Probaility

Short Answer Type Questions

1. A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

Sol. Let p be the probability that B gets selected.

P (Exactly one of A, B is selected) = 0.6 (given)

P (A is selected, B is not selected; B is selected, A is not selected) = 0.6

$P(A \cap B') + P(A' \cap B) = 0.6$

$\Rightarrow P(A)P(B') + P(A')P(B) = 0.6$

$\Rightarrow (0.7)(1-p) + (0.3)p = 0.6$

$\Rightarrow p = 0.25$

Thus, the probability that B gets selected is 0.25.

2. The probability of simultaneous occurrence of at least one of two events A and B is p. If the probability that exactly one of A, B occurs is q, then prove that $P(A') + P(B') = 2 - 2p + q$.

Sol. Since P (exactly one of A, B occurs) = q(given), we get

$P(A \cup B) - P(A \cap B) = q$

$\Rightarrow p - P(A \cap B) = q$

$\Rightarrow P(A \cap B) = p - q$

$\Rightarrow 1 - P(A' \cup B') = p - q$

$\Rightarrow P(A' \cup B') = 1 - p + q$

$\Rightarrow P(A') + P(B') - P(A' \cap B') = 1 - p + q$

$\Rightarrow P(A') + P(B') = (1 - p + q) + P(A' \cap B')$

$= (1 - p + q) + (1 - P(A \cup B))$

$= (1 - p + q) + (1 - p)$

$= 2 - 2p + q.$

3. 10% of the bulbs produced in a factory are of red colour and 2% are red and defective. If one bulb is picked up at random, determine the probability of its being defective if it is red.

Sol. Let A and B be the events that the bulb is red and defective, respectively.

$P(A) = \frac{10}{100} = \frac{1}{10},$

$P(A \cap B) = \frac{2}{100} = \frac{1}{50}$

$$P(B \mid A)=\frac{P(A\cap B)}{P(A)}=\frac{1}{50}\times\frac{10}{1}=\frac{1}{5}$$

Thus, the probability of the picked up bulb of its being defective, if it is red, is $\frac{1}{5}$.

4. Two dice are thrown together. Let A be the event 'getting 6 on the first die' and B be the event 'getting 2 on the second die'. Are the events A and B independent?

Sol. A = {(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

B = {(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)}

$A\cap B=\{(6,\ 2)\}$

$$P(A)=\frac{6}{36}=\frac{1}{6},\ P(B)=\frac{1}{6},\ P(A\cap B)=\frac{1}{36}$$

Events A and B will be independent if

$P(A\cap B)=P(A)P(B)$

i.e., LHS $=P(A\cap B)=\frac{1}{36}$, RHS $=P(A)P(B)=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$

Hence, A and B are independent.

5. A committee of 4 students is selected at random from a group consisting 8 boys and 4 girls. Given that there is at least one girl on the committee, calculate the probability that there are exactly 2 girls on the committee.

Sol. Let A denote the event that at least one girl will be chosen, and B the event that exactly 2 girls will be chosen. We require $P(B\mid A)$.

Since A denotes the event that at least one girl will be chosen, A' denotes that no girl is chosen, i.e., 4 boys are chosen. Then

$$P(A')=\frac{{}^{8}C_4}{{}^{12}C_4}=\frac{70}{495}=\frac{14}{99}$$

$$P(A)=1-\frac{14}{99}\ \frac{85}{99}$$

Now $P(A\cap B)=P$ (2boys and 2 girls) $=\frac{{}^{8}C_2.{}^{4}C_2}{{}^{12}C_4}$

$$=\frac{6\times 28}{495}=\frac{56}{165}$$

Thus $P(B|A)=\frac{P(A\cap B)}{P(A)}=\frac{56}{165}\times\frac{99}{85}=\frac{168}{425}$

6. Three machines E_1, E_2, E_3 in a certain factory produce 50%, 25% and 25%, respectively, of the total daily output of electric tubes. It is known that 4% of the tubes produced one each of machines E_1 and E_2 are defective, and that 5%

of those produced on E_3 are defective. If one tube is picked up at random from a day's production, calculate the probability that it is defective.

Sol. Let D be the event that the picked-up tube is defective.

Let A_1, A_2 *and* A_3 be the events that the tube is produced on machines E_1, E_2 and E_3 respectively.

$$P(D)=P(A_1)P(D|A_1)+P(A_2)P(D|A_2)+P(A_3)P(D|A_3)$$

$$P(A_1)=\frac{50}{100}=\frac{1}{2}, P(A_2)=\frac{1}{4}, P(A_3)\frac{1}{4}$$

Also $P(D|A_1)=P(D|A_2)=\frac{4}{100}=\frac{1}{25}$

$$P(D|A_3)=\frac{5}{100}=\frac{1}{20}$$

Putting these values in (1), we get

$$P(D)=\frac{1}{2}\times\frac{1}{25}+\frac{1}{4}\times\frac{1}{25}+\frac{1}{4}\times\frac{1}{20}$$

$$=\frac{1}{50}+\frac{1}{100}+\frac{1}{80}=\frac{17}{400}=.0425$$

7. Find the probability that in 10 throws of a fair die a score which is a multiple of 3 will be obtained in at least 8 of the throws.

Sol. Here success is a score which is a multiple of 3 i.e., 3 or 6.

Therefore, $p(3 \, or \, 6)=\frac{2}{6}\;\frac{1}{3}$

The probability of r success in 10 throws is given by

$$P(r)={}^{10}C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{10-r}$$

Now P (at least 8 successes) = P (8) + P (9) + P (10)

$$={}^{10}C_8\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2+{}^{10}C_9\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right)^1+{}^{10}C_{10}\left(\frac{1}{3}\right)^{10}$$

$$=\frac{1}{3^{10}}[45\times4+10\times2+1]=\frac{201}{3^{10}}.$$

8. A discrete random variable X has the following probability distribution:

X	**1**	**2**	**3**	**4**	**5**	**6**	**7**
P(x)	**C**	**2C**	**2C**	**3C**	**C^2**	**$2C^2$**	**$7C^2 + C$**

Find the value of C. Also find the mean of the distribution.

