Ch-7 Discrete Time Fourier Analysis

CTFS — DTFS

CTFT → LT — DTFT → ZT

$x(t)$ CTFS

c_n

con. — t — f

P & C — D & AP

CTFT

$n\omega_0 \to \omega_0$ D → C

con. — t — f

AP & C — C & AP

C		D
t	→	n
T	→	N
t_0	→	n
$\int$	→	$\sum$
$\frac{d}{dt}$	→	∇
n	→	k

DTFS

c_k

$x[n]$ t — f

P & D — D & P

DTFT

AP & D — C & P

C → AP
AP → C
Dual

D → P
P → D
Dual

	t	f
CTFS	P & C	D & AP
CTFT	AP & C	C & AP
DTFS	P & D	D & P
DTFT	AP & D	C & P

Dual pairs

$\boxed{1}$ CTFS P & C ⤫ D & AP
DTFT AP&D ⤫ C & P

$\boxed{2}$ CTFT AP & C ⟷ C & AP → self dual

$\boxed{3}$ DTFS P & D ⟷ D & P → self dual

Q:- Match List-II with List-I

List-I	List-II (f)
A. Fourier Series	1. Discrete, Periodic
B. Fourier Transform	2. Continuous, Periodic
C. Discrete time fourier series	3. Discrete, Aperiodic
D. Discrete time fourier transform	4. Continuous, Aperiodic

Sol:- A - (3), B - (4), C - (1), D - (2)

CTFS

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$\omega_0 \downarrow \frac{2\pi}{T}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$\omega_0 = \frac{2\pi}{T}$$

$$\Omega = \frac{2\pi}{N}$$

$$e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k$$
$$= 1$$

AP

DTFS

$$\boxed{x[n] = \sum_{k=0}^{N-1} C_k e^{jk\left(\frac{2\pi}{N}\right)\cdot n}}$$

$$x[n+N] = x[n]$$

$$x[n+N] = \sum_{k=0}^{N-1} C_k e^{jk\left(\frac{2\pi}{N}\right)\cdot(n+N)}$$

$$= \sum_{k=0}^{N-1} C_k \cdot e^{jk\left(\frac{2\pi}{N}\right)\cdot n} \cdot e^{jk\left(\frac{2\pi}{\cancel{N}}\right)\cdot \cancel{N}}$$

$$x[n+N] = \sum_{k=0}^{N-1} C_k e^{jk\left(\frac{2\pi}{N}\right)\cdot n}$$

$$= x[n]$$

∴ It is periodic

P

$$C_k = \frac{1}{N}\sum_{k=0}^{N-1} x[n] \cdot e^{-jk\left(\frac{2\pi}{N}\right)\cdot n}$$

Periodic N — Periodic N — Periodic N

CTFT (ω)

$$X[j\omega] = X[\omega] = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X[\omega]\, e^{j\omega t}\, d\omega$$

DTFT (Ω)

$$X[j\Omega] = X[\Omega] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi}\sum_{n=-\pi}^{\pi} X[\Omega] e^{j\Omega n}$$

DTFT

t	f
D	P
A	C

CTFT

$x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

IFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{+j\omega t} d\omega$$

DTFT

$x[n]$

$$X[\Omega] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

IDTFT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{+j\omega n} d\omega$$

Existence of DTFT

- In CTFT, it was value of $x(t)$ should be absolutely integrable. i.e. $\int_{-\infty}^{\infty} x(t) dt < \infty$ then $|X(\omega)| < \infty$

- A sufficient condition for convergence of DTFT is

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

- Some sequences are not absolutely summable but they are square summable.
- There are some signals which are absolutely summable nor have a finite energy but still have DTFT.

Q:- Find DTFT of following signal:-

1) $x[n] = a^n u[n]$, $|a|<1$

$$X(\Omega) = X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n}$$

$$= 1 + a^1 e^{-j\omega} + a^2 e^{-j\omega 2} + \cdots\cdots$$

$$= 1 + (ae^{-j\omega})^1 + (ae^{-j\omega})^2 + \cdots\cdots$$

$$\boxed{X[e^{j\omega}] = \frac{1}{1 - ae^{-j\omega}} , \ |a|<1}$$

$$X[e^{j\omega}] = \frac{1}{1 - a[\cos\omega \cdot j\sin\omega]}$$

$$= \frac{1}{1 - a\cos\omega + aj\sin\omega} \times \frac{1 - a\cos\omega - aj\sin\omega}{1 - a\cos\omega - aj\sin\omega}$$

$$= \frac{1 - a\cos\omega - aj\sin\omega}{1 - 2a\cos\omega + a^2}.$$

$$X[e^{j\omega}] = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} - j\frac{a\sin\omega}{1 - 2a\cos\omega + a^2}$$

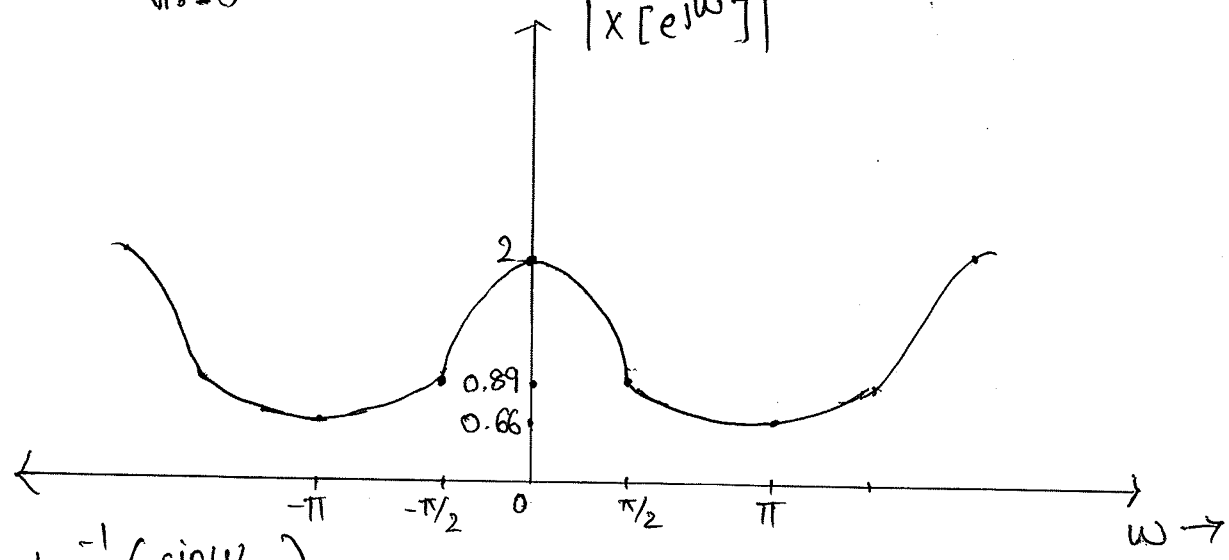
$$|X[e^{j\omega}]| = \frac{\sqrt{(1 - a\cos\omega)^2 + a^2\sin^2\omega}}{1 - 2a\cos\omega + a^2}$$

$$\boxed{|X[e^{j\omega}]| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}} \qquad \boxed{\phi = -\tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right)}$$

