 1$

$\lim_{x \to k} f(x) = \lim_{x \to k}(x^2) = 1^2 = 1 = f(k)$

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number greater than 1.

Therefore, x = 1 is the only point of discontinuity of f.

Q. 13

Is the function defined by f (x) = $\begin{cases} x+5, if\, x \leq 1 \\ x-5, if\, x > 1 \end{cases}$ a continuous function?

Answer:

The given function is f (x) = $\begin{cases} x+5, if\, x \leq 1 \\ x-5, if\, x > 1 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 1, or k = 1 or k > 1

Now,

Case I: k < 1

Then, f(k) = k + 5

$\lim_{x\to k} f(x) = \lim_{x\to k}(x+5)$ = k + 5 = f(k)

Thus, $\lim_{x\to k} f(x) = f(k)$

Hence, f is continuous at all real number less than 1.

Case II: k = 1

Then, f(k) = f (1) = 1 + 5 = 6

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+5)$ = 1 + 5 = 6

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x-5)$ = 1 - 5 = -4

= $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$

Hence, f is not continuous at x = 1.

Case III: k > 1

Then, f(k) = k -5

$\lim_{x \to k} f(x) = \lim_{x \to k}(x - 5)$ = k - 5

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all real number greater than 1.

Therefore, x = 1 is the only point of discontinuity of f.

Q. 14

Discuss the continuity of the function f, where f is defined by

f (x) = $\begin{cases} 3 if 0 \leq x \leq 1 \\ 4 if 1 < x < 3 \\ 5 if 3 \leq x \leq 10 \end{cases}$

Answer:

The given function is f (x) = $\begin{cases} 3 if 0 \leq x \leq 1 \\ 4 if 1 < x < 3 \\ 5 if 3 \leq x \leq 10 \end{cases}$

The function f is defined at all points of the interval [0,10].

Let k be the point in the interval [0,10].

Then, we have 5 cases i.e., 0≤ k < 1, k = 1, 1 < k < 3, k = 3 or 3 < k ≤ 10.

Now, Case I: 0≤ k < 1

Then, f(k) = 3

$\lim_{x \to k} f(x) = \lim_{x \to k}(3)$ = 3= f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous in the interval [0,10).

Case II: k = 1

f (1) = 3

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(3)$ = 3

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(4)$ = 4

= $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$

Hence, f is not continuous at x = 1.

Case III: 1 < k < 3

Then, f(k) = 4

$\lim_{x\to k} f(x) = \lim_{x\to k}(4)$ = 4 = f(k)

Thus, $\lim_{x\to k} f(x) = f(k)$

Hence, f is continuous in (1, 3).

Case IV: k = 3

$\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(4)$ = 4

$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(5)$ = 5

= $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$

Hence, f is not continuous at x = 3.

Case V: 3 < k ≤ 10

Then, f(k) = 5

$\lim_{x\to k} f(x) = \lim_{x\to k}(5)$ = 5 = f(k)

Thus, $\lim_{x\to} f(x) = f(k)$

Hence, f is continuous at all points of the interval (3, 10].

Therefore, x = 1 and 3 are the points of discontinuity of f.

Q. 15

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, if x \leq 0 \\ 0 if 0 \leq x \leq 1 \\ 4x if x > 1 \end{cases}$$

Answer:

The given function is $f(x) = \begin{cases} 2x, if x \leq 0 \\ 0 if 0 \leq x \leq 1 \\ 4x if x > 1 \end{cases}$

The function f is defined at all points of the real line.

Then, we have 5 cases i.e., k < 0, k = 0, 0 < k < 1, k = 1 or k < 1.

Now, Case I: k < 0

Then, f(k) = 2k

$\lim_{x \to k} f(x) = \lim_{x \to k}(2x)$ = 2k= f(k)

Thus, $\lim_{x \to k} f(x) = f\ (k)$

Hence, f is continuous at all points x, s.t. x < 0.

Case II: k = 0

f (0) = 0

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x)$ = 2 × 0 = 0

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(0)$ = 0

= $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k)$

Hence, f is continuous at x = 0.

Case III: 0 < k < 1

Then, f(k) = 0

$\lim_{x \to k} f(x) = \lim_{x \to k}(0)$ = 0 = f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous in (0, 1).

Case IV: k = 1

Then f(k) = f (1) = 0

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(0)$ = 0

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4x)$ = 4 × 1 = 4

= $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$

Hence, f is not continuous at x = 1.

Case V: k < 1

Then, f(k) = 4k

$\lim_{x \to k} f(x) = \lim_{x \to k}(4x)$ = 4k = f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all points x, s.t. x > 1.

Therefore, x = 1 is the only point of discontinuity of f.

Q. 16

f (x) = $\begin{cases} -2, i\ fx \leq -1 \\ 2x, i\ f \leq x \leq 1 \\ 2, i\ fx > 1 \end{cases}$

Discuss the continuity of the function f, where f is defined by

Answer:

The given function is f (x) = $\begin{cases} -2, i\ f x \leq -1 \\ 2x, i\ f \leq x \leq 1 \\ 2, i\ f x > 1 \end{cases}$

The function f is defined at all points of the real line.

Then, we have 5 cases i.e., k < -1, k = -1, -1 < k < 1, k = 1 or k > 1.

Now, Case I: k < 0

Then, f(k) = -2

$\lim_{x \to k} f(x) = \lim_{x \to k} f(x)$ = -2 = f (k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous at all points x, s.t. x < -1.

Case II: k = -1

f(k) = f (=1) = -2

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2)$ = -2

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x)$ = 2 × (-1) = -2

= $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k)$

Hence, f is continuous at x = -1.

Case III: -1 < k < 1

Then, f(k) = 2k

$\lim_{x \to k} f(x) = \lim_{x \to k} (-2)$ = 2k = f(k)

Thus, $\lim_{x \to k} f(x) = f(k)$

Hence, f is continuous in (-1, 1).

Case IV: k = 1

Then f (k) = f (1) = 2 × 1 = 2

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x)$ = 2 × 1 = 2

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2)$ = 2

= $\lim_{x\to k^-} f(x) = \lim_{x\to k^+} f(x) = f(x)$

Hence, f is continuous at x = 1.

Case V: k > 1

Then, f(k) = 2

$\lim_{x\to k} f(x) = \lim_{x\to k}(2)$ = 2 = f(k)

Thus, $\lim_{x\to k} f(x) = f(k)$

Hence, f is continuous at all points x, s.t. x > 1.

Therefore, f is continuous at all points of the real line.

Q. 17

Find the relationship between a and b so that the function f defined by

f (x) = $\begin{cases} ax+1, if\ x \le 3 \\ bx+3, if\ x > 3 \end{cases}$ is continuous at x = 3.

Answer:

It is given function is f (x) = $\begin{cases} ax+1, if\ x \le 3 \\ bx+3, if\ x > 3 \end{cases}$

It is given that f is continuous at x = 3, then, we get,

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ = f (3) … (1)

And

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$ = 3a + 1

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$ = 3b + 1

f (3) = 3a + 1

Thus, from (1), we get,

3a + 1 = 3b + 3 = 3a + 1

⇒ 3a +1 = 3b + 1

⇒ 3a = 3b +

⇒ a = b +

Therefore, the required the relation is a = b + $\frac{2}{3}$.

Q. 18

For what value of λ is the function defined by f (x) = $\begin{cases} \lambda(x^2 - 2x), if\ x \leq 0 \\ 4x + 1, if\ x > 0 \end{cases}$ Continuous at x = 0? What about continuity at x = 1?

Answer:

It is given that f (x) = $\begin{cases} \lambda(x^2 - 2x), if\ x \leq 0 \\ 4x + 1, if\ x > 0 \end{cases}$

It is given that f is continuous at x = 0, then, we get,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ = f (0)

And

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\lambda(x^2 - 2x)\right) = \lim_{x \to 0^-} \left(\lambda(0^2 - 2 \times 0)\right)$ = 0

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1)$ = 4 × 0 + 1 = 1

$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Thus, there is no value of for which f is continuous at x = 0

f (1) = 4x + 1 = 4 × 1 + 1 = 5

$\lim_{x \to 1}(4x + 1)$ = 4 × 1 + 1 = 5

Then, $\lim_{x \to 1} f(x) = f\ (1)$

Hence, for any values of, f is continuous at x = 1

Q. 19

Show that the function defined by g(x) = x – [x] is discontinuous at all integral points. Here [x] denotes the greatest integer less than or equal to x.

Answer:

It is given that g(x) = x – [x]

We know that g is defined at all integral points.

Let k be ant integer.

Then,

g(k) = k – [-k] = k + k = 2k

$\lim_{x \to k^-} g(k) = \lim_{x \to k^-}(x - [x])$

$\lim_{x \to k^-}(x) = \lim_{x \to k^-}[x]$ = k – (k – 1) = 1

And

$\lim_{x \to k^+} g(x) = \lim_{x \to k^+}(x - [x])$

$\lim_{x \to k^+}(x) - \lim_{x \to k^+}[x]$ = k – k = 0

$\therefore \lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$

Therefore, g is discontinuous at all integral points.

Q. 20 Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Answer:

It is given that $f(x) = x^2 - \sin x + 5$

We know that f is defined at $x = \pi$

So, at $x = \pi$,

$f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi 2 + 5$

Now, $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} f(x^2 - \sin x + 5)$

Let put $x = \pi + h$

If $x \to \pi$, then we know that $k \to 0$

$\therefore \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5)$

$= \lim_{k \to 0} f[(\pi + k)^2 - \sin(\pi + k) + 5]$

$= \lim_{k \to 0} f(\pi + k)^2 - \lim_{k \to 0} \sin(\pi + k) + \lim_{k \to 0} 5$

$= (\pi + 0)^2 - \lim_{k \to 0} [\sin \pi \cos k + \cos \pi \sin k] + 5$

$= \pi^2 - \lim_{k \to 0} \sin \pi \cos k - \lim_{k \to 0} \cos \pi \sin k + 5$

$= \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5$

$= \pi^2 - 0 \times 1 (-1) \times 0 + 5$

$= \pi^2 + 5$

Thus, $\lim_{x \to \pi} f(x) f(\pi)$

Therefore, the function f is continuous at $x = \pi$.

Q. 21 Discuss the continuity of the following functions:

(a) f (x) = sin x + cos x

(b) f (x) = sin x – cos x

(c) f (x) = sin x. cos x

Answer:

We know that g and k are two continuous functions, then,

g + k, g – k and g. k are also continuous.

First we have to prove that g(x) = sin x and k(x) = cos x are continuous functions.

Now, let g(x) = sin x

We know that g(x) = sin x is defined for every real number.

Let h be a real number. Now, put x = h + k

So, if x → h and k → 0

g(h) = sin h

$\lim_{x\to h} g(x) = \lim_{x\to h} \sin x$

$= \lim_{x\to 0} \sin(h + k)$

$= \lim_{x\to 0} [\sin h \cos k + \cos h \sin k]$

= sinhcos0 + coshsin0

= sin h + 0

= sin h

Thus, $\lim_{x\to h} g(x) = g(h)$

Therefore, g is a continuous function … (1)

Now, let k(x) = cos x

We know that k(x) = cos x is defined for every real number.

Let h be a real number. Now, put x = h + k

So, if x → h and k → 0

Now k (h) = cos h

$\lim_{x \to h} k(x) = \lim_{x \to h} \cos x$

$= \lim_{k \to 0} \cos(h + k)$

$= \lim_{x \to 0} [ci\ sh \cos k - \sin h \sin k]$

= coshcos0 - sinhsin0

= cos h - 0

= cos h

Thus, $\lim_{x \to h} k(x) = k\ (h)$

Therefore, k is a continuous function … (2)

So, from (1) and (2), we get,

(a) f(x) = g(x) + k(x) = sin x + cos x is a continuous function.

(b) f(x) = g(x) - k(x) = sin x – cos x is a continuous function.

(c) f(x) = g(x) × k(x) = sin x × cos x is a continuous function.

Q. 22 Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Answer:

We know that if g and h are two continuous functions, then,

(i) $\frac{h(x)}{g(x)}$, g (x) ≠ 0 is continuous

(ii) $\frac{1}{g(x)}$, g (x) ≠ 0 is continuous

(iii) $\frac{1}{g\ (x)}$, g (x) ≠ 0 is continuous

So, first we have to prove that g(x) = sin x and h(x) = cos x are continuous functions.

Let g(x) = sin x

We know that g(x) = sin x is defined for every real number.

Let h be a real number. Now, put x = k + h

So, if x → k and h → 0

g(k) = sink

$\lim_{x \to k} g(x) = \lim_{x \to k} \sin x$

$= \lim_{h \to 0} \sin(k + h)$

$= \lim_{h \to 0} [\sin k \cos h + \cos k \sin h]$

= sinkcos0 + cosksin0

= sink + 0

= sink

Thus, $\lim_{x \to k} g(x) = g(k)$

Therefore, g is a continuous function … (1)

Let h(x) = cos x

We know that h(x) = cos x is defined for every real number.

Let k be a real number. Now, put x = k + h

So, if x → k and h → 0

h(k) = sink

$\lim_{x \to k} h(x) = \lim_{x \to k} \cos x$

= $\lim_{h \to 0} \cos(k + h)$

= $\lim_{h \to 0} [\cos k \cos h - \sin k \sin h]$

= coskcos0 - sinksin0

= cos k - 0

= cos k

Thus, $\lim_{x \to k} h(x) = h(k)$

Therefore, g is a continuous function … (2)

So, from (1) and (2), we get,

Cosec x = $\frac{1}{\sin x}$, sin x ≠ 0 is continuous

= cosec x, ≠ nπ (n ϵ Z) 0 is continuous

Thus, cosecant is continuous except at x = np, (n ϵ Z)

Sec x = $\frac{1}{\cos x}$, cos x ≠ 0 is continuous

= sex x, x ≠ (2n + 1) $\frac{\pi}{2}$ (n ϵ Z) is continuous

Thus, secant is continuous except at x = (2n + 1) $\frac{\pi}{2}$, (n ϵ Z)

Cot x = $\frac{\cos x}{\sin x}$, sin x ≠ 0 is continuous

= cot x, x ≠ nπ (n ϵ Z) 0 is continuous

Thus, cotangent is continuous except at x = np, (n ϵ Z)

Q. 23

Find all points of discontinuity of f, where f (x) = $\begin{cases} \frac{\sin x}{x},\ i\ fx\ < 0 \\ x + 1, i\ fx \geq 0 \end{cases}$

Answer:

It is given that f (x) = $\begin{cases} \frac{\sin x}{x}, if x < 0 \\ x + 1, if x \geq 0 \end{cases}$

We know that f is defined at all points of the real line.

Let k be a real number.

Case I: k < 0,

Then f(k) = $\frac{\sin k}{k}$

$\lim_{x \to k} f(x) = \lim_{x \to k} \left(\frac{\sin x}{x}\right) = \frac{\sin k}{k}$

= $\lim_{x \to k} f(x) = f(k)$

Thus, f is continuous at all points x that is x < 0.

Case II: k > 0,

Then f(k) = c + 1

$\lim_{x \to k} f(x) = \lim_{x \to k}(x + 1)$ = k + 1

= $\lim_{x \to k} (x) = f(k)$

Thus, f is continuous at all points x that is x > 0.

Case III: k = 0

Then f(k) = f (0) = 0 + 1 = 1

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\sin x}{x}\right)$ = 1

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1)$ = 1

= $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(x)$

Hence, f is continuous at x = 0.

Therefore, f is continuous at all points of the real line

Q. 24 Determine if f defined by f (x) = $\begin{cases} x^2 \sin\frac{1}{x}, i\ fx \ \neq 0 \\ 0, i\ fx = 0 \end{cases}$ is a continuous function?

Answer:

It is given that f (x) = $\begin{cases} x^2 \sin\frac{1}{x} \ \ i\ fx \ \neq 0 \\ 0, i\ fx = 0 \end{cases}$

We know that f is defined at all points of the real line.

Let k be a real number.

Case I: k ≠ 0,

Then f(k) = $k^2 \sin\frac{1}{k}$

$\lim_{x\to k} f\ (x) = \lim_{x\to k}\left(x^2 \sin\frac{1}{x}\right)$ = $k^2 \sin\frac{1}{k}$

$\therefore \lim_{x\to k} f\ (x) = f(k)$

Thus, f is continuous at all points x that is x ≠ 0.

Case II: k = 0

Then f(k) = f (0) = 0

$\lim_{x\to 0^-} f\ (x)$ = $\lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$

We know that -1 ≤ x ≤ 1, x ≠ 0

⇒ $x^2 \leq x^2 \sin\frac{1}{x} \leq 0$

⇒ $\lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$ = 0

⇒ $\lim_{x\to 0^-} f\ (x)$ = 0

Similarly, $\lim_{x\to 0^+} f\ (x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$\lim_{x \to 0^-} f\ (x) = f\ (0) = \lim_{x \to 0^+} f\ (x)$

Therefore, f is continuous at x = 0.

Therefore, f has no point of discontinuity.

Q. 26 Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

f (x) = $\begin{cases} \frac{k \cos x}{\pi - 2x}, i\ fx \neq \frac{\pi}{2} \\ \qquad at\ x = \frac{\pi}{2} \\ 3, i\ fx = \frac{\pi}{2} \end{cases}$

Answer:

It is given that f (x) = $\begin{cases} \frac{k \cos x}{\pi - 2x}, i\ fx\ \neq \frac{\pi}{2} \\ \qquad at\ x = \frac{\pi}{2} \\ 3, i\ fx = \frac{\pi}{2} \end{cases}$

Also, it is given that function f is continuous at x = $\frac{\pi}{2}$,

So, if f is defined at x = $\frac{\pi}{2}$ and if the value of the f at x = equals the limit of $\frac{\pi}{2}$ f at x = $\frac{\pi}{2}$.

We can see that f is defined at x = $\frac{\pi}{2}$ and f $\left(\frac{\pi}{2}\right)$ = 3

$\lim_{x \to \frac{\pi}{2}} f\ (x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$

Now, let put x = $\frac{\pi}{2} + h$

Then, x → $\frac{\pi}{2}$ = h → 0∴ $\lim_{x \to \frac{\pi}{2}} f\ (x) = \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$

$= k\lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2}. 1 = \frac{k}{2}$

$\therefore \lim_{x \to \frac{\pi}{2}} f\ (x) = f\ \left(\frac{\pi}{2}\right)$

$\Rightarrow \frac{k}{2} = 3$

$\Rightarrow$ k = 6

Therefore, the value of k is 6.

