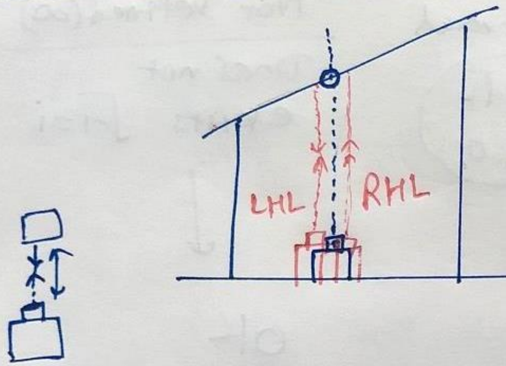
LIMITS & DERIVATIVES

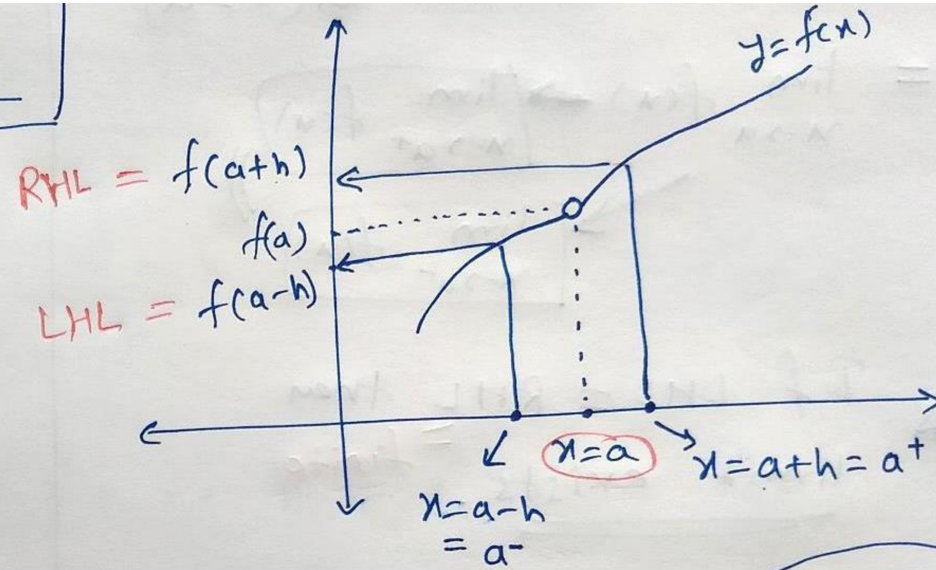
Limit:

machine.

$d = 5 \times t$



Definition. :→ limit of a function $y = f(x)$ at $x = a$ is the value of function in the neighborhood of $x = a$.



$h \approx 0$ $\quad$ $h = 0.000000$

$h \to 0$

RHL = Right Hand limit $= \lim\limits_{x \to a^+} f(x)$

$= \lim\limits_{x \to a+h} f(x) = f(a^+) = \lim\limits_{h \to 0} f(a+h)$

LHL = Left Hand limit $= \lim\limits_{x \to a^-} f(x)$

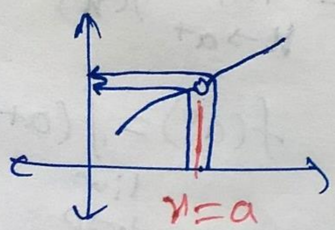
$= \lim\limits_{x \to a-h} f(x) = f(a^-) = \lim\limits_{h \to 0} f(a-h)$

★ limit of a function at $(x=a)$

$$= \lim_{x\to a} f(x) \rightarrow \boxed{\lim_{x\to a^+} f(x)}, \quad \boxed{\lim_{x\to a^-} f(x)}$$

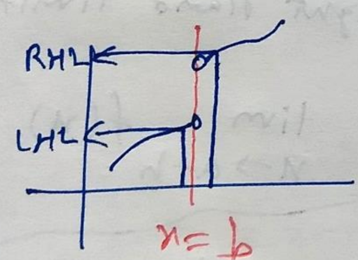
★ If LHL = RHL then limit exists. = finite

★ If LHL ≠ RHL then limit does not exists.



LHL = RHL

Limit exists



LHL ≠ RHL

limit Does not exist

Values → $f(x) = 0$

Normal	Not Defined (∞)	Indeterminate Forms
5, $\sqrt{2}$, 3, 0,	Does not exists $\sqrt{-1} = i$	
↓	↓	↓
OK	OK	Dangerous

★ Chapter-13 Limit

Indeterminate Forms

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty \cdot 0,\ \infty - \infty,\ 0^0,\ \infty^0,\ 1^\infty$$

Not exact

$$\boxed{(\text{exact } 1)^\infty = 1}$$

e.g. $\frac{0}{0}$

Find the value of function $f(x)=\dfrac{x^2-9}{x-3}$ at $x=3$.

Domain $x-3\neq 0$

$x\neq 3$

we can not find the value of $f(x)$ at $x=3$ but we can find (value of $f(x)$ in the neighborhood of $x=3$.)

→ limit at $x=3$

$$f(3)=\frac{(3)^2-9}{(3)-3}=\frac{9-9}{3-3}=\frac{0}{0}$$

Indeterminate form.

$$f(x)=\frac{x^2-9}{x-3}$$

By Factorisation

$$f(x)=\frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$$

$x=3$

$$f(x)=x+3$$

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^2-9}{x-3} \quad \frac{0}{0}$$

$$=\lim_{x\to 3}(x+3)$$

$$=3+3$$

$$=6$$

$RHL=3^{+}+3$
$=6^{+}$

$LHL=3^{-}+3$
$=6^{-}$

Algebra of limits

(i) $$\lim_{x\to a}\left(f(x) \pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

(ii) $$\lim_{x\to a}\left(f(x) \times g(x)\right) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

(iii) $$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

(iv) $$\lim_{x\to a}\left(K \cdot f(x)\right) \longrightarrow K \cdot \lim_{x\to a} f(x)$$

(v) $$\lim_{x\to a}\left(f(x)\right)^{g(x)} = \left(\lim_{x\to a} f(x)\right)^{\left(\lim_{x\to a} g(x)\right)}$$

Limit for Polynomial Functions. = Continuous

$f(x) = 3x+1$

→ do not calculate limits Differently ~~LHL~~, ~~RHL~~

→ For limit, Directly put the value of x.

e.g. $f(x) = 3x+1$

limit at $x = 2$ = ?

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (3x+1)$

$= 3(2)+1 = 7$ ✓

Limit for Rational F^n. $= \dfrac{f(x)}{g(x)}$

$\left(\dfrac{0}{0}\right)$ → By Factorisation

→ By Standard Form

$$\boxed{\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}}$$

e.g. $\dfrac{2x+3}{x-1}$, $\dfrac{x^2+3x+5}{x^3-9}$

Proof: of $\lim_{x \to a} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}$

LHS $= \lim_{x \to a} \frac{x^n - a^n}{x-a}$ Factorise $\frac{0}{0}$

$= \lim_{x \to a} \frac{\cancel{(x-a)}\left(x^{n-1} + x^{n-2} \cdot a^1 + x^{n-3} \cdot a^2 + \cdots + x^0 \cdot a^{n-1}\right)}{\cancel{(x-a)}}$

$= \underline{a^{n-1}} + \underline{a^{n-2} \cdot a^1} + \underline{a^{n-3} \cdot a^2} + \cdots + a^0 \cdot a^{n-1}$

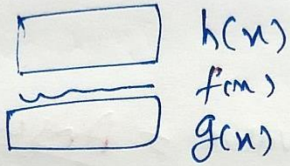
$= \underbrace{\underline{a^{n-1}} + \underline{a^{n-1}} + \underline{a^{n-1}} + \cdots + \underline{a^{n-1}}}_{n \text{ times}}$

$= n \cdot a^{n-1}$

e.g. $\lim_{x \to 3} \frac{x^2 - 9}{x-3} = n \cdot a^{n-1} = 2 \cdot 3^{2-1} = 2 \times 3^1 = 6$

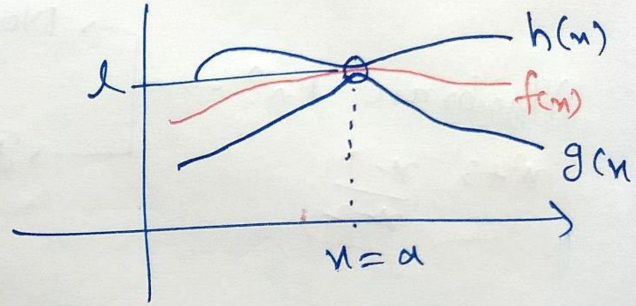
$n=2$ $a=3$

Sandwich Theorem :



Always

$$g(x) \le f(x) \le h(x)$$



$$\Rightarrow \lim_{x\to a} g(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} h(x)$$

Tough

$l \qquad l \qquad l$

Easy $\qquad$ Easy

$\theta \to 0$

$\sin\theta \to \theta$

By Sandwich theorem

Important Result

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$ ☆

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\frac{\sin 0}{0} = \frac{0}{0}$$

$$\lim_{f(x)\to 0} \frac{\sin f(x)}{f(x)} = 1$$

$$\boxed{\lim_{x\to 0} \cos x = 1.}$$

Final - Conclusion:

Algebraic F^n (x^2+3, $\frac{x-1}{x-2}$)
- → Normal value = Answer ✓
- → Indeterminate Form ($\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0\times\infty$)
  - → By Factorisation
  - → By Standard Form $\lim_{x\to a} \frac{x^n-a^n}{x-a} = n.a^{n-1}$

Trigonometric F^n.
- → Normal Value ✓
- → Indeterminate Form ($\frac{0}{0}, ---$) → By Standard Form $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Piecewise F^n → $\boxed{LHL}$ & $\boxed{RHL}$ →
- If $LHL = RHL$ (= finite) → Limit exists
- If $LHL \neq RHL$ → Limit Does not exists.

$$f(x) = \begin{cases} x+3, & x \geq 5 \text{ RHL} \\ x-1, & x < 5 \text{ LHL} \end{cases}$$

$\lim_{x\to 5} f(x)$
- $LHL = f(5^-) = 4$
- $RHL = f(5^+) = 8$ #

Exercise - 12.1

Function $y = f(x)$ (limit at $x = a$) $\lim_{x \to a}$ → put $x = a$ in $y = f(x)$ (Directly) →

→ Normal No. $(2, 3, -1, \frac{1}{2}, 0, \infty \ldots)$ = Answer

→ Indeterminate Form $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty\right)$ → ways

- Factorisation
- Standard forms
  - $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$
  - $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Q.1 $\lim_{x \to 3} x+3$

(Directly put $x=3$)

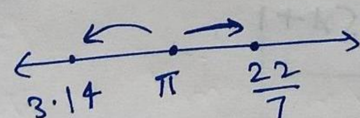
$= 3+3$

$= 6$

Q.2 $\lim_{x \to \pi} \left(x - \frac{22}{7}\right)$

Put $x = \pi$ Directly

$= \pi - \frac{22}{7}$



Q.3 $\lim_{r \to 1} \pi r^2$

Put $r = 1$

$= \pi(1)^2$

$= \pi$

Q.4 $\lim_{x \to 4} \frac{4x+3}{x-2}$

$x = 4$ put.

$= \frac{16+3}{4-2} = \frac{19}{2}$

Q.5 $\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Put $x = -1$

$= \frac{(-1)^{10} + (-1)^5 + 1}{(-1) - 1}$

$= \frac{\cancel{1} - \cancel{1} + 1}{-1-1} = \frac{1}{-2}$ ✓

Q.6 $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

Put $x = 0$ $\frac{1^5 - 1}{0} = \frac{0}{0}$ = Indeterminate form

$(x+1)^5 \rightarrow$ Expand

$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10.x^2 + 5x + 1$

$(x+1)^2.(x+1)^3$ $^5C_0$

$\Rightarrow \lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

$= \lim_{x \to 0} \frac{(x^5 + 5.x^4 + 10.x^3 + 10.x^2 + 5.x + \cancel{1}) - \cancel{1}}{x}$

$= \lim_{x \to 0} (x^4 + 5x^3 + 10x^2 + 10x + 5)$

$= 0 + 0 + 0 + 0 + 5$ Put $x = 5$

$= 5$

Q.7 $\lim_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 - 4}$

Put $x = 2$ Direct $\dfrac{12 - 2 - 10}{4 - 4} = \dfrac{0}{0}$ form

(Factorisation)

$= \lim_{x \to 2} \dfrac{3x^2 - 6x + 5x - 10}{(x-2)(x+2)}$

$= \lim_{x \to 2} \dfrac{3x(x-2) + 5(x-2)}{(x-2)(x+2)}$

$= \lim_{x \to 2} \dfrac{\cancel{(x-2)}(3x+5)}{\cancel{(x-2)}(x+2)}$

Put $x = 2$

$= \dfrac{6+5}{2+2} = \dfrac{11}{4}$

Q.8 $\lim_{x \to 3} \dfrac{x^4 - 81}{2x^2 - 5x - 3}$ $81 = 3^4$

Put $x = 3$ Directly $\dfrac{3^4 - 81}{2(3)^2 - 5(3) - 3} = \dfrac{0}{0}$ Form

$= \lim_{x \to 3} \dfrac{(x^2)^2 - (3^2)^2}{2x^2 - 6x + x - 3}$

$= \lim_{x \to 3} \dfrac{(x^2 - 3^2)(x^2 + 3^2)}{2x(x-3) + 1(x-3)}$

$= \lim_{x \to 3} \dfrac{\cancel{(x-3)}(x+3)(x^2+9)}{\cancel{(x-3)}(2x+1)}$

$= \lim_{x \to 3} \dfrac{(x+3)(x^2+9)}{2x+1}$

$= \dfrac{6 \times 18}{7} = \dfrac{108}{7}$

Q.9 $\lim_{x\to 0} \frac{ax+b}{cx+1}$

Put $x=0$

$= \frac{0+b}{0+1} = b$

Q.10 $\lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Put $z=1$ $\frac{1-1}{1-1} = \frac{0}{0}$ Form

Standard Form

$$\boxed{\lim_{x\to a} \frac{x^n-a^n}{x-a} = n.a^{n-1}}$$

$\lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

$(z-1)$ multiply & Divide

$= \lim_{z\to 1} \frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} \times \frac{z-1}{z-1}$

$= \lim_{z\to 1} \frac{z^{1/3}-1^{1/3}}{z-1} \times \left(\frac{z-1}{z^{1/6}-1}\right)$

$n.a^{n-1}$ $\frac{1}{n.a^{n-1}}$

$= \left[\frac{1}{3}.(1)^{1/3-1}\right] \times \left(\frac{1}{\frac{1}{6}.1^{1/6-1}}\right)$

$= \frac{1}{3} \times (6)$

$= 2$

Q.11 $\lim_{x\to 1} \frac{ax^2+bx+c}{cx^2+bx+a}$

$a+b+c \neq 0$

$x=1$ put

$\Rightarrow \frac{a+\cancel{b+c}}{c+\cancel{b+a}} = 1$

Q.12 $\lim_{x\to -2} \left(\frac{\frac{1}{x}+\frac{1}{2}}{x+2}\right)$

put $x=-2$

$\frac{0}{0}$

$= \lim_{x\to -2} \frac{\left(\frac{\cancel{2+x}}{2x}\right)}{\cancel{(x+2)}}$

$= \lim_{x\to -2} \left(\frac{1}{2x}\right)$

$= \left(\frac{1}{-4}\right)$ ✓

Q.13 $\lim_{x\to 0} \frac{\sin ax}{bx}$

Put $x=0$

$\frac{\sin 0}{0} = \frac{0}{0}$ form

Standard Form

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \lim_{x\to 0} \frac{\sin ax}{bx} \times \frac{a}{a}$

$= \left(\lim_{x\to 0} \frac{\sin ax}{ax}\right) \times \frac{a}{b} = 1\times\frac{a}{b} = \frac{a}{b}$ ✓

Q.14 $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$, $a, b \neq 0$

$= \lim_{x\to 0} \frac{\sin ax}{ax} \times \frac{bx}{\sin bx} \times \frac{a\cancel{x}}{b\cancel{x}}$

$= 1 \times 1 \times \frac{a}{b}$

$= \frac{a}{b}$ ✓

Q.15 $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

$\frac{\sin(\pi-\pi)}{\pi \cdot (\pi - \pi)} = \frac{\sin 0}{\pi \times 0} = \frac{0}{0}$

By Standard Form

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$= \frac{1}{\pi} \times 1 = \frac{1}{\pi}$

Q.16 $\lim_{x \to 0} \frac{\cos x}{\pi - x}$

Put $x = 0$ Direct

$= \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$

Q.17 $\lim_{x \to 0} \frac{(\cos 2x) - 1}{\cos x - 1}$

Put $x = 0$

$\frac{\cos 0 - 1}{\cos 0 - 1} = \frac{1-1}{1-1} = \frac{0}{0}$

$\cos 2x = 1 - 2\sin^2 x$

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

$= \lim_{x \to 0} \frac{(\cancel{1} - 2\sin^2 x) - \cancel{1}}{(\cancel{1} - 2\sin^2 \frac{x}{2}) - \cancel{1}} = \lim_{x \to 0} \frac{\cancel{-2}\sin^2 x}{\cancel{-2}\sin^2 \frac{x}{2}}$

$= \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{\frac{x}{2} \times 2}{\sin \frac{x}{2}} \right)^2$

1 1

$= (1 \times 1 \times 2)^2$

$= 4$

Q.18 $\lim_{x\to 0} \frac{ax + x\cos x}{b\sin x}$ $\frac{0}{0}$ form

Std form $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= \lim_{x\to 0} \frac{\left(\frac{ax + x\cos x}{x}\right)}{\left(b\frac{\sin x}{x}\right)}$ $\frac{\sin x}{x} \to 1$