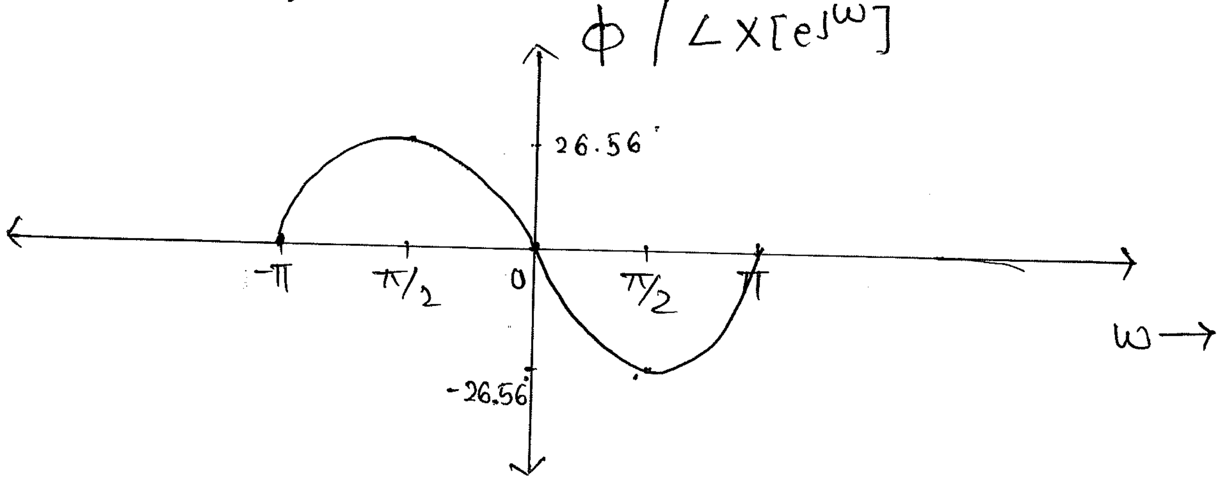
Now, magnitude and phase plot

let $a = 1/2$, $|a|<1$

$$|X[e^{j\omega}]| = \frac{1}{\sqrt{1.25-\cos\omega}}$$



$$\phi = -\tan^{-1}\left(\frac{\sin\omega}{2-\cos\omega}\right)$$

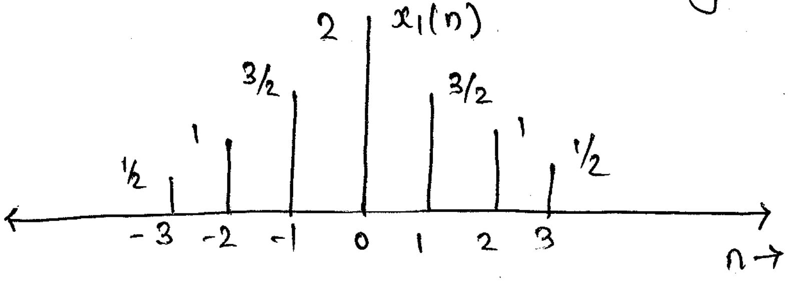


The signal is a periodic signal with fundamental period 2π

$$X[e^{j\omega}] = X[e^{j(\omega+2\pi)}] = X[e^{j\omega}\cdot e^{j2\pi}] = X[e^{j\omega}]$$

1:- Find F.T. of the following signal.

(a)



- Here, discrete time fourier transform will be applied

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-3}^{3} x(n) e^{-j\omega n}$$

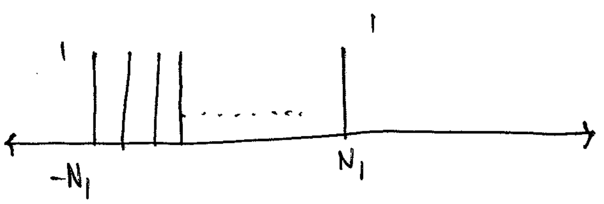
$$= \frac{1}{2} e^{+j\omega 3} + e^{+j\omega 2} + \frac{3}{2} e^{+j\omega} + 2 + \frac{3}{2} e^{-j\omega} + e^{-j\omega 2} + \frac{1}{2} e^{-j\omega 3}$$

$$= 2 + \frac{3}{2}\left[e^{j\omega} + e^{-j\omega}\right] + \frac{1}{2}\left[e^{j\omega 3} - e^{-j\omega 3}\right] + e^{j\omega 2} + e^{-j\omega 2}$$

$$X[e^{j\omega}] = 2 + 3\cos\omega + \cos 3\omega + \cancel{\tfrac{2}{2}} \cos 2\omega \cdot 2$$

→ periodic signal (ω = fundamental period).

(b)



$$X[e^{j\omega}] = \sum_{n=-N_1}^{N_1} 1 \cdot e^{-j\omega n}$$

$$\left[\sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r-1}\right]$$

$$= \frac{(e^{-j\omega})^{(N_1+1)} - (e^{-j\omega})^{-N_1}}{e^{-j\omega} - 1} = \frac{e^{j\omega N_1} - e^{-j\omega(N_1+1)}}{1 - e^{-j\omega}}$$

$$= \frac{\cancel{e^{-j\frac{\omega}{2}}}\left[e^{j\omega N_1}\cdot e^{j\frac{\omega}{2}} - e^{-j\omega(N_1+1)}\cdot e^{j\frac{\omega}{2}}\right]}{\cancel{e^{-j\omega/2}}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

$$= \frac{e^{j\omega N_1}\cdot e^{j\omega/2} - e^{-j\omega(N_1)}\cdot e^{-j\omega}\cdot e^{j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{e^{j\omega N_1}\cdot e^{j\omega/2} - e^{-j\omega N_1}\cdot e^{-j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

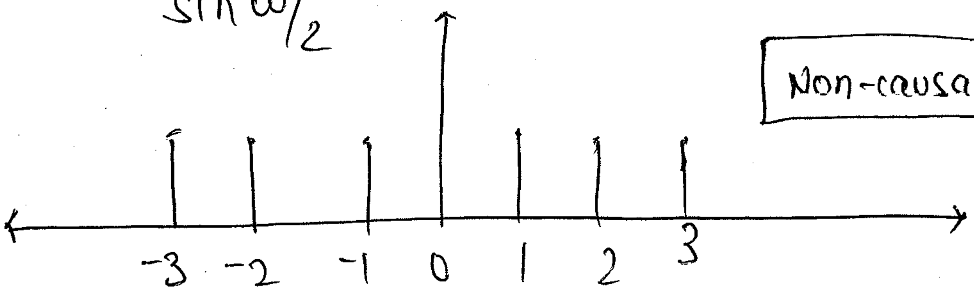
$$= \frac{e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$\boxed{X(z) = \frac{\sin\omega(N_1+1/2)}{\sin\omega/2}}$$

Q:- Determine whether or not the discrete time sequence with given frequency response are causal.