Sol. Since $\sum p_i = 1$, we have

$C+2C+2C+3C+C^2+2C^2+7C^2+C=1$

$i.e., 10C^2+9C-1=0$

$i.e., (10C-1)(C+1)=0$

$\Rightarrow C=\frac{1}{10},\ C=-1$

Therefore, the permissible value of $C=\frac{1}{10}$ (Why?)

Mean = $\sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{7} x_i p_i$

$=1+\frac{1}{10}+2\times\frac{2}{10}+3\times\frac{2}{10}+4\times\frac{3}{10}+5\left(\frac{1}{10}\right)^2+6\times2\left(\frac{1}{10}\right)^2+7\left(7\left(\frac{1}{10}\right)^2+\frac{1}{10}\right)$

$=\frac{1}{10}+\frac{4}{10}+\frac{6}{10}+\frac{12}{10}+\frac{5}{100}+\frac{12}{100}+\frac{49}{100}+\frac{7}{10}$

$=3.66.$

Long Answer Type Questions

9. **Four balls are to be drawn without replacement from a box containing 8 red and 4 white balls. If X denotes the number of red ball drawn, find the probability distribution of X.**

Sol. Since 4 balls have to be drawn, therefore, X can take the values 0, 1, 2, 3, 4.

$P(X=0)$ = P (no red ball) = P (4 white balls)

$\frac{{}^4C_4}{{}^{12}C_4}=\frac{1}{495}$

$P(X=1)$ = P (1 red ball and 3 white balls)

$\frac{{}^8C_1\times{}^4C_3}{{}^{12}C_4}=\frac{32}{495}$

$P(X=2)$ = P (2 red balls and 2 white balls)

$\frac{{}^8C_2\times{}^4C_2}{{}^{12}C_4}=\frac{168}{495}$

$P(X=3)$ = P (3 red balls and 1 white ball)

$\frac{{}^8C_3\times{}^4C_1}{{}^{12}C_4}=\frac{224}{495}$

$P(X=4)$ = P (4 red balls) $= \frac{^8C_4}{^{12}C_4} = \frac{70}{495}$.

Thus, the following is the required probability distribution of X

X	0	1	2	3	4
P(X)	$\frac{1}{495}$	$\frac{32}{495}$	$\frac{168}{495}$	$\frac{224}{495}$	$\frac{70}{495}$

10. Determine variance and standard deviation of the number of heads in three tosses of a coin.

Sol. Let X denote the number of heads tossed. So, X can take the values 0, 1, 2, 3. When a coin is tossed three times, we get

Sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$P(X=0)$ = P (no head) = P (TTT) $= \frac{1}{8}$

$P(X=1)$ = P (one head) = P (HTT, THT, TTH) $= \frac{3}{8}$

$P(X=2)$ = P (two heads) = P (HHT, HTH, THH) $= \frac{3}{8}$

$P(X=3)$ = P (three heads) = P (HHH) $= \frac{1}{8}$

Thus the probability distribution of X is:

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Variance of X $= \sigma^2 = \sum x_i^2 p_i - u^2$, (1)

where μ is the mean of X given by

$\mu = \sum x_i p_i = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$

$= \frac{3}{2}$ (2)

Now $\sum x_i^2 p_i = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3$ (3)

From (1), (2) and (3), we get

$$\sigma^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

Standard deviation $= \sqrt{\sigma^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

11. Refer to Example 6. Calculate the probability that the defective tube was produced on machine E1.

Sol. Now, we have to find $P(A_1/D)$.

$$P(A_1/D) = \frac{P(A_1 \cap D)}{P(D)} = \frac{P(A_1)P(D/A_1)}{P(D)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{25}}{\frac{17}{400}} = \frac{8}{17}.$$

12. A car manufacturing factory has two plants, X and Y. Plant X manufactures 70% of cars and plant Y manufactures 30%. 80% of the cars at plant X and 90% of the cars at plant Y are rated of standard quality. A car is chosen at random and is found to be of standard quality. What is the probability that it has come from plant X?

Sol. Let E be the event that the car is of standard quality. Let B_1 *and* B_2 be the events that the car is manufactured in plants X and Y, respectively. Now

$$P(B_1) = \frac{70}{100} = \frac{7}{10}, P(B_2) = \frac{30}{100} = \frac{3}{10}$$

$P(E|B_1)$ = Probability that a standard quality car is manufactured in plant

$$= \frac{80}{100} = \frac{8}{10}$$

$$P(E|B_2) = \frac{90}{100} = \frac{9}{10}$$

$P(B_1|E)$ = Probability that a standard quality car has come from plant X

$$= \frac{P(B_1) \times P(E|B_1)}{P(B_1).P(E|B_1) + P(B_2).P(E|B_2)}$$

$$= \frac{\frac{7}{10} \times \frac{8}{10}}{\frac{7}{10} \times \frac{8}{10} + \frac{3}{10} \times \frac{9}{10}} = \frac{56}{83}$$

Hence the required probability is $\frac{56}{83}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 13 to 17.

13. **Let A and B be two events. If** $P(A)=0.2,\ P(B)=0.4,\ P(AB)=0.6,$ **then** $P(A|B)$ **is equal to**

(A) 0.8 **(B) 0.5** **(C) 0.3** **(D) 0**

Sol. The correct answer is (D). From the given data $P(A)+P(B)=P(A\cup B)$.

This shows that $P(A\cap B)=0$. Thus, $P(A|B)=\frac{\text{P(A}\cap\text{B)}}{\text{P(B)}}=0.$

14. **Let A and B be two events such that** $P(A)=0.6,\ P(B)=0.2,$ **and** $P(A|B)=0.5$**.** **Then** $P(A'|B')$ **equals**

(A) $\frac{1}{10}$

(B) $\frac{3}{10}$

(C) $\frac{3}{8}$

(D) $\frac{6}{7}$

Sol. The correct answer is (C). $P(A\cap B)=P(A|B)P(B)$

$=0.5\times 0.2=0.1$

$$\text{P(A'|B')}=\frac{\text{P(A'}\cap\text{B')}}{\text{P(B')}}=\frac{\text{P[(A}\cup\text{B')]}}{\text{P(B')}}=\frac{1-\text{P(A}\cup\text{B)}}{1-\text{P(B)}}$$

$$=\frac{1-\text{P(A)}-\text{P(B)}+\text{P(A}\cap\text{B)}}{1-0.2}=\frac{3}{8}.$$

15. **If A and B are independent events such that** $0<P(A)<1$ *and* $0<P(B)<1,$ **then which of the following is not correct?**

(A) A *and* B **are mutually exclusive**

(B) A *and* B' **are independent**

(C) A' *and* B **are independent**

(D) A' *and* B' **are independent**

Sol. The correct answer is (A).