$= \lim_{x\to 0} \frac{a + \cos x}{b \times \left(\frac{\sin x}{x}\right)}$

$= \frac{a + \cos 0}{b \times 1} = \frac{a+1}{b}$

$1 - \cos^2 x = \sin^2 x$

Q.19 $\lim_{x\to 0} x\sec x$

$x = 0$ put

$= 0 \times \sec 0$

$= 0 \times 1 = 0$

Q.21 $\lim_{x\to 0} (\operatorname{cosec} x - \cot x)$

put $x = 0$

$= \operatorname{cosec} 0 - \cot 0$

$= \infty - \infty$

Indeterminate

$= \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \left(\frac{1-\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \frac{(1-\cos x)}{\sin x} \cdot \frac{(1+\cos x)}{(1+\cos x)}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{\sin x \cdot (1+\cos x)}$

Q.20 $\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{\left(\frac{\sin ax + bx}{x}\right)}{\left(\frac{ax + \sin bx}{x}\right)}$

$= \lim_{x\to 0} \frac{\frac{\sin ax}{ax} \times a + b}{a + \frac{\sin bx}{bx} \times b}$

$= \frac{1 \times a + b}{a + 1 \times b} = \frac{a+b}{a+b} = 1$

$= \lim_{x\to 0} \frac{\sin^{\cancel{2}} x}{\cancel{\sin x} \cdot (1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin x}{1+\cos x}$

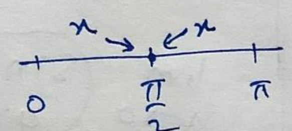
$= \frac{\sin 0}{1+\cos 0} = \frac{0}{1+1} = \frac{0}{2} = 0$

Q.22 $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}}$

Std Form $\boxed{\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1}$

Put $x = \frac{\pi}{2}$ $\quad \dfrac{\tan \cancel{2}\left(\frac{\pi}{\cancel{2}}\right)}{\frac{\pi}{2} - \frac{\pi}{2}} = \dfrac{0}{0}$ Form

$\lim\limits_{x \to \frac{\pi}{2}}$



$x = \frac{\pi}{2} + h$ $\quad$ $(h \to 0)$ $\quad$ $(h \approx 0)$

$h = 0.000000001$

Substitute

$= \lim\limits_{\left(\frac{\pi}{2} + h\right) \to \frac{\pi}{2}} \left(\dfrac{\tan 2\left(\frac{\pi}{2} + h\right)}{\left(\frac{\pi}{\cancel{2}} + h\right) - \frac{\pi}{\cancel{2}}} \right)$

$= \lim\limits_{h \to 0} \dfrac{\tan(\pi + 2h)}{h}$ $\quad$ $\tan(\pi + \theta) = \tan\theta$

$= \lim\limits_{h \to 0} \dfrac{\tan 2h}{h}$

$= \lim\limits_{h \to 0} \dfrac{(\tan 2h)}{h}$

$= \lim\limits_{h \to 0} 2 . \dfrac{\sin 2h}{2h . \cos 2h}$

$\downarrow$ 1

$= \dfrac{2 \times 1}{\cos 0} = \dfrac{2}{1}$

$= 2$

Piecewise Functions

$$f(x) = \begin{cases} x^2+3, & x \geq a \\ 2x-1, & x < a \end{cases}$$

Critical point

limit at critical point

LHL — $x=a$ — RHL

a^- a^+

$x<a$ $x>a$

$2a-1$ a^2+3

LHL = RHL ✓

LHL ≠ RHL ✗

limit at non-critical point

$x=b$

Directly put

Q.23

$$f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$$

Critical point $= 0$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} 3(x+1)$$

Directly put

$$= 3(1+1)$$

$$= 3\times 2 = 6$$

$$\lim_{x\to 0} f(x)$$

Critical point

Left Right

0^- 0^+

0

LHL

$$\lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} (2x+3)$$

$$= 3$$

RHL

$$\lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^-} 3(x+1)$$

$$= 3(0+1)$$

$$= 3$$

$x=0$

Here LHL = RHL $= 3$

$$\boxed{\lim_{x\to 0} f(x) = 3}$$

Q.24

$$f(x) = \begin{cases} x^2-1, & x \leq 1 \\ -x^2-1, & x > 1 \end{cases}$$

Critical point $= 1$

$$\lim_{x \to 1} f(x)$$

LHL	RHL
$= \lim_{x \to 1^-} f(x)$	$= \lim_{x \to 1^+} f(x)$
$= \lim_{x \to 1^-} (x^2-1)$	$= \lim_{x \to 1^+} (-x^2-1)$
$= (1)^2 - 1$	$= -(1)^2 - 1$
$= 1-1$	$= -1-1$
$= 0 = \text{LHL}$	$= -2 = \text{RHL}$

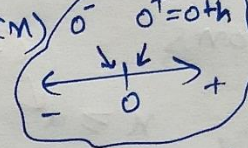
$\text{LHL} \neq \text{RHL}$

$\therefore \lim_{x \to 1} f(x)$ does not exist

Q.25

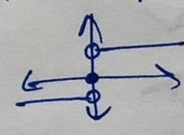
$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ $\cancel{x \geq 0}$

$$\lim_{x \to 0} f(x)$$



Critical point $= 0$

LHL	RHL
$= \lim_{x \to 0^-} f(x)$	$= \lim_{x \to 0^+} f(x)$
$= \lim_{x \to 0^-} \dfrac{\lvert x \rvert}{x}$	$= \lim_{x \to 0^+} \dfrac{\lvert x \rvert}{x}$
$0^- = 0-h$ ($h>0$) ($h \to 0$)	$0^+ = 0+h$
$= \lim_{h \to 0} \dfrac{\lvert -h \rvert}{-h}$	$= \lim_{h \to 0} \dfrac{\lvert 0+h \rvert}{0+h}$
$= \lim_{h \to 0} \dfrac{h}{-h}$	$= \lim_{h \to 0} \dfrac{h}{h}$
$= -1 = \text{LHL}$	$= 1 = \text{RHL}$



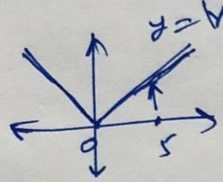
$\text{LHL} \neq \text{RHL}$

$\therefore$ Limit $\rightarrow$ Not exist

$\boxed{Q.27}$ $f(x) = |x| - 5$

$\lim_{x \to 5} f(x)$

$|x|$ changes its nature at $x = 0$



$\lim_{x \to 5} f(x) = |5| - 5$

$= 5 - 5$

$= 0$

$\boxed{Q.28}$

$$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$$

$a, b = ?$

Given $\underbrace{\lim_{x \to 1} f(x)}_{\text{limit neighborhood}} = \underbrace{f(1)}_{\text{Exact}} = 4$

LHL = RHL

$\boxed{LHL = RHL = 4}$

$\Rightarrow \lim_{x \to 1^-} \boxed{f(x)} = \lim_{x \to 1^+} \boxed{f(x)} = 4$

$\Rightarrow \lim_{x \to 1^-} (a + bx) = \lim_{x \to 1^+} (b - ax) = 4$

$\Rightarrow a + b = b - a = 4$

$\Rightarrow a + b = 4$ $\quad$ $a + \cancel{b} = \cancel{b} - a$

$\Rightarrow \boxed{b = 4}$ $\quad$ $\Rightarrow 2a = 0 \Rightarrow \boxed{a = 0}$

$a = 0$
$b = 4$

Q.29 $f(x) = \underline{(x-a_1)(x-a_2)\cdots(x-a_n)}$

$$\lim_{x\to a_1} f(x) = \lim_{x\to a_1} (x-a_1)(x-a_2)\cdots(x-a_n)$$

$$= \underbrace{(a_1-a_1)}_{0}(a_1-a_2)\cdots(a_1-a_n) = 0$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} (x-a_1)(x-a_2)\cdots(x-a_n)$$

$$= (a-a_1)(a-a_2)\cdots(a-a_n)$$

Q.30

$$f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$$

For what value(s) of a does $\lim_{x\to a} f(x)$ exist?

$\because$ Here $|x|+1$ & $|x|-1$ are algebraic functions, therefore these functions will be continuous in their domain

$\Rightarrow$ ~~They will be~~

$\Rightarrow$ limit will exists at every point in their domain.

But Critical point $x=0$

(we have to check only on this point)

$$\boxed{\lim_{x\to 0} f(x)}$$

LHL RHL

$$LHL = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} (|x|+1)$$

$$= |0^-| + 1$$

$$= 0 + 1$$

$$= 1$$

$$RHL = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (|x|-1)$$

$$= |0| - 1$$

$$= -1$$

At $x=0$, $LHL \neq RHL$

$\therefore$ limit does not exist at $x=0$.

$\frac{0}{0}$ $\frac{\infty}{\infty}$ $0.\infty$ $\infty - \infty$

$\times$

$x \to 0-h$

$\therefore \lim_{x \to a} f(x)$ exists at every $a \in R - \{0\}$

~~Q.30~~ $\boxed{Q.31}$ $\lim_{x \to 1} \left(\frac{f(x) - 2}{x^2 - 1} \right) = \pi$ $\quad \lim_{x \to 1} f(x) = ?$

$$\Rightarrow \frac{\lim_{x \to 1} f(x) - 2}{\lim_{x \to 1} (x^2) - 1} = \pi$$

$$\frac{\lim_{x \to 1} f(x) - 2}{1-1} = \pi$$

$$\frac{\lim_{x \to 1} f(x) - 2}{0} = \pi$$

$\Rightarrow$ If $\lim_{x \to 1} f(x) \neq 2$

$$\therefore \frac{0 \neq \lim_{x \to 1} f(x) - 2}{0 = \lim_{x \to 1} x^2 - 1} = \infty \neq \pi$$

Impossible

$\therefore \lim_{x \to 1} f(x)$ should be equal to '2'

$\frac{0}{0}$

Q.32

$$f(x) = \begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

Integers

$m, n = ?$

$\lim_{x\to 0} f(x)$ exist

$\lim_{x\to 1} f(x)$ exist

$\because \lim_{x\to 0} f(x)$ exists.