(a) $H(\omega) = \dfrac{\sin\left(\frac{7\omega}{2}\right)}{\sin(\omega/2)}$

Sequence $\xleftrightarrow{\text{DTFT}}$ $\dfrac{\sin \omega\left(N_1 + \frac{1}{2}\right)}{\sin \omega/2}$ $\quad$ $N_1 = 3$



$\boxed{\text{Non-causal}}$

Property of DTFT:-

① Linearity

If $x_1[n] \longleftrightarrow X_1[e^{j\omega}]$

$x_2[n] \longleftrightarrow X_2[e^{j\omega}]$

then

$\alpha_1 x_1[n] + \beta x_2[n] \longleftrightarrow \alpha_1 X_1[e^{j\omega}] + \beta X_2[e^{j\omega}]$

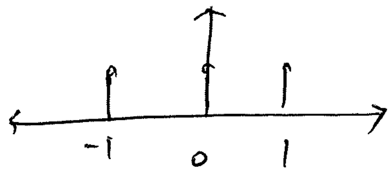
② Time shifting

If $x[n] \longleftrightarrow X[e^{j\omega}]$

$x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X[e^{j\omega}]$

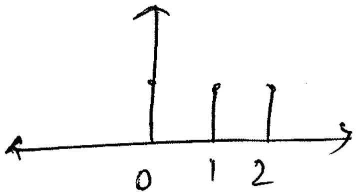
Q(b): $H[\omega] = \dfrac{\sin(3\omega/2)}{\sin(\omega/2)} e^{-j\omega}$. Check causality

$h(n) \longrightarrow -1$ to 1



non-causal

but $h(n-1)$



causal

NOTE:

$\delta(t) \xleftrightarrow{CTFT} 1$

$\delta[n] \xleftrightarrow{DTFT} 1$

(c) $H(\omega) = e^{-j3\omega} + e^{j2\omega}$

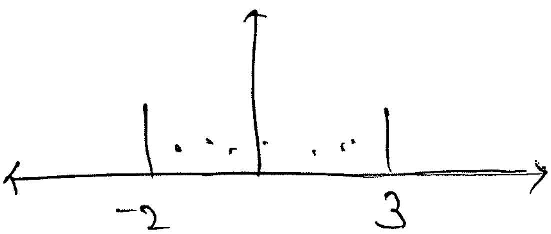
$H(\omega) = 1 \cdot e^{-j3\omega} + e^{j2\omega} \cdot 1$

$1 \xleftrightarrow{DTFT} \delta[n]$

$\delta[n] \longleftrightarrow 1$

$h[n] = \delta[n]\big|_{n=n-3} + \delta[n]\big|_{n=n+2}$

$h[n] = \delta[n-3] + \delta[n+2]$



non-causal

③ Frequency Shift:

If $x[n] \longleftrightarrow X[e^{j\omega}]$

then $e^{j\omega_0 n} x[n] \longleftrightarrow X[e^{j(\omega-\omega_0)}] = X[\Omega-\Omega_0]$

$e^{j\Omega_0 n} x[n]$

④ Time Scaling

If $x[n] \longleftrightarrow X[e^{j\omega}]$

then $x\left[\frac{n}{k}\right] \longleftrightarrow X\left[e^{j(\omega k)}\right]$

⑤ Time Reversal

If $x[n] \longleftrightarrow X[e^{j\omega}]$

$x[-n] \longleftrightarrow X[e^{-j\omega}]$

⑥ Modulation property

$x[n] \cdot \cos\omega_c n \longleftrightarrow \frac{X(\omega-\omega_c)+X(\omega+\omega_c)}{2}$

⑦ Frequency differentiation

$n \cdot x[n] \longleftrightarrow j\frac{dX[e^{j\omega}]}{d\omega}$

⑧ Conjugate property

$x^*[n] \longleftrightarrow X^*[e^{j\omega}] \mid X^*[-\Omega]$

⑨ Convolution

$x_1[n] * x_2[n] \longleftrightarrow X_1[e^{j\omega}] \cdot X_2[e^{j\omega}]$

⑩ Multiplication

$$x_1(n)\cdot x_2(n) \longleftrightarrow \frac{1}{2\pi}\left[X_1(e^{j\omega}) * X_2(e^{j\omega})\right]$$

⑪ ~~Frequen~~ First Difference

$$x[n]-x[n-1] \longleftrightarrow [1-e^{-j\omega}]\cdot X[e^{j\omega}]$$

⑫ Parseval's Theorem

$$x[n] \longleftrightarrow X[e^{j\omega}]$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad , \quad E_{x[n]} = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X[e^{j\omega}]|^2 d\omega$$

Q:- Let $x[n] = \left(\frac{1}{2}\right)^n u[n]$, $y[n] = x^2[n]$ and $Y(e^{j\omega})$ be the D.T.F.T of $y(n)$ then $Y[e^{j0}]$ is ____.

Sol^n:-

$$x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$$

$$X[e^{j\omega}] = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$x[n]\cdot x[n] = X[e^{j\omega}] * X[e^{j\omega}]$$

$$Y[e^{j\omega}] = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n}$$

$$Y[e^{j\omega}] = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\cdot e^{-j\omega n}$$

$$Y[e^{j0}] = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\cdot 1$$

$$Y[e^{j0}] = \frac{1}{1-1/4}$$

$$Y[e^{j0}] = \frac{4}{3} = 1.333$$

Q:- Given $X[e^{j\omega}] = \cos^3 3\omega$. Find the sum

$$S = \sum_{n=-\infty}^{\infty} (-1)^n \cdot x[n]$$

Sol^n: $X[e^{j\omega}] = \cos^3 3\omega$

$\cos 3\theta = 3\cos\theta - 4\cos^3\theta$

$\cos^3\theta = \dfrac{3\cos\theta - \cos 3\theta}{4}$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} \underline{x[n]\, e^{-j\omega n}}$$

$\xrightarrow{\omega=\pi} (-1)^n x[n]$

$$X[e^{j\pi}] = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\pi}$$

$$X[e^{j\pi}] = \sum_{-\infty}^{\infty} (-1)^n \cdot x[n]$$

$$X[e^{j\omega}]\Big|_{\omega=\pi} = \sum_{n=-\infty}^{\infty} (-1)^n \cdot x[n] = \cos^3(3\pi) = \underline{\underline{-1}}$$

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$\downarrow$

$-\pi$ to π

(DC gain $\omega = 0$) (high freq. gain Put $\omega = \pi$)

$$X[e^{j0}] = \sum_{n=-\infty}^{\infty} x[n] \cdot 1$$

HF. gain

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi n}$$

$$X[n] = \frac{1}{2\pi} \int_{2\pi} X[e^{j\omega}] \cdot e^{+j\omega n} \, d\omega$$

$n = 0$

$$x[0] = \frac{1}{2\pi} \int_{2\pi} X[e^{j\omega}] \, d\omega$$

$$\int_{2\pi} X[e^{j\omega}] \, d\omega = 2\pi \, x[0]$$

Q:- Find the DC high frequency gain

$$h[n] = \{\underset{\underset{n=0}{\uparrow}}{1}, 2, 3, 4\}$$

$$H[e^{j\omega}] = 1 + 2e^{-j\omega} + 3e^{-j\omega 2} + 4e^{-j3\omega}$$

D.C. gain $= 10$ $(\omega = 0)$

H.F. gain $= -2$ $(\omega = \pi)$

$$H[e^{j\omega}] = 1 + 2(-1) + 3(1) + 4(-1) = -2$$

Q:- Find the F.T. of

$$y[n] = \left(\frac{1}{4}\right)^n \cdot u(n-3)$$

Sol:- $y[n] = \left(\frac{1}{4}\right)^{(n-3)+3} \cdot u(n-3)$

$$y[n] = \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^{n-3} \cdot u(n-3) = \frac{1}{64} \left(\frac{1}{4}\right)^{n-3} \cdot u(n-3)$$

$$\Big\updownarrow \text{F.T.}$$

$$Y[e^{j\omega}] = \frac{1}{64}\left(\frac{e^{-j3\omega}}{1-\frac{1}{4}e^{-j\omega}}\right)$$