16. **Let X be a discrete random variable. The probability distribution of X is given below:**

X	30	10	-10
P(X)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{2}$

Then E (X) is equal to

(A) 6 (B) 4 (C) 3 (D) –5

Sol. The correct answer is (B).

$$E(X)=30\times\frac{1}{5}+10\times\frac{3}{10}-10\times\frac{1}{2}=4.$$

17. Let X be a discrete random variable assuming values $x_1, x_2, \ldots. x_n$ with probabilities $p_1, p_2, \ldots. p_n$, respectively. Then variance of X is given by

(A) $E\left(X^2\right)$

(B) $E\left(X^2\right)+E\left(X\right)$

(C) $E\left(X^2\right)-\left[E\left(X\right)\right]^2$

(D) $\sqrt{E(X^2)-[E(X)]^2}$

Sol. The correct answer is (C).

Fill in the blanks in Examples 18 and 19

18. If A and B are independent events such that $P\left(A\right)=p,\ P\left(B\right)=2p$ and P(Exactly one of A, B) $=\frac{5}{9}$, then p = ______________

Sol. $p=\frac{1}{3},\frac{5}{12}\ \left[(1-p)(2p)+p(1-2p)=3p-4p^2=\frac{5}{9}\right]$

19. If A and B' are independent events then $P(A'\cup B)=1-$ _______

Sol. $P\left(A'\cup B\right)=1-P\left(A\cap B'\right)=1-P\left(A\right)P\left(B'\right)$ (since A and B' are independent).

State whether each of the statement in Examples 20 to 22 is True or False

20. Let A and B be two independent events. Then $P\left(A\cap B\right)=P\left(A\right)+P\left(B\right)$.

Sol. False, because $P\left(A\cap B\right)=P\left(A\right).P\left(B\right)$ when events A and B are independent.

21. Three events A, B and C are said to be independent if $P\left(A\cap B\cap C\right)=P\left(A\right)P\left(B\right)P\left(C\right)$.

Sol. False. Reason is that A, B, C will be independent if they are pairwise independent and $P\left(A\cap B\cap C\right)=P\left(A\right)P\left(B\right)P\left(C\right)$.

22. One of the condition of Bernoulli trials is that the trials are independent of each other.

Sol. True.

Probability

Objective Type Questions

Choose the correct answer from the given four options in each of the exercises from 56 to 82.

56. **If** $P(A)=\frac{4}{5}$**, and** $P(A\cap B)=\frac{7}{10}$**, then** $P(B|A)$ **is equal to**

(A) $\frac{1}{10}$

(B) $\frac{1}{8}$

(C) $\frac{7}{8}$

(D) $\frac{17}{20}$

Sol. (C) $\because P(A)=\frac{4}{5}, P(A\cap B)=\frac{7}{10}$

$\therefore P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{7/10}{4/5}=\frac{7}{8}$

57. **If** $P(A\cap B)=\frac{7}{10}$ **and** $P(B)=\frac{17}{20}$**, then** $P(A/B)$ **equals**

(A) $\frac{14}{17}$

(B) $\frac{17}{20}$

(C) $\frac{7}{8}$

(D) $\frac{1}{8}$

Sol. (A) Here, $P(A\cap B)=\frac{7}{10}$ and $P(B)=\frac{17}{20}$

$\therefore P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{7/10}{17/20}=\frac{14}{17}$

58. **If** $P(A)=\frac{3}{10}$**,** $P(B)=\frac{2}{5}$ **and** $P(A\cup B)=\frac{3}{5}$**, then** $P(B/A)+P(A/B)$ **equals**

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{5}{12}$

(D) $\frac{7}{2}$

Sol. (D) Here, $P(A)=\frac{3}{10}, P(B)=\frac{2}{5}$ and $P(A\cup B)=\frac{3}{5}$

$$P(B/A)+P(A/B)=\frac{P(B\cap A)}{P(A)}+\frac{P(A\cap B)}{P(B)}$$

$$=\frac{P(A)+P(B)-P(A\cup B)}{P(A)}+\frac{P(A)+P(B)-P(A\cup B)}{P(B)}$$

$$\begin{bmatrix}\because P(A\cup B)=P(A)+P(B)-P(A\cap B)\\ i.e., P(A\cap B)=P(A)+P(B)-P(A\cup B)\end{bmatrix}$$

$$=\frac{\frac{3}{10}+\frac{2}{5}-\frac{3}{5}}{\frac{3}{10}}+\frac{\frac{3}{10}+\frac{2}{5}-\frac{3}{5}}{\frac{2}{5}}$$

$$=\frac{\frac{1}{10}}{\frac{3}{10}}+\frac{\frac{1}{10}}{\frac{2}{5}}=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$$

59. **If** P(A) = $\frac{2}{5}$**,** P(B) = $\frac{3}{10}$ **and** P(A $\cap$ B) = $\frac{1}{5}$**, then P(A' | B').P(B' | A') is equal to**

(A) $\frac{5}{6}$

(B) $\frac{5}{7}$

(C) $\frac{25}{42}$

(D) 1

Sol. (C) Here, $P(A)=\frac{2}{5}, P(B)=\frac{3}{10}$ and $P(A\cap B)=\frac{1}{5}$

$$P(A'/B')=\frac{P(A'\cap B')}{P(B')}=\frac{1-P(A\cup B)}{1-P(B)}$$

$$=\frac{1-[P(A)+P(B)-P(A\cap B)]}{1-P(B)}$$

$$= \frac{1-\left(\frac{2}{5}+\frac{3}{10}-\frac{1}{5}\right)}{1-\frac{3}{10}}$$

$$= \frac{1-\left(\frac{4+3-2}{10}\right)}{\frac{7}{10}} = \frac{1-\frac{1}{2}}{\frac{7}{10}} = \frac{5}{7}$$