LHL = RHL

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0^-} (mx^2+n) = \lim_{x\to 0^+} (nx+m)$

$\Rightarrow m(0)^2+n = n(0)+m$

$\Rightarrow n = m$

$\lim_{x\to 1} f(x)$ exists

LHL = RHL

$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow \lim_{x\to 1^-} n(x)+m = \lim_{x\to 1^+} (nx^3+m)$

$\Rightarrow n(1)+m = n(1)+m$

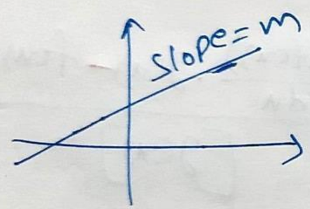
$\Rightarrow n+m = n+m$

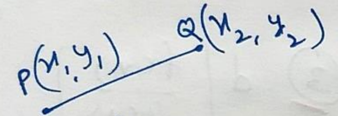
This is always true for every integral value of m & n.

Derivatives ⟶ Rate of change

Slope

Straight line $y = mx + c$

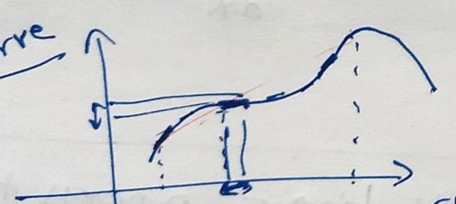


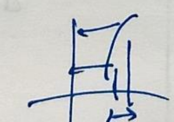


$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

$$\text{Slope} = m = \frac{\text{change in 'y'}}{\text{change in 'x'}} = \frac{\text{change in output}}{\text{change in input}}$$

For curve



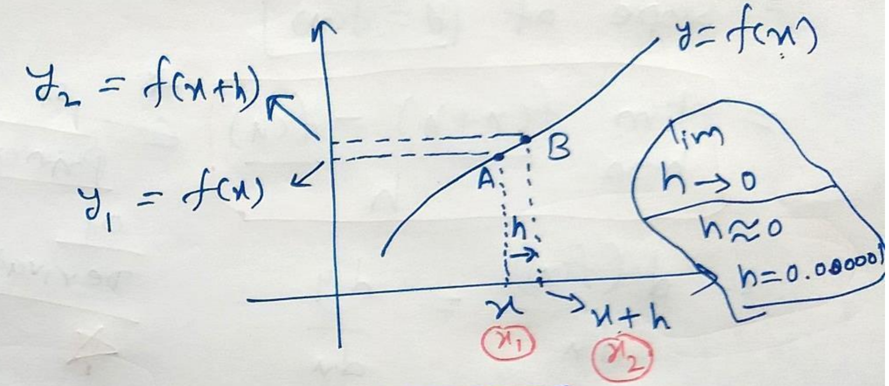


instantaneous Slope $= \dfrac{\text{change in output}}{\text{very very small change in input}}$

⇕

Derivative

Derivative of a Function $\boxed{y = f(x)}$



Derivative of $\boxed{y = f(x)}$

= rate of change of $\boxed{y = f(x)}$

$= \dfrac{\text{change in output}}{\text{v.v. small change in input}}$

$= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{(x+h) - x}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

☆ Derivative of $\boxed{y = f(x)}$

= Slope of $\boxed{y = f(x)}$

$$= \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \leftarrow 1^{st} \text{ Principle of Derivative}$$

☆

$$= \frac{d(f(x))}{dx} = \frac{dy}{dx}$$

$$= f'(x) = y'$$

☆ Derivative (slope) at particular Point $x = a$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \left.\frac{d(f(x))}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

$$= f'(a)$$

Algebra of Derivatives

① $$\frac{d[f(x) \pm g(x)]}{dx} = \frac{d f(x)}{dx} \pm \frac{d(g(x))}{dx}$$

② $$\frac{d[K \cdot f(x)]}{dx} = K \frac{d(f(x))}{dx}$$

③ $$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{\frac{d(f(x))}{dx} \cdot g(x) - f(x) \cdot \frac{d(g(x))}{dx}}{[g(x)]^2}$$

④ $$\frac{d(f(x) \cdot g(x))}{dx} = \frac{d f(x)}{dx} \cdot g(x) + f(x) \cdot \frac{d(g(x))}{dx}$$

$\boxed{\text{Differentiation} = \text{process to find derivative}}$

$$\star \quad (u.v)' = u'.v + u.v'$$

$$\star \quad \left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

Important Derivatives

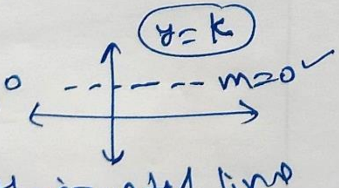
$$\frac{d(x^n)}{dx} = n.x^{n-1}$$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\text{constant})}{dx} = 0$$

$$K = K.x^0$$



Derivative of $y = f(x) = x^n$ (using 1st Principle)

$$\frac{d(x^n)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$(x+h)^n = {}^nc_0 . x^n . h^0 + {}^nc_1 . x^{n-1} . h^1 + \dots + {}^nc_n . x^0 . h^n$$

Binomial Theorem

$${}^nc_r = \frac{n!}{(n-r)!\, r!}$$

$${}^nc_0 = 1 = {}^nc_n$$

$${}^nc_1 = n = {}^nc_{n-1}$$

$$= x^n + n.x^{n-1}.h + \dots + h^n$$

$$\frac{d(x^n)}{dx} = \lim_{h\to 0} \frac{[\cancel{x^n} + n.x^{n-1}.h + \dots + h^n] - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \left(n.x^{n-1} + \frac{n(n-1)}{2}.x^{n-2}.h + \dots + h^{n-1}\right)$$

$$= n.x^{n-1}$$

Derivative of $y = \sin x = f(x)$

(Using first principle)

$$\frac{d(\sin x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\boxed{\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right) \cdot \cos\left(\frac{A+B}{2}\right)}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{\cancel{x}+h-\cancel{x}}{2}\right) \cdot \cos\left(\frac{x+h+x}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \cos\left(x + \frac{h}{2}\right)$$

$$\downarrow$$

$$1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$= 1 \times \cos(x + 0)$$

$$= \cos x$$

$$\boxed{\frac{d(\sin x)}{dx} = \cos x}$$

~~Find~~

e.g. find Derivative of '$x^2 \cdot \sin x$'

$\underbrace{x^2}_{u} \times \underbrace{\sin x}_{v}$

$(u.v)' = u'.v + u.v'$

$u = x^2$

$v = \sin x$

$u = x^2 \longrightarrow u' = \frac{d(x^2)}{dx} = 2.x^{2-1}$

$= 2.x^1$

$= 2x$

$v = \sin x \longrightarrow v' = \frac{d(\sin x)}{dx} = \cos x$

$$\frac{d(x^2 \cdot \sin x)}{dx} = \underline{(2x).\sin x + x^2 \cdot \cos x}$$

e.g. find derivative of '$\tan x$'

$\tan x = \frac{\sin x}{\cos x}$

$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

$$\frac{d(\tan x)}{dx} = \frac{d\left(\frac{\sin x}{\cos x}\right)}{dx}$$

$$= \frac{(\sin x)'\cos x - (\sin x).(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - (\sin x).(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Exercise 12.2

Q.1 Derivative of $x^2-2 = f(x)$ at $x=10$.

By first principle of Derivative

$$\left.\frac{d(x^2-2)}{dx}\right|_{x=10} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$

$$= \lim_{h\to 0}\frac{[(10+h)^2-2]-[(10)^2-2]}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{100}+h^2+20h-\cancel{2}-\cancel{100}+\cancel{2}}{h}$$

$$= \lim_{h\to 0}(h+20)$$

$$= 0+20 = 20$$

$$\frac{d(x^2-2)}{dx} = (2x^{2-1}-0) = 2x \rightarrow 20$$

$x=10$

Shortcut

Q.2 Derivative of $\underset{f(x)}{=}99x$ at $x=100$

$$\left.\frac{d(99x)}{dx}\right|_{x=100} = \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$

$$= \lim_{h\to 0}\frac{99(100+h)-99\times 100}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{99\times 100}+99\times h-\cancel{99\times 100}}{h}$$

$$= 99$$

$\boxed{Q.3}$ Derivative of 'x' $= f(x)$ at $x=1$.

$$\left.\frac{d(x)}{dx}\right|_{x=1} = \lim_{h\to 0} \frac{f(\cancel{x}1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{1}+h) - (\cancel{1})}{h}$$

$$= 1$$

$\boxed{Q.4}$ "Derivative Using First Principle".