Q:- $y_2[n] = \delta[n-n_0]$

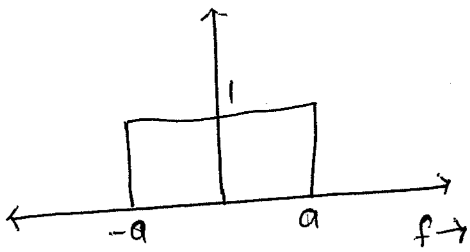
$$\Big\updownarrow \text{F.T.}$$

$$Y_2[e^{j\omega}] = 1\cdot e^{-jn_0\omega}$$

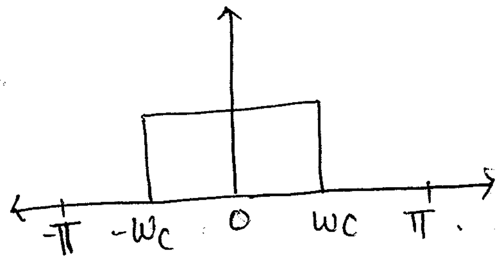
Q:- y_3

$\boxed{\text{NOTE:-}}$

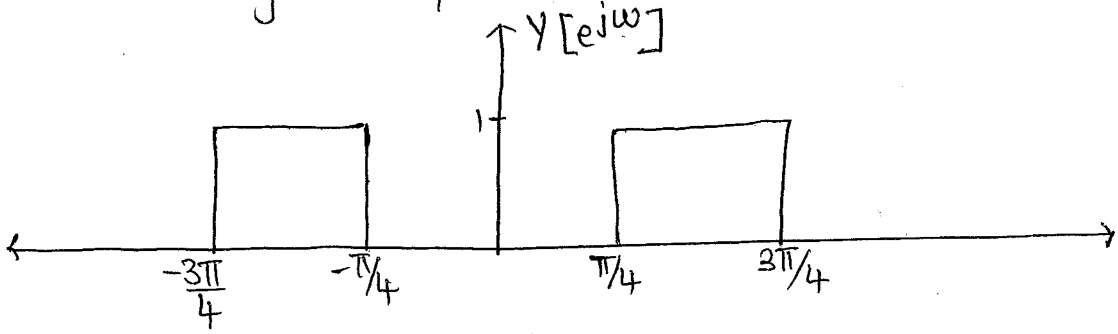
$$\frac{\sin at}{\pi t} \xleftrightarrow{\text{CTFT}}$$



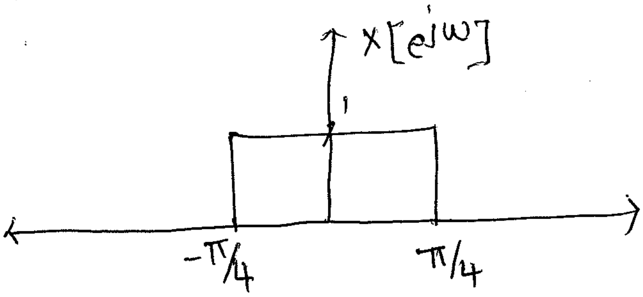
$$\frac{\sin \omega_c n}{\pi n} \xleftrightarrow{\text{DTFT}}$$



Q:- Find the signal as spectrum is shown in figure



Sol^n:-

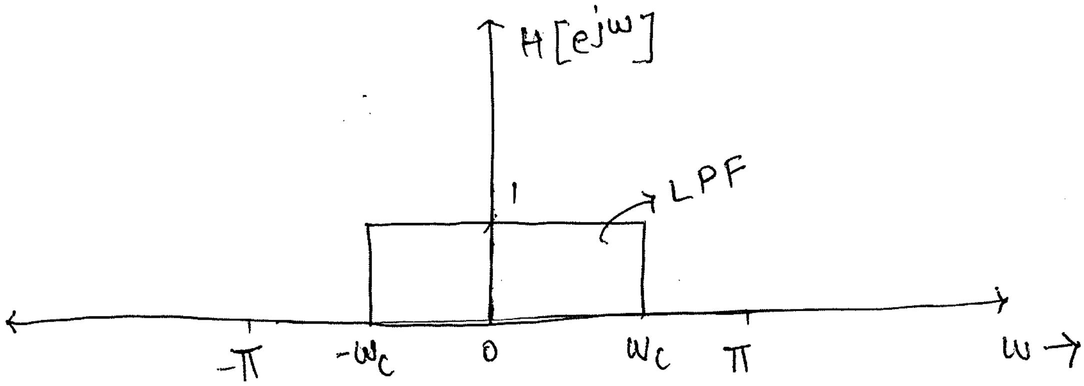


$$x[n] = \frac{\sin \frac{\pi}{4} n}{\pi n}$$

Now, shift $x[n]$ by $\pi/2$ So by using modulation property

$$y[n] = 2\frac{\sin \frac{\pi}{4} n}{\pi n} \cdot \cos \frac{\pi}{2} n$$

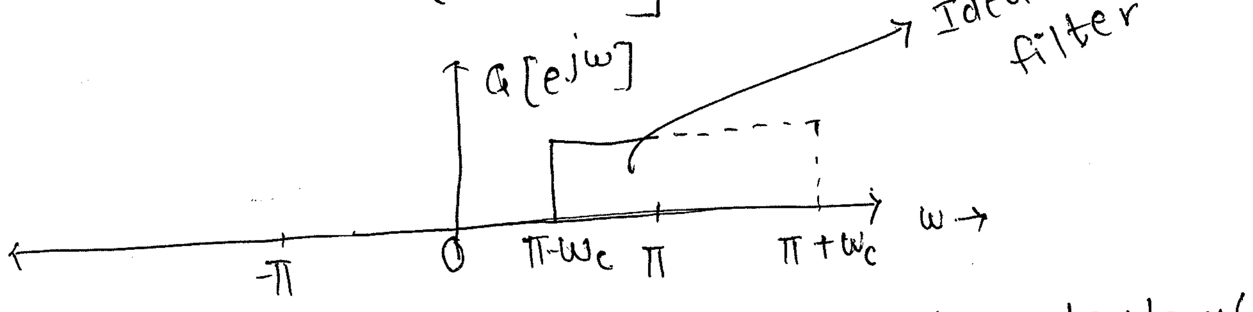
Q:- Let $h[n]$ is the impulse response of low pass filter with cut-off frequency ω_c. What type of filter has unit impulse response as $g[n] = (-1)^n h[n]$