And $P(B'/A') = \frac{P(B'\cap A')}{P(A')} = \frac{1-P(A\cup B)}{1-P(A)}$

$$= \frac{1-\frac{1}{2}}{1-\frac{2}{5}} = \frac{1/2}{3/5} = \frac{5}{6} \qquad \left[\because P(A\cup B) = \frac{1}{2}\right]$$

$$\therefore P(A'/B')\cdot P(B'/A') = \frac{5}{7}\cdot\frac{5}{6} = \frac{25}{42}$$

60. If A and B are two events such that $P(A) = \frac{1}{2}$**,** $P(B) = \frac{1}{3}$**,** $P(A/B) = \frac{1}{4}$**, then** $P(A'\cap B')$ **equals**

(A) $\frac{1}{12}$

(B) $\frac{3}{4}$

(C) $\frac{1}{4}$

(D) $\frac{3}{16}$

Sol. (C) Here, $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and $P(A/B) = \frac{1}{4}$

$\because P(A/B) = \frac{P(A\cap B)}{P(B)}$

$\Rightarrow P(A\cap B) = P(A/B)\cdot P(B) = \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}$

Now, $P(A'\cap B') = 1 - P(A\cup B)$

$= 1-[P(A)+P(B)-P(A\cap B)]$

$= 1-\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{12}\right] = 1-\left[\frac{6+4-1}{12}\right]$

$=1-\frac{9}{12}=\frac{3}{12}=\frac{1}{4}$

61. **If** $P(A)=0.4,\ P(B)=0.8$ **and** $P(B|A)=0.6,$ **then** $P(A\cup B)$ **is equal to**

(A) 0.24

(B) 0.3

(C) 0.48

(D) 0.96

Sol. (D) Here, $P(A)=0.4,\ P(B)=0.8$ **and** $P(B|A)=0.6,$

$\because P(B/A)=\frac{P(B\cap A)}{P(A)}$

$\Rightarrow P(B\cap A)=P(B/A)\cdot P(A)$

$=0.6\times 0.4=0.24$

$\because P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=0.4+0.8-0.24$

$=1.2-0.24=0.96$

62. **If A and B are two events and** $\mathrm{A}\neq\phi,\mathrm{B}\neq\phi$**, then**

(A) $P(A/B)=P(A).P(B)$

(B) $\mathrm{P(A/B)}=\frac{\mathrm{P(A\cap B)}}{\mathrm{P(B)}}$

(C) $P(A/B).P(B/A)=1$

(D) $P(A/B)=P(A)/P(B)$

Sol. (B) If $A\neq\phi$ and $B\neq\phi$, then $P(A/B)=\frac{P(A\cap B)}{P(B)}$

63. **A and B are events such that** $P(A)=0.4,\ P(B)=0.3$ **and** $P(A\cup B)=0.5$**, Then** $P(B'\cap A)$ **equals**

(A) $\frac{2}{3}$

(B) $\frac{1}{2}$

(C) $\frac{3}{10}$

(D) $\frac{1}{5}$

Sol. (D) Here, $P(A)=0.4,\ P(B)=0.3$ and $P(A\cup B)=0.5$

$\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = 0.4 + 0.3 - 0.5 = 0.2$

$\because P(B' \cap A) = P(A) - P(A \cap B)$

$= 0.4 - 0.2 = 0.2 = \frac{1}{5}$

64. You are given that A and B are two events such that $\text{P(B)}=\frac{3}{5}$**,** $\text{P(A/B)}=\frac{1}{2}$ **and** $\text{P}(\text{A} \cup \text{B})=\frac{4}{5}$**, then P(A) equals**

(A) $\frac{3}{10}$

(B) $\frac{1}{5}$

(C) $\frac{1}{2}$

(D) $\frac{3}{5}$

Sol. (C) Here, $P(B) = \frac{3}{5}, P(A/B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$

$\because P(A/B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow \frac{1}{2} = \frac{P(A \cap B)}{3/5}$

$\Rightarrow P(A \cap B) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$

And $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \frac{4}{5} - P(A) + \frac{3}{5} - \frac{3}{10}$

$\therefore P(A) = \frac{4}{5} - \frac{3}{5} + \frac{3}{10} = \frac{8 - 6 + 3}{10} = \frac{1}{2}$

65. In Exercise 64 above, $P(B \mid A')$ **is equal to**

(A) $\frac{1}{5}$

(B) $\frac{3}{10}$

(C) $\frac{1}{2}$

(D) $\frac{3}{5}$

Sol. (D) $P(B/A')=\frac{P(B\cap A')}{P(A')}=\frac{P(B)-P(B\cap A)}{1-P(A)}$

$=\frac{\frac{3}{5}-\frac{3}{10}}{1-\frac{1}{2}}=\frac{\frac{6-3}{10}}{\frac{1}{2}}=\frac{6}{10}=\frac{3}{5}$

66. **If** $\text{P(B)}=\frac{3}{5}$**,** $\text{P(A/B)}=\frac{1}{2}$ **and** $\text{P}(\text{A}\cup\text{B})=\frac{4}{5}$**, then** $\text{P}(\text{A}\cup\text{B})'+\text{P}(\text{A}'\cup\text{B})=$

(A) $\frac{1}{5}$

(B) $\frac{4}{5}$

(C) $\frac{1}{2}$

(D) 1

Sol. (D) Here, $P(B)=\frac{3}{5}, P(A/B)=\frac{1}{2}$

And $P(A\cup B)=\frac{4}{5}$

Since, $P(A/B)=\frac{P(A\cap B)}{P(B)}$

$\Rightarrow P(A\cap B)=P(A/B)\cdot P(B)$

$=\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}$

Also, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow P(A)=\frac{4}{5}-\frac{3}{5}+\frac{3}{10}=\frac{1}{2}$