Q. 27 Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

$$f(x) = \begin{cases} kx^2, if\ x \leq 2 \\ \quad at\ x = 2 \\ 3, if\ x > 2 \end{cases}$$

Answer:

It is given that $f(x) = \begin{cases} kx^2, if\ x \leq 2 \\ \quad at\ x = 2 \\ 3, if\ x > 2 \end{cases}$

Also, it is given that function f is continuous at x = 2,

So, if f is defined at x = 2 and if the value of the f at x = 2 equals the limit of f at x = 2.

We can see that f is defined at x = 2 and

$f(2) = k(2)^2 = 4k$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 0^+} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2^-} (kx^2) = \lim_{x \to 0^+} (3)$

$\Rightarrow k \times 2^2 = 3 = 4k$

$\Rightarrow 4k = 3 = 4k$

$\Rightarrow 4k = 3$

$\Rightarrow$ k = $\frac{3}{4}$

Therefore, the required value of k is $\frac{3}{4}$.

Q. 28 Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

f (x) = $\begin{cases} kx + 1 \text{ , } i\ fx \ \le \pi \\ \cos x, i\ fx > \pi \end{cases}$ at x = π

Answer:

It is given that f (x) = $\begin{cases} kx + 1 \text{ , } i\ fx \ \le \pi \\ cos\, x, i\ fx > \pi \end{cases}$ at x = π

Also, it is given that function f is continuous at x = k,

So, if f is defined at x = p and if the value of the f at x = k equals the limit of f at x = k.

We can see that f is defined at x = p and

f(π) = kπ + 1

$\lim_{x\to\pi^-} f\ (x) = \lim_{x\to\pi^+} f\ (x) = f(\pi)$

$\Rightarrow \lim_{x\to\pi^-}(kx + 1) = \lim_{x\to\pi^+}(\cos x)$ = kπ + 1

$\Rightarrow$ kπ + 1 = cosπ = kπ + 1

$\Rightarrow$ kπ + 1 = -1 = kπ + 1

$\Rightarrow$ k = $-\frac{2}{\pi}$

Therefore, the required value of k is -$\frac{2}{\pi}$.

Q. 29

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

f (x) = $\begin{cases} kx + 1, if x \leq 5 \\ 3x - 5, if x > 5 \end{cases}$ at x = 5

Answer:

It is given that f (x) = $\begin{cases} kx + 1, if x \leq 5 \\ 3x - 5, if x > 5 \end{cases}$ at x = 5

Also, it is given that function f is continuous at x = 5,

So, if f is defined at x = 5 and if the value of the f at x = 5 equals the limit of f at x = 5.

We can see that f is defined at x = 5 and

f (5) = kx + 1 = 5k + 1

$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f\ (\pi)$

= $\lim_{x \to 5^-} (kx + 1) = \lim_{x \to 5^+} (3x - 5)$ = 5k + 1

⇒ 5k + 1 = 15 -5 = 5k + 1

⇒ 5k + 1 = 10

⇒ 5k = 9

⇒ k = $\frac{9}{5}$

Therefore, the required value of k is $\frac{9}{5}$.

Q. 30

Find the values of a and b such that the function defined by

f (x) = $\begin{cases} 5, if x \leq 2 \\ ax + b, if 2 \leq x \leq 10 \\ 21, if x \geq 10 \end{cases}$ is a continuous function.

Answer:

It is given function is f (x) = $\begin{cases} 5, if x \le 2 \\ ax + b, if 2 \le x \le 10 \\ 21, if x \ge 10 \end{cases}$

We know that the given function f is defined at all points of the real line.

Thus, f is continuous at x = 2, we get,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{x \to 2^-} (5) = \lim_{x \to 2^+} (ax + b) = 5$

⇒ 5 = 2a + b = 5

⇒ 2a + b = 5 … (1)

Thus, f is continuous at x = 10, we get,

$\lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$

$= \lim_{x \to 10^-} (ax + b) = \lim_{x \to 10^+} (21) = 21$

⇒ 10a + b = 21 =21

⇒ 10a + b = 21 … (2)

On subtracting eq. (1) from eq. (2), we get,

8a = 16

⇒ a = 2

Thus, putting a = 2 in eq. (1), we get,

2 × 2 + b = 5

⇒ 4 + b = 5

⇒ b = 1

Therefore, the values of a and b for which f is a continuous function are 2 and 1 resp.

Q. 31

Show that the function defined by f (x) = cos (x^2) is a continuous function.

Answer:

It is given function is f(x) = cos (x^2)

This function f is defined for every real number and f can be written as the composition of two function as,

f = goh, where, g(x) = cos x and h(x) = x^2

First we have to prove that g(x) = cos x and h(x) = x^2 are continuous functions.

We know that g is defined for every real number.

Let k be a real number.

Then, g(k) =cos k

Now, put x = k + h

If x → k, then h → 0

$\lim_{x\to k} g\ (k) = \lim_{x\to k} \cos x$

$= \lim_{h\to 0} \cos(k+h)$

$= \lim_{h\to 0} \cos[\cos k \cos h - \sin k \sin h]$

$= \lim_{h\to 0} \cos k \cos h - \lim_{h\to 0} \sin k \sin h$

= coskcos0 – sinksin0

= cos k × 1 – sin × 0

= cos k

$\therefore \lim_{x\to k} g(x) = g(k)$

Thus, g(x) = cos x is continuous function.

Now, $h(x) = x^2$

So, h is defined for every real number.

Let c be a real number, then $h(c) = c^2$

$$\lim_{x \to c} h(x) = \lim_{x \to c} x^2$$

$$\lim_{x \to c} h(x) = h(c)$$

Therefore, h is a continuous function.

We know that for real valued functions g and h,

Such that (fog) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Q. 32

Show that the function defined by f (x) = | cos x| is a continuous function.

Answer:

It is given that f(x) = |cos x|

The given function f is defined for real number and f can be written as the composition of two functions, as

f = goh, where g(x) = |x| and h(x) = cos x

First we have to prove that g(x) = |x| and h(x) = cos x are continuous functions.

g(x) = |x| can be written as

$$g(x) = \begin{cases} -x, if\ x < 0 \\ x, if x \geq 0 \end{cases}$$

Now, g is defined for all real number.

Let k be a real number.

Case I: If k < 0,

Then g(k) = -k

And $\lim_{x \to k} g(x) = \lim_{x \to k}(-x)$ = - k

Thus, $\lim_{x \to k} g(x) = g\ (k)$

Therefore, g is continuous at all points x, i.e., x > 0

Case II: If k > 0,

Then g(k) = k and

$\lim_{x \to k} g(x) = \lim_{x \to k} x$ = k

Thus, $\lim_{x \to k} g(x) = g\ (k)$

Therefore, g is continuous at all points x, i.e., x < 0.

Case III: If k = 0,

Then, g(k) = g (0) = 0

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x)$ = 0

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x)$ = 0

∴ $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x)$ = g (0)

Therefore, g is continuous at x = 0

From the above 3 cases, we get that g is continuous at all points.

h(x) = cos x

We know that h is defined for every real number.

Let k be a real number.

Now, put x = k + h

If x → k, then h → 0

$\lim_{x \to k} h(x) = \lim_{x \to k} \cos x$

$= \lim_{h \to 0} \cos(k + h)$

$= \lim_{h \to 0} \cos[\cos k \cos h - \sin k \sin h]$

$= \lim_{h \to 0} \cos k \cos h \ - \lim_{h \to 0} \sin k \sin h$

= coskcos0 – sinksin0

= cos k × 1 – sin × 0

= cos k

$\therefore \lim_{x \to k} h(k) = h\ (k)$

Thus, h(x) = cos x is continuous function.

We know that for real valued functions g and h, such that (goh) is defined at k, if g is continuous at k and if f is continuous at g(k),

Then (fog) is continuous at k.

Therefore, f(x) = (gof)(x) = g(h(x)) = g (cos x) = |cos x| is a continuous function.

Q. 33 Examine that sin | x| is a continuous function.

Answer:

It is given that f(x) = sin |x|

The given function f is defined for real number and f can be written as the composition of two functions, as

f = goh, where g(x) = |x| and h(x) = sin x

First we have to prove that g(x) = |x| and h(x) = sin x are continuous functions.

g(x) = |x| can be written as

g (x) = $\begin{cases} -x, i\ f x < 0 \\ x, i\ f x \ \geq 0 \end{cases}$

Now, g is defined for all real number.

Let k be a real number.

Case I: If k < 0,

Then g(k) = -k

And $\lim_{x \to k} g(x) = \lim_{x \to k} (-x) = -\text{k}$

Thus, $\lim_{x \to k} g(x) = g\ (k)$

Therefore, g is continuous at all points x, i.e., x > 0

Case II: If k > 0,

Then g(k) = k and

$\lim_{x \to k} g(x) = \lim_{x \to k} x = \text{k}$

Thus, $\lim_{x \to k} g\ (x) = g\ (k)$

Therefore, g is continuous at all points x, i.e., x < 0.

Case III: If k = 0,

Then, g(k) = g (0) = 0

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$

$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g\ (0)$

Therefore, g is continuous at x = 0

From the above 3 cases, we get that g is continuous at all points.

h(x) = sin x

We know that h is defined for every real number.

Let k be a real number.

Now, put x = k + h

If x → k, then h → 0

$\lim_{x\to k} h(x) = \lim_{x\to k} \sin x$

$= \lim_{h\to 0} \sin(k+h)$

$= \lim_{h\to 0}[\sin k \cos h + \cos k \sin h\ ]$

$= \lim_{h\to 0} \sin k \cos h + \lim_{h\to 0} \cos k \sin h$

= sinkcos0 + cosksin0

= sink

$\therefore \lim_{x\to k} h(x) = g(k)$

Thus, h(x) = cos x is continuous function.

We know that for real valued functions g and h, such that (goh) is defined at k, if g is continuous at k and if f is continuous at g(k),

Then (fog) is continuous at k.

Therefore, f(x) = (gof)(x) = g(h(x)) = g (sin x) = |sin x| is a continuous function.

Q. 34

Find all the points of discontinuity of f defined by f (x) = | x| – | x + 1|.

Answer:

It is given that f(x) = |x| - |x + 1|

The given function f is defined for real number and f can be written as the composition of two functions, as

f = goh, where g(x) = |x| and h(x) = |x + 1|

Then, f = g - h

First we have to prove that g(x) = |x| and h(x) = |x + 1| are continuous functions.

g(x) = |x| can be written as

g (x) = $\begin{cases} -x, i\ fx < 0 \\ x, i\ fx \geq 0 \end{cases}$

Now, g is defined for all real number.

Let k be a real number.

Case I: If k < 0,

Then g(k) = -k

And $\lim_{x \to k} g\ (x) = \lim_{x \to k} (-x) = -$k

Thus, $\lim_{x \to k} g\ (x) = g\ (k)$

Therefore, g is continuous at all points x, i.e., x > 0

Case II: If k > 0,

Then g(k) = k and

$\lim_{x \to k} g(x) = \lim_{x \to k} x$ = k

Thus, $\lim_{x \to k} g\ (x) = g\ (k)$

Therefore, g is continuous at all points x, i.e., x < 0.

Case III: If k = 0,

Then, g(k) = g (0) = 0

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$

$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = 0$

Therefore, g is continuous at x = 0

From the above 3 cases, we get that g is continuous at all points.

g(x) = |x + 1| can be written as

$$g(x) = \begin{cases} -(x+1), if x < -1 \\ x+1, if x \geq -1 \end{cases}$$

Now, h is defined for all real number.

Let k be a real number.

Case I: If k < -1,

Then h(k) = -(k + 1)

And $\lim_{x \to k} h(x) = \lim_{x \to k} [-(x+1)] = -(k+1)$

Thus, $\lim_{x \to k} h(x) = h(k)$

Therefore, h is continuous at all points x, i.e., x < -1

Case II: If k > -1,

Then h(k) = k + 1 and

$\lim_{x \to k} h(x) = \lim_{x \to k} (x+1) = k+1$

Thus, $\lim_{x \to k} h(x) = h(k)$

Therefore, h is continuous at all points x, i.e., x > -1.

Case III: If k = -1,

Then, h(k) = h (-1) = -1 + 1 = 0

$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-}[-(x+1)] = -$ (1 + 1) = 0

$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+}(x+1)$ = (- 1 + 1) = 0

$\therefore \lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x)$ = h (-1)

Therefore, g is continuous at x = -1

From the above 3 cases, we get that h is continuous at all points.

Hence, g and h are continuous function.

Therefore, f = g – h is also a continuous function.

Exercise 5.2

Q. 1 Differentiate the functions with respect to x. sin $(x^2 + 5)$

Answer:

Given: sin $(x^2 + 5)$

Let y = sin $(x^2 + 5)$

$= \frac{dy}{dx} = \frac{d}{dx}\sin(x^2 + 5)$

= cos $(x^2 + 5)$. $\frac{d}{dx}$ sin $(x^2 + 5)$

= cos $(x^2 + 5)$. $\left[\frac{d}{dx}(x)^2 + \frac{d}{dx}(5)\right]$

= cos $(x^2 + 5)$. (2x + 0)

= cos $(x^2 + 5)$. (2x)

= 2x.cos $(x^2 + 5)$

Q. 2 Differentiate the functions with respect to x. cos (sin x)

Answer:

Given: cos (sin x)

Let y = cos (sin x)

$= \frac{dy}{dx} = \frac{d}{dx}$ (cos (sin x))

= - sin (sin x). $\frac{d}{dx}$ (sin x)

= -sin (sin x). Cos x

= -cos x. sin (sin x)

Q. 3 Differentiate the functions with respect to x. sin (ax + b)

Answer:

Given: sin (ax + b)

Let y = sin (ax + b)

$= \frac{dy}{dx} = \frac{d}{dx}$ (sin (ax + b))

= cos (ax + b). $\frac{d}{dx}$ (ax + b)

= cos (ax + b). $\left(\frac{d}{dx}(ax + \frac{d}{dx}(b)\right)$

= cos (ax + b). (a + 0)

= cos (ax + b). (a)

= a. cos (ax + b)

Q. 4

Differentiate the functions with respect to x.

Sec $\left(\tan\left(\sqrt{x}\right)\right)$

Answer:

Given: sec (tan(√x))

Let y= sec (tan(√x))

$= \frac{dy}{dx} = \frac{d}{dx}\left(sec\left(tan\left(\sqrt{x}\right)\right)\right)$

= sec (tan ($\sqrt{x}$)). tan (tan ($\sqrt{x}$)) $\left(\frac{d}{dx}\left(\tan\sqrt{x}\right)\right)$

= sec (tan ($\sqrt{x}$)). tan (tan ($\sqrt{x}$)). Sec^2 ($\sqrt{x}$).$\frac{d}{dx}\left(\sqrt{x}\right)$

= sec (tan ($\sqrt{x}$)).tan (tan ($\sqrt{x}$)).sec^2 ($\sqrt{x}$)).$\frac{1}{2\left(\sqrt{x}\right)}$

$= \frac{1}{2\left(\sqrt{x}\right)}$ (sec (tan ($\sqrt{x}$)).tan (tan ($\sqrt{x}$)).sec^2 ($\sqrt{x}$))

Q. 5Differentiate the functions with respect to x.

$\frac{\sin(ax+b)}{\cos(cx+d)}$

Answer:

Given: $\frac{\sin(ax+b)}{\cos(cx+d)}$

Let y = $\frac{\sin(ax+b)}{\cos(cx+d)}$

$= \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin(ax+b)}{\cos(cx+d0}\right)$

We know that $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vd(u)-ud(v)}{v^2}$

$= \frac{[\cos(cx+d).d(\sin(ax+b))-\sin(\mathrm{ax+b}).\mathrm{d}(\cos(cx+d))]}{[\cos(cx+d)]^2}$

$= \frac{[\cos(cx+d).(\cos(ax+b)).d(ax+b)-\sin(ax+b).(-\sin(cx+d)d(cx+d)]}{[\cos(cx+d]^2}$

$= \frac{[\cos(cx+d).(\cos(ax+b)).(a)-\sin(ax+b).(-\sin(cx+d)(c)]}{[\cos(cx+d)]^2}$

$= \frac{[\mathrm{a}\cos(cx+d)\cos(ax+b)]}{[\cos(cx+d)]^2} + \frac{[c\sin(cx+d)\sin(ax+b)}{[\cos(cx+d)]^2}$

$= \frac{[a\cos(ax+b)]}{[\cos(cx+d)]} + \frac{[c\sin(cx+d)\sin(ax+b)]}{[\cos(cx+d)]]\cos(cx+d)]}$

= a cos (ax + b) sec (cx + d) + c sin (ax + b) tan (cx + d) sec (cx + d)

Q. 6 Differentiate the functions with respect to x.

cos x^3. sin^2 (x^5)

Answer:

Given: cos x^3. sin^2 (x^5)

Let y = cos x^3. sin^2 (x^5)

$= \frac{dy}{dx} = \frac{d}{dx}$ (cos x^3.sin^2 (x^5))

We know that, $\frac{dy}{dx}$ (u. v) = u. d (v) + v. d (u)

= $\cos x^3 .\frac{d}{dx} \sin^2 (x^5) + \sin^2 (x^5). \frac{d}{dx} (\cos x^3)$

= $\cos x^3. 2 \sin (x^5). \left(\frac{d}{dx}\sin(x^5)\right) + \sin2 (x^5). (-\sin x^3). \left(\frac{d}{dx}x^3\right)$

= $\cos x^3. 2 \sin (x^5). \text{Cos} (x^5) \left(\frac{d}{dx}x^5\right) + \sin2 (x^5). (-\sin x^3). (3x^2)$

= $\cos x^3.2\sin(x^5). \cos(x^5) (5x^4) + \sin2(x^5). (-\sin x^3). (3x^2)$

= $10x^4.\cos x^3.\sin(x^5). \cos (x^5) - (3x^2). \sin2 (x^5). (\sin x^3)$

Q. 8 Differentiate the functions with respect to x.

$\cos(\sqrt{x})$

Answer:

Given: $\cos \sqrt{x}$

Let y = $\cos \sqrt{x}$

$= \frac{dy}{dx} = \frac{d}{dx}(\cos \sqrt{x})$

$= -\sin\left(\sqrt{x}\right). \left(\frac{d}{dx}\sqrt{x}\right)$

$= -\sin\left(\sqrt{x}\right). \frac{1}{2}. \left(x^{-\frac{1}{2}}\right)$

$= -\sin\left(\sqrt{x}\right). \frac{1}{2\sqrt{x}}$

$= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$

Q. 9 Prove that the function f given by f (x) = | x – 1|, x ∈ R is not differentiable at $x = 1$.

Answer:

Given: f(x)=|x-1|, x ∈R

because a function f is differentiable at a point x=c in its domain if both its limits as:

$\lim_{h\to 0^-} \frac{[f\ (c+h)-f(c)]}{h}$ and $\lim_{h\to 0^+} \frac{[f(c+h)-f(c)]}{h}$ are finite and equal.