(i) $x^3 - 27 = f(x)$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^3 - 27] - [x^3 - 27]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + h^3 + 3x^2h + 3xh^2 - \cancel{27} - \cancel{x^3} + \cancel{27}}{h}$$

$$\boxed{(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2}$$

$(x+h)^3 = - - -$

$$= \lim_{h\to 0} \frac{h^3 + 3x^2h + 3xh^2}{h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} (h^2 + 3x^2 + 3xh)$$

$$= 0 + 3x^2 + 0 = 3x^2$$

4 (ii) $(x-1)(x-2) = f(x)$

$$\Rightarrow f(x) = x^2 - 3x + 2$$

$$\frac{d(f(x))}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(1^{st} principle)

$$= \lim_{h \to 0} \frac{\left[(x+h)^2 - 3(x+h) + 2\right] - \left[x^2 - 3x + 2\right]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 - 3x - 3h + \cancel{2} \; \overset{+3x-\cancel{2}}{-\cancel{x^2}}}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (2x + h - 3) = 2x - 3 \text{ Ans.}$$

4 (iii) $\frac{1}{x^2} = f(x)$

$$\frac{d(f(x))}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{\frac{x^2 - (x+h)^2}{x^2 \cdot (x+h)^2}}{\frac{h}{1}} \right\}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x^2} - h^2 - 2hx}{h \cdot x^2 \cdot (x+h)^2}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}(h + 2x)}{\cancel{h} \cdot (x^2) \cdot (x+h)^2}$$

$$= \frac{-2x}{x^2 \cdot x^2} = -\frac{2}{x^3}$$

④ (IV) $\dfrac{x+1}{x-1} = f(x)$

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{x+h+1}{(x+h-1)} - \dfrac{x+1}{(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(x^2 - x + hx - h + x - 1) - (x^2 + x \overset{+hx+h}{} - x - 1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} \cancel{-x} + \cancel{hx} - h \cancel{+x} \cancel{-1} \cancel{-x^2} \cancel{-x} \cancel{+x} \cancel{+1} \cancel{-hx} \overset{-h}{\cancel{+h}}}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2}{(x+\underset{0}{h}-1)(x-1)}$$

$$= \frac{-2}{(x-1)^2}$$

Q.5 Prove $\boxed{f'(1) = 100 \cdot f'(0)}$ $(1 = x^0)$

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x^1 + 1$$

$f'(x) = ?$ $\boxed{\frac{d(x^n)}{dx} = n \cdot x^{n-1}}$ $\frac{d(\text{Const})}{dx} = 0$

$$f'(x) = \frac{100 \cdot x^{99}}{100} + \frac{99 \cdot x^{98}}{99} + \cdots + \frac{2 \cdot x^1}{2} + 1 \cdot x^0 + 0$$

$$\boxed{f'(x) = x^{99} + x^{98} + \cdots + x + 1}$$

$$f'(1) = \underbrace{1 + 1 + \cdots + 1 + 1}_{100 \text{ times}} = 100$$

$$f'(0) = 0 + 0 + \cdots + 0 + 1 = 1$$

$$f'(1) = 100 = 100 \times 1 = 100 \times f'(0)$$

Q.6 a = some fixed real no.

$$f(x) = x^n + a x^{n-1} + a^2 x^{n-2} + \cdots + a^{n-1} x + a^n$$

$$\frac{d(f(x))}{dx} = \frac{d(x^n + \cdots + a^n)}{dx}$$

$$= n \cdot x^{n-1} + (n-1) \cdot a \cdot x^{n-2} + (n-2) a^2 \cdot x^{n-3} + \cdots + a^{n-1} \cdot 1 + 0$$

Q.7 $a, b \rightarrow$ constants

(i) $(x-a)(x-b) = f(x)$

$\Rightarrow f(x) = x^2 - ax - bx + ab$

$f(x) = x^2 - x(a+b) + ab$

$$\frac{d(f(x))}{dx} = 2x - 1 \cdot (a+b) + 0$$

$$= 2x - a - b$$

(ii) $f(x) = (ax^2+b)^2$

$f(x) = (ax^2+b)\cdot(ax^2+b)$

$\boxed{(u \cdot v)' = u'.v + u.v'}$

$$\frac{d(f(x))}{dx} = (a.2x'+0)\cdot(ax^2+b) + (ax^2+b)\cdot(a.2x'+0)$$

$= (ax^2+b).\{2ax+2ax\}$

$= 4ax.(ax^2+b).$

(iii) $\frac{u}{v} = \frac{x-a}{x-b} = f(x)$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$\frac{d(f(x))}{dx} = f'(x) = \frac{(1-0).(x-b)-(x-a).(1-0)}{(x-b)^2}$$

$$f'(x) = \frac{(x-b)-(x-a)}{(x-b)^2}$$

$$f'(x) = \frac{\cancel{x}-b-\cancel{x}+a}{(x-b)^2}$$

$$\boxed{f'(x) = \frac{a-b}{(x-b)^2}}$$

$\boxed{Q.8}$

$a =$ some constant

$$x^n \to n.x^{n-1}$$

$$x^1 \to 1.x^{1-1} = 1.x^0 = 1 \times 1 = 1$$

$$f(x) = \frac{x^n - a^n}{x - a} = \frac{u}{v}$$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$\frac{d(f(x))}{dx} = f'(x) = \frac{(n.x^{n-1} - 0).(x-a) - (x^n - a^n).(1-0)}{(x-a)^2}$$

$$= \frac{n.x^{n-1}.(x-a) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{n.x^n - n.a.x^{n-1} - x^n + a^n}{(x-a)^2}$$

$$= \frac{(n-1)x^n - nax^{n-1} + a^n}{(x-a)^2}$$

Q.9 (i) $2x - \frac{3}{4} = f(x)$ $\quad \frac{d(x^n)}{dx} = n.x^{n-1}$

Derivative: $f'(x) = \frac{d\left(2x - \frac{3}{4}\right)}{dx}$

$f'(x) = 2\frac{d(x^1)}{dx} - \frac{d(3/4)}{dx} \to 0$

$= 2 \times 1 . x^{1-1} - 0$

$= 2 \times x^0 = 2$

(ii) $(5x^3 + 3x - 1)(x - 1) = f(x)$

$f(x) = 5x^4 - 5x^3 + 3x - 3 - x + 1$

$f(x) = 5x^4 - 5x^3 + 2x - 2$

$f'(x) = 5.4.x^3 - 5.3.x^2 + 2 - 0$

$\boxed{f'(x) = 20x^3 - 15x^2 + 2}$

(iii) $x^{-3}(5 + 3x^1) = f(x)$

$\Rightarrow f(x) = 5x^{-3} + 3x^{-2}$ $\quad -3-1 = -4$

Derivative

$\Rightarrow f'(x) = (-3).5x^{-4} + (-2)3x^{-3}$

$f'(x) = -15x^{-4} - 6x^{-3}$

(iv) $f(x) = x^5 . (3 - 6x^{-9})$

$f(x) = 3x^5 - 6 . x^{-4}$

Derivative

$\Rightarrow f'(x) = 5.(3x^4) - (-4).6.x^{-5}$

$\boxed{f'(x) = 15x^4 + 24x^{-5}}$

(V) $x^{-4} . (3 - 4x^{-5}) = f(x)$

$f(x) = 3x^{-4} - 4 . x^{-9}$

$f'(x) = (-4).3x^{-5} - (-9).4.x^{-10}$

$f'(x) = -12x^{-5} + 36x^{-10}$

(VI) $f(x) = \dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$f'(x) = \dfrac{(0).(x+1) - 2(1+0)}{(x+1)^2}$

$- \dfrac{(2x)(3x-1) - (x^2).(3\times1-0)}{(3x-1)^2}$

$\Rightarrow f'(x) = \left(\dfrac{-2}{(x+1)^2}\right) - \left(\dfrac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right)$

$\Rightarrow f'(x) = \left(\dfrac{-2}{(x+1)^2}\right) - \left(\dfrac{3x^2 - 2x}{(3x-1)^2}\right)$

Q.10 Derivative of $\cos x$ (by first principle)

$f(x) = \cos x$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\frac{d(f(x))}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\Rightarrow \frac{d(\cos x)}{dx} = \lim_{h \to 0} \frac{-2\sin\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{\cancel{x}+h-\cancel{x}}{2}\right)}{h}$$

$$\boxed{\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(x+\frac{h}{2}\right) \cdot \sin\frac{h}{2}}{h}$$

Indeterminate $\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{h \to 0} \left(-\sin\left(x+\frac{h}{2}\right) \cdot \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)} \right)$$

$\to 1$

$$= -\sin(x+0) \times 1$$

$$= -\sin x$$

$$\boxed{\frac{d(\cos x)}{dx} = -\sin x}$$

Q.11 Find the Derivatives

(i) $f(x) = \sin x \cdot \cos x$

$(u.v)' = u'.v + u.v'$

$$\frac{d(f(x))}{dx} = f'(x) = \frac{d(\sin x \cdot \cos x)}{dx}$$

$$= \frac{d(\sin x)}{dx} \cdot \cos x + \sin x \cdot \frac{d(\cos x)}{dx}$$

$$= \cos x \cdot \cos x + \sin x \cdot (-\sin x)$$

$$= \cos^2 x - \sin^2 x$$

$$= \cos 2x$$

(ii) $f(x) = \sec x = \frac{1}{\cos x} = \frac{u}{v}$

$$f'(x) = \frac{d\left(\frac{1}{\cos x}\right)}{dx} \qquad \left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$= \frac{(0).\cos x - (1).(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \sec x \cdot \tan x$$