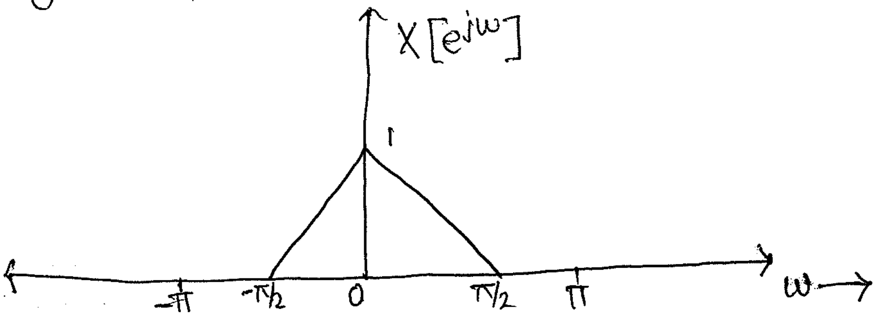
$$g[n] = (-1)^n \cdot h[n]$$

$$g[n] = e^{j\pi n} h[n]$$

$$G[e^{j\omega}] = H[e^{j(\omega-\pi)}]$$

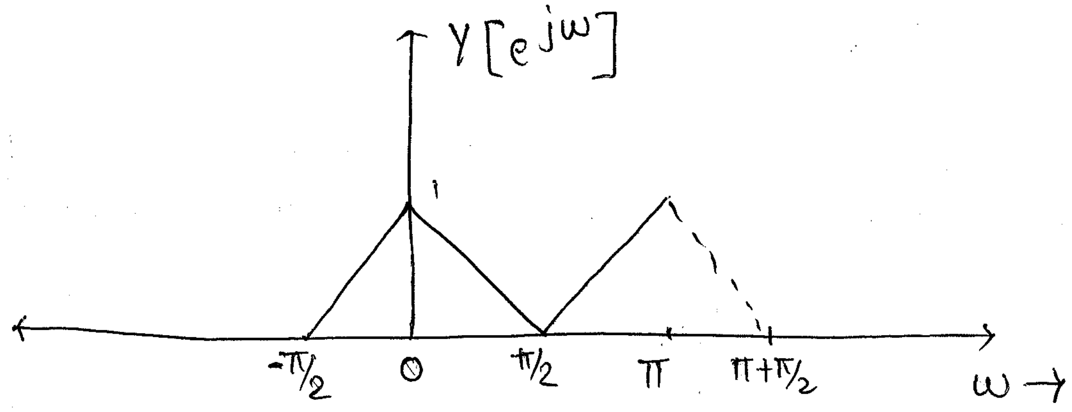


Q:- A discrete system with input $x(n)$ and output $y(n)$ are related as $y(n) = x(n) + (-1)^n x(n)$ If the input spectrum $X[e^{j\omega}]$ is shown in figure the output spectrum at $\omega = 0$ and $\omega = \pi$ are ?



$y[n] = x[n] + (-1)^n x[n]$

$Y[e^{j\omega}] = X[e^{j\omega}] + X[e^{j(\omega-\pi)}]$



$Y[e^{j\omega}]\Big|_{\omega=0} = 1 \quad , \quad Y[e^{j\omega}]\Big|_{\omega=\pi} = 1$

Q:- Find the inverse fourier transform of

$$Y[e^{j\omega}] = \frac{1}{1 - \frac{1}{2} e^{-j10\omega}}$$

$$y[n/k] \longleftrightarrow Y[e^{jk\omega}]$$

We know that,

$$a^n . u(n) \longleftrightarrow \frac{1}{1-ae^{-j\omega}} \quad ; |a|<1$$

$$\left(\frac{1}{2}\right)^n . u(n) \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\boxed{\left(\frac{1}{2}\right)^{\frac{n}{10}} . u\left(\frac{n}{10}\right)} \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-j10\omega}}$$

Q:- If DTFT of $x[n] = \left(\frac{1}{5}\right)^n \cdot u[n+2] \xleftrightarrow{DTFT} X[e^{j\omega}]$

Find the sequence $y[n]$ which has DTFT $Y[e^{j\omega}] = X[e^{j2\omega}]$

$$Y[e^{j\omega}] = X[e^{j2\omega}]$$

$$\Rightarrow y[n] = x\left[\frac{n}{2}\right] = \left(\frac{1}{5}\right)^{n/2} \cdot u\left(\frac{n}{2}+2\right)$$

Q:- Find the DTFT of $y[n] = n \cdot a^n u[n]$

$$y[n] = a^n \cdot u[n]$$

$$Y[e^{j\omega}] = \frac{1}{1-ae^{-j\omega}}$$

Now, $y[n] = n \cdot a^n \cdot u[n]$

$$= j \frac{d}{d\omega} \frac{1}{(1-ae^{-j\omega})}$$

$$= j \frac{+ae^{-j\omega} \cdot (-j)}{(1-ae^{-j\omega})^2}$$

$$Y[e^{j\omega}] = \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

Q:- Find the DTFT of $x[n] = n \cdot e^{\frac{jn\pi}{8}} \alpha^{n-3} u[n-3]$

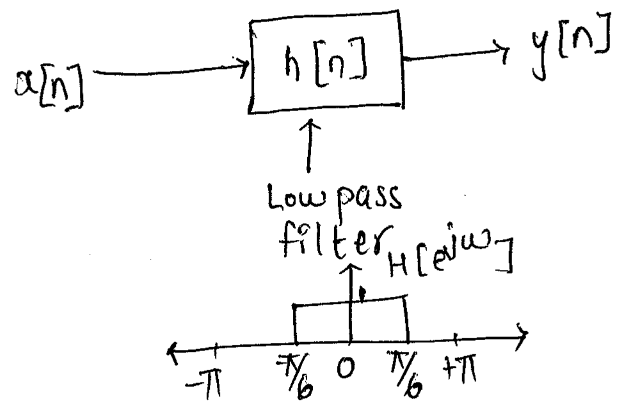
$$x'[n] = n\, \alpha^{n-3} \cdot u[n-3]$$

$$X'[e^{j\omega}] = j \frac{d}{d\omega}\left(\frac{e^{-j3\omega}}{1-\alpha e^{-j\omega}}\right) \qquad X[e^{j\omega}] = j\frac{d}{d\omega} \frac{e^{-j3(\omega-\frac{\pi}{8})}}{1-\alpha e^{-j(\omega-\pi/8)}}$$

Now, time shifting

$$e^{j\omega_0 n} \longleftrightarrow X[e^{j(\omega-\omega_0)}]$$

Q:- Consider $x[n] = \sin\left(\frac{n\pi}{8}\right) - 2\cos\left(\frac{n\pi}{4}\right)$. If the impulse response is $h[n] = \frac{\sin\left(\frac{\pi n}{6}\right)}{\pi n}$. Then the output $y[n]$ is ___.



Only between 0 to $\pi/6$ will be passed

$x[n] = \sin\frac{n\pi}{8} - 2\cos\frac{n\pi}{4}$

✓ ✗

$$\boxed{y[n] = \sin\frac{n\pi}{8}}$$

Q:- Design 3.5 FIR filter with $h[n] = \{\alpha, \underset{\uparrow}{\beta}, \alpha\}$ and the magnitude response blocks the frequency $f = 1/3$ and the frequency at $f = 1/8$ unity gain. What is the DC gain of filter?

→ $H[e^{j\omega}] = \alpha \cdot e^{+j\omega} + \beta + \alpha e^{-j\omega}$, $H[e^{j\omega}]\big|_{f=0} = ?$ DC gain

$H[e^{j\omega}]\big|_{f=1/3} = 0 \Rightarrow \alpha e^{j/3} + \beta + \alpha e^{-j/3} = 0$

$\beta + \alpha 2\left(\frac{e^{j/3} + e^{-j/3}}{2}\right) = 0$

$H[e^{j\omega}]\big|_{f=1/8} = 1$

$\beta + 2\alpha\cos\omega\big|_{f=1/3}$

$\beta + 2\alpha\cos\frac{2\pi}{3} = 0$

$$\boxed{\alpha = \beta}$$

$$H[e^{j\omega}]\Big|_{f=1/8} = 1$$

$$= \beta + \alpha e^{j\omega} + \alpha e^{-j\omega}$$

$$= \beta + 2\alpha \cos \omega$$

$$= \sqrt{2}\alpha + \beta = 1$$

$$\alpha = \beta = \frac{1}{1+\sqrt{2}}$$

$$H[e^{j\omega}]\Big|_{f=0} = 3 \cdot \alpha = \frac{3}{1+\sqrt{2}} = 1.24$$

In CTFT,

$$x(t) \longrightarrow \underset{H(\omega)}{\boxed{h(t)}} \longrightarrow y(t)$$

① $x(t) = e^{j\omega_1 t} \longrightarrow y(t) = e^{j\omega_1 t} H(\omega_1)$

In LT,

$$x(t) \longrightarrow \boxed{h(s)} \longrightarrow y(t)$$

$x(t) = e^{s_1 t} \longrightarrow y(t) = e^{s_1 t} H[s_1]$

In DTFT,

$$x[n] \longrightarrow \underset{H[e^{j\omega}]}{\boxed{h[n]}} \longrightarrow y[n]$$