$\therefore P(A\cup B)'=1-P(A\cup B)=1-\frac{4}{5}=\frac{1}{5}$

And $P(A'\cup B)=1-P(A-B)=1-P(A\cap B')$

$=1-P(A)\cdot P(B')$

$=1-\frac{1}{2}\cdot\frac{2}{5}=\frac{4}{5}$

$\Rightarrow P(A\cup B')+P(A'\cup B)=\frac{1}{5}+\frac{4}{5}=\frac{5}{5}=1$

67. **Let** $P(A)=\frac{7}{13}$**,** $P(B)=\frac{9}{13}$ **and** $P(A\cap B)=\frac{4}{13}$**. Then P(A'/B) is equal to**

(A) $\frac{6}{13}$

(B) $\frac{4}{13}$

(C) $\frac{4}{9}$

(D) $\frac{5}{9}$

Sol. (D) Here, $P(A)=\frac{7}{13}, P(B)=\frac{9}{13}$ and $P(A\cap B)=\frac{4}{13}$

$\because P(A'/B)=\frac{P(A'\cap B)}{P(B)}=\frac{P(B)-P(A\cap B)}{P(B)}$

$=\frac{\frac{9}{13}-\frac{4}{13}}{\frac{9}{13}}=\frac{\frac{5}{13}}{\frac{9}{13}}=\frac{5}{9}$

68. **If A and B are such events that P(A) > 0 and P(B)≠1, then P(A'/B') equals.**

(A) $1-P(A/B)$

(B) $1-P(A'/B)$

(C) $\frac{1-P(A\cup B)}{P(B')}$

(D) $P(A')/P(B')$

Sol. (C) $\because P(A)>0$ and $P(B)\neq 1$

$P(A'/B')=\frac{P(A'\cap B')}{P(B')}=\frac{1-P(A\cup B)}{P(B')}$

69. **If A and B are two independent events with** $P(A)=\frac{3}{5}$ **and** $P(B)=\frac{4}{9}$**, then** $P(A'\cap B')$ **equals**

(A) $\frac{4}{15}$

(B) $\frac{8}{45}$

(C) $\frac{1}{3}$

(D) $\frac{2}{9}$

Sol. (D) $P(A'\cap B')=1-P(A\cup B)$

$=1-[P(A)+P(B)-P(A\cap B)]$

$=1-\left[\frac{3}{5}+\frac{4}{9}-\frac{3}{5}\times\frac{4}{9}\right]$ $[\because P(A\cap B)=P(A)\cdot P(B)]$

$=1-\left[\frac{27+20-12}{45}\right]=1-\frac{35}{45}=\frac{10}{45}=\frac{2}{9}$

70. If two events are independent, then

(A) they must be mutually exclusive

(B) the sum of their probabilities must be equal to 1

(C) (A) and (B) are both are correct

(D) None of the above is correct

Sol. (D) If two events A and B are independent, then we know that

$P(A\cap B)=P(A)\cdot P(B), P(A)\neq 0, P(B)\neq 0$

Since, A and B have a common outcome.

Further, mutually exclusive events never have a common outcome.

In other words, two independents events having non-zero probabilities of occurrence cannot be mutually exclusive and conversely, i.e., two mutually exclusive events having non-zero probabilities of outcome cannot be independent.

71. Let A and B be two events such that $\text{P(A)}=\frac{3}{8}$**,** $\text{P(B)}=\frac{5}{8}$ **and** $\text{P}(A\cup B)=\frac{3}{4}$**. Then** $P(A|B).P(A'/B)$ **is equal to**

(A) $\frac{2}{5}$

(B) $\frac{3}{8}$

(C) $\frac{3}{10}$

(D) $\frac{6}{25}$

Sol. (D) Here, $P(A)=\frac{3}{8}, P(B)=\frac{5}{8}$ and $P(A\cup B)=\frac{3}{4}$

$\because P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow P(A\cap B)=\frac{3}{8}+\frac{5}{8}-\frac{3}{4}=\frac{3+5-6}{8}=\frac{2}{8}=\frac{1}{4}$

$\because P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{1/4}{5/8}=\frac{8}{20}=\frac{2}{5}$

And $P(A'/B)=\dfrac{P(A'\cap B)}{P(B)}=\dfrac{P(B)-P(A\cap B)}{P(B)}$

$$=\frac{\frac{5}{8}-\frac{1}{4}}{\frac{5}{8}}=\frac{\frac{5-2}{8}}{\frac{5}{8}}=\frac{3}{5}$$

$$\therefore P(A/B)\cdot P(A'/B)=\frac{2}{5}\cdot\frac{3}{5}=\frac{6}{25}$$

72. If the events A and B are independent, then $P(A\cap B)$ is equal to

(A) $P(A)+P(B)$

(B) $P(A)-P(B)$

(C) $P(A).P(B)$

(D) $P(A)/P(B)$

Sol. (C) If A and B are independent, then $P(A\cap B)=P(A)\cdot P(B)$

73. Two events E and F are independent. If $P(E)=0.3,\ P(E\cup F)=0.5,$ then $P(E\,|\,F)-P(F\,|\,E)$ equals

(A) $\frac{2}{7}$

(B) $\frac{3}{35}$

(C) $\frac{1}{70}$

(D) $\frac{1}{7}$

Sol. (C) Here $P(E)=0.3,\ P(E\cup F)=0.5,$

Let P(F) = x

$\therefore P(E\cup F)=P(E)+P(F)-P(E\cap F)$

$=P(E)+P(F)-P(E)\cdot P(F)$

$\Rightarrow 0.5=0.3+x-0.3x$

$\Rightarrow x=\dfrac{0.5-0.3}{0.7}=\dfrac{2}{7}=P(F)$

$\therefore P(E/F)-P(F/E)=\dfrac{P(E\cap F)}{P(F)}-\dfrac{P(F\cap E)}{P(E)}$

$$= \frac{P(E \cap F) \cdot P(E) - P(F \cap E) \cdot P(F)}{P(E) \cdot P(F)}$$

$$= \frac{P(E \cap F)[P(E) - P(F)]}{P(E \cap F)} = P(E) - P(F)$$

$$= \frac{3}{10} - \frac{2}{7} = \frac{21-20}{70} = \frac{1}{70}$$

74. A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement the probability of getting exactly one red ball is

(A) $\frac{45}{196}$

(B) $\frac{135}{392}$

(C) $\frac{15}{56}$

(D) $\frac{15}{29}$

Sol. (C) Probability of getting exactly one red (R) ball $= P_R \cdot P_{\bar{R}} \cdot P_{\bar{R}} + P_{\bar{R}} \cdot P_R \cdot P_{\bar{R}} + P_{\bar{R}} \cdot P_{\bar{R}} \cdot P_R$