(iii) $f(x) = 5\sec x + 4\cos x$

$$f'(x) = \frac{d(5\sec x + 4\cos x)}{dx}$$

$$= \frac{d(5\sec x)}{dx} + \frac{d(4\cos x)}{dx}$$

$$= 5 \cdot \frac{d(\sec x)}{dx} + 4 \frac{d(\cos x)}{dx}$$

By (ii) Part

$$= 5 \cdot (\sec x \cdot \tan x) - 4\sin x$$

(iv) $f(x) = \operatorname{Cosec} x = \frac{1}{\sin x} = \frac{u}{v}$

$$f'(x) = \frac{(0).\sin x - (1).(\cos x)}{\sin^2 x} \qquad \left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \cdot \operatorname{Cosec} x$$

(v) $f(x) = 3\cot x + 5\operatorname{Cosec} x$

$$f(x) = \frac{3\cos x}{\sin x} + \frac{5}{\sin x}$$

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$f'(x) = \frac{3\, d\left(\frac{\cos x}{\sin x}\right)}{dx} + \frac{5 \cdot d\left(\frac{1}{\sin x}\right)}{dx}$$

$$= 3 \cdot \frac{(-\sin x) \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$+ 5\left(-\cot x \cdot \operatorname{Cosec} x\right)$$

$$= -3(\sin^2 x + \cos^2 x)/\sin^2 x$$
$$- 5 \operatorname{cosec} x \cdot \cot x$$

$$= -3\operatorname{Cosec}^2 x$$
$$- 5\operatorname{Cosec} x \cdot \cot x$$

(vi) $f(x) = 5\sin x - 6\cos x + 7$

$\frac{d(\sin x)}{dx} = \cos x$

$\frac{d(\cos x)}{dx} = -\sin x$

$\frac{d(\text{constant})}{dx} = 0$

$f'(x) = \frac{d(5\sin x - 6\cos x + 7)}{dx}$

$= 5.(\cos x) - 6(-\sin x) + 0$

$= 5\cos x + 6\sin x$

(vii) $f(x) = 2\tan x - 7\sec x$

$f(x) = \frac{2\sin x}{\cos x} - \frac{7}{\cos x}$

$f'(x) = 2.\frac{d\left(\frac{\sin x}{\cos x}\right)}{dx} - 7.\frac{d\left(\frac{1}{\cos x}\right)}{dx}$

$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

$f'(x) = 2.\frac{\cos x.\cos x - (\sin x).(-\sin x)}{\cos^2 x}$

$-7(\sec x.\tan x)$

$= 2.\frac{(\cos^2 x + \sin^2 x)}{\cos^2 x} - 7\sec x.\tan x$

$= 2\sec^2 x - 7\sec x.\tan x$

DERIVATIVES

- First Principle of Derivatives

$$\frac{d(f(x))}{dx} = f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $(u \pm v)' = u' \pm v'$

- $(u.v)' = u'.v + u.v'$

  $(u.v.w)' = u'.v.w + u.v'.w$
  $+ u.v.w'$

- $(K\,u)' = K.u'$

  Constant function

- $\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$

Important Derivatives :

- $\frac{d(x^n)}{dx} = n.x^{n-1}$

- $\frac{d(\sin x)}{dx} = \cos x$

- $\frac{d(\cos x)}{dx} = -\sin x$

- $\frac{d(\tan x)}{dx} = \sec^2 x$

- $\frac{d(\sec x)}{dx} = \sec x . \tan x$

- $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

- $\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x . \cot x$

- $\dfrac{d(\sin^{-1}x)}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

- $\dfrac{d(\cos^{-1}x)}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

- $\dfrac{d(\tan^{-1}x)}{dx} = \dfrac{1}{1+x^2}$

- $\dfrac{d(\cot^{-1}x)}{dx} = \dfrac{-1}{1+x^2}$

- $\dfrac{d(\sec^{-1}x)}{dx} = \dfrac{1}{|x|\sqrt{x^2-1}}$

- $\dfrac{d(\operatorname{cosec}^{-1}x)}{dx} = \dfrac{-1}{|x|\sqrt{x^2-1}}$

- $\dfrac{d(\log x)}{dx} = \dfrac{d(\log_e x)}{dx} = \dfrac{d(\ln(x))}{dx} = \dfrac{1}{x}$

Logarithmic Fn $\boxed{e = \text{euler' No.} = 2.718\ldots}$

- $\dfrac{d(e^x)}{dx} = e^x$
- $\dfrac{d(a^x)}{dx} = a^x \cdot \log_e a$

Exponential Fn

- $\dfrac{d(\text{constant})}{dx} = 0$

- $\dfrac{d\left(\frac{1}{x}\right)}{dx} = -\dfrac{1}{x^2}$

- $\dfrac{d(\sqrt{x})}{dx} = \dfrac{1}{2\sqrt{x}}$

- $\dfrac{d(x)}{dx} = 1$

$\frac{1}{x} = x^{-1}$ x^n

$\sqrt{x} = x^{\frac{1}{2}}$

Note: $\frac{d(x^n)}{dx} = n \cdot x^{n-1}$

$\frac{d(x)}{dx} = 1$, $\frac{d(X)}{dX} = 1$, $\frac{d(y)}{dy} = 1$

$\frac{d(f(x))}{d(f(x))} = 1$ $\boxed{\frac{d(\sin X)}{dX} = \cos X}$

Also.

- $\frac{d\,(f(x))^n}{d(f(x))} = n \cdot (f(x))^{n-1}$

$\sqrt{x}$ $\sqrt{\sin x}$

- $\frac{d\,(\sin f(x))}{d(f(x))} = \cos(f(x))$

e.g. $\frac{d\,(\cos(\sin x))}{d(\sin x)} = -\sin(\sin x)$

$\frac{d(\cos x)}{dx} = -\sin x$

Chain Rule:

function in the function
↓
Composite Function

e.g. $\sqrt{\sin x}$, $\sin(x^2)$

$\sin(\cos x)$, $\cos(\tan x)$

$f(x)$ $g(x)$ $= f(g(x))$

$(\sin x)^n = x^n$ sin x

e.g. Find the derivative of $\sin(\cos x)$.

e.g. $f(x) = \sin(\cos x)$

Find $f'(x)$.

$$\frac{d(f(x))}{dx} = \frac{d(\sin(\cos x))}{dx}$$

$$= \frac{d(\sin(\cos x))}{d(\cos x)} \cdot \frac{d(\cos x)}{dx}$$

$$= \cos(\cos x) \cdot (-\sin x)$$

e.g. If $f(x) = \sin(\tan(\sqrt{x}))$

find $f'(x)$.

Ans. $$\frac{d(f(x))}{dx} = \frac{d(\sin(\tan(\sqrt{x})))}{dx}$$

$$= \frac{d(\sin(\tan\sqrt{x}))}{d(\tan\sqrt{x})} \cdot \frac{d(\tan\sqrt{x})}{d(\sqrt{x})} \cdot \frac{d(\sqrt{x})}{dx}$$

$$= \cos(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

Chain Rule Fast

$$\frac{d[f(g(h(x)))]}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Composite Fn.

e.g. $f(x) = \sin(\cos x)$

$f'(x) = \cos(\cos x) \cdot (-\sin x)$

e.g. $f(x) = \sin(\tan(\sqrt{x}))$

$f'(x) = \cos(\tan\sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

e.g. $f(x) = \sqrt{\sin(x^2)}$

$x^n \to nx^{n-1}$

$= \left(\sin(x^2)\right)^{\frac{1}{2}}$

$x^{\frac{1}{2}}$

$f'(x) = \frac{1}{\cancel{2}\sqrt{\sin(x^2)}} \cdot \cos(x^2) \cdot \cancel{2}x$

$= \frac{x \cdot \cos(x^2)}{\sqrt{\sin(x^2)}}$

Miscellaneous Exercise 12.3

First Principle of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Q.1 (i) $f(x) = -x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-x} - h + \cancel{x}}{h}$$

$$= -1$$

(ii) $f(x) = (-x)^{-1} = \frac{1}{(-x)} = -\frac{1}{x}$

~~lim~~

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{-1}{x+h}\right) - \left(\frac{-1}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(-\frac{1}{x+h} + \frac{1}{x}\right)}{h}$$

$$\frac{\frac{a}{b}}{\frac{c}{b}}$$

$$= \lim_{h \to 0} \frac{\left(\frac{\cancel{-x} + \cancel{x} + h}{x(x+h)}\right)}{\left(\frac{h}{1}\right)}$$

$$= \lim_{h \to 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x(x+0)} = \frac{1}{x^2}$$

(iii) $f(x) = \sin(x+1)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+1+h) - \sin(x+1)}{h}$$

Standard form.