If $x[n] = e^{j\omega_1 n} \longrightarrow y[n] = e^{j\omega_1 n} \cdot H[e^{j\omega_1}]$

Q:- In a LTI system if $h(n) = \begin{cases} 4\sqrt{2} & ; n = 2, -2 \\ -2\sqrt{2} & ; n = 1, -1 \\ 0 & ; \text{otherwise} \end{cases}$

find the o/p when input applied is $x(n) = e^{jn\pi/4}$

Sol^n:-

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$H[e^{j\omega}]$$

if $x[n] = e^{j\omega_0 n} \longrightarrow y[n] = e^{j\omega_1 n} \cdot H[e^{j\omega_1}]$

$$H[e^{j\omega}] = 8\sqrt{2}\cos 2\omega - 4\sqrt{2}\cos\omega$$

$$y[n] = e^{jn\pi/4} \; H[e^{j\pi/4}]$$

$$y[n] = e^{jn\pi/4} \cdot (-4)$$

*Digital Filters

① $y[n] = x[n] - x[n-1]$

$$H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]}$$

$$x[n] \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow y[n]$$

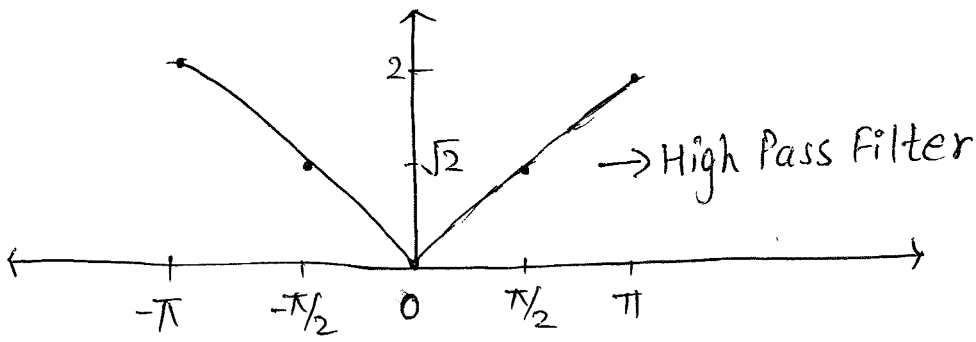
$$Y[e^{j\omega}] = X[e^{j\omega}]\,[1 - e^{-j\omega}]$$

$$H[e^{j\omega}] = 1 - e^{-j\omega}$$

$$|H[e^{j\omega}]|\Big|_{\omega=0} = 1 - e^{-j0} = 1 - 1 = 0$$

$$|H[e^{j\omega}]|\Big|_{\omega=\pi/2} = 1 - e^{-j\pi/2} = |1 + j| = \sqrt{2}$$

$$H[e^{j\omega}]\Big|_{\omega=\pi} = 1 - (-1) = 2$$



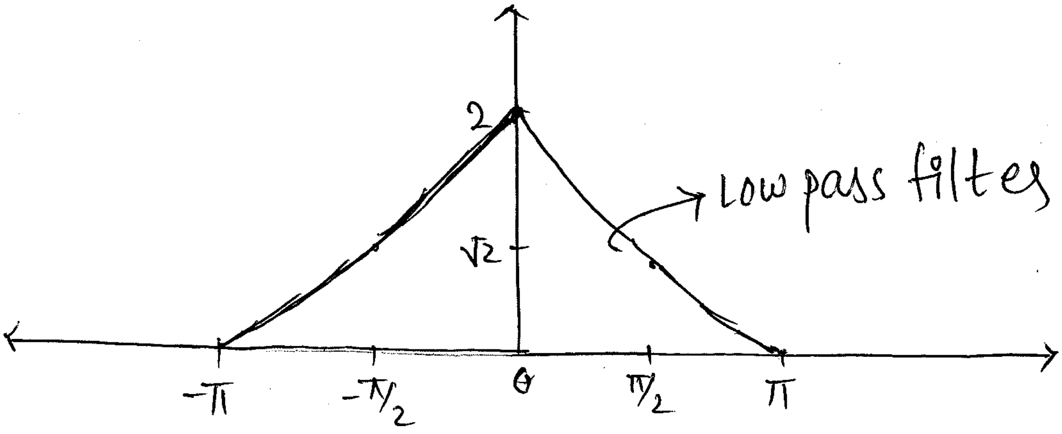
② $y[n] = x[n] + x[n-1]$

$$Y[e^{j\omega}] = X[e^{j\omega}]\,[1 + e^{-j\omega}]$$

$$H[e^{j\omega}] = 1 + e^{-j\omega}$$

$$H[e^{j\omega}]\Big|_{\omega=0} = 1 + 1 = 2 \quad , \quad H[e^{j\omega}]\Big|_{\omega=\pi/2} = |1 + j| = \sqrt{2}$$