Now, to check the differentiability of the given function at x=1,

Let we consider the left hand limit of function f at x=1

$= \lim_{h\to 0^-} \frac{[f(1+h)-f(1)]}{h}$

$= \lim_{h\to 0^-} [|1+h-1|-|1-1|]$

$= \lim_{h\to 0^-} \frac{[|h|-0]}{h}$

$= \lim_{h\to 0^-} \frac{[-h]}{h}$ because, {h < 0 ⇒ |h|= -h}

= -1

Now, let we consider the right hand limit of function f at x=1

$= \lim_{h\to 0^+} [f(1+h)-f(1)]$

$= \lim_{h\to 0^+} [|1+h-1|-|1-1|]$

$= \lim_{h\to 0^+} [|h|-0]$

$= \lim_{h\to 0^+} \frac{[h]}{h}$ because, {h>0 ⇒ |h|= h}

= 1

Because, left hand limit is not equal to right hand limit of function f at x=1, so f is not differentiable at x=1.

Q. 10 Prove that the greatest integer function defined by f (x) = [x], 0 < x < 3 is not differentiable at x = 1 and x = 2.

Answer:

Given: f(x) =[x], 0 < x <3

because a function f is differentiable at a point x=c in its domain if both its limits as:

$\lim_{h\to 0^-} \frac{[f(c+h)-f(c)]}{h}$ and $\lim_{h\to 0^+} \frac{[f\ (c+h)-f(c)]}{h}$ are finite and equal.

Now, to check the differentiability of the given function at x=1,

Let we consider the left-hand limit of function f at x=1

$= \lim_{h\to 0^-} \frac{[f(1+h)-f(1)]}{h}$

$= \lim_{h\to 0^-} \frac{[|1+h|-|1|]}{h}$

$= \lim_{h\to 0^-} \frac{[1+h-1-1]}{h}$

$= \lim_{h\to 0^-} \frac{[h-1]}{h}$ because, {h<0=> |h|= -h}

$= -\frac{1}{0} = \infty$

Let we consider the right hand limit of function f at x=1

$= \lim_{h\to 0^+} \frac{[f(1+h)-f(1)]}{h}$

$= \lim_{h\to 0^+} \frac{[1+h]-[1]}{h}$

$= \lim_{h\to 0^+} \frac{[1-1]}{h}$

$= \lim_{h\to 0^+} \frac{[0]}{h}$

= 0

Because, left hand limit is not equal to right hand limit of function f at x=1, so f is not differentiable at x=1.

Let we consider the left hand limit of function f at x=2

$$= \lim_{h \to 0^-} \frac{[f(2+h) - f(2)]}{h}$$

$$= \lim_{h \to 0^-} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^-} \frac{[2+h-1-2]}{h}$$

$$= \lim_{h \to 0^-} \frac{[h+1-2]}{h}$$

$$= -\frac{1}{0} = \infty$$

Now, let we consider the right hand limit of function f at x=2

$$= \lim_{h \to 0^+} \frac{[f(2+h) - f(2)]}{h}$$

$$= \lim_{h \to 0^+} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \to 0^+} \frac{[2-2]}{h}$$

$$= \lim_{h \to 0^+} \frac{[0]}{h}$$

$= 0$

Because, left hand limit is not equal to right hand limit of function f at x=2, so f is not differentiable at x=2.

Exercise 5.3

Q. 1n Find dy /dx in the following:

$2x + 3y = \sin x$

Answer:

It is given that $2x + 3y = \sin x$

Differentiating both sides w.r.t. x, we get,

$= \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin x)$

$= 2 + 3\frac{dy}{dx} = \cos x$

$= 3\frac{dy}{dx} = \cos x - 2$

$= \frac{dy}{dx} = \frac{\cos x - 2}{3}$

Q. 2 Find dy/dx in the following:

$2x + 3y = \sin y$

Answer:

It is given that $2x + 3y = \sin y$

Differentiating both sides w.r.t. x, we get,

$= \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y)$

$= 2 + 3\frac{dy}{dx} = \cos y\,\frac{dy}{dx}$

$= 2 = (\cos y - 3)\frac{dy}{dx}$

$= \frac{dy}{dx} = \frac{2}{(\cos y - 3)}$

Q. 3 Find dy/dx in the following:

$ax + by^2 = \cos y$

Answer:

It is given that $ax + by^2 = \cos y$

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}(ax + by^2) = \frac{d}{dx}(\cos y)$

$= \frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$

$= a + b\frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y)$

$= a + b \times 2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$

$= (2by + \sin y)\frac{dy}{dx} = -a$

$= \frac{dy}{dx} = \frac{-a}{(2by + \sin y)}$

Q. 4 Find dy/dx in the following:

$xy + y^2 = \tan x + y$

Answer:

It is given that $xy + y^2 = \tan x + y$

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)$

$= \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$

$= \left[y\frac{dy}{dx}(x) + x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 + \frac{dy}{dx}$

$= y.1 + x\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$

$= (x + 2y - 1)\frac{dy}{dx} = \sec 2 - y$

$= \frac{dy}{dx} = \frac{sec^2 x - y}{(x+2y-1)}$

Q. 5 Find dy/dx in the following:

$x^2 + xy + y^2 = 100$

Answer:

It is given that $x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)$

$= \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$

$= 2x + \left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$

$= 2x + y.1 + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$= 2x + y + (x + 2y)\frac{dy}{dx} = 0$

$= \frac{dy}{dx} = -\frac{2x+y}{x+2y}$

Q. 6 Find dy/dx in the following:

$x^3 + x^2y + xy^2 + y^3 = 81$

Answer:

It is given that $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) = \frac{d}{dx}(81)$

$= \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2\text{y}) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = 0$

$= 3x^2 + \left[y \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}\right] + \left[y^2 \frac{d}{dx}(x) + x \frac{d}{dx}(y^2)\right] + 3y^2 \frac{dy}{dx} = 0$

$= 3x^2 + \left[y.2x + x^2 \frac{dy}{dx}\right] + \left[y^2.1 + x.2y.\frac{dy}{dx}\right] + 3y^2 \frac{dy}{dx} = 0$

$= (x^2 + 2xy + 3y^2) \frac{dy}{dx} + (3x^2 + 2xy + y^2) = 0$

$= \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$

Q. 7

Find dy/dx in the following:

$\sin 2y + \cos xy = \pi$

Answer:

It is given that $\sin 2y + \cos xy = \pi$

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}(\sin 2y + \cos xy) = \frac{d}{dx}(\pi)$

$= 2 \sin y \cos y \frac{dy}{dx} - \sin xy \left[y \frac{d}{dx}(x) + x \frac{dy}{dx}\right]$

$= 2 \sin y \cos y \frac{dy}{dx} - \sin xy \left[y.1 + x \frac{dy}{dx}\right] = 0$

$= 2 \sin y \cos y \frac{dy}{dx} - y \sin xy - x \sin x y \frac{dy}{dx} = 0$

$= (2 \sin y \cos y - x \sin xy) \frac{dy}{dx} = y \sin xy$

$= (\sin 2y - x \sin xy) \frac{dy}{dx} = y \sin xy$

$= \frac{dy}{dx} = \frac{y \sin xy}{(\sin 2y - x \sin xy)}$

Q. 8

Find dy/dx in the following:

sin2 x + cos2 y = 1

Answer:

It is given that sin2 x + cos2 y = 1

Differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}$ (sin2x + cos2y) = $\frac{d}{dx}$ (1)

= $\frac{d}{dx}$ (sin2x) + $\frac{d}{dx}$ (cos2x) = 0

= 2 sin x. $\frac{d}{dx}$ (sin x) + 2 cos y. $\frac{d}{dx}$ (cos y) = 0

= 2 sin x cos x + 2 cos y (- sin y). $\frac{dy}{dx}$ = 0

= sin 2x – sin 2y $\frac{dy}{dx}$ = 0

= $\frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$

Q. 9

Find dy/dx in the following:

y = sin^{-1} $\left(\frac{2x}{1+x^2}\right)$

Answer:

Let x = tan A

then, A = tan^{-1}x

= $\frac{dA}{DX} = \frac{1}{1+x^2}$

y = sin^{-1} $\left(\frac{2\tan A}{1+tan^2 A}\right)$

also, we know $\left[\sin 2A = \frac{2\tan A}{1+tan^2 A}\right]$

And = y = sin^{-1} (sin 2A)

= y = 2A

= $\frac{dy}{dx} = 2\frac{dA}{dx}$ [by chain rule[

= $\frac{dy}{dx} = \frac{2}{1+x^2}$

Q. 10

Find dy/dx in the following:

y = $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Answer:

It is given that:

y = $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Assumption: Let x = tan θ, putting it in y, we get,

y = $\tan^{-1}\left(\frac{3\tan\theta-tan^3\theta}{1-3tan^2\theta}\right)$

we know by the formula that, tan $3x = \frac{3\tan x - tan^3 x}{1-3tan^2 x}$

Putting this in y, we get, $y = \tan^{-1}(\tan 3\theta)$

$y = 3(\tan^{-1}x)$

Differentiating both sides, we get, $\frac{dy}{dx} = \frac{3}{1+x^2}$

Q. 11 Find dy/dx in the following:

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer:

It is given that,

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= \cos y = \frac{1-x^2}{1+x^2}$

$= \frac{1-tan^2\frac{y}{2}}{1+tan^2\frac{y}{2}} = \frac{1-x^2}{1+x^2}$

On comparing both sides, we get,

$\tan\frac{y}{2} = x$

Now, differentiating both sides, we get,

$\text{Sec}^2\left(\frac{y}{2}\right).\frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$

$= \sec^2\frac{y}{2} \times \frac{1}{2}\frac{dy}{dx} = 1$

$= \frac{dy}{dx} = \frac{2}{sec^2\frac{y}{2}}$

$= \frac{dy}{dx} = \frac{2}{1+tan^2\frac{y}{2}}$

$= \frac{dy}{dx} = \frac{2}{1+x^2}$

Q. 12 Find dy/dx in the following: $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Answer:

It is given that $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= \sin y = \frac{1-x^2}{1+x^2}$

$= (1 + x^2) \sin y = 1 - x^2$

$= (1 + \sin y)\, x^2 = 1 - \sin y$

$= x^2 = \frac{1-\sin y}{1+\sin y}$

Now, we can change the numerator and the denominator,

$1 = \sin 2\,\frac{y}{2} + \cos 2\,\frac{y}{2}$

We know that we can write, and

Sin $y = 2\sin\frac{y}{2}$. Cos $\frac{y}{2}$

Therefore, by applying the formula: $(a + b)^2 = a^2 + b^2 + 2ab$ and $(a - b)^2 = a^2 + b^2 - 2ab$, we get,

$= x^2 = \frac{\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2} + \sin\frac{y}{2}\right)^2}$

$= x = \frac{\cos\frac{y}{2} - \sin\frac{y}{2}}{\cos\frac{y}{2} + \sin\frac{y}{2}}$

Dividing the numerator and denominator by cos (y/2), we get,

$= x\ \frac{1 - \tan\frac{y}{2}}{1 + \tan\frac{y}{2}}$

Now, we know that:

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

$= x \tan\left(\frac{\pi}{4} - \frac{y}{2}\right)$

Now, differentiating both sides, we get,

$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)\right)$

$= 1 = \sec^2\left(\frac{\pi}{2} - \frac{y}{2}\right) \times \frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$

$= 1 = \left[1 + tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\right] . \left(-\frac{1}{2}\frac{dy}{dx}\right)$

$= 1 = [1 + x^2] . \left(-\frac{1}{2}\frac{dy}{dx}\right)$

$= \frac{dy}{dx} = \frac{-2}{1+x^2}$

Q. 13

Find dy/dx in the following:

y = cos^{-1} $\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Answer:

It is given that y = cos^{-1} $\left(\frac{2x}{1+x^2}\right)$

= cos $y = \frac{2x}{1+x^2}$

Differentiating both sides w.r.t. x, we get,

$-\text{Sin } y \frac{dy}{dx} = \frac{(1+x^2).\frac{d}{dx}(2x)-2x.\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$

$= \sqrt{1-cos^2y}\frac{dy}{dx} = \frac{(1+x^2)\times 2-2x.2x}{(1+x^2)^2}$

$= \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\frac{dy}{dx} = \left[\frac{(1-x^2)}{(1+x^2)}\right]$

$= \sqrt{\frac{(1-x^2)^2-4x^2}{(1+x^2)^2}}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$

$= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$

$= \frac{1-x^2}{1+x^2}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$

$= \frac{dy}{dx} = \frac{-2}{1+x^2}$

Q. 14 Find dy/dx in the following:

y = sin^{-1} $\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Answer:

It is given that y = sin^{-1} $\left(2x\sqrt{1-x^2}\right)$

= sin y = 2x $\sqrt{1-x^2}$

Differentiating both sides w.r.t. x, we get,

Cos $y \frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dy}{dx}\right]$

= $\sqrt{1-sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1+x^2}\right]$

= $\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$

= $\sqrt{1-4x^2(1-x^2)}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$

= (1 – 2x2)$\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$

= $\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Q. 15 Find dy/dx in the following:

y = sec^{-1} $\left(\frac{1}{2x^2+1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Answer:

It is given that y = sec^{-1} $\left(\frac{1}{2x^2+1}\right)$

= sec $y = \frac{1}{2x^2+1}$

= cos $y = 2x^2 + 1$

= $2x^2 = 1 + \cos y$

$= 2x^2 = 2\cos 2\frac{y}{2}$

$= x = \cos\frac{y}{2}$

Differentiating w.r.t. x, we get,

$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos\frac{y}{2}\right)$

$= 1 = -\sin\frac{y}{2}.\frac{d}{dx}\left(\frac{y}{2}\right)$

$= \frac{-1}{\sin\frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$

$= \frac{dy}{dx} = \frac{-2}{\sin\frac{y}{2}} = \frac{-2}{\sqrt{1-cos^2\frac{y}{2}}}$

$= \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$

Exercise 5.4

Q. 1 Differentiate the following w.r.t. x: $\frac{e^x}{\sin x}$

Answer:

Let y = $\frac{e^x}{\sin x}$

By using the quotient rule, we get

$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}$

= $\frac{\sin x . e^x - e^x .(\cos x)}{\sin^2 x}$

= $\frac{e^x(\sin x - \cos x)}{\sin^2 x}$

Q. 2 Differentiate the following w.r.t. x:

$e^{\sin^{-1} x}$

Answer:

Let y = $e^{\sin^{-1} x}$

Now, by using the chain rule, we get,

$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin^{-1} x})$

= $\frac{dy}{dx} = e^{\sin^{-1} x} . \frac{d}{dx}(\sin^{-1} x)$

= $e^{\sin^{-1} x} . \frac{21}{\sqrt{1-x^2}}$

= $\frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$

Thus, $\frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$

Q. 3 Differentiate the following w.r.t. x:

e^{x^3}

Answer:

Let y = e^{x^3}

So, by using the chain rule, we get,

$\frac{dy}{dx} = \frac{d}{dx}(e^{x^3})$

= $e^{x^3} . \frac{d}{dx}(x^3)$

= $e^{x^3} . 3x^2$

= $3x^2 e^{x^3}$

Q. 4 Differentiate the following w.r.t. x:

sin ($\tan^{-1}$ e^{-x})

Answer:

Let y = sin ($\tan^{-1}$ e^{x})

So, by using chain rule, we get

$\frac{dy}{dx} = \frac{d}{dx}$ [sin ($\tan^{-1}$ e^{-x})]

= cos ($\tan^{-1}$ e^{-x}).$\frac{d}{dx}$ ($\tan^{-1}$ e^{-x})

= cos ($\tan^{-1}$ e^{-x}). $\frac{1}{1+(e^{-x})^2} . \frac{d}{dx}$ (e^{-x})

= $\frac{\cos(tan^{-1}e^{-x})}{1+e^{-2x}}$.e-x.$\frac{d}{dx}$ (-x)

= $\frac{e^{-x}\cos(tan^{-1}e^{-x})}{1+e^{-2x}} . (-1)$

= $\frac{-e^{-x}\cos(tan^{-1}e^{-x})}{1+e^{-2x}}$

Q. 5 Differentiate the following w.r.t. x:

$\log(\cos e^x)$

Answer:

Let $y = \log(\cos e^x)$

So, by using the chain rule, we get,

$\frac{dy}{dx} = \frac{d}{dx}(\log(\cos e^x))$

$= \frac{1}{\cos e^x} \cdot \frac{d}{dx}(\cos e^x)$

$= \frac{1}{\cos e^x} \cdot (-\sin e^x) \cdot \frac{d}{dx}(e^x)$

$= \frac{-\sin e^x}{\cos e^x} \cdot e^x$

$= -e^x \tan e^x$

Q. 6 Differentiate the following w.r.t. x:

$e^x + e^{x^2} + \cdots + e^{x^5}$

Answer:

Let $y = e^x + e^{x^2} + \cdots + e^{x^5}$

$= \frac{d}{dx}\left(e^x + e^{x^2} + \cdots + e^{x^5}\right)$

$= \frac{d}{dx}(e^x) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) + \frac{d}{dx}\left(e^{x^5}\right)$

$= e^x + e^{x^2}.2x + e^{x^3}.3x^2 + e^{x^4}.4x^3 + e^{x^5}.5x^4$

$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$

Q. 7 Differentiate the following w.r.t. x:

$\sqrt{e^{\sqrt{x}}}, x > 0$

Answer:

Let $y = \sqrt{e^{\sqrt{x}}}$

Then, $y^2 = e^{\sqrt{x}}$

Now, differentiating both sides we get,

$2y \frac{dy}{dx} = e^{\sqrt{x}} \frac{d}{dx}\left(\sqrt{x}\right)$

$= e^{\sqrt{x}} \frac{1}{2} . \frac{1}{\sqrt{x}}$

$= \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$

$= \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$

$= \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$

Q. 8 Differentiate the following w.r.t. x:

$\log^{(\log x)}$, $_x > 1$

Answer:

let $_y = \log(\log x)$

So, by using chain rule, we get,

$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x))$

$= \frac{1}{\log x} . \frac{d}{dx}(\log x)$

$= \frac{1}{\log x} . \frac{1}{x}$

$= \frac{1}{x \log x}$

Q. 9 Differentiate the following w.r.t. x:

$\frac{\cos x}{\log x}, x > 0$

Answer:

Let $y = \frac{\cos x}{\log x}, x > 0$

So, by using the quotient rule, we get,

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x)\times\log x - \cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{-\sin x log\ x - \cos x\times\frac{1}{x}}{(\log x)^2}$$

$$= \frac{-[x\log x.\sin x + \cos x]}{x(\log x)^2}$$

Q. 10 Differentiate the following w.r.t. x:

cos (log x + e^x), x > 0

Answer:

Let y = cos (log$x$ + e^x)

So, by using chain rule, we get,

$$\frac{dy}{dx} = -\sin(\log x + \ e^x).\frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x.\left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)\right]$$

$$= -\sin(\log x + \ e^x).\left(\frac{1}{x} + e^x\right)$$

$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x)$$

Exercise 5.5

Q. 1 Differentiate the functions given in w.r.t. x.

$\cos x. \cos 2x. \cos 3x$

Answer:

Given: $\cos x. \cos 2x. \cos 3x$

Let $y = \cos x. \cos 2x. \cos 3x$

Taking log on both sides, we get

$\log y = \log(\cos x. \cos 2x. \cos 3x)$

$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$

Now, differentiate both sides with respect to x

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\log(\cos x) + \frac{d}{dx}\log(\cos 2x) + \frac{d}{dx}(\log\cos 3x)$$

$$= \frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}.\frac{d}{dx}(\cos x) + \frac{1}{\cos 2x}.\frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x}\frac{d}{dx}(\cos 3x)$$

$$= \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}.\frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x}\frac{d}{dx}(3x)\right]$$

$$= \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x[\tan x + \tan 2x\ (2) + \tan 3x(3)]$$

$$= \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x[\tan x + 2\tan 2x + 3\tan 3x]$$

Q. 2 Differentiate the functions given in w.r.t. x.

$(\log x)^{\cos x}$

Answer:

Given: $(\log x)^{\cos x}$

Let $y = (\log x)^{\cos x}$

Taking log on both sides, we get

$\log y = \log (\log x)^{\cos x}$

$\Rightarrow \log y = \cos x. \log (\log x)$

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log y) = \frac{d}{dx}[\cos x. \log(\log x)]$

$= \frac{1}{y}\frac{dy}{dx} = \cos x. \frac{d}{dx}(\log(\log x)) + \log(\log x). \frac{d}{dx}(\cos x)$

$= \frac{dy}{dx} = y\left[\cos x. \frac{1}{\log x}. \frac{d}{dx}(\log x) + \log(\log x). (-\sin x)\right]$

$= \frac{dy}{dx} = (\log x)^{\cos x}\left[\cos x. \frac{1}{\log x}. \frac{1}{x} + \log(\log x). (-\sin x)\right]$

$= \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x.\log x} - (\sin x). \log(\log x)\right]$

Q. 4 Differentiate the functions given in w.r.t. x.

$x^x - 2^{\sin x}$

Answer:

Given: $x^x - 2^{\sin x}$

Let $y = x^x - 2^{\sin x}$

Let y = u - v

$\Rightarrow$ u = xx and v = $2^{\sin} x$

For, u = x^x

Taking log on both sides, we get

$\log u = \log x^x$

$\Rightarrow \log u = x. \log(x)$

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log u) = \frac{d}{dx}[x.\log(x)]$

$= \frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(x)$

$= \frac{du}{dx} = u\left[x.\frac{1}{x} + \log x.(1)\right]$

$= \frac{du}{dx} = x^x(1 + \log x)$

For, v = 2sin x

Taking log on both sides, we get

log v = log $2^{\sin x}$

⇒log v = sin x. log (2)

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log v) = \frac{d}{dx}[\sin x.\log(2)]$

$= \frac{1}{v}\frac{dv}{dx} = \log 2.\frac{d}{dx}(\sin x)$

$= \frac{dv}{dx} = v[\log 2.(\cos x)]$

$= \frac{dv}{dx} = 2^{\sin x}.\cos xlog\ 2$

Because, y = u - v

$= \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

dy/dx = x^x (1 + log$x$) - 2sin$x$.cos$x$.log2

Q. 5 Differentiate the functions given in w.r.t. x.

$(x + 3)^2. (x + 4)^3. (x + 5)^4$

Answer:

Given: $(x + 3)^2. (x + 4)^3. (x + 5)^4$

Let y= $(x + 3)^2 . (x + 4)^3 . (x + 5)^4$

Taking log on both sides, we get

$\log y = \log ((x + 3)^2 . (x + 4)^3 . (x + 5)^4)$

$\Rightarrow \log y = \log (x + 3)^2 + \log (x + 4)^3 + \log (x + 5)^4$

$\Rightarrow \log y = 2.\log (x + 3) + 3.\log (x + 4) + 4.\log (x + 5)^4$

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log y) = \frac{d}{dx}(2.\log(x+3)) + \frac{d}{dx}(3.\log(x+4)) + \frac{d}{dx}(4.\log(x+5))$

$= \frac{1}{y}\frac{dy}{dx} = 2.\frac{1}{x+3}.\frac{d}{dx}(x+3) + 3.\frac{1}{x+4}.\frac{d}{dx}(x+4) + 4.\frac{1}{x+5}.\frac{d}{dx}(x+5)$

$= \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$

$= \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$

$= \frac{dy}{dx} = (x+3)^1(x+4)^2(x+5)^3[2(x^2+9x+20) + 3(x^2+8x+15) + 4(x^2+7x+12)]$

$= (x + 3)(x + 4)^2 (x + 5)^3 (9x^2 + 70x + 133)$

Q. 6 Differentiate the functions given in w.r.t. x.

$\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Answer:

Given: $\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Let y= $\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

Also, Let y = u + v

$= u\left(x+\frac{1}{x}\right)^{x}$ and v = $x^{\left(1+\frac{1}{x}\right)}$

for, u = $\left(x+\frac{1}{x}\right)^{x}$

Taking log on both sides, we get

log u = log $\left(x+\frac{1}{x}\right)^{x}$

= log u = x. log $\left(x+\frac{1}{x}\right)$

Now, differentiate both sides with respect to x

$$\frac{d}{dx}(\log u)=\frac{d}{dx}\left[x.\log\left(x+\frac{1}{x}\right)\right]$$

$$=\frac{1}{u}-\frac{du}{dx}=x.\frac{d}{dx}\left(\log\left(x+\frac{1}{x}\right)\right)+\log\left(x+\frac{1}{x}\right).\frac{d}{dx}(x)$$

$$=\frac{du}{dx}=u\left[x.\frac{1}{\left(x+\frac{1}{x}\right)}.\frac{d}{dx}\left(x+\frac{1}{x}\right)+\log\left(x+\frac{1}{x}\right)\right]$$

$$=\frac{du}{dx}=u\left[x.\frac{1}{\left(x+\frac{1}{x}\right)}.\left(\frac{dx}{dx}+\frac{d}{dx}\left(\frac{1}{x}\right)\right)+\log\left(x+\frac{1}{x}\right)\right]$$

$$=\frac{du}{dx}=u\left[\frac{x}{\left(x+\frac{1}{x}\right)}.\left(1-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\right]$$

$$=\frac{du}{dx}=u\left[\frac{x}{\left(x+\frac{1}{x}\right)}.\left(\frac{x^2-1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\right]$$

$$=\frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\left(\frac{x^2-1}{x^2+1}\right)+\log\left(x+\frac{1}{x}\right)\right]$$

for, v = $x^{\left(1+\frac{1}{x}\right)}$

Taking log on both sides, we get

log v = log $x^{\left(1+\frac{1}{x}\right)}$

$= \log v = \left(1 + \frac{1}{x}\right). \log x$

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log v) = \frac{d}{dx}\left[\left(1 + \frac{1}{x}\right). \log x\right]$

$= \frac{1}{v}\frac{dv}{dx} = \log x. \frac{d}{dx}\left(1 + \frac{1}{x}\right) + \left(1 + \frac{1}{x}\right). \frac{d}{dx}(\log x)$

$= \frac{dv}{dx} = v\left[\log x. \left(0 - \frac{1}{x^2}\right) + \left(1 + \frac{1}{x}\right). \frac{1}{x}\right]$

$= \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[-\frac{\log x}{x^2} + \left(\frac{1}{x} + \frac{1}{x^2}\right)\right]$

$= \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\frac{-\log x + x + 1}{x^2}\right]$

$= \frac{dv}{dx} = x^{\left(1+\frac{1}{x}\right)}\left[\frac{x + 1 - \log x}{x^2}\right]$

Because, y = u + v

$= \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x\left[\left(\frac{x^2 - 1}{x^2 + 1}\right) + \log\left(x + \frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left[\frac{x + 1 - \log x}{x^2}\right]$

Q. 7 Differentiate the functions given in w.r.t. x.

$(\log x)^{x} + x^{\log x}$

Answer:

Given: $(\log x)^{x} + x^{\log x}$

Let y= $(\log x)\, x + x^{\log x}$

Let y = u + v

⇒ u = $(\log x)^{x}$ and v = $x^{\log x}$

For, u = $(\log x)^{x}$

Taking log on both sides, we get

log u =log (log x) ^{x}

⇒log u = x.log (log(x))

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log u) = \frac{d}{dx}[x.\log(\log x)]$

$= \frac{1}{u} - \frac{du}{dx} = x.\frac{d}{dx}\log(\log x)) + \log(\log x).\frac{d}{dx}(x)$

$= \frac{du}{dx} = u\left[x.\frac{1}{\log x}\frac{d}{dx}(\log x) + \log(\log x).(1)\right]$

$= \frac{du}{dx} = (\log x)^{x}\left[\frac{x}{\log x}.\frac{1}{x} + \log(\log x).(1_\right]$

$= \frac{du}{dx} = (\log x)^{x}\left[\frac{1+\log(\log x).(\log x)}{\log x}\right]$

$= \frac{du}{dx} = (\log x)^{x-1}[1 + \log x.\log(\log x)]$

For, v = x $^{\log x}$

Taking log on both sides, we get

log v =log (x log x)

⇒log v = log x. log x

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log v) = \frac{d}{dx}[(\log x)^{2}]$

$= \frac{1}{v}\frac{dv}{dx} = 2.\log x\frac{d}{dx}(\log x)$

$= \frac{dv}{dx} = v\left[2.\frac{\log x}{x}\right]$

$= \frac{dv}{dx} = x^{\log x}\left[2.\frac{\log x}{x}\right]$

$= \frac{dv}{dx} = 2.x^{\log x-1}.\log x$

Because, y = u + v

$= \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= \frac{dy}{dx} = (\log x)^{x-1}[1 + \log x.\log(\log x)] + 2.x^{\log x-1}.\log x$

Q. 8 Differentiate the functions given in w.r.t. x.

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Answer:

Given: $(\sin x)^x + s\,i\,\bar{n}^1\sqrt{x}$

Let y= $(\sin x)^x + s\,i\,\bar{n}^1\sqrt{x}$

Let y = u + v

= u = $(\sin x)^x$ and v = $s\,i\,\bar{n}^1\sqrt{x}$

for, u = $(\sin x)^x$

Taking log on both sides, we get

log u = log $(\sin x)^x$

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log u) = \frac{d}{dx}[x.\log(\sin x)$

$= \frac{1}{u}\frac{du}{dx} = x.\frac{d}{dx}\log(\sin x)) + \log(\sin x).\frac{d}{dx}(x)$

$= \frac{du}{dx} = u\left[x.\frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log(\sin x).(1)\right]$

$= \frac{dy}{dx} = (\sin x)^x\left[\frac{x}{\sin x}.\cos x + \log(\sin x).(1)\right]$

$= \frac{dy}{dx} = (\sin x)^x[x.\cot x + \log\sin x]$

for, v = $sin^{-1}\sqrt{x}$

Now, differentiate both sides with respect to x

$= \frac{dv}{dx} = \frac{d}{dx}\left[sin^{-1}\sqrt{x}\right]$

$= \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}\left(\sqrt{x}\right)$

$= \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2(\sqrt{x})}$

$= \frac{dv}{dx} =$

$\frac{1}{2\sqrt{x-x^2}}$Because, y = u + v

$= \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= \frac{dy}{dx} = (\sin x)^x\,[x.\cot x + \log \sin x] + \frac{1}{2\sqrt{x-x^2}}$

Q. 9 Differentiate the functions given in w.r.t. x.

$x^{\sin x} + (\sin x)^{\cos x}$

Answer:

Given: $x^{\sin x} + (\sin x)^{\cos x}$

Let y= $x^{\sin x} + (\sin x)^{\cos x}$

Let y = u + v

⇒ u = $x^{\sin x}$ and v = $(\sin x)^{\cos x}$

For, u = $x^{\sin x}$

Taking log on both sides, we get

log u =log (x sin x)

⇒log u = sin x.log(x)

Now, differentiate both sides with respect to x

$=\frac{d}{dx}(\log u) = \frac{d}{dx}[\sin x . \log x$

$= \frac{1}{u}\frac{du}{dx} = \sin x . \frac{d}{dx}(\log x) + \log x . \frac{d}{dx}(\sin x)$

$= \frac{du}{dx} = u\left[\sin x . \frac{1}{x} + \log x . \cos x\right]$

$= \frac{du}{dx} = (x)^{\sin x}\left[\frac{\sin x}{x} + \log x . \cos x\right]$

For, v = $(\sin x)^{\cos x}$

Taking log on both sides, we get

log v =log $(\sin x)^{\cos x}$

⇒log v = cos x. log (sin x)

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log v) = \frac{d}{dx}[\cos x . \log(\sin x)]$

$= \frac{1}{v}\frac{dv}{dx} = \cos x . \frac{d}{dx}\log(\sin x) + \log \sin x . \frac{d}{dx}(\cos x)$

$= \frac{dv}{dx} = v\left[\cos x . \frac{1}{\sin x} . \frac{d}{dx}(\sin x) + \log(\sin x) . (-\sin x)\right]$

$= \frac{dv}{dx} = (\sin x)^{\cos x}\left[\frac{\cos x}{\sin x} . \cos x + \log \sin x . (-\sin x)\right]$

$= \frac{dy}{dx} = (\sin x)^{\cos x}[\cot x . \cos x - \sin x . \log \sin x]$Because, y = u + v

$= \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= \frac{dy}{dx} = (x)^{\sin x}\left[\frac{\sin x}{x} + \log x . \cos x\right] + (\sin x)^{\cos x}[\cot x . \cos x - \sin x . \log \sin x]$

Q. 10 Differentiate the functions given in w.r.t. x.

$x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Answer:

Given: $x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Let y= $x^{x\cos x}+\frac{x^2+1}{x^2-1}$

Let y = u + v

= u = $x^{x\cos x}$ and v = $\frac{x^2+1}{x^2-1}$

for, u = $x^{x cos\, x}$

Taking log on both sides, we get

log u = log $x^{x\cos x}$

⇒log u = x. cos x.log

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log u)=\frac{d}{dx}[x.\cos x.\log x]$

= $\frac{1}{u}\frac{du}{dx}=\cos x log\ x.\frac{d}{dx}(x)+x.\log x.\frac{d}{dx}(\cos x)+x.\cos x.\frac{d}{dx}(\log x)$

= $\frac{du}{dx}=u\left[\cos x.\log x+x.\log x(-\sin x)+x.\cos x.\left(\frac{1}{x}\right)\right]$

= $\frac{du}{dx}=x^{x\cos x}[\cos x.\log x-x.\log x.\sin x+\cos x]$

= $\frac{dy}{dx}=x^{x\cos x}[\cos x(1+\log x)-x.\log x.\sin x]$

for, v = $\frac{x^2+1}{x^2-1}$

Taking log on both sides, we get

log v = log $\left(\frac{x^2+1}{x^2-1}\right)$

⇒ log v = log $(x^2 + 1)$ – log $(x^2 – 1)$

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log v) = \frac{d}{dx}[\log(x^2+1) - \log(x^2-1)]$

$= \frac{1}{v}\frac{dy}{dx} = \frac{1}{x^2+1}.\frac{d}{dx}(x^2) - \frac{1}{x^2-1}.\frac{d}{dx}(x^2)$

$= \frac{dy}{dx} = v.\left[\frac{1}{x^2+1}.(2x) - \frac{1}{x^2-1}.(2x)\right]$

$= \frac{dy}{dx} = \left(\frac{x^2+1}{x^2-1}\right).\left[\frac{2x(x^2-1)-2x(x^2+1)}{(x^2+1)(x^2-1)}\right]$

$= \frac{dy}{dx} = \left(\frac{x^2+1}{x^2-1}\right).\left[\frac{-4x}{(x^2+1)(x^2-1)}\right]$

$= \frac{dy}{dx} = \left[\frac{-4x}{(x^2-1)^2}\right]$

Because, y = u + v

$= \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= \frac{dy}{dx} = x^{x\cos x}[\cos x(1+\log x) - x.\log x.\sin x] - \left[\frac{4x}{(x^2-1)^2}\right]$

Q. 11 Differentiate the functions given in w.r.t. x.

$(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Answer:

Given: $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Let y= $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Let y = u + v

= u = $(x\cos x)^x$ and v = $(x\sin x)^{\frac{1}{x}}$

for, u = $(x\cos x)^x$

Taking log on both sides, we get

$\log u = \log (x \cos x)^{x}$

$\Rightarrow \log u = x. \log (x \cos x)$

$\Rightarrow \log u = x (\log x + \log (\cos x))$

$\Rightarrow \log u = x (\log x) + x (\log (\cos x))$

Now, differentiate both sides with respect to x

$= \frac{dy}{dx}(\log x) = \frac{d}{dx}[x.\log(x)] + \frac{d}{dx}[x.\log(\cos x)]$

$= \frac{1}{u}\frac{du}{dx} = \left\{x.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(x)\right\} + \left\{x.\frac{d}{dx}(\log\cos x) + \log\cos x.\frac{d}{dx}(x)\right\}$

$= \frac{du}{dx} = u\ \left[\left\{x.\frac{1}{x} + \log x.(1)\right\} + \left\{x.\frac{1}{\cos x}.\frac{d}{dx}(\cos x) + \log\cos x.(1)\right\}\right]$

Taking log on both sides, we get

$\log v = \log (x \sin x)^{\frac{1}{x}}$

Now, differentiate both sides with respect to x

$\frac{d}{dx}(\log v) = \frac{d}{dx}\left[\frac{1}{x}.(\log x)\right] + \frac{d}{dx}\left[\frac{1}{x}.\log(\sin x)\right]$

$= \frac{1}{v}\frac{dy}{dx} = \left\{\frac{1}{x}.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}\left(\frac{1}{x}\right)\right\} + \left\{\frac{1}{x}.\frac{d}{dx}(\log\sin x) + \log\sin x.\frac{d}{dx}\left(\frac{1}{x}\right)\right\}$

$= \frac{dy}{dx} = v\left[\left\{\frac{1}{x}.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}\left(\frac{1}{x}\right)\right\} + \left\{\frac{1}{x}.\frac{d}{dx}(\log\sin x) + \log\sin x.\frac{d}{dx}\left(\frac{1}{x}\right)\right\}\right]$