$$= \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{2}{6} + \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{5}{6}$$

$$= \frac{15}{4 \cdot 7 \cdot 6} + \frac{15}{4 \cdot 7 \cdot 6} + \frac{15}{4 \cdot 7 \cdot 6}$$

$$= \frac{5}{56} + \frac{5}{56} + \frac{5}{56} = \frac{15}{56}$$

75. Refer to Question 74 above. The probability that exactly two of the three balls were red, the first ball being red, is

(A) $\frac{1}{3}$

(B) $\frac{4}{7}$

(C) $\frac{15}{28}$

(D) $\frac{5}{28}$

Sol. (B) Let E_1 = Event that first ball being red

And E_2 = Event that exactly two of three balls being red

$\therefore P(E_1) = P_R \cdot P_R \cdot P_R + P_R \cdot P_R \cdot P_{\overline{R}} + P_R \cdot P_{\overline{R}} \cdot P_R + P_R \cdot P_{\overline{R}} \cdot P_{\overline{R}}$

$= \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} + \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} + \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6}$

$= \frac{60+60+60+30}{336} = \frac{210}{336}$

$P(E_1 \cap E_2) = P_R \cdot P_{\overline{R}} \cdot P_R + P_R \cdot P_R \cdot P_{\overline{R}}$

$= \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{120}{336}$

$\therefore P(E_2 / E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{120/336}{210/336} = \frac{4}{7}$

76. Three persons, A, B and C, fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is

(A) 0.024

(B) 0.188

(C) 0.336

(D) 0.452

Sol. (B) Here, $P(A) = 0.4, P(\overline{A}) = 0.6, P(B) = 0.3, P(\overline{B}) = 0.7,$

$P(C) = 0.2$ And $P(\overline{C}) = 0.8$

$\therefore$ Probability of two hits $= P_A \cdot P_B \cdot P_{\overline{C}} + P_A \cdot P_{\overline{B}} \cdot P_C + P_{\overline{A}} \cdot P_B \cdot P_C$

$= 0.4 \times 0.3 \times 0.8 + 0.4 \times 0.7 \times 0.2 + 0.6 \times 0.3 \times 0.2$

$= 0.096 + 0.056 + 0.036 = 0.188$

77. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{2}{3}$

(D) $\frac{4}{7}$

Sol. (D) Here, S={(B,B,B, (G,G,G), (B,G,G), (G,B,G), (G,G,B), (G,B,B), (B,G,B), (B,B,G)}

E_1 = Event that a family has at least one girl, then

E_1 = {(G, B, B), (B, G, B), (B, B, G), (G, G, B), (B, G, G), (G, B, G), (G, G, G)}

E_2 = Event that the eldest child is a girl, then

E_2 = {(G, B, B), (G, G, B), (G, B, G), (G, G, G)}

$\therefore E_1 \cap E_2 = \{(G,B,B),(G,G,B),(G,B,G),(G,G,G)\}$

$\therefore P(E_2 / E_1) = \dfrac{P(E_1 \cap E_2)}{P(E_1)} = \dfrac{4/8}{7/8} = \dfrac{4}{7}$

78. A die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card is

(A) $\dfrac{1}{2}$

(B) $\dfrac{1}{4}$

(C) $\dfrac{1}{8}$

(D) $\dfrac{3}{4}$

Sol. (C) Let E_1 = Event for getting an even number on the die

And E_2 = Event that a spade card is selected

$\therefore P(E_1) = \dfrac{3}{6} = \dfrac{1}{2}$ and $P(E_2) = \dfrac{13}{52} = \dfrac{1}{4}$

Then, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = \dfrac{1}{2} \cdot \dfrac{1}{4} = \dfrac{1}{8}$

79. A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

(A) $\dfrac{3}{28}$

(B) $\dfrac{2}{21}$

(C) $\dfrac{1}{28}$

(D) $\dfrac{167}{168}$

Sol. (A) Probability of drawing 2 green balls and one blue ball

$= P_G \cdot P_G \cdot P_B + P_B \cdot P_G \cdot P_G + P_G \cdot P_B \cdot P_G$

$= \dfrac{3}{8} \cdot \dfrac{2}{7} \cdot \dfrac{2}{7} + \dfrac{2}{8} \cdot \dfrac{3}{7} \cdot \dfrac{2}{6} + \dfrac{3}{8} \cdot \dfrac{2}{7} \cdot \dfrac{2}{6}$

$= \dfrac{1}{28} + \dfrac{1}{28} + \dfrac{1}{28} = \dfrac{3}{28}$

80. A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, the probability that both are dead is

(A) $\frac{33}{56}$

(B) $\frac{9}{64}$

(C) $\frac{1}{14}$

(D) $\frac{3}{28}$

Sol. (D) Required probability $= P_D \cdot P_D = \frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}$

81. Eight coins are tossed together. The probability of getting exactly 3 heads is

(A) $\frac{1}{256}$

(B) $\frac{7}{32}$

(C) $\frac{5}{32}$

(D) $\frac{3}{32}$

Sol. (B) We know that, Probability distribution $P(X = r) = {}^nC_r (p)^r q^{n-1}$

Here, n = 8, r = 3, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$\therefore$ Required probability $= {}^8C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{8-3} = \frac{8!}{5!3!}\left(\frac{1}{2}\right)^8$

$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} \cdot \frac{1}{16 \cdot 16} = \frac{7}{32}$

82. Two dice are thrown. If it is known that the sum of numbers on the dice was less than 6, the probability of getting a sum 3, is