$$H[e^{j\omega}]\Big|_{\omega=\pi} = \cancel{8}\,0$$



(3) $h[n] = \delta[n] - \delta[n-2]$

$H[e^{j\omega}] = 1 - e^{-j2\omega}$

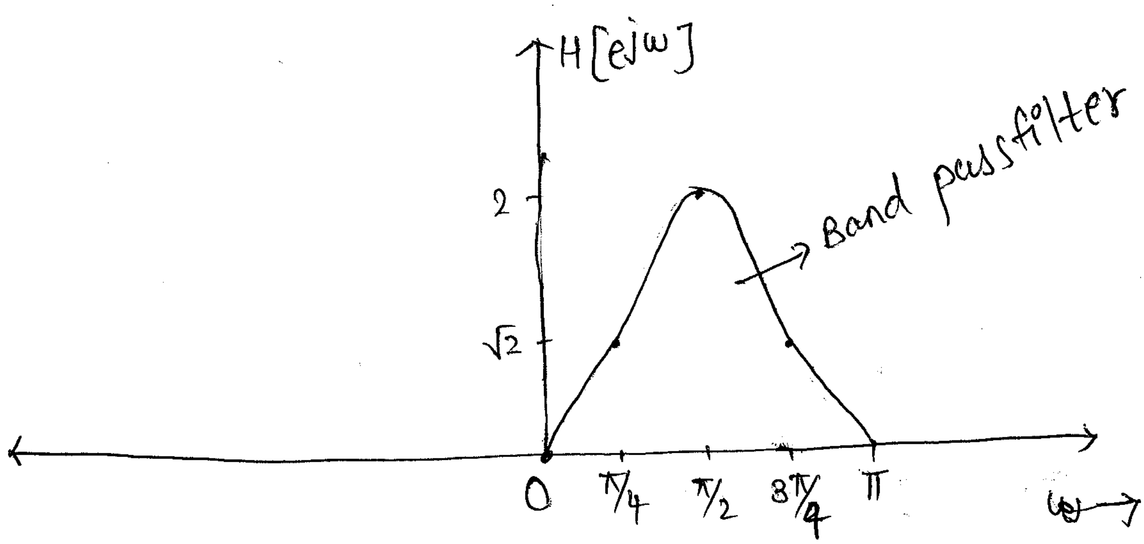
$|H[e^{j\omega}]|_{\omega=0} = 0$

$|H[e^{j\omega}]|_{\omega=\frac{3\pi}{4}} = |1+j| = \sqrt{2}$

$|H[e^{j\omega}]|_{\omega=\pi/4} = |1+j| = \sqrt{2}$

$|H[e^{j\omega}]|_{\omega=\pi} = 1-(1) = 0$

$|H[e^{j\omega}]|_{\omega=\pi/2} = 1-(-1) = 2$



(4) $h[n] = \delta[n] + \delta[n-2]$

$H[e^{j\omega}] = 1 + e^{-j2\omega}$

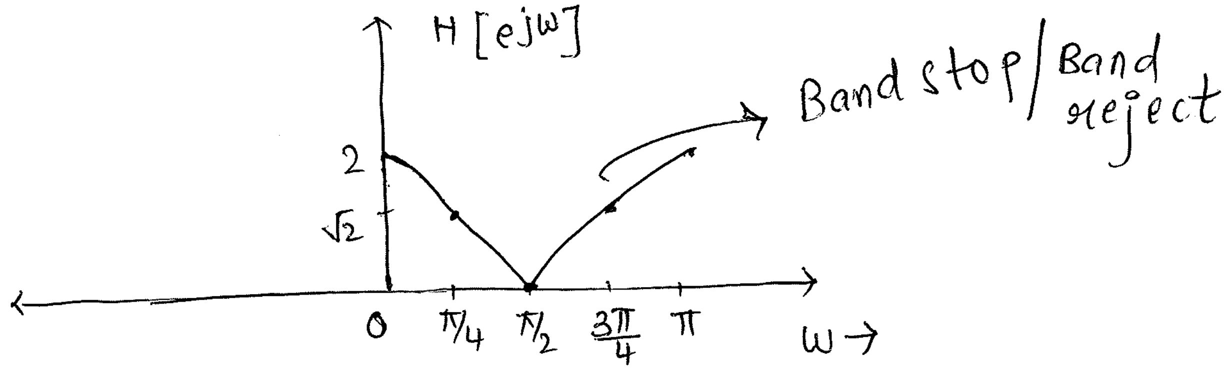
$|H[e^{j\omega}]|_{\omega=0} = 2$

$|H[e^{j\omega}]|_{\omega=\pi/2} = 0$

$|H[e^{j\omega}]|_{\omega=\pi/4} = 1+j = \sqrt{2}$

$|H[e^{j\omega}]|_{\omega=3\pi/4} = \sqrt{2}$

$|H[e^{j\omega}]|_{\omega=\pi} = 2$



Q:- Consider the system describe by eqⁿ $y[n] = ay[n-1] + bx[n] + x[n-1]$ where a and b are real. Find the relation between a and b such that $|H[e^{jw}]| = 1$

$$y[n] = ay[n-1] + bx[n] + x[n-1]$$

$$Y[e^{jw}] - a e^{-jw} Y[e^{jw}] = X[e^{jw}]\,[b + e^{-jw}]$$

$$Y[e^{jw}]\,[1 - a e^{-jw}] = X[e^{jw}]\,[b + e^{-jw}]$$

$$Y[e^{jw}] = X[e^{jw}] \left(\frac{b + e^{-jw}}{1 - a e^{-jw}} \right)$$

$$H[e^{jw}] = \frac{b + e^{-jw}}{1 - a e^{-jw}}$$

$$|H[e^{jw}]|^2 = 1$$

$$H[e^{jw}] \cdot H^*[e^{jw}] = |H[e^{jw}]|^2 = 1$$

$$\left(\frac{b + e^{-jw}}{1 - a e^{-jw}} \right) \left(\frac{b + e^{jw}}{1 - a e^{jw}} \right) = 1$$

$$\frac{b^2 + b e^{jw} + b e^{-jw} + e^0}{1 + a^2 \cancel{e^{2jw}} - a e^{jw} - a e^{-jw}} = 1 \quad \ldots \text{So, } a = -b$$

But,

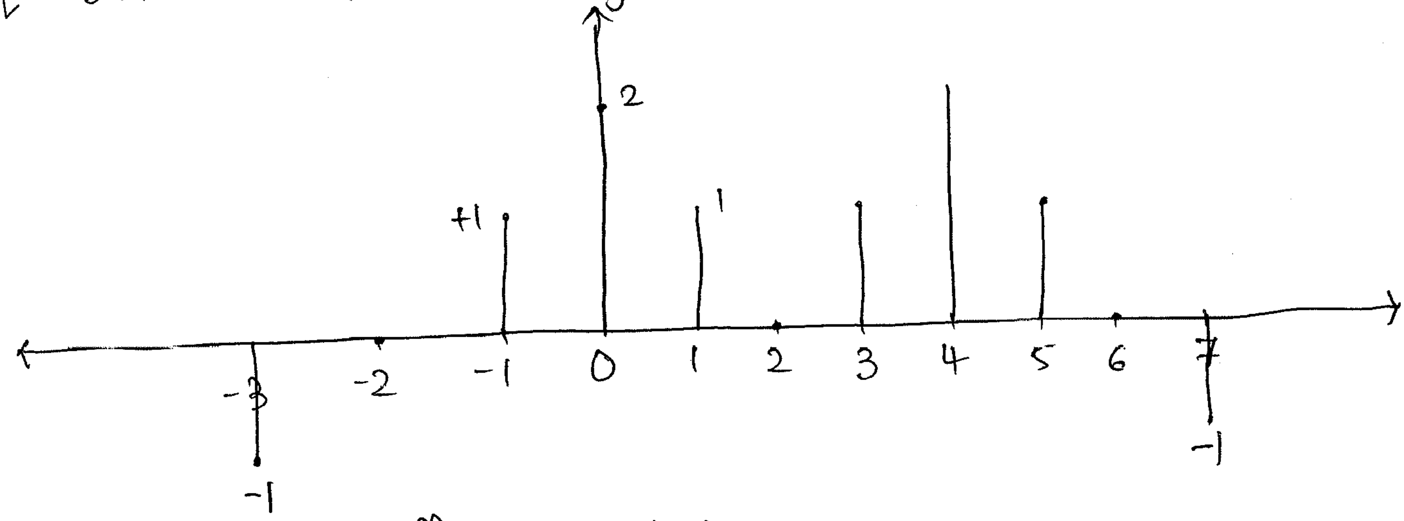
$$|H[e^{j\omega}]| = 1 \ , \forall \omega$$

$$|H[e^{j\omega}]|_{\omega=0} = \frac{b + e^{-j\omega}}{1 - a e^{-j\omega}} = \frac{b+1}{1-a} = 1$$

$$b + 1 = 1 - a$$

$$\boxed{a = -b}$$

Q:- For the signal shown in figure find the following quantity ~~DTFT~~ without using DTFT.



$$\rightarrow X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X[e^{j0}] = \sum_{n=-\infty}^{\infty} x[n]$$

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X[e^{j\omega}] \cdot e^{+j\omega n} d\omega$$

(a) $X[e^{j0}] = 6$

(b) $X[e^{j\pi}] = \sum_{n=-\infty}^{\infty} (-1)^n \cdot x[n]$

$$= \cancel{1} \cancel{-1} + \cancel{2} - \cancel{1} - 1 + 2 - 1 + \cancel{1} = 2$$

(c) $\int_{-\pi}^{\pi} X[e^{j\omega}] \cdot d\omega = 2\pi x(0) = 2\pi \cdot 2 = 4\pi$

(d) $\int_{-\pi}^{\pi} X[e^{j\omega}] \cdot e^{j2\omega} d\omega = 2\pi x(2) = 0$

(e) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega =$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega \quad (\because \text{Parseval's Theorem})$$

$= 2\pi [1+1+4+1+1+4+1+1]$

$= 2\pi(14) = 28\pi$

(f) $\int_{-\pi}^{\pi} \left| \frac{d}{d\omega} X[e^{j\omega}] \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |n x(n)|^2$