$= \frac{dy}{dx} = (x\sin x)^{\frac{1}{x}}\left[\left\{\frac{1}{x^2}(1 - \log x)\right\} + \left\{\frac{\cos x}{x.\sin x}. -\frac{\log\sin x}{x^2}\right\}\right]$

$= \frac{dy}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1-\log x}{x^2} + \frac{\cot x}{x} - \frac{\log \sin x}{x^2}\right]$

$= \frac{dy}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1-\log x + x \cot x - \log \sin x}{x^2}\right]$

$= \frac{dy}{dx} = (x \sin x)^{\frac{1}{x}} \left[\frac{1 + x \cot x - \log(x.\sin x)}{x^2}\right]$

$= \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$= \frac{dy}{dx} = (x \cos x)^{x} [1 - x.\tan x + \log(x.\cos x)] +$
$(x \sin x)^{\frac{1}{x}} \left[\frac{1 + x \cot x - \log(x.\sin x)}{x^2}\right]$

Q. 12 Find dy/dx of the functions.

$x^y + y^x = 1$

Answer:

Given: $x^y + y^x = 1$

Let y= $x^y + y^x = 1$

Let u = x^y and v = y^x

Then, ⇒ u + v = 1

$= \frac{du}{dx} + \frac{dv}{dx} = 0$

For, u = xy

Taking log on both sides, we get

Log u =log xy

⇒log u = y.log(x)

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log u) = \frac{d}{dx}[y.\log(x)]$

$= \frac{1}{u}\frac{du}{dx} = \left\{y.\frac{d}{dx}(\log x) + \log x.\frac{d}{dx}(y)\right\}$

$= \frac{du}{dx} = u\left[y.\frac{1}{x} + \log x.\left(\frac{dy}{dx}\right)\right]$

$= \frac{dy}{dx} = x^y\left[\frac{y}{x} + \log x.\left(\frac{dy}{dx}\right)\right]$

For, v = y^x

Taking log on both sides, we get

Log v =log y^x

⇒log v = x.log(y)

Now, differentiate both sides with respect to x

$= \frac{d}{dx}(\log v) = \frac{d}{dx}[x.\log(y)]$

$= \frac{1}{v}\frac{dv}{dx} = \left\{x.\frac{d}{dx}(\log y) + \log y.\frac{d}{dx}x\right\}$

$= \frac{dv}{dx} = v\left[x.\frac{1}{y}.\frac{dy}{dx} + \log y.\left(\frac{dy}{dx}\right)\right]$

$= \frac{dy}{dx} = y^x\left[\frac{x}{y}.\frac{dy}{dx} + \log y\right]$

because, $\frac{du}{dx} + \frac{dv}{dx} = 0$

so, $x^y\left[\frac{y}{x} + \log x.\left(\frac{dy}{dx}\right)\right] + y^x\left[\frac{x}{y}.\frac{dy}{dx} + \log y\right] = 0$

$= (x^y \log x + xy^{x-1}).\frac{dy}{dx} + (yx^{y-1} + y^x \log y) = 0$

$= (x^y \log x + xy^{x-1}).\frac{dy}{dx} = -(yx^{y-1} + y^x \log y)$

$= \frac{dy}{dx} = -\frac{(yx^{y-1} + y^x \log y)}{(x^y \log x + xy^{x-1})}$

Q. 13 Find dy/dx of the functions.

$y^x = x^y$

Answer:

Given: $y^x = x^y$

Taking log on both sides, we get

log yx =log x^y

⇒x log y = y log x

Now, differentiate both sides with respect to x

$$x.\frac{d}{dx}\log y + \log y.\frac{d}{dx}x = y.\frac{d}{dx}\log x + \log x.\frac{d}{dx}y$$

$$x.\frac{1}{y}.\frac{dy}{dx} + \log y.(1) = y.\frac{1}{x} + \log x.\frac{dy}{dx}$$

$$\frac{x}{y}.\frac{dy}{dx} - \log x.\frac{dy}{dx} = y.\frac{1}{x} - \log y$$

$$= \frac{dy}{dx}\left(\frac{x}{y} - \log x\right) = \frac{y - x\log y}{x}$$

$$= \frac{dy}{dx}\left(\frac{x - y\log x}{y}\right) = \frac{y - x\log y}{x}$$

$$= \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

Q. 14 Find dy/dx of the functions.

(cos x) ^{y} = (cos y) ^{x}

Answer:

Given: (cos x) ^{y} = (cos y) x

Taking log on both sides, we get

log (cos x) ^{y} =log (cos y) ^{x}

⇒y log (cos x) = x log (cos y)

Now, differentiate both sides with respect to x

$y.\frac{d}{dx}\log(\cos x)+\log(\cos x).\frac{d}{dx}y=x.\frac{d}{dx}\log(\cos y)+\log\cos y.\frac{d}{dx}x$

$= y.\frac{1}{\cos x}.\frac{d}{dx}(\cos x)+\log(\cos x).\frac{dy}{dx}=x.\frac{1}{\cos y}.\frac{d}{dx}(\cos y)+$
$\log(\cos y).\frac{dy}{dx}$

$=\frac{y}{\cos x}.(-\sin x)+\log(\cos x).\frac{dy}{dx}=\frac{x}{\cos y}.(-\sin y).\frac{dy}{dx}+$
$\log(\cos y).(1)$

$=\frac{dy}{dx}\left(\frac{x.\sin y}{\cos y}+\log(\cos x)\right)=y.\frac{\sin x}{\cos x}+\log(\cos y)$

$=\frac{dy}{dx}(x\tan x+\log(\cos x))=y.\tan x+\log(\cos y)$

$=\frac{dy}{dx}=\left(\frac{y.\tan x+\log(\cos y)}{x.\tan x+\log(\cos x)}\right)$

Q. 15 Find dy/dx of the functions.

$xy = e^{(x-y)}$

Answer:

Given: $xy = e^{(x-y)}$

Taking log on both sides, we get

$\log(xy) = \log(e(x-y))$

$\Rightarrow \log x + \log y = (x - y)\log e$

$\Rightarrow \log x + \log y = (x - y).1$

$\Rightarrow \log x + \log y = (x - y)$

Now, differentiate both sides with respect to x

$\frac{d}{dx}\log x+\frac{d}{dx}\log y=\frac{d}{dx}x-\frac{d}{dx}y$

$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$

$\left(1+\frac{1}{y}\right)\frac{dy}{dx} = 1 - \frac{1}{x}$

$\frac{1+y}{y}\frac{dy}{dx} = \frac{x-1}{x}$

$\frac{dy}{dx} = \frac{y(x-1)}{x(1+y)}$

Q. 16 Find the derivative of the function given by f (x) = (1 + x) (1 + x^2) (1 + x^4) (1 + x^8) and hence find f ′ (1).

Answer:

Given: f (x) = (1 + x) (1 + x^2) (1 + x^4) (1 + x^8)

Taking log on both sides, we get

log f (x) =log (1 + x) + log (1 + x^2) + log (1 + x^4) + log (1 + x^8)

Now, differentiate both sides with respect to x

$\frac{d}{dx}\log f(x) = \frac{d}{dx}\log(1+x) + \frac{d}{dx}\log(1+x^2) + \frac{d}{dx}\log(1+x^4) + \frac{d}{dx}\log(1+x^8)$

$= \frac{1}{f(x)}.\frac{d}{dx}[f(x)]$

$= \frac{1}{1+x}.\frac{d}{dx}(1+x) + \frac{1}{1+x^2}\frac{d}{dx}(1+x^2) + \frac{1}{1+x^4}\frac{d}{dx}(1+x^4) + \frac{1}{1+x^8}\frac{d}{dx}(1+x^8)$

= f' (x) = f (x) = $\left[\frac{1}{1+x} + \frac{1}{1+x^2}.(2x) + \frac{1}{1+x^4}.(4x^3) + \frac{1}{1+x^8}(8x^7)\right]$

$= f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$

$= f'(x) = (1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1} + \frac{2(1)}{1+1} + \frac{4(1)^3}{1+(1)^4} + \frac{8(1)^7}{1+(1)^8}\right]$

$= f'(1) = (2)(2)(2)(2)\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$

$= f'(1) = 16\left[\frac{1+2+4+8}{2}\right]$

$= f'(1) = 16\left(\frac{15}{2}\right)$

$= f'(1) = 120$

Q. 17 Differentiate $(x^2 – 5x + 8)$ $(x^3 + 7x + 9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

Answer:

Given: $(x^2 – 5x + 8)$ $(x^3 + 7x + 9)$

Let $y = (x^2 – 5x + 8)$ $(x^3 + 7x + 9)$

(i) By applying product rule differentiate both sides with respect to x

$\frac{dy}{dx} = \frac{dy}{dx}(x^2 - 5x + 8)(x^3 + 7x + 9)$

$= \frac{dy}{dx} = (x^3 + 7x + 9).\frac{d}{dx}(x^2 - 5x + 8) + (x^2 - 5x + 8).\frac{d}{dx}(x^3 + 7x + 9)$

$= \frac{dy}{dx} = (x^3 + 7x + 9).(2x - 5) + (x^2 - 5x + 8).(3x^2 + 7)$

$= \frac{dy}{dx} = 2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56$

$= \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \ldots..(1)$

(ii) by expanding the product to obtain a single polynomial

y = (x^2 – 5x + 8) (x^3 + 7x + 9)

y = x^5 + 7x^3 + 9x^2 - 5x^4 – 35x^2 - 45x + 8x^3 + 56x + 72

y = x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72

Now, differentiate both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) - \frac{d}{dx}(5x^4) + \frac{d}{dx}(15x^3) - \frac{d}{dx}(26x^2) + \frac{d}{dx}(11x) + \frac{d}{dx}(72)$$

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \ldots (2)$$

(iii) by logarithmic differentiation

y = (x^2 – 5x + 8) (x^3 + 7x + 9)

Taking log on both sides, we get

log y = log ((x^2 – 5x + 8) (x^3 + 7x + 9))

log y = log (x^2 – 5x + 8) + log (x^3 + 7x + 9)

Now, differentiate both sides with respect to x

$$\frac{dy}{dx}(\log y) = \frac{d}{dx}\log(x^2 - 5x + 8) + \frac{d}{dx}\log(x^3 + 7x + 9)$$

$$= \frac{1}{y}\frac{d}{dx}(y) = \left[\frac{1}{(x^2-5x+8)}.\frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{(x^3+7x+9)}.\frac{d}{dx}(x^3 + 7x + 9)\right]$$

$$= \frac{1}{y}\frac{d}{dx}(y) = \left[\frac{1}{(x^2-5x+8)}.(2x - 5) + \frac{1}{(x^3+7x+9)}.(3x^2 + 7)\right]$$

$$= \frac{d}{dx}(y) = y.\left[\frac{(2x-5)}{(x^2-5x+8)} + \frac{(3x^2+7)}{(x^3+7x+9)}\right]$$

$$= \frac{d}{dx}(y) = y.\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+9)}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$= \frac{d}{dx}(y) = y.\left[\frac{2x^4+14x^2+18x-5x^3-35x-45+3x^4-15x^3+24x^2+7x^2-35x+56}{(x \quad -5x+8)(x \quad +7x+9)}\right]$

$= \frac{d}{dx}(y) = (x^2 - 5x + 8)(x^3 + 7x + 9).\left[\frac{5x^4-20x^3-45x^2-52x+11}{(x^2-5x+8)(x^3+7x+9)}\right]$

$= \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11 \ldots ..(3)$

From equation (i), (ii) and (iii), we can say that value of given function after differentiating by all the three methods is same.

Q. 18 If u, v and w are functions of x, then show that

$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

in two ways – first by repeated application of product rule, second by logarithmic differentiation.

Answer:

To prove: $\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

Let y=u. v. w=u. (v. w)

(a) by applying product rule differentiate both sides with respect to x

$\frac{dy}{dx} = (v.w).\frac{du}{dx} + u.\frac{d}{dx}(v.w)$

$= \frac{dy}{dx} = (v.w).\frac{du}{dx} + u.\left[v.\frac{d}{dx}(w) + w.\frac{d}{dx}(v)\right]$

$= \frac{dy}{dx} = (v.w).\frac{du}{dx} + (u.v).\frac{dw}{dx} + (u.w.).\frac{dv}{dx}$

(b) Taking log on both sides, we get

as, y=u. v. w

log y = log (u. v. w)

log y = log u + log v + log w

Now, differentiate both sides with respect to x

$$= \frac{d}{dx}(\log y) = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$$

$$= \frac{1}{y}.\frac{d}{dx}(y) = \frac{1}{u}.\frac{d}{dx}(u) + \frac{1}{v}.\frac{d}{dx}(v) + \frac{1}{w}.\frac{d}{dx}(w)$$

$$= \frac{dy}{dx}(y) = y\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}.\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx}\right]$$

$$= \frac{dy}{dx} = u.v.w\left[\frac{1}{u}.\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}.\frac{dw}{dx}\right]$$

$$= \frac{dy}{dx} = v.w.\frac{du}{dx} + u.w.\frac{dv}{dx} + u.v.\frac{dw}{dx}$$

From equation (i), (ii) and (iii), we can say that value of given function after differentiating by all the three methods is same.

Exercise 5.6

Q. 1 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

$x = 2at^2$, $y = at^4$

Answer:

It is given that

$x = 2at^2$, $y = at^4$

So, now

$$\frac{dx}{dt} = \frac{d(2at^2)}{dt}$$

$$= 2a\,\frac{d(t^2)}{dt}$$

$= 2a.2t$

$= 4at$ ………… (1)

And

$$\frac{dy}{dt} = \frac{d(at^4)}{dt}$$

$$= a\,\frac{d(t^4)}{dt}$$

$= a.4.t^3$

$= 4at^3$………… (2)

Therefore, form equation (1) and (2). we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$$

Hence, the value of $\frac{dy}{dx}$ is t^2

Q. 2 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

$x = a \cos \theta, y = b \cos \theta$

Answer:

It is given that

$x = a \cos \theta, y = b \cos \theta$

Then, we have

$$\frac{dx}{d\theta} = \frac{d(a\cos \theta)}{d\theta}$$

$= a(-\sin\theta)$

$= -a \sin \theta$………… (1)

$$\frac{dy}{d\theta} = \frac{d(b \cos\theta)}{d\theta}$$

$= b (-\sin \theta)$

$= -b \sin \theta$ …… (2)

From equation (1) and (2), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b \sin \theta}{-a \sin \theta} = \frac{b}{a}$$

Hence, the value of $\frac{dy}{dx}$ is $\frac{b}{a}$

Q. 3 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

$x = \sin t, y = \cos 2t$

Answer:

It is given that

$x = \sin t, y = \cos 2t$

Then, we have

$\frac{dx}{dt} = \frac{d(\sin t)}{dt}$

= cost ………… (1)

$\frac{dy}{dx} = \frac{d(\cos 2t)}{dt} = -\sin 2t \frac{d(2t)}{dt}$

= -2sin2t………… (2)

So, equation (1) and (2), we get

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{\cos t}$

= $\frac{-2.2\sin t\cos t}{\cos t}$, Since sin2t = 2sintcost

= -4sint

Hence, the value of $\frac{dy}{dx}$ is -4sint

Q. 4 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

x = 4t, y = 4/t

Answer:

It is given that

x = 4t, y = $\frac{4}{t}$

Then, we have

$\frac{dx}{dt} = \frac{d(4t)}{dt}$

= 4 ……… (1)

$\frac{dy}{dt} = \frac{d\left(\frac{4}{t}\right)}{dt} = 4\frac{-1}{t^2} = \frac{-4}{t^2}$ ………… (2)

Therefore, from equation (1) and (2), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-4}{t^2}}{4} = \frac{-1}{t^2} = -4\sin t$$

Hence, the value of $\frac{dy}{dx}$ is $\frac{-1}{t^2}$

Q. 5 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

x = cos θ – cos 2θ, y = sin θ – sin 2θ

Answer:

It is given that

x = cos θ – cos 2θ, y = sin θ – sin 2θ

Then, we have

$$\frac{dx}{d\theta} = \frac{d(\cos\theta - \cos 2\theta)}{d\theta}$$

$$= \frac{d(\cos\theta)}{d\theta} - \frac{d(\cos 2\theta)}{d\theta}$$

= -sin θ – (-2sin2θ)

= 2sin2θ – sin θ …………… (1)

$$\frac{dy}{d\theta} = \frac{d(\sin\theta - \sin\theta)}{d\theta}$$

$$= \frac{d(\sin\theta)}{d\theta} - \frac{d(\sin\theta)}{d\theta}$$

= cos θ – 2cos2θ

= -b sin θ ………… (2)

From equation (1) and (2), we get,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Hence, the value of $\frac{dy}{dx}$ is $\frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$

Q. 6 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

x = a (θ – sin θ), y = a (1 + cos θ)

Answer:

It is given that

x = a (θ – sin θ), y = a (1 + cos θ)

Then, we have

$$\frac{dx}{d\theta} = a\left[\frac{d(\theta)}{d\theta} - \frac{d(\sin\theta)}{d\theta}\right]$$

= a(1-cosθ) …………………… (1)

$$\frac{dy}{d\theta} = a\left[\frac{d(1)}{d\theta} - \frac{d(\cos\theta)}{d\theta}\right]$$

= a [0 + (-sin θ)]

= -a sin θ …………………… (2)

From equation (1) and (2), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2sin^2\frac{\theta}{2}}$$

$$= \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Hence, the value of $\frac{dy}{dx}$ is $-\cot\frac{\theta}{2}$

Q. 8 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

x = a $\left(\cos t + \log \tan \frac{t}{2}\right)$ y = a sin t

Answer:

It is given that

x = a $\left(\cos t + \log \tan \frac{t}{2}\right)$ y = a sin t

Then, we have

$$\frac{dx}{dt} = a\left[\frac{d(\cos t)}{dt} + \frac{d\left(\log \tan\frac{t}{2}\right)}{dt}\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\frac{d\left(\tan\frac{t}{2}\right)}{dt}\right]$$

$$= a\left[-\sin t + \cot\frac{t}{2}.sec^2\frac{t}{2}\frac{d\left(\frac{t}{2}\right)}{dt}\right]$$

$$= a\left[-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \times \frac{1}{cos^2\frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{2}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1-sin^2 t}{\sin t}\right]$$

$$= a\frac{cos^2 t}{\sin t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$\frac{dy}{dt} = a\frac{d(\sin t)}{dt}$$

= a cos t…………………… (2)

From equation (1) and (2), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\left(a\frac{cos^2 t}{\sin t}\right)}$$

$= \frac{\sin t}{\cos t}$

= tan t

Hence, the value of $\frac{dy}{dx}$ is tan t

Q. 9 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

x = a sec θ, y = b tan θ

Answer:

It is given that

x

= a sec θ, y = b tan θ

Then, we have

$$\frac{dx}{d\theta} = a\frac{d(\sec\theta)}{d\theta}$$

= a sec θ tan θ…………………… (1)

$$\frac{dy}{d\theta} = b\frac{d(\tan\theta)}{d\theta}$$

= bsec2θ…………………… (2)

From equation (1) and (2), we get,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{bsec^2\theta}{\text{asec}\,\theta\tan\theta}$$

$= \frac{b}{a}\sec\theta\cot\theta$

$= \frac{b\cos\theta}{a\cos\theta\sin\theta}$

$= \frac{b}{a} \times \frac{1}{\sin\theta}$

$= \frac{b}{a}\ cosec\theta$

Hence, the value of $\frac{dy}{dx}$ is $cosec\theta$

Q. 10 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

x = a (cos θ + θ sin θ), y = a (sin θ – θ cos θ)

Answer:

It is given that

x = a (cos θ + θ sin θ), y = a (sin θ – θ cos θ)

Then, we have

$\frac{dx}{d\theta} = a\left[\frac{d(\cos\theta)}{d\theta} + \frac{d(\theta\sin\theta)}{d\theta}\right]$

$= a\left[-\sin\theta + \frac{\theta d(\sin\theta)}{d\theta} + \sin\theta\frac{d(\theta)}{d\theta}\right]$

= a [-sin θ + θ cos θ + sin θ]

= a θ cos θ…………………… (1)

$\frac{dy}{d\theta} = a\left[\frac{d(\sin\theta)}{d\theta} - \frac{d(\theta\cos\theta)}{d\theta}\right]$

$= a\left[\cos\theta - \left\{\frac{\theta d(\cos\theta)}{d\theta} + \cos\theta\frac{d(\theta)}{d\theta}\right\}\right]$

= a [cos θ + θ sin θ – cos θ]

= a θ sin θ…………………… (2)

From (1) and (2) we get,

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta \sin\theta}{a\theta \cos\theta}$

= tan θ

Hence, the value of $\frac{dy}{dx}$ is tan θ

Q. 11 If x and y are connected parametrically by the equations given in without eliminating the parameter, Find dy/dx.

If, x = $\sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$ show that $\frac{dy}{dx} = -\frac{y}{x}$

Answer:

It is given that

x = $\sqrt{a^{\sin^{-1}t}}$, y = $\sqrt{a^{\cos^{-1}t}}$

Now,

x = $\sqrt{a^{\sin^{-1}t}} = x = \left(a^{\sin^{-1}t}\right)^{\frac{1}{2}} = x = a^{\frac{1}{2}\sin^{-1}t}$

Similarly, y = $\sqrt{a^{\cos^{-1}t}} = y\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}} = y = a^{\frac{1}{2}\cos^{-1}t}$

Let us consider,

x = $a^{\frac{1}{2}\sin^{-1}t}$

Taking Log on both sides, we get

log x = $\frac{1}{2}\sin^{-1}t \log a$

Therefore, $\frac{1}{x}.\frac{dx}{dt} = \frac{1}{2}\log a.\frac{d(\sin^{-1}t)}{dt}$

= $\frac{dx}{dt} = \frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}$

= $\frac{dx}{dt} = \frac{x \log a}{2\sqrt{1-t^2}}$ … (1)

Now, Consider

y = $a^{\frac{1}{2}cos^{-1}t}$

Taking Log on both sides, we get

log y = $\frac{1}{2}cos^{-1}t \log a$

Therefore, $\frac{1}{y}.\frac{dy}{dt} = \frac{1}{2}\log a.\frac{d(cos^{-1}t)}{dt}$

= $\frac{dy}{dt} = \frac{y}{2}\log a.\frac{-1}{\sqrt{1-t^2}}$

= $\frac{dy}{dt} = \frac{-y\log a}{2\sqrt{1-t^2}}$ …………….. (2)

So, from equation (1) and (2), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{-y\log a}{2\sqrt{1-t^2}}}{\frac{x\log a}{2\sqrt{1-t^2}}} = -\frac{y}{x}$$

Therefore, L.H.S. = R.H.S.

Hence Proved

Exercise 5.7

Q. 1 Find the second order derivatives of the function

$x^2 + 3x + 2$

Answer:

Let us take $y = x^2 + 3x + 2$

Now,

$$\frac{dy}{dx} = \frac{d(x^2)}{dx} + \frac{d(3x)}{dx} + \frac{d(2)}{dx}$$

$= 2x + 3$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d(2x+3)}{dx} = \frac{d(2x)}{dx} + \frac{d(3)}{dx}$$

$= 2 + 0$

$= 2$

Q. 2 Find the second order derivatives of the function

x^{20}

Answer:

Let us take $y = x^{20}$

Now,

$$\frac{dy}{dx} = \frac{d(x^{20})}{dx}$$

$= 20x^{19}$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d(20x^{19})}{dx} = 20\,\frac{d(x^{19})}{dx}$$

$= 20 \times 19 \times x^{18}$

$= 380\, x^{18}$

Q. 3 Find the second order derivatives of the function

x. cos x

Answer:

: Let us take y= x. cos x

Now,

$\frac{dy}{dx} = \frac{d(x \cos x)}{dx}$

$= \cos x \frac{d(x)}{dx} + x \frac{d(\cos x)}{dx}$

= cosx.1 + x (-sin x)

= cos x –x sin x

Therefore,

$\frac{d^2y}{dx^2} = \frac{d(\cos x - x \sin x)}{dx}$

$= \frac{d(\cos x)}{dx} - \frac{d(x \sin x)}{dx}$

$= -\sin x - \left[\sin x . \frac{d(x)}{dx} + x . \frac{d(\sin x)}{dx}\right]$

= -sin x – (sin x + x cos x)

= - (x cos x + 2sinx)

Q. 4 Find the second order derivatives of the function

log x

Answer:

Let us take y = logx

Now,

$\frac{dy}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$

Therefore,

$\frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{x}\right)}{dx} = \left(-\frac{1}{x^2}\right)$

Q. 5 Find the second order derivatives of the function

$x^3 \log x$

Answer:

Let us take $y = x^3 \log x$

Now,

$\frac{dy}{dx} = \frac{d(x^3 \log x)}{dx}$

$= \log x. \frac{d(x^3)}{dx} + x^3. \frac{d(\log x)}{dx}$

$= \log x.3x^2 + x^3.1/x$

$= \log x .3x^2 + x^2$

$= x^2(1 + 3\log x)$

Therefore,

$\frac{d^2y}{dx^2} = \frac{d[x^2(1+3 \log x)]}{dx}$

$= (1 + 3 \log x). \frac{d(x^2)}{dx} + x^2 \frac{d(1+3 \log x)}{dx}$

$= (1 + 3 \log x). 2x + x^2. \frac{3}{x}$

$= 2x + 6x\log x + 3x$

$= 5x + 6x\log x$

= x (5 + 6logx)

Q. 6 Find the second order derivatives of the function

e^x sin 5x

Answer:

Let us take y = e^x sin 5x

Now,

$$\frac{dy}{dx} = \frac{d(e^x \sin 5x)}{dx}$$

$$= \sin 5x.\frac{d(e^x)}{dx} + e^x.\frac{d(\sin 5x)}{dx}$$

$$= \sin 5x.e^x + e^x.\cos 5x.\frac{d(5x)}{dx}$$

= e^xsin5x+e^xcos5x.5

= e^x (sin5x + 5cos5x)

$$\frac{d^2y}{dx^2} = \frac{d[e^x(\sin 5x + 5\cos 5x)]}{dx}$$

$$= (\sin 5x + 5\cos 5x).\frac{d(e^x)}{dx} + e^x.\frac{d(\sin 5x + 5\cos 5x)}{dx}$$

$$= (\sin 5x + 5\cos 5x)e^x + e^x\left[\cos 5x.\frac{d(5x)}{dx} + 5(-\sin 5x).\frac{d(5x)}{dx}\right]$$

= e^x (sin5x + 5cos5x) + e^x (5cos5x – 25sin5x)

= e^x (10cos5x – 24sin5x)

= 2e^x (5cos5x – 12sin5x)

Q. 7 Find the second order derivatives of the function

e^{6x} cos 3x

Answer:

Let us take y = e^{6x} cos 3x

Now,

$\frac{dy}{dx} = \frac{d(e^{6x}\cos 3x)}{dx}$

$= \cos 3x.\frac{d(e^{6x})}{dx} + e^{6x}\frac{d(\cos 3x)}{dx}$

$= \cos 3x.\ e^{6x}.\frac{d(6x)}{dx} + e^{6x}.(-\sin 3x).\frac{d(3x)}{dx}$

= 6e6xcos3x - 3e6xsin3x

$\frac{d^2y}{dx^2} = \frac{d[6e^{6x}\cos 3x - 3e^{6x}\sin 3x]}{dx}$

$= 6.\frac{d(e^{6x}\cos 3x)}{dx} - 3.\frac{d(e^{6x}\sin 3x)}{dx}$

$= 6.[6e^{6x}\cos 3x - 3e^{6x}\sin 3x] - 3\left[\sin 3x.\frac{d(e^{6x})}{dx} + e^{6x}\frac{d(\sin 3x)}{dx}\right]$

$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 3[\sin 3x.e^{6x}.6 + e^{6x}.\cos 3x.3]$

$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$

$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$

$= 9e^{6x}(3\cos 3x - 4\sin 3x)$

Q. 8 Find the second order derivatives of the function

$\tan^{-1} x$

Answer:

Let us take $y = \tan^{-1} x$ Now,

$\frac{dy}{dx} = \frac{d(tan^{-1})}{dx} = \frac{1}{1+x^2}$

$\frac{d^2y}{dx^2} = \frac{d\left[\frac{1}{1+x^2}\right]}{dx}$

$= \frac{d(+x^2)^{-1}}{dx} = (-1).(1+x^2).\frac{d(1+x^2)}{dx}$

$= \frac{1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2}$

Q. 9 Find the second order derivatives of the function

log (log x)

Answer:

Let us take y = log (log x)

Now,

$\frac{dy}{dx} = \frac{d[\log(\log x)]}{dx}$

$= \frac{1}{\log x} . \frac{d(\log x)}{dx} = \frac{1}{x \log x}$

$= (x \log x)^{-1}$

$\frac{d^2y}{dx^2} = \frac{d(x \log x)^{-1}}{dx}$

$= (-1). (x \log x)^{-2} . \frac{d(x \log x)}{dx}$

$= \frac{-1}{(xlog\ x)^2} . \left[\log x . \frac{d(x)}{dx} + x . \frac{d(\log x)}{dx}\right]$

$= \frac{-1}{(x \log x)^2} . \left[\log x . 1 + x . \frac{1}{x}\right]$

$= \frac{-(1+\log x)}{(x \log x)^2}$

Q. 10 Find the second order derivatives of the function

sin (log x)

Answer:

Let us take y = sin (log x)

Now,

$\frac{dy}{dx} = \frac{d[\sin(\log x)]}{dx}$

$= \cos(\log x).\ \frac{d(\log x)}{dx}$

$= \frac{\cos(\log x)}{x}$

Then

$\frac{d^2y}{dx^2} = \frac{d\left(\frac{\cos(\log x)}{x}\right)}{dx}$

$= \frac{x.\frac{d[\cos(l\,ogx)]}{dx} - \cos(\log x).\frac{d(x)}{dx}}{x^2}$

$= \frac{x.\left[-\sin(\log x).\frac{d(\log x)}{dx}\right] - \cos(\log x).1}{x^2}$

$= \frac{-x\sin(\log x).\frac{1}{x}.\cos(\log x)}{x^2}$

$= \frac{-\sin(\log x) + \cos(\log x)}{x^2}$

Q. 11 If y = 5 cos x – 3 sin x, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer:

It is given that y = 5 cos x – 3 sin x

Now, on differentiating we get,

$\frac{dy}{dx} = \frac{d[5\cos x - 3\sin x]}{dx}$

$= \frac{d(5\cos x)}{dx} - \frac{d(3\sin x)}{dx}$

$= \frac{5d(\cos 5x)}{dx} - \frac{3d(\sin x)}{dx}$

= 5(-sin x) – 3(cos x)

= -(5sinx + cos x)

Then,

$\frac{d^2y}{dx^2} = \frac{d(-(5\sin x + \cos x))}{dx}$

$= -\left[5.\frac{d(\sin x)}{dx} + 3.\frac{d(\cos x)}{dx}\right]$

= - [5cosx +3(-sin x)]

= -[5cosx-3sinx]

= -y

Therefore,

$\frac{d^2y}{dx^2} + y = 0$

Hence Proved.

Q. 12 If y = $\cos^{-1}$ x, Find d^2y/dx^2 in terms of y alone.

Answer:

It is given that y = $\cos^{-1}$ x

Now,

$\frac{dy}{dx} = \frac{d(cos^{-1})}{dx} = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}}$

Therefore,

$\frac{d^2y}{dx^2} = \frac{d(-(1-x^2)^{-\frac{1}{2}})}{dx}$

$= -\left(-\frac{1}{2}\right).(1-x^2)^{-\frac{3}{2}}.\frac{d(1-x^2)}{dx}$

$= \frac{1}{2\sqrt{1-x^2}^3} \times (-2x)$

$$\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)}^3} \ \ldots\ldots\ldots \ (1)$$

Now it is given that $y = \cos^{-1} x$

$\Rightarrow x = \cos y$

Now putting the value of x in equation (1), we get

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{1-cos^2 y}^3}$$

$$= \frac{-\cos y}{\sqrt{sin^2 y}^3}$$

$$= \frac{-\cos y}{(\sin y)^3} = \frac{-\cos y}{\sin y} . \frac{1}{sin^2 y}$$

$$= \frac{d^2y}{dx^2} = -\cot y . cosec^2 y$$

Q. 13 If y = 3 cos (log x) + 4 sin (log x), show that $x^2 y_2 + xy_1 + y = 0$

Answer:

It is given that y = 3 cos (log x) + 4 sin (log x)

Now, on differentiating we get,

$$\frac{dy}{dx} = \frac{d(3\cos(\log x)) + 4\sin(\log x))}{dx}$$

$$= 3. \frac{d(\cos(\log x))}{dx} + 4. \frac{d(\sin(\log x))}{dx}$$

$$= 3. \left[-\sin(\log x). \frac{d(\log x)}{dx}\right] + 4. \left[\cos(\log x). \frac{d(\log x)}{dx}\right]$$

$$= \frac{dy}{dx} = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

Again differentiating we get,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)}{dx}$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Therefore,

$x^2 y_2 + xy_1 + y$

$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) +$
$3\cos(\log x) + 4\sin(\log x)$

= -sin(logx) – 7cos(logx) + 4cos(logx) – 3sin(logx) + 3cos(logx) + 4sin(logx)

= 0

So, $x^2 y_2 + xy_1 + y = 0$

Hence Proved

Q. 14

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer:

According to given equation, we have,

$y = Ae^{mx} + Be^{nx}$

Then, $\frac{dy}{dx} = \frac{d(Ae^{mx} + Be^{nx})}{dx}$

$= A.\frac{d(e^{mx})}{dx} + B.\frac{d(e^{nx})}{dx}$

= A.$e^{mx} \frac{d(mx)}{dx} + B.e^{nx} \frac{d(nx)}{dx}$

= $Ame^{mx} + Bne^{nx}$

Now, on again differentiating we get,

$\frac{d^2y}{dx^2} = \frac{d(Am\,e^{mx}+Bne^{nx})}{dx}$

= Am.$\frac{d(e^{mx})}{dx} + Bn.\frac{d(e^{nx})}{dx}$

= Am.$e^{mx} \frac{d(mx)}{dx} + Bn.e^{nx} \frac{d(nx)}{dx}$

= $Am^2e^{mx} + Bn^2e^{nx}$

$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

= $Am^2e^{mx} + Bn^2e^{nx} - (m+n)(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$

= $Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$

= 0

= $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Hence Proved

Q. 15 If y = $500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Answer:

According to given equation, we have,

y = $500e^{7x} + 600e^{-7x}$

$\frac{dy}{dx} = \frac{d\left(500e^{7x}+600e^{-7x}\right)}{dx}$

= 500. $\frac{d\left(e^{7x}\right)}{dx} + 600.\frac{d(-7x)}{dx}$

$= 500.e^{7x} \frac{d(7x)}{dx} + 600.e^{-7x} \frac{d(-7x)}{dx}$

$= 3500e^{7x} - 4200e^{-7x}$

Now, on again differentiating we get,

$\frac{d^2y}{dx^2} = \frac{d(3500e^{7x} - 4200e^{-7x})}{dx}$

$= 3500.\frac{d(e^{7x})}{dx} - 4200\frac{d(e^{-7x})}{dx}$

$= 3500\, e^{7x} \frac{d(7x)}{dx} - 42500e^{-7x} \frac{d}{dx(-7x)}$

$= 7\times3500.e^{7x} + 7\times4200.e^{-7x}$

$= 49\times500e^{7x} + 49\times600e\text{-}^{7x}$

$= 49(500e^{7x} + 600e^{-7x})$

$= 49y$

$\therefore \frac{d^2y}{dx^2} = 49\, y$

Hence Proved

Q. 16 If $e^y (x + 1) = 1$, show that=

Answer:

It is given that

$e^y (x + 1) = 1$

$= e^y = \frac{1}{x+1}$

Now, taking logarithm on both the sides we get,

$y = \log \frac{1}{x+1}$

On differentiating both sides, we get,

$\frac{dy}{dx} = (x+1)\frac{d\left(\frac{1}{x+1}\right)}{dx}$

$= (x + 1).\frac{-1}{(x+1)^2} = \frac{-1}{x+1}$

Again, on differentiating we get,

$\therefore \frac{d^2y}{dx^2} = -\frac{d\left(\frac{1}{x+1}\right)}{dx}$

$= -\left(\frac{d^2y}{dx^2}\right) = \frac{1}{(x+1)^2}$

$= \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

$= \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Hence Proved

Q. 17 If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

Answer:

: It is given that

$y = (\tan^{-1} x)^2$

On differentiating we get,

$\frac{dy}{dx} = \frac{d[(tan^{-1}x)^2]}{dx}$

$= 2 \tan^{-1} x \frac{d[tan^{-1}x]}{dx}$

$= 2 \tan^{-1}x \frac{1}{1+x^2}$

$= (1 + x^{2)} \frac{dy}{dx} = 2 \tan^{-1} x$

Again differentiating, we get,

$(1 + x^2)\ \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 2\left(\frac{1}{1+x^2}\right)$

$= (1 + x^2)^2\ \frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} = 2$

So, $(1+x^2)2y^2 + 2x(1+x^2)\ y_1 = 2$

where, $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

Hence Proved

Exercise 5.8

Q. 1 Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Answer:

The given function is $f(x) = x^2 + 2x - 8$ and $x \in [-4, 2]$.