(A) $\frac{1}{18}$

(B) $\frac{5}{18}$

(C) $\frac{1}{5}$

(D) $\frac{2}{5}$

Sol. (C) Let E_1 = Event that the sum of numbers on the dice was less than 6

And E_2 = Event that the sum of numbers on the dice is 3

$\therefore E_1 = \{(1,4),(4,1),(2,3),(3,2),(2,2),(1,3),(3,1),(1,2),(2,1),(1,1)\}$

$\Rightarrow n(E_1) = 10$

And $E_2 = \{(1,2),(2,1)\} \Rightarrow n(E_2) = 2$

$\therefore$ Required probability $= \frac{2}{10} = \frac{1}{5}$

83. Which one is not a requirement of a binomial distribution?

(A) There are 2 outcomes for each trial

(B) There is a fixed number of trials

(C) The outcomes must be dependent on each other

(D) The probability of success must be the same for all the trials

Sol. (C) We know that, in a Binomial distribution,

(i) There are 2 outcomes of each trail.

(ii) There is a fixed number of trails.

(iii) The probability of success must be the same for all the trails.

84. Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability, that both cards are queens, is

(A) $\frac{1}{13} \times \frac{1}{13}$

(B) $\frac{1}{13} + \frac{1}{13}$

(C) $\frac{1}{13} \times \frac{1}{17}$

(D) $\frac{1}{13} \times \frac{4}{51}$

Sol. (A) Required probability $= \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{13} \times \frac{1}{13}$ [with replacement]

85. The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

(A) $\frac{7}{64}$

(B) $\frac{7}{128}$

(C) $\frac{45}{1024}$

(D) $\frac{7}{41}$

Sol. (B) We know that, $P(X=r)={}^{n}C_{r}(p)^{r}(q)^{n-1}$

Here, $n=10, p=\frac{1}{2}, q=\frac{1}{2}$

And $r\geq 8$ *i.e.*, $r=8,9,10$

$\Rightarrow P(X=r)=P(r=8)+P(r=9)=P(r=10)$

$={}^{10}C_{8}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{10-8}+{}^{10}C_{9}\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)+{}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$

$=\frac{10!}{8!2!}\left(\frac{1}{2}\right)^{10}+\frac{10!}{9!1!}\left(\frac{1}{2}\right)^{10}+\left(\frac{1}{2}\right)^{10}$

$=\left(\frac{1}{2}\right)^{10}\cdot[45+10+1]=\left(\frac{1}{2}\right)^{10}\cdot 56$

$=\frac{1}{16\cdot 16}\cdot 56=\frac{7}{128}$

86. The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

(A) ${}^{5}C_{4}(0.7)^{4}(0.3)$

(B) ${}^{5}C_{1}(0.7)(0.3)^{4}$

(C) ${}^{5}C_{4}(0.7)(0.3)^{4}$

(D) $(0.7)^{4}(0.3)$

Sol. (A) Here, $\overline{p}=0.3\Rightarrow p=0.7$ and q = 0.3, n = 5 and r = 4

$\therefore$ Required probability $={}^{5}C_{4}(0.7)^{4}(0.3)$

87. The probability distribution of a discrete random variable X is given below:

X	2	3	4	5
P(X)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

The value of k is

(A) 8

(B) 16

(C) 32

(D) 48

Sol. (C) We know that, $\sum P(X)=1$

$$\Rightarrow \frac{5}{k}+\frac{7}{k}+\frac{9}{k}+\frac{11}{k}=1$$

$$\Rightarrow \frac{32}{k}=1$$

$$\therefore k=32$$

88. For the following probability distribution:

X	-4	-3	-2	-1	0
P(X)	0.1	0.2	0.3	0.2	0.2

E(X) is equal to:

(A) 0

(B) –1

(C) –2

(D) –1.8

Sol. (D) $E(X)=\sum XP(X)$

$=-4\times(0.1)+(-3\times0.2)+(-2\times0.3)+(-1\times0.2)+(0\times0.2)$

$=-0.4-0.6-0.6-0.2=-1.8$

89. For the following probability distribution

X	1	2	3	4
P(X)	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

$E\left(X^2\right)$ **is equal to**

(A) 3

(B) 5

(C) 7

(D) 10

Sol. (D) $E(X^2)=\sum X^2P(X)=1\cdot\frac{1}{10}+4\cdot\frac{1}{5}+9\cdot\frac{3}{10}+16\cdot\frac{2}{5}$

$$=\frac{1}{10}+\frac{4}{5}+\frac{27}{10}+\frac{32}{5}$$

$$=\frac{1+8+27+64}{10}=10$$

90. Suppose a random variable X follows the binomial distribution with parameters n and p, where $0<p<1$. If $P(x=r)/P(x=n-r)$ is independent of n and r, then p equals

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{5}$

(D) $\frac{1}{7}$

Sol. (A) $\because P(X=r)\,{}^{n}C_{r}(p)^{r}(q)^{n-r}=\frac{n!}{(n-r)!r!}(p)^{r}(1-p)^{n-r}$ $[\because q=1-p]$ ...(i)

$P(X=0)=(1-p)^{n}$

And $P(X=n-r)={}^{n}C_{n-r}(p)^{n-r}(q)^{n-(n-r)}$

$=\frac{n!}{(n-r)!r!}(p)^{n-r}(1-p)^{+r}$ $[\because q=1-p][\because {}^{n}C_{r}={}^{n}C_{n-r}]$ ...(ii)

Now, $\frac{P(x=r)}{P(x=n-r)}=\frac{\frac{n!}{(n-r)!r!}p^{r}(1-p)^{n-r}}{\frac{n!}{(n-r)!r!}p^{n-r}(1-p)^{+r}}$ [using Eqs. (i) and (iii)]

$=\left(\frac{1-p}{p}\right)^{n-r}\times\frac{1}{\left(\frac{1-p}{p}\right)^{r}}$

Above expression is independent of n and r, if $\frac{1-p}{p}=1\Rightarrow\frac{1}{p}=2\Rightarrow p=\frac{1}{2}$

91. In a college, 30% students fail in physics, 25% fail in mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in physics if she has failed in mathematics is