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

$$\boxed{\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right) \cdot \cos\left(\frac{A+B}{2}\right)}$$

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{\cancel{x}+\cancel{1}+h-\cancel{x}-\cancel{1}}{2}\right) \cdot \cos\left(\frac{2x+2+h}{2}\right)}{h}$$

Indeterminate form

$$= \lim_{h \to 0} \frac{2\sin\left(\frac{h}{2}\right) \cdot \cos\left(x+1+\frac{h}{2}\right)}{h}, \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0} \boxed{\frac{\sin\left(\frac{h}{2}\right) \cdot \cos\left(x+1+\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}} = 1 \times \cos(x+1+0) = \cos(x+1)$$

1

(iv) $f(x) = \cos\left(x - \frac{\pi}{8}\right)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos\left(x - \frac{\pi}{8} + h\right) - \cos\left(x - \frac{\pi}{8}\right)}{h}$$

$$\boxed{\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)}$$

$$= \lim_{h \to 0} \frac{-2 \cdot \sin\left(\frac{2x - \frac{2\pi}{8} + h}{2}\right) \cdot \sin\left(\frac{\cancel{x} - \cancel{\frac{\pi}{8}} + h - \cancel{x} + \cancel{\frac{\pi}{8}}}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2 \cdot \sin\left(x - \frac{\pi}{8} + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-\sin\left(x - \frac{\pi}{8} + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = -\sin\left(x - \frac{\pi}{8} + 0\right) \times 1 = -\sin\left(x - \frac{\pi}{8}\right)$$

$\to 1$

$\boxed{Q.2}$

Derivative

$$f(x) = x + a$$

$$f'(x) = \frac{d(x+a)}{dx}$$

$$= \frac{d(x)}{dx} + \frac{d(a)}{dx}$$

$$= 1 + 0$$

$$= 1$$

$\boxed{Q.3}$

$$f(x) = [p(x) + q] \cdot \left[\frac{r}{x} + s\right]$$

$$f(x) = pr + ps.x + \frac{qr}{x} + q.s$$

$$f'(x) = \frac{d\left(pr + ps.x + \frac{qr}{x} + q.s\right)}{dx}$$

$$f'(x) = 0 + ps.\frac{d(x)}{dx} + qr\frac{d\left(\frac{1}{x}\right)}{dx} + 0$$

$$= ps.(1) + qr.\left(\frac{-1}{x^2}\right)$$

$$= ps - \frac{qr}{x^2}$$

$$\frac{d(x^n)}{dx} = n.x^{n-1}$$

$$\frac{1}{x} = x^{-1}$$

$$= -1.x^{-1-1}$$

$$= -x^{-2} = \frac{-1}{x^2}$$

$\boxed{Q.4}$ $f(x) = (ax+b).(cx+d)^2$

I-method.

$f(x) = (ax+b).(cx+d).(cx+d)$

$(u.v.w)'$

$f'(x) = (a.1+0).(cx+d).(cx+d)$

$+ (ax+b).(c.1+0).(cx+d)$ ✓

$+ (ax+b).(cx+d).(c.1+0)$ ✓

$= a.(cx+d)^2 + 2.c.(ax+b).(cx+d)$

✓

$(x)^2 \rightarrow 2.x^{2-1} = 2x$

$x^n \rightarrow n.x^{n-1}$

II-method. (By chain Rule)

$f(x) = (ax+b).(cx+d)^2$

$(u.v)'$

$f'(x) = (a+0).(cx+d)^2.$

$+ (ax+b).\frac{d(cx+d)^2}{dx}$

$f'(x) = a.(cx+d)^2 + (ax+b).2(cx+d).(c.1+0)$

$f'(x) = a.(cx+d)^2 + 2c(ax+b).(cx+d)$

$\boxed{Q.5}$ $f(x) = \dfrac{ax+b}{cx+d} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(ax+b)'.(cx+d) - (ax+b).(cx+d)'}{(cx+d)^2}$$

$$f'(x) = \frac{(a\cdot 1+0)(cx+d) - (ax+b).(c+0)}{(cx+d)^2}$$

$$f'(x) = \frac{a\cancel{cx} + ad - a\cancel{cx} - bc}{(cx+d)^2}$$

$$f'(x) = \frac{ad-bc}{(cx+d)^2}$$

$\boxed{Q.6.}$ $f(x) = \dfrac{1+\frac{1}{x}}{1-\frac{1}{x}} = \dfrac{\frac{x+1}{\cancel{x}}}{\frac{x-1}{\cancel{x}}}$

$$f(x) = \frac{x+1}{x-1} = \frac{u}{v}$$

$$f'(x) = \frac{(1+0).(x-1) - (x+1).(1-0)}{(x-1)^2}$$

$$= \frac{\cancel{x}-1-\cancel{x}-1}{(x-1)^2}$$

$$f'(x) = \frac{-2}{(x-1)^2}$$

$\boxed{Q.7}$ $f(x) = \dfrac{1}{(ax^2+bx+c)} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(0).(ax^2+bx+c) - 1.(a.2.x+b.1+0)}{(ax^2+bx+c)^2}$$

$x^2 \longrightarrow x^n$ $\quad \dfrac{d(x^n)}{dx} = n.x^{n-1}$

$$f'(x) = \frac{0 - 2ax - b}{(ax^2+bx+c)^2}$$

$$f'(x) = \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

$\boxed{Q.8}$. $f(x) = \dfrac{ax+b}{px^2+qx+r} = \dfrac{u}{v}$

$$f'(x) = \frac{(a+0).(px^2+qx+r) - (ax+b).(p2x+q+0)}{(px^2+qx+r)^2}$$

$$\Rightarrow f'(x) = \frac{a(px^2+qx+r) - (ax+b).(2px+q)}{(px^2+qx+r)^2}$$

$$\Rightarrow f'(x) = \frac{apx^2 + \cancel{aqx} + ar - 2apx^2 - \cancel{aqx} - 2bpx - bq}{(px^2+qx+r)^2}$$

$$\Rightarrow f'(x) = \frac{-apx^2 - 2bpx + ar - bq}{(px^2+qx+r)^2}$$

$\boxed{Q.9}$ $f(x) = \dfrac{px^2 + qx + r}{ax + b} = \dfrac{u}{v}$

$\left(\dfrac{u}{v}\right)' = \dfrac{u'.v - u.v'}{v^2}$

$f'(x) = \dfrac{(2px + q + 0).(ax+b) - (px^2 + qx + r).(a + 0)}{(ax+b)^2}$

$f'(x) = \dfrac{apx^2 + 2bpx - ar + bq}{(ax+b)^2}$

$\dfrac{d(x^n)}{dx} = n.x^{n-1}$

$\boxed{Q.10}$ $f(x) = \dfrac{a}{x^4} - \dfrac{b}{x^2} + \cos x$

$f(x) = a.(x^{-4}) - b.(x^{-2}) + (\cos x)$

$f'(x) = a.\left(-4.x^{-4-1}\right) - b\left(-2.x^{-2-1}\right) + (-\sin x)$

$f'(x) = -\dfrac{4a}{x^5} + \dfrac{2b}{x^3} - \sin x$

$\boxed{Q.11}$

$f(x) = 4\sqrt{x} - 2$

$f(x) = 4(x)^{\frac{1}{2}} - 2$

$f'(x) = 4.\frac{1}{2}.x^{\frac{1}{2}-1} - 0$

$f'(x) = 2.x^{-\frac{1}{2}}$

$f'(x) = \dfrac{2}{x^{\frac{1}{2}}}$

$\boxed{f'(x) = \dfrac{2}{\sqrt{x}}}$

$\boxed{Q.12}$ $f(x) = (ax+b)^n$

$f'(x) = \dfrac{d(ax+b)^n}{dx}$ $\qquad \dfrac{d(x^n)}{dx} = n \cdot x^{n-1}$

Chain Rule

$= \dfrac{d(ax+b)^n}{d(ax+b)} \cdot \dfrac{d(ax+b)}{dx}$

$= n \cdot (ax+b)^{n-1} \cdot (a \times 1 + 0)$

$= na(ax+b)^{n-1}$

II - method, (fast) - Chain Rule.

$f(x) = (ax+b)^n$

$\Rightarrow f'(x) = n \cdot (ax+b)^{n-1} \cdot (a \times 1 + 0) = na(ax+b)^{n-1}$

$\boxed{Q.13}$ $f(x) = (ax+b)^n \cdot (cx+d)^m$

$(u \cdot v)' = u' \cdot v + u \cdot v'$

$f'(x) = \dfrac{d(ax+b)^n}{dx} \cdot (cx+d)^m + (ax+b)^n \cdot \dfrac{d(cx+d)^m}{dx}$

$= n(ax+b)^{n-1} \cdot (a \times 1 + 0) \cdot (cx+d)^m$

$+ (ax+b)^n \cdot m \cdot (cx+d)^{m-1} \cdot (c \times 1 + 0)$

$= na(ax+b)^{n-1} \cdot (cx+d)^m$

$+ m.c\,(ax+b)^n \cdot (cx+d)^{m-1}$

$= (ax+b)^{n-1} \cdot (cx+d)^{m-1} \cdot \left\{ \begin{array}{l} na(cx+d) \\ + mc(ax+b) \end{array} \right\}$