By Rolle's Theorem, for a function $f: [a, b] \to R$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f(a) = f(b)

Then there exists some c in (a, b) such that $f'(c) = 0$.

As $f(x) = x^2 + 2x - 8$ is a polynomial function,

(a) f(x) is continuous in [-4, 2]

(b) $f'(x) = 2x + 2$

So, f(x) is differentiable in (-4, 2).

(c) $f(a) = f(-4) = (-4)^2 + 2(-4) - 8 = 16 - 8 - 8 = 16 - 16 = 0$

$f(b) = f(2) = (2)^2 + 2(2) - 8 = 4 + 4 - 8 = 8 - 8 = 0$

Hence, f(a) = f(b).

∴ There is a point $c \in (-4, 2)$ where $f'(c) = 0$.

$f(x) = x^2 + 2x - 8$

$f'(x) = 2x + 2$

$f'(c) = 0$

$\Rightarrow f'(c) = 2c + 2 = 0$

$\Rightarrow 2c = -2$

⇒ c = -2 /2

⇒ c = -1 where c = -1 ∈ (-4, 2)

Hence, Rolle's Theorem is verified.

Q. 2 Examine if Rolle's theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's theorem from these examples?

(i) f (x) = [x] for x ∈ [5, 9]

(ii) f (x) = [x] for x ∈ [– 2, 2]

(iii) f (x) = x^2 – 1 for x ∈ [1, 2]

Answer:

By Rolle's Theorem, for a function f: [a, b] → R, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f(a) = f(b)

Then there exists some c in (a, b) such that f '(c) = 0.

If a function does not satisfy any of the above conditions, then Rolle's Theorem is not applicable.

(i) f (x) = [x] for x ∈ [5, 9]

As the given function is a greatest integer function,

(a) f(x) is not continuous in [5, 9]

(b) Let y be an integer such that y ∈ (5, 9)

Left hand limit of f(x) at x = y: $\lim_{h\to 0^-} \frac{f\ (y+h)-f(y)}{h} = \lim_{h\to 0^-} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^-} \frac{y-1-y}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^+}\frac{y-y}{h} = \lim_{h\to 0^+}\frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x = y is not equal, f(x) is not differentiable at x=y.

So, f(x) is not differentiable in [5, 9]

(c) f(a)= f (5) = [5] = 5

f(b) = f (9) = [9] = 9

f(a) ≠ f(b)

Here, f(x) does not satisfy the conditions of Rolle's Theorem.

Rolle's Theorem is not applicable for f(x) = [x] for x ∈ [5, 9].

(ii) f (x) = [x] for x ∈ [– 2, 2]

As the given function is a greatest integer function,

(a) f(x) is not continuous in [-2, 2]

(b) Let y be an integer such that y ∈ (-2, 2)

Left hand limit of f(x) at x = y:

$$\lim_{h\to 0^-}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^-}\frac{y-1-y}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^+}\frac{y-y}{h} = \lim_{h\to 0^+}\frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x = y is not equal, f(x) is not differentiable at x = y.

So, f(x) is not differentiable in (-2, 2)

(c) f(a)= f (-2) = [-2] = -2

f(b) = f (2) = [2] = 2

f(a) ≠ f(b)

Here, f(x) does not satisfy the conditions of Rolle's Theorem.

Rolle's Theorem is not applicable for f(x) = [x] for x ∈ [-2, 2].

(iii) f (x) = $x^2 - 1$ for x ∈ [1, 2]

As the given function is a polynomial function,

(a) f(x) is continuous in [1, 2]

(b) f'(x) = 2x

So, f(x) is differentiable in [1, 2]

(c) f(a) = f (1) = $1^2 - 1 = 1 - 1 = 0$

f(b) = f (2) = $2^2 - 1 = 4 - 1 = 3$

f(a) ≠ f(b)

Here, f(x) does not satisfy a condition of Rolle's Theorem.

Rolle's Theorem is not applicable for f(x) = $x^2 - 1$ for x ∈ [1, 2].

Q. 3

If f: [– 5, 5] → R is a differentiable function and if f′(x) does not vanish anywhere, then prove that f (– 5) ≠ f (5).

Answer:

Given: f: [-5, 5] → R is a differentiable function.

Mean Value Theorem states that for a function f: [a, b] → R, if

(a)f is continuous on [a, b]

(b)f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that

We know that a differentiable function is a continuous function.

So,

(a) f is continuous on [-5, 5]

(b) f is differentiable on (-5, 5)

∴ By Mean Value Theorem, there exists c ∈ (-5, 5) such that

⇒ f’ (c) = $\frac{f(5)-f(-5)}{5-(-5)}$

⇒ 10 f'(c) = f (5) – f (-5)

It is given that f'(x) does not vanish anywhere.

∴ f'(c) ≠ 0

10 f'(c) ≠ 0

f (5) – f (-5) ≠0

f (5) ≠ f (-5)

Hence proved.

By Mean Value Theorem, it is proved that f (5) ≠ f (-5).

Q. 4 Verify Mean Value Theorem, if f (x) = $x^2 - 4x - 3$ in the interval [a, b], where a = 1 and b =4.

Answer:

Given: f(x) = $x^2 - 4x - 3$ in the interval [1, 4]

Mean Value Theorem states that for a function f: [a, b] → R, if

(a)f is continuous on [a, b]

(b)f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that f ‘ (c) = $\frac{f(b)-f(a)}{b-a}$

As f(x) is a polynomial function,

(a) f(x) is continuous in [1, 4]

(b) f'(x) = 2x – 4

So, f(x) is differentiable in (1, 4).

$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1}$

f (4) = 4^2 – 4(4) – 3 = 16 – 16 – 3 = -3

f (1) = 1^2 – 4(1) – 3 = 1 – 4 – 3 = -6

$= \frac{f(4)-f(1)}{4-1} = \frac{-3-(-6)}{4-1} = \frac{3}{3} = 1$

∴ There is a point c ∈ (1, 4) such that f'(c) = 1

⇒ f'(c) = 1

⇒ 2c – 4 = 1

⇒ 2c = 1+4 =5

⇒ c = 5/2 where c ∈ (1,4)

The Mean Value Theorem is verified for the given f(x).

Q. 5 Verify Mean Value Theorem, if f (x) = $x^3 - 5x^2 - 3x$ in the interval [a, b], where a = 1 and b = 3. Find all c ∈ (1, 3) for which f′(c) = 0.

Answer:

Given: f(x) = $x^3 - 5x^2 - 3x$ in the interval [1, 3]

Mean Value Theorem states that for a function f: [a, b] → R, if

(a)f is continuous on [a, b]

(b)f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that f ‘ (c) = $\frac{f(b)-f(a)}{b-a}$

As f(x) is a polynomial function,

(a) f(x) is continuous in [1, 3]

(b) f'(x) = 3x2 - 10x - 3

So, f(x) is differentiable in (1, 3).

$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1}$

f (3) = $3^3 - 5(3)^2 - 3(3)$ = 27 – 45 - 9 = -27

f (1) = $1^3 - 5(1)^2 - 3(1)$ = 1 – 5 - 3 = -7

$\Rightarrow \frac{f(3)-f(1)}{3-1} = \frac{-27-(-7)}{3-1} = \frac{-20}{0} = -10$

∴ There is a point c ∈ (1, 4) such that f'(c) = -10

⇒ f'(c) = -10

⇒ $3c^2 - 10c - 3 = -10$

⇒ $3c^2 - 10c + 7 = 0$

⇒ $3c^2 - 3c - 7c + 7 = 0$

⇒ 3c (c – 1) – 7(c – 1) = 0

⇒ (c – 1) (3c – 7) = 0

⇒ c = 1, 7/3 where c = 7/3 ∈ (1, 3)

The Mean Value Theorem is verified for the given f(x) and c = 7/3 ∈ (1, 3) is the only point for which f'(c) = 0.

Q. 6 Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer:

Mean Value Theorem states that for a function f: [a, b] → R, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that f' (c) = $\frac{f(b)-f(a)}{b-a}$

If a function does not satisfy any of the above conditions, then Mean Value Theorem is not applicable.

(i) f (x) = [x] for x ∈ [5, 9]

As the given function is a greatest integer function,

(a)f(x) is not continuous in [5, 9]

(b) Let y be an integer such that y ∈ (5, 9)

Left hand limit of f(x) at x = y:

$$\lim_{h\to 0^-}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^-}\frac{y-1-y}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^+}\frac{y-y}{h} = \lim_{h\to 0^+}\frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x=y is not equal, f(x) is not differentiable at x=y.

So, f(x) is not differentiable in [5, 9].

Here, f(x) does not satisfy the conditions of Mean Value Theorem.

Mean Value Theorem is not applicable for f(x) = [x] for x ∈ [5, 9].

(ii) f (x) = [x] for x ∈ [– 2, 2]

As the given function is a greatest integer function,

(a)f(x) is not continuous in [-2, 2]

(b) Let y be an integer such that y ∈ (-2, 2)

Left hand limit of f(x) at x = y:

$$\lim_{h\to 0^-}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^-}\frac{y-1-y}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^+} \frac{y-y}{h} = \lim_{h\to 0^+} \frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x=y is not equal, f(x) is not differentiable at x=y.

So, f(x) is not differentiable in (-2, 2)

Here, f(x) does not satisfy the conditions of Mean Value Theorem.

Mean Value Theorem is not applicable for f(x) = [x] for x ∈ [-2, 2].

(iii) f (x) = x2 – 1 for x ∈ [1, 2]

As the given function is a polynomial function,

(a) f(x) is continuous in [1, 2]

(b) f'(x) = 2x

So, f(x) is differentiable in [1, 2].

Here, f(x) satisfies the conditions of Mean Value Theorem.

So, Mean Value Theorem is applicable for f(x).

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1}$$

f (2) = 2^2 – 1 = 4 -1 = 3

f (1) = 1^2 – 1 = 1 – 1 = 0

$$\Rightarrow \frac{f(2)-f(1)}{2-1} = \frac{3-0}{2-1} = \frac{3}{1} = 3$$

∴ There is a point c ∈ (1, 2) such that f'(c) = 3

⇒ f'(c) = 3

⇒ 2c = 3

⇒ c = 3/2 where c ∈ (1, 2)

Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Miscellaneous Exercise

Q. 1 Differentiate w.r.t. x the function

$(3x^2 - 9x + 5)^9$

Answer:

Let $y = (3x^2 - 9x + 5)^9$

If $u = v(w(x))$

Then using chain rule $\frac{du}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$

∴ Differentiating y w.r.t. x using chain rule

$\frac{dy}{dx} = \frac{d}{dx}(3x^2 - 9x + 5)^9$

$= 9\,(3x^2 - 9x + 5)^8 \times \frac{d}{dx}(3x2 - 9x + 5)$

$= 9(3x^2 - 9x + 5)^8 \times (6x - 9)$

$=9(3x^2 - 9x + 5)^8 \times 3(2x - 3)$

$=27(3x^2 - 9x + 5)^8\,(2x - 3)$

$\therefore \frac{dy}{dx} = 27\,(3x^2 - 9x + 5)^8\,(2x - 3)$

Q. 2 Differentiate w.r.t. x the function

$\sin^3 x + \cos^6 x$

Answer:

Let $y = \sin^3 x + \cos^6 x$

Differentiating both sides with respect to x

$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\cos^6 x)$ $\because \frac{d}{dx}(\sin x) = \cos x$ & $\frac{d}{dx}(\cos x) = -\sin x$
]

$= 3 \sin 2x \times \frac{d}{dx} (\sin x) + 6 \cos 5 x \times \frac{d}{dx} (\cos x)$

$= 3 \sin^2 x \times \cos x + 6 \cos^5 x \times (- \sin x)$

$= 3 \sin x \cos x (\sin x - 2 \cos^4 x)$

$\therefore \frac{dy}{dx} = 3 \sin x \cos x (\sin x - 2 \cos^4 x)$

Q. 3 Differentiate w.r.t. x the function

$(5x)^{3\cos 2x}$

Answer:

Let $y = (5x)^{3\cos 2x}$

Then $\log y = \log (5x)^{3\cos 2x}$

$\Rightarrow \log y = 3\text{cox } 2x \times \log 5x$

Differentiating both sides with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = 3 \left[\log 5x \times \frac{d}{dx}(\cos 2x) + \cos 2x \times \frac{d}{dx}(\log 5x)\right]$

$\left[\therefore \frac{d}{dx}(uv) = u \times \frac{dv}{dx} + v \times \frac{du}{dx}\right]$

$= \frac{dy}{dx} = 3y \left[\log 5x(-2 \sin 2x) \times \frac{d}{dx}(2x) + \cos 2x \times \frac{1}{5x} \times \frac{d}{dx}(5x)\right]$

$= \frac{dy}{dx} = 3y[-2 \sin 2x \log 5x +] \frac{\cos 2x}{x}$

$= \frac{dy}{dx} = y \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x\right]$

$= \frac{dy}{dx} = (5x)^{3 \cos 2x} \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x\right]$

$\because \frac{dy}{dx} = (5x)^{3 \cos 2x} \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x\right]$

Q. 4 Differentiate w.r.t. x the function

$\text{Sin}^{-1} \left(x\sqrt{x}\right), 0 \le x \le 1$

Answer:

Let y = sin-1 $\left(x\sqrt{x}\right), 0 \leq x \leq 1$

Differentiating both sides with respect to x, we get

Using chain rule we get

$\frac{dy}{dx} = \frac{d}{dx}$ sin^{-1} $\left(x\sqrt{x}\right)$

$= \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$

$= \frac{dy}{dx} = \frac{1}{\sqrt{1-x^3}} \times \frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{dy}{dx} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$

$= \frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{\sqrt{1-x^3}}}$

$\therefore \frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$

Q. 5 Differentiate w.r.t. x the function

$\frac{cos^{-1}}{\sqrt{2x+7}}, -2 < x < 2$

Answer:

Let y = $\frac{cos^{-1}}{\sqrt{2x+7}}, -2 < x < 2$

Differentiating both sides with respect to x, we get

Using Quotient rule

$\frac{dy}{dx} = \frac{\sqrt{2x+7}\frac{d}{dx}\left(cos^{-1}\frac{x}{2}\right) - \left(cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{d}{dx}\left(\frac{x}{2}\right)\right] - \left(cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{2x+7}$$

$$\frac{dy}{dx} = \frac{\sqrt{2x+7} \times -\frac{1}{\sqrt{4-x^2}} - \left(cos^{-1}\frac{x}{2}\right) \times \frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$\frac{dy}{dx} = -\frac{\sqrt{2x+7}}{\sqrt{4-x^2} \times (2x+7)} - \frac{cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$\therefore \frac{dy}{dx} = -\left[\frac{1}{\sqrt{4-x^2} \times \sqrt{2x+7}} + \frac{cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Q. 6 Differentiate w.r.t. x the function

Cot^{-1} $\left(\frac{\sqrt{1+si\,nx} + \sqrt{1-si\,nx}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right), 0 < x < \frac{\pi}{2}$

Answer:

Let y = cot^{-1} $\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right), 0 < x < \frac{\pi}{2}$

$$\frac{\sqrt{1+si\,nx} + \sqrt{1-si\,nx}}{\sqrt{1+si\,nx} - \sqrt{1-si\,nx}}$$

$$= \frac{\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+si\,nx} - \sqrt{1-si\,nx}\right)\left(\sqrt{1+\sin x} - \sqrt{1-\sin x}\right)}$$

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$$

$$= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)(1-\sin x)}$$

$$= \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \frac{2 + 2\sqrt{1-si\,n^2x}}{2\sin x}$$

$$= \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} = \frac{1+\cos x}{\sin x} = \frac{2cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = cot\,\frac{x}{2}$$

Substituting the value of $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \cot\frac{x}{2}$ in y.

$\therefore$ y = $\cot^{-1}\left(\cot\frac{x}{2}\right)$

⇒ y = x/2

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

Q. 7

Differentiate w.r.t. x the function

$(\log x)^{\log x}$, x > 1

Answer:

Let y = $(\log x)^{\log x}$, x > 1

Taking logarithm on both sides

⇒ log y = log (log x) log x = log x × log (log x)

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\log x \times \log(\log x)]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x) \times \frac{d}{dx}(\log x) + \log x \times \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x) \times \frac{1}{x} + \log x \times \frac{1}{\log x} \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Q. 8 Differentiate w.r.t. x the function

cos (a cos x + b sin x), for some constant a and b.