$= 2\pi [9+0+1+0+1+0+9+64+25+49]$

$= 2\pi [64+45+49]$

$= 2\pi [158]$

$= 316\pi$

Distortionless Transmission

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]$$

$$H[e^{j\omega}]$$

$$y[n] = k \cdot x[n-n_0]$$

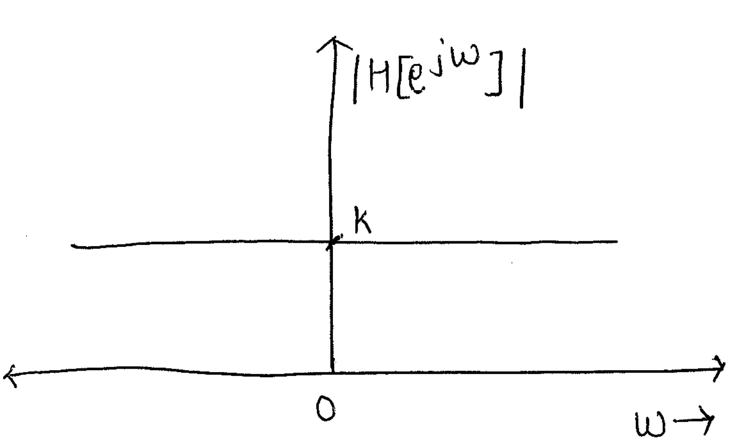
$$Y[e^{j\omega}] = k \cdot e^{-j\omega n_0} \cdot X[e^{j\omega}]$$

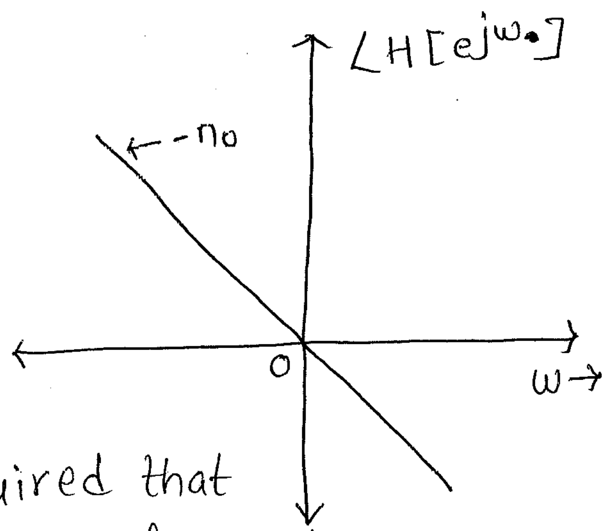
$$H[e^{j\omega}] = \frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = k \cdot e^{-j\omega n_0}$$

$$H[e^{j\omega}] = k \cdot e^{-j\omega n_0}$$

$$|H[e^{j\omega}]| = \text{Magnitude} = |ke^{-j\omega n_0}| = |k| = k$$

$$\phi = \angle H[e^{-j\omega}] = -\omega n_0 = -\Omega n_0$$





- In many applications it is required that output waveform or signal should be replica of input
- Transmission is said to be distortionless if input $x(n)$ and output $y(n)$ satisfy the condition

$$y[n] = k \cdot x[n-n_0]$$

$$H[e^{j\omega}] = k \cdot e^{-j\omega n_0}$$

$\therefore$ The equivalent condition in frequency domain is

(i) Magnitude of $|H[e^{j\omega}]|$ should be constant

$$\text{Magn.} = |H[e^{j\omega}]| = k$$

(ii) The phase response $\angle H[e^{j\omega}]$ must be linear function of frequency ω

$$\phi = \angle H[e^{-j\omega}] = -\omega n_0 = -\Omega n_0$$

Phase Delay

- Time delay experienced by single frequency signal when the signal passes through system is referred to as phase delay and is given by

$$\boxed{\tau_p(\Omega) = \frac{-\angle H[e^{j\omega_0}]}{\omega_0}}$$

Group Delay

- Time delay experienced by group of frequency when group of signal frequencies that contains components with different frequencies [not harmonically related] passes through a system is refered to as group delay.

$$\boxed{\tau_g(\omega) = -\frac{d}{d\omega} \angle H(e^{j\omega})}$$

Energy spectral density

- According to Parseval's theorem total energy is given by

$$E_x = \sum_{-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X[e^{j\omega}]|^2 \, d\omega$$

ESD : $\frac{\text{Energy}}{\text{Bandwidth}}$. Energy per unit bandwidth is known as energy spectral density as it is denoted by $S_x(\Omega)$.

$$\therefore \boxed{S_x(\Omega) = |X[\Omega]|^2}$$

Power spectral density

- PSD is power per unit bandwidth

$$S_x(\Omega) = |X[\Omega]|^2$$

Relation between output PSD and input PSD signal.

$$X[e^{j\omega}] \longrightarrow \boxed{H[e^{j\omega}]} \longrightarrow Y[e^{j\omega}]$$

$$S_x[e^{j\omega}] = |X[e^{j\omega}]|^2$$

$$S_Y[e^{j\omega}] = |X[e^{j\omega}]|^2$$

$$y[n] = x[n] * h[n]$$

$$Y[e^{j\omega}] = X[e^{j\omega}] \cdot H[e^{j\omega}]$$

$$|Y[e^{j\omega}]|^2 = |H[e^{j\omega}]|^2 \cdot |X[e^{j\omega}]|^2$$

$$\boxed{S_Y[e^{j\omega}] = |H[e^{j\omega}]|^2 \cdot S_x[e^{j\omega}]}$$

Z-Transform

- Limitation of DTFT

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

- DTFT exists only for absolutely summable signals.
- DTFT doesn't exists for exponential or even linearly growing signal.
- DTFT method can be applied only for BIBO stable system. It can't be used for unstable or marginally stable system.

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}$$

$x[n] \longrightarrow$ unstable

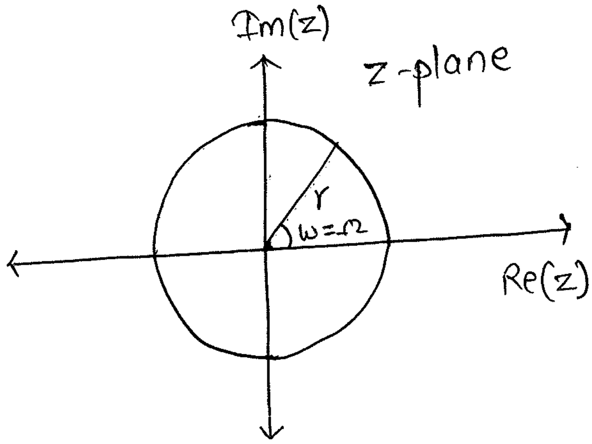
$\downarrow$

$x[n] \cdot r^{-n}$

$\infty \quad \cdot 0 \quad = 0$

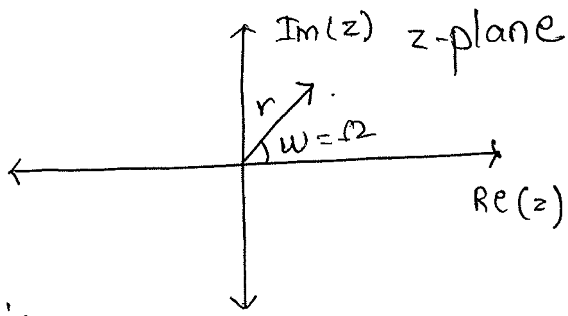
$\downarrow$

stable



$$\therefore X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n] \cdot r^{-n} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \, (re^{+j\omega})^{-n}$$



Let $z = re^{j\omega}$

$$\therefore \boxed{X[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}} \quad ; \quad z = re^{j\omega}$$

$|z| = r$

$\angle z = \omega = \Omega$

$\downarrow$

Z-Transform