$\boxed{Q.14}$ $f(x) = \sin(x+a)$

Chain Rule

$\Rightarrow f'(x) = \cos(x+a)\cdot(1+0)$

$\Rightarrow f'(x) = \cos(x+a)$ ✓

$\boxed{Q.15}$ $f(x) = \operatorname{cosec} x \cdot \cot x$

$(u \cdot v)' = u'.v + u.v'$

$$f'(x) = \frac{d(\operatorname{cosec} x \cdot \cot x)}{dx}$$

$$= \frac{d(\operatorname{cosec} x)}{dx} \cdot \cot x + \operatorname{cosec} x \cdot \frac{d(\cot x)}{dx}$$

$$= (-\operatorname{cosec} x \cdot \cot x) \cdot \cot x + \operatorname{cosec} x \cdot (-\operatorname{cosec}^2 x)$$

$$= -\operatorname{cosec} x \cdot \cot^2 x - \operatorname{cosec}^3 x$$

$\boxed{Q.16}$ $f(x) = \dfrac{\cos x}{1+\sin x} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(-\sin x)\cdot(1+\sin x) - \cos x\cdot(0+\cos x)}{(1+\sin x)^2}$$

$$f'(x) = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$f'(x) = -\left(\frac{+\sin x + (1)}{(1+\sin x)^2}\right)$$

$$= -\frac{\cancel{(1+\sin x)}^1}{(1+\sin x)^{\cancel{2}}}$$

$$= -\frac{1}{(1+\sin x)}$$ ✓

$\boxed{Q.17}$ $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{(\cos x - \sin x).(\sin x - \cos x) - (\sin x + \cos x).(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= -\left\{\frac{\sin^2 x + \cos^2 x - \cancel{2\sin x \cos x} + \sin^2 x + \cos^2 x + \cancel{2\sin x . \cos x}}{(\sin x - \cos x)^2}\right\}$$

$$= -\frac{2}{(\sin x - \cos x)^2}$$

$\boxed{Q.18}$ $f(x) = \dfrac{\sec x - 1}{\sec x + 1} = \dfrac{u}{v}$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{[\sec x . \tan x - 0] . (\sec x + 1) - (\sec x - 1) . [\sec x . \tan x + 0]}{(\sec x + 1)^2}$$

$$= \frac{\cancel{\sec^2 x . \tan x} + \sec x . \tan x - \cancel{\sec^2 x . \tan x} + \sec x . \tan x}{(\sec x + 1)^2}$$

$$= \frac{2 \sec x . \tan x}{(\sec x + 1)^2}$$

Q.19

$$f(x) = \sin^n x$$

$$f(x) = (\sin x)^n$$

Chain Rule

$$f'(x) = \frac{d\,(\sin x)^n}{dx}$$

$$= n.(\sin x)^{n-1} \times \cos x$$

$$= n.\cos x.(\sin x)^{n-1}$$

Q.20

$$f(x) = \frac{a + b\sin x}{c + d\cos x} \quad \frac{u}{v}$$

$$\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}$$

$$f'(x) = \frac{\frac{d(a+b\sin x)}{dx}.(c+d\cos x) - (a+b\sin x).\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$= \frac{(0+b\cos x).(c+d\cos x) - (a+b\sin x).[0+d(-\sin x)]}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + bd\cos^2 x + ad\sin x + bd\sin^2 x}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd(1)}{(c+d\cos x)^2}$$

Q.21 $f(x) = \frac{\sin(x+a)}{\cos x}$ $\frac{u}{v}$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$f'(x) = \frac{\frac{d(\sin(x+a))}{dx} . \cos x - \sin(x+a) . \frac{d(\cos x)}{dx}}{(\cos x)^2}$$

$$= \frac{\cos(x+a).(1+0) . \cos x - \sin(x+a) . (-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos(x+a) . \cos x + \sin(x+a) . \sin x}{(\cos x)^2}$$

$\cos A . \cos B + \sin A . \sin B = \cos(A-B)$

$$= \frac{\cos(\cancel{x}+a-\cancel{x})}{(\cos x)^2} = \frac{\cos a}{(\cos x)^2}$$

$\boxed{Q.22}$ $f(x) = \overbrace{x^4}^{u} . \overbrace{(5\sin x - 3\cos x)}^{v}$

$(u.v)' = u'.v + u.v'$

$$f'(x) = \frac{d(x^4)}{dx} . (5\sin x - 3\cos x) + x^4 . \frac{d(5\sin x - 3\cos x)}{dx}$$

$$f'(x) = 4.x^3.(5\sin x - 3\cos x) + x^4.(5.\cos x + 3\sin x)$$

$$= x^3.\left\{ \begin{array}{l} 20\sin x - 12\cos x \\ + 5x\cos x + 3x\sin x \end{array} \right\}$$

$\boxed{Q.23}$ $f(x) = (x^2+1).\cos x$

$\boxed{(u.v)' = u'.v + u.v'}$

$$f'(x) = (2.x + 0).\cos x + (x^2+1).(-\sin x)$$

$$= 2x\cos x - \sin x.(x^2+1)$$

$\boxed{Q.24}$ $f(x) = (ax^2 + \sin x) \cdot (p + q\cos x)$

$$\boxed{(u \cdot v)' = \underline{u'} \cdot \underline{v} + \underline{u} \cdot \underline{v'}}$$

$$f'(x) = (a \cdot 2x + \cos x) \cdot (p + q\cos x)$$
$$+ (ax^2 + \sin x) \cdot (0 + q(-\sin x))$$

$$f'(x) = (2ax + \cos x) \cdot (p + q\cos x)$$
$$- q\sin x \cdot (ax^2 + \sin x)$$

$\boxed{Q.25}$

$$f(x) = (x+\cos x)\,(x-\tan x)$$

$$(u.v)' = u'.v + u.v'$$

$$f'(x) = \frac{d\,(x+\cos x)}{dx}.(x-\tan x) + (x+\cos x).\frac{d\,(x-\tan x)}{dx}$$

$$= (1-\sin x).(x-\tan x) + (x+\cos x).(1-\sec^2 x)$$

$\boxed{Q.26}$

$$f(x) = \frac{4x+5\sin x}{3x+7\cos x} \quad \left(\frac{u}{v}\right)$$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$f'(x) = \frac{(4+5\cos x).(3x+7\cos x) - (4x+5\sin x).(3-7\sin x)}{(3x+7\cos x)^2}$$

$$= \frac{(4+5\cos x).(3x+7\cos x) - (4x+5\sin x).(3-7\sin x)}{(3x+7\cos x)^2}$$

$$= \frac{\cancel{12x} + 28\cos x + 15x\cos x + 35\cos^2 x - \cancel{12x} + 28x\sin x - 15\sin x + 35\sin^2 x}{(3x+7\cos x)^2}$$

$$= \frac{28\cos x + 15x\cos x + 28x\sin x - 15\sin x + 35}{(3x+7\cos x)^2}$$

$\boxed{Q.27}$ $f(x) = \dfrac{x^2 \cdot \cos\frac{\pi}{4}}{\sin x}$ $\quad \dfrac{u}{v}$

$\left(\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \underline{\underline{\text{Constant}}} \right)$

$$\frac{d f(x)}{dx} = \frac{d\left(\dfrac{x^2 \cdot \cos\frac{\pi}{4}}{\sin x} \right)}{dx}$$

$$= \cos\frac{\pi}{4} \cdot \frac{d\left[\dfrac{x^2}{\sin x} \right]}{dx} \quad \frac{u}{v}$$

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \cos\frac{\pi}{4} \cdot \frac{(2x) \cdot \sin x - x^2 \cdot (\cos x)}{\sin^2 x}$$

$$= \cos\frac{\pi}{4} \cdot \left(\frac{2x \sin x - x^2 \cos x}{\sin^2 x} \right)$$

$\boxed{Q.28}$ $f(x) = \dfrac{x}{1 + \tan x}$ $\quad \dfrac{u}{v}$

$$f'(x) = \frac{(1) \cdot (1 + \tan x) - (x) \cdot (0 + \sec^2 x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2} \checkmark$$

$\boxed{Q.29}$ $f(x) = (x + \sec x) \cdot (x - \tan x)$

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

$$f'(x) = (1 + \sec x \cdot \tan x) \cdot (x - \tan x)$$
$$+ (x + \sec x) \cdot (1 - \sec^2 x)$$

(30) $f(x) = \dfrac{x}{\sin^n x}$ $\left(\dfrac{u}{v}\right)$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'.v - u.v'}{v^2}}$$

$$f'(x) = \frac{d\left(\frac{x}{\sin^n x}\right)}{dx}$$

$$= \frac{\frac{d(x)}{dx}.\sin^n x - x.\frac{d(\sin^n x)}{dx}}{(\sin^n x)^2}$$

$$= \frac{1.\sin^n x - x.(n.\cos x.(\sin x)^{n-1})}{(\sin^n x)^2}$$

$$= \frac{\sin^n x - nx.\cos x.\sin^{n-1} x}{\sin^{2n}.x}$$

$\sin^n x = (\sin x)^n$

Chain Rule

$x^n \to n.x^{n-1}$

$$\frac{d(\sin x)^n}{dx} = n.(\sin x)^{n-1}.\cos(x).1$$

$$= n.\cos x.(\sin x)^{n-1}$$

$$= \frac{\cancel{\sin^{n-1} x}(\sin x - nx.\cos x)}{\cancel{(\sin x)^{n-1}}.(\sin x)^{n+1}}$$

$$= \frac{\sin x - nx.\cos x}{(\sin x)^{n+1}}$$