Answer:

Let y = cos (a cos x + b sin x)

a and b are some constants

y = cos (a cos x + b sin x)

Differentiating both sides with respect to x, we get

Using chain rule

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$

$\Rightarrow \frac{dy}{dx} = -\sin(a\cos + bs\,i\,nx) \times \frac{d}{dx}(a\cos x + b\sin x)$

$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\ \times [a\ (-\sin x) + b\cos x]$

$\therefore \frac{dy}{dx} = (a\sin x - b\cos x)\ \times \sin(a\cos x + b\sin x)$

Q. 9 Differentiate w.r.t. x the function

$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer:

Let $y = \sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Taking logarithm both sides, we get

log y = log [(sin x – cos x) (sin x – cos x)]

⇒ log y = (sin x – cos x) × log (sin x – cos x)

Differentiating both sides with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[(\sin x - \cos x) \times \log(\sin x - \cos x)]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos) + (\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos x)$

$\Rightarrow \frac{dy}{dx} = y\left[\log(\sin x - \cos x) \times (\cos x + \sin x) + (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times \frac{d}{dx}(\sin x - \cos x)\right]$

$\Rightarrow \frac{dy}{dx} = y\left[\log(\sin x - \cos x) \times (\cos x + \sin x) + (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times \frac{d}{dx}(\sin x - \cos x)\right]$

$\Rightarrow \frac{dy}{dx} = y[(\cos x + \sin x)\log(\sin x - \cos x) + (\cos x + \sin x)]$

$\therefore \frac{dy}{dx} =$ (sin x – cos x) (sin x – cos x) (cos x + sin x) [1 + log (sin x – cos x)]

Q. 10 Differentiate w.r.t. x the function

$x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

Answer:

Let $y = x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

And let $x^x = u$, $x^a = v$, $a^x = w$ and $a^a = s$

Then y = u + v + w + s

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \dots$ (I)

Now,

$u = x^x$

Taking logarithm both sides, we get

$\log u = \log x^x$

$\Rightarrow$ log u = x log x

Differentiating both sides w.r.t. x

$$\Rightarrow \frac{1}{u}\frac{dy}{dx} = \log x \times \frac{d}{x}(x) + x \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = u\left[\log x + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \ldots \text{(II)}$$

$v = x^a$

Differentiating both sides with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \ \ldots\ldots\ldots\ldots \text{(III)}$$

$w = a^x$

Taking logarithm both sides

log w = log ax

log w = x log a

Differentiating both sides with respect to x

$$\frac{1}{w}\frac{dy}{dx} = \log a \times \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w \log a$$

$$\Rightarrow \frac{dy}{dx} = a^x \log a \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(IV)}$$

$s = a^a$

Differentiating both sides with respect to x

$$\frac{ds}{dx} = 0 \ \ldots\ldots\ldots\ldots \text{(V)}$$

Putting (II), (III), (IV) and (V) in (I)

$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a + 0$

$\therefore \frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a$

Q. 11 Differentiate w.r.t. x the function

$x^{x^2-3} + (x-3)^{x^2}$, for x > 3

Answer:

Let y = $x^{x^2-3} + (x-3)^{x^2}$

And let x^{x^2-3} = u & $(x-3)^{x^2}$ = v

∴ y = u + v

Differentiating both sides w.r.t. x we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ …….. (I)

Now,

u = x^{x^2-3}

Taking logarithm both sides

log u = log $x^{x^2 3}$

⇒ log u = (x2 – 3) log x

Differentiating w.r.t. x, we get

$\frac{1}{u}\frac{dy}{dx} = \log x \times \frac{d}{dx}(x^2-3) + (x^2-3) \times \frac{d}{dx}(\log x)$

$\Rightarrow \frac{dy}{dx} = u\ \left[\log x \times 2x + (x^2-3) \times \frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x \log x\right]$ …………… (II)

Also,

v = $(x-3)^{x^2}$

Taking logarithm both sides

log v = log $(x-3)^{x^2}$

⇒ log v = x^2 log (x – 3)

Differentiating both sides w.r.t. x

$\frac{1}{v}\frac{dv}{dx} = loh\ (x-3) \times \frac{d}{dx}(x^2) \times \frac{d}{dx}[\log(x-3)]$

$\Rightarrow \frac{dv}{dx} = v\left[\log(x-3) \times 2x + x^2 \times \frac{1}{(x-3)} \times \frac{d}{dx}(x-3)\right]$

$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[2x\log(x-3) + \frac{x^2}{(x-3)} \times 1\right]$

$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{(x-3)} + 2x\log(x-3)\right]$ ……………. (II)

Substituting (II) and (III) in (I)

$\therefore \frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{(x-3)} + 2x\log(x-3)\right]$

Q. 12 Find dy/dx, if y = 12 (1 – cos t), x = 10 (t – sin t),

Answer:

To find $\frac{dy}{dx}$ we need to find out $\frac{dx}{dt}$ and $\frac{dy}{dt}$

So, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given, y = 12 (1 – cos t) and x = 10 (t – sin t)

x = 10 (t – sin t)

Differentiating with respect to t.

$\frac{dx}{dt} = \frac{d}{dt}[10\ (t - \sin t)]$

$\Rightarrow \frac{dx}{dt} = 10 \times \frac{dy}{dx}(t - \sin t) = 10\ (1 - \cos t)$

y = 12 (1 – cos t)

Differentiating with respect to t.

$\frac{dy}{dt} = \frac{d}{dx}[12\ (1 - \cos t)]$

$\Rightarrow \frac{dy}{dx} = 12 \times \frac{dy}{dx}(1 - \cos t) = 12\ \times [0 - (-\sin t)] = 12 \sin t$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1-\cos t)} = \frac{12\times 2 \sin\frac{t}{2}\cos\frac{t}{2}}{10\times 2 sin^2\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$

$\therefore \frac{dy}{dx} = \frac{6}{5}\cot\frac{t}{2}$

Q. 13 Find dy/dx, if y = $\sin^{-1}$ x + $\sin^{-1}$ $\sqrt{1-x^2}$, 0 < x < 1

Answer:

Given,

y = $\sin^{-1}$ x + $\sin^{-1}$ $\sqrt{1-x^2}$

Differentiating with respect to x

$\frac{dy}{dx} = \frac{d}{dx}\left[sin^{-1}x + sin^{-1}\sqrt{1-x^2}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(sin^{-1}x) + \frac{d}{dx}\left(sin^{-1}\sqrt{1-x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-x^2)}} \times \frac{d}{x}\left(\sqrt{1-x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \times \frac{1}{2\sqrt{1-x^2}} \times \frac{d}{dx}(1-x^2)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}} \times (-2x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dx} = 0$

Q. 14 If x $\sqrt{1+y}$ + y $\sqrt{1+x}$ = 0,for , – 1 < x < 1, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Answer:

Given, x $\sqrt{1+y}$ + y $\sqrt{1+x}$ = 0

x $\sqrt{1+y}$ = -y $\sqrt{1+x} = 0$

Now, squaring both sides, we get

$\Rightarrow \left(x\sqrt{1+y}\right)^2 = (-y\sqrt{1+x})^2$

$\Rightarrow x^2 (1 + y) = y^2 (1+x)$

$\Rightarrow x^2 + x^2y = y^2 + y^2x$

$\Rightarrow x^2 – y^2 = xy^2 – x^2y$

$\Rightarrow (x + y) (x – y) = xy (y – x)$

$\Rightarrow x + y = –xy$

$\Rightarrow y + xy = –x$

$\Rightarrow y (1 + x) = –x$

$\Rightarrow y = -\frac{x}{(1+x)}$

Differentiating both sides with respect to x, we get

$y = -\frac{x}{(1+x)}$

Using Quotient Rule

$$y = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x^2)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Hence, Proved

Q. 15 If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Answer:

Given, $(x - a)^2 + (y - b)^2 = c^2$

Differentiating with respect to x, we get

$$\frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] = \frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a) \times \frac{d}{dx}(x-a) + 2(y-b) \times \frac{d}{dx}(y-b) = 0$$

$$\Rightarrow 2(x-a) \times 1 + 2(y-b) \times \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{(x-a}{y-b}$$

Differentiating again with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{(x-a)}{y-b}\right]$$

Using Quotient Rule

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)\times\frac{d}{dx}(x-a)-(x-a)\times\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)-(x-a)\times\frac{d}{dx}}{(y-b)^2}\right]$$

Substituting the value of dy/dx in the above equation

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)-(x-a)\times\left\{-\frac{(x-a)}{y-b}\right\}}{(y-b)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{c^3}{(y-b)^3}\right]}{\frac{c^2}{(y-b)^3}} = -c$$

$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}}$ = - c, which is independent of a and b

Hence, Proved

Q. 16 If cos y = x cos (a + y), with cos a ≠ ± 1, prove that $\frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$

Answer

Given, cos y = x cos (a + y)

Differentiating both sides with respect to x

$$\frac{d}{dy}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\times\frac{d}{dx}(x) + x\times\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \ \ldots\ldots\ldots\ldots\ (I)$$

Since, cos y = x cos (a + y) ⇒ x = cos y/cos (a + y)

Substituting the value of x in (I)

$$\left[\frac{\cos y}{\cos(a+y)} \times \sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y \times \sin(a+y) - \sin y \times \cos(a+y)]\frac{dy}{dx} = \cos\,(a+y) \times \cos\,(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = cos\,^2(a+y)$$

$$\Rightarrow \sin a \times \frac{dy}{dx} = cos^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$$

Hence, proved

Q. 17 If x = a (cos t + t sin t) and y = a (sin t – t cos t), find d^2y/dx^2.

Answer:

Given, x = a (cos t + t sin t) and y = (sin t – t cos t)

To find $\frac{dy}{dx}$ we need to find out $\frac{dx}{dt}$ and $\frac{dy}{dt}$

So, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}}{dt} \times \frac{dt}{dx}$

x = a (cos t + t sin t)

Differentiating with respect to t.

$$\frac{dx}{dt} = \frac{d}{dt}[a\,(\cos t + t\sin t)]$$

$$\Rightarrow \frac{dx}{dt} = a \times \frac{d}{dt}(\cos t + t\sin t) = a[-\sin t + \sin t\ \times \frac{d}{dt}(t) + t \times \frac{d}{dt}(\sin t)]$$

$$\Rightarrow \frac{dx}{dt} = a[-\sin t + \sin t + t\cos t] = at\cos t$$

y = a (sin t – t cos t)

Differentiating with respect to t.

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t - t\cos t) = a\left[\cos t - \left\{\cos t \times \frac{d}{dt}(t) + t \times \frac{d}{dt}(cos\, t\right\}\right]$$

$$\Rightarrow \frac{dy}{dt} = a[\cos t - \cos t + t\sin t] = a\ t\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\ t\sin t}{a\ t\cos t} = \tan t$$

Differentiating dy/dx with respect to t

$$\frac{\frac{dy}{dx}}{dt} = \frac{d}{dt}(\tan t) = \ sec^2 t$$

And $\frac{dt}{dx} = \frac{1}{a\ t\cos t} = \frac{\sec t}{a\ t}$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = sec^2 t \times \frac{\sec t}{a\ t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{sec^3 t}{a\ t}$$

Q. 18 If f (x) = $|x|^3$, show that f″(x) exists for all real x and find it.

Answer:

$$|x| = \begin{cases} x, i\ fx \geq 0 \\ -x, i\ fx < 0 \end{cases}$$

When, x ≥ 0,

f(x) = |x|^3 = x^3

So, f'(x) = $3x^2$

And f''(x) = d(f'(x))/dx = 6x

∴ f''(x) = 6x

When x < 0,

f(x) = |x|^3 = $(-x)^3 = -x^3$

f'(x) = $-3x^2$

f''(x) = – 6x

∴ f'' (x) = $\begin{cases} 6x, x \geq 0 \\ -6x, x < 0 \end{cases}$

Q. 19 Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Answer:

To prove: P(n): $\frac{d}{dx}(x^n)$ = nx^{n-1} for all positive integers n

For n = 1,

LHS = $\frac{d}{dx}(x)$ = 1

RHS = 1 × x1 – 1 = 1

So, LHS = RHS

∴ P (1) is true.

∴ P(n) is true for n = 1

Let P(k) be true for some positive integer k.

i.e. P(k) = $\frac{dy}{dx}(x^k) = kx^{k-1}$

Now, to prove that P (k + 1) is also true

RHS = (k + 1) x (k + 1) – 1

LHS = $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \times x^k)$

= $x^k \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(x^k)$

= $x^k \times 1 + x \times k \times x^{k-1}$

= $x^k + kx^k$

= $(k + 1) \times x^k$

= $(k + 1)x^{(k+1)-1}$

∴ LHS = RHS

Thus, P (k + 1) is true whenever P(k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Q. 20 Using the fact that sin (A + B) = sin A Cos B + cos A Sin B and the differentiation, obtain the sum formula for cosines.

Answer:

sin (A + B) = sin A Cos B + cos A Sin B

Differentiating with respect to x, we get

$\frac{d}{dx}$ [sin (A + B)] = $\frac{d}{dx}$ (sin A cos B) + $\frac{d}{dx}$ (cos A sin B)

$\Rightarrow \cos(A + B)\frac{d}{dx}(A + B) = \cos B\frac{d}{dx}(\sin A) + \sin A\frac{d}{dx}(\cos B) + \sin B\frac{d}{dx}(\cos A) + \cos A\frac{d}{dx}(\sin B)$

$\Rightarrow$ cos (A + B) $\frac{d}{dx}$ (A + B) = cos B cos A $\frac{dA}{dx}$ + sin A (-sin B)$\frac{dB}{dx}$ + sin B (-sin A) $\frac{dA}{dx}$ + cos A cos B $\frac{dB}{dx}$

$\Rightarrow$ cos (A + B) × $\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$ = (cos A cos B – sin A sin B) × $\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$

∴ cos (A + B) = cos A Cos B – sin A Sin B

Q. 21 Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Answer:

Considering the function

f(x) = |x| + |x + 1|

The above function f is continuous everywhere, but is not differentiable at x = 0 and x = – 1

$$f(x) = \begin{cases} -x - (x+1), x \leq -1 \\ -x + (x+1), -1 < x < 0 \\ x + (x+1), x \geq 0 \end{cases}$$

$$= \begin{cases} -2x - 1, x \leq -1 \\ 1, -1 < x < 0 \\ 2x + 1, x \geq 0 \end{cases}$$

Now, checking continuity

CASE I: At x < – 1

f(x) = – 2x – 1

f(x) is a polynomial

$\Rightarrow$ f(x) is continuous [∵ Every polynomial function is continuous]

CASE II: x > 0

f(x) = 2x + 1

f(x) is a polynomial

⇒ f(x) is continuous [∵ Every polynomial function is continuous]

CASE III: At – 1 < x < 0

f(x) = 1

f(x) is constant

⇒ f(x) is continuous

CASE IV: At x = – 1

$$f(x) = \begin{cases} -2x - 1, x \leq -1 \\ 1, -1 < x < 0 \\ 2x + 1, x \geq 0 \end{cases}$$

A function will be continuous at x = – 1

If LHL = RHL = f (– 1)

i.e. $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$

LHL = $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -2x - 1$

Putting x = – 1

LHL = – 2 × (– 1) – 1 = 2 – 1 = 1

RHL = $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 1 = 1$

f(x) = – 2x – 1

f (– 1) = – 2 × (– 1) – 1 = 2 – 1 = 1

so, LHL = RHL = f (– 1)

⇒ f is continuous.

CASE V: At x = 0

$$f(x) = \begin{cases} -2x - 1, x \leq -1 \\ 1, -1 < x < 0 \\ 2x + 1, x \geq 0 \end{cases}$$

A function will be continuous at x = 0

If LHL = RHL = f (0)

i.e. $\lim_{x\to 0^{-1}} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

LHL = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1$

RHL = $\lim_{x\to 0^+} \quad = \lim_{x\to 0^+} 2x + 1$

Putting x = 0

RHL = 2 × 0 + 1 = 1

f(x) = 2x + 1

f (0) = 2 × 0 + 1 = 0 + 1 = 1

so, LHL = RHL = f (0)

⇒ f is continuous.

Thus f(x) = |x| + |x + 1| is continuous for all values of x.

Checking differentiability

CASE I: At x < – 1

f(x) = – 2x – 1

f'(x) = – 2

f(x) is polynomial.

⇒ f(x) is differentiable

CASE II: At x > 0

f(x) = 2x + 1

f'(x) = 2

f(x) is polynomial.

⇒ f(x) is differentiable

CASE III: At – 1 < x < 0

f(x) = 1

f(x) is constant.

⇒ f(x) is differentiable

CASE IV: At x = – 1

$$f(x) = \begin{cases} -2x-1, x \le -1 \\ 1, -1 < x < 0 \\ 2x+1, x \ge 0 \end{cases}$$

f is differentiable at x = – 1 if

LHD = RHD = f' (– 1)

i.e. $\lim_{h \to -1^-} \frac{f(-1)-f(-1-h)}{h} = \lim_{h \to -1^+} \frac{f(-1+h)-f(-1)}{h} = f'(-1)$

LHD = $\lim_{h \to -1^-} \frac{f(-1)-f(-1-h)}{h} = \lim_{h \to -1^-} \frac{-2\times(-1)-1-(-2\times(-1-h)-1)}{h} = \lim_{h \to -1^-} \frac{2-1(2+2h-1)}{h}$

LHD = $\lim_{h \to -1^-} \frac{1-2h-1}{h} = \lim_{h \to -1^-} \frac{-2h}{h} = -2$

RHD = $\lim_{h \to -1^+} \frac{f(-1+h)-f(-1)}{h} = \lim_{h \to -1^+} \frac{1-(-2\times(-1)-1}{h} = \lim_{h \to -1^+} \frac{1-1}{h} = 0$

Since, LHD ≠ RHD

∴ f is not differentiable at x = – 1

CASE V: At x = 0

f (x) = $\begin{cases} -2x-1, x \le -1 \\ 1, -1 < x < 0 \\ 2x+1, x \ge 0 \end{cases}$

f is differentiable at x = 0 if

LHD = RHD = f' (0)

i.e. $\lim_{h \to 0^-} \frac{f(0)-f(0-h)}{h} = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = f'(0)$

LHD = $\lim_{h \to 0^-} \frac{f(0)-f(0-h)}{h} = \lim_{h \to 0^-} \frac{2\times 0+1-1}{h} = 0$

RHD = $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{2\times(0+h)+1-(2\times 0+1)}{h} = \lim_{h \to -1^+} \frac{2h+1-1}{h} = 0$

Since, LHD ≠ RHD

∴ f is not differentiable at x = 0

So, f is not differentiable at exactly two-point x = 0 and x = 1, but continuous at all points.

Q. 22

If $\begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$

Answer:

Let y = $\begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$

Differentiation of determinant u = $\begin{vmatrix} e & f & g \\ h & i & j \\ k & l & m \end{vmatrix}$ is given b

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(e) & \frac{d}{dx}(f) & \frac{d}{dx}(g) \\ h & i & j \\ k & l & m \end{vmatrix} + \begin{vmatrix} e & f & g \\ \frac{d}{dx}(h) & \frac{d}{dx}(i) & \frac{d}{dx}(j) \\ k & l & m \end{vmatrix} + \begin{vmatrix} e & f & g \\ h & i & j \\ \frac{d}{dx}(k) & \frac{d}{dx}(l) & \frac{d}{dx}(m) \end{vmatrix}$$

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ 1 & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(m) & \frac{d}{dx}(n) \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ \frac{d}{dx}(a) & \frac{d}{dx}(b) & \frac{d}{dx}(c) \end{vmatrix}$$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 1 & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

Since, a, b, c and l, m, n are constants so, their differentiation is zero.

Also in a determinant if all the elements of row or column turns to be zero then the value of determinant is zero.

$$\therefore \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} = 0 \text{ and } \begin{vmatrix} f(x) & g(x) & f(x) \\ 1 & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$$

Hence, proved.

Q. 23 If, Y = $e^{a\cos^{-1}x}, -1 \leq x \leq 1$ show that (1 – x2) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Answer:

Given, y = $e^{a\cos^{-1}x}$

Taking logarithm both sides, we get

log y = log $e^{a\cos^{-1}x}$

⇒ log y = a cos – 1x log e

⇒ log y = a cos – 1x [log e = 1]

Differentiating both sides with respect to x

$\frac{1}{y}\frac{dy}{dx} = a \times -\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = -\frac{ay}{\sqrt{1-x^2}}$

Squaring both sides

$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$

⇒ (1 – x2) $\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Differentiating both sides

$\Rightarrow \left(\frac{dy}{dx}\right)^2 \frac{dy}{dx}(1-x^2) + (1-x^2) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}(y^2)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + (1-x^2) \times 2 \times \frac{d}{dx} \times \frac{d^2y}{dx^2} = a^2 \times 2y \times \frac{dy}{dx}$

$\Rightarrow -x \times \frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2y$

$\therefore (1 - x^2)\frac{d^2y}{dx^2} - x \times \frac{dy}{dx} - a^2y = 0$

Hence, proved