(A) $\frac{1}{10}$

(B) $\frac{2}{5}$

(C) $\frac{9}{20}$

(D) $\frac{1}{3}$

Sol. (B) Here, $P_{(Ph)}=\frac{30}{100}=\frac{3}{10},P_{(M)}=\frac{25}{100}=\frac{1}{4}$

And $P_{(M \cap Ph)} = \frac{10}{100} = \frac{1}{10}$

$\therefore P\left(\frac{Ph}{M}\right) = \frac{P(Ph \cap M)}{P(M)} = \frac{1/10}{1/4} = \frac{2}{5}$

92. A and B are two students. Their chances of solving a problem correctly are $\frac{1}{3}$ and $\frac{1}{4}$, respectively. If the probability of their making a common error is, $\frac{1}{20}$ and they obtain the same answer, then the probability of their answer to be correct is

(A) $\frac{1}{12}$

(B) $\frac{1}{40}$

(C) $\frac{13}{120}$

(D) $\frac{10}{13}$

Sol. (D) Let E_1 = Event that both A and B solve in the problem

$\therefore P(E_1) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$.

Let E_2 = Event that both A and B got incorrect solution of the problem

$\therefore P(E_2) = \frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$

Let E = Event that they got same answer

Here, $P(E/E_1) = 1, P(E/E_2) = \frac{1}{20}$

$\therefore P(E_1/E) = \frac{P(E_1 \cap E)}{P(E)} = \frac{P(E_1) \cdot P(E/E_1)}{P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)}$

$= \frac{\frac{1}{12} \times 1}{\frac{1}{12} \times 1 + \frac{1}{2} \times \frac{1}{20}} = \frac{1/2}{\frac{10+3}{120}} = \frac{120}{12 \times 3} = \frac{10}{13}$

93. A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(A) $\left(\frac{9}{10}\right)^5$

(B) $\frac{1}{2}\left(\frac{9}{10}\right)^4$

(C) $\frac{1}{2}\left(\frac{9}{10}\right)^5$

(D) $\left(\frac{9}{10}\right)^5+\frac{1}{2}\left(\frac{9}{10}\right)^4$

Sol. (D) Here, $n=5, p=\frac{10}{100}=\frac{1}{10}$ and $q=\frac{9}{10}$

$r\le 1$

$\Rightarrow r=0,1$

Also, $P(X=r)={}^nC_r p^r q^{n-r}$

$\therefore P(X=r)=P(r=0)+P(r=1)$

$={}^5C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^5+{}^5C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^4$

$=\left(\frac{9}{10}\right)^5+5\cdot\frac{1}{10}\cdot\left(\frac{9}{10}\right)^4$

$=\left(\frac{9}{10}\right)^5+\frac{1}{2}\left(\frac{9}{10}\right)^4$

State True or False for the statements in each of the Exercises 94 to 103.

94. Let P(A) > 0 and P(B) > 0. Then A and B can be both mutually exclusive and independent.

Sol. False

95. If A and B are independent events, then A’ and B’ are also independent.

Sol. True

96. If A and B are mutually exclusive events, then they will be independent also.

Sol. False

97. Two independent events are always mutually exclusive.

Sol. False

98. If A and B are two independent events then $P(A \text{ and } B)=P(A).P(B)$**.**

Sol. True

99. Another name for the mean of a probability distribution is expected value.

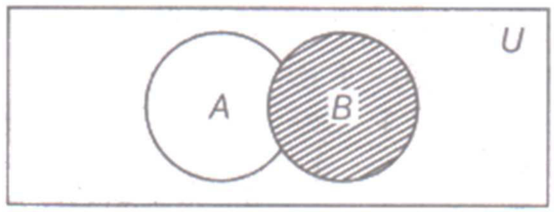
Sol. True

$E(X)=\sum XP(X)=\mu$

100. If A and B′ are independent events, then $P(A'\cup B)=1-P(A)P(B')$

Sol. True

$P(A'\cup B) = 1 - P(A\cap B') = 1 - P(A)P(B')$



$(A' \cup B)$

101. If A and B are independent, then
P (exactly one of A, B occurs) = $P(A)P(B') + P(B)\ P(A')$

Sol. Ture

102. If A and B are two events such that $P(A) > 0\ and\ P(A) + P(B) > 1,$ **then**

$$P(B/A) \geq 1 - \frac{P(B')}{P(A)}$$

Sol. False

$$\therefore P(B/A) = \frac{P(A\cap B)}{P(A)}$$

$$= \frac{P(A) + P(B) - P(A\cup B)}{P(B)} > \frac{1 - P(A\cup B)}{P(A)}$$

103. If A, B and C are three independent events such that P(A) = P(B) = P(C) = p, then
P (At least two of A, B, C occur) $= 3p^2 - 2p^3$

Sol. True
P (at least two of A, B and C occur)

$$= p\times p\times(1-p) + (1-p)\cdot p\cdot p + p(1-p)\cdot p + p\cdot p\cdot p$$

$$= p^2[1-p+1-p+1-p+p]$$

$$= p^2(3-3p) + p^3$$

$$= 3p^2 - 3p^3 + p^3 = 3p^2 - 2p^3$$

Fill in the blanks in each of the following questions:

104. If A and B are two events such that

$P(A/B) = p, P(A) = p, P(B) = \frac{1}{3}$ **And** $P(A\cup B) = \frac{5}{9}$, **then p = ____________**

Sol. Here, $P(A) = p, P(B) = \frac{1}{3}$ and $P(A\cup B) = \frac{5}{9}$

$$\because P(A/B) = \frac{P(A\cap B)}{P(B)} = p \Rightarrow P(A\cap B) = \frac{p}{3}$$

And $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$\Rightarrow \frac{5}{9} = p + \frac{1}{3} - \frac{p}{3} \Rightarrow \frac{5}{9} - \frac{1}{3} = \frac{2p}{3}$$

$$\Rightarrow \frac{5-3}{9} = \frac{2p}{3} \Rightarrow \quad p = \frac{2}{9} \times \frac{3}{2} = \frac{1}{3}$$

105. If A and B are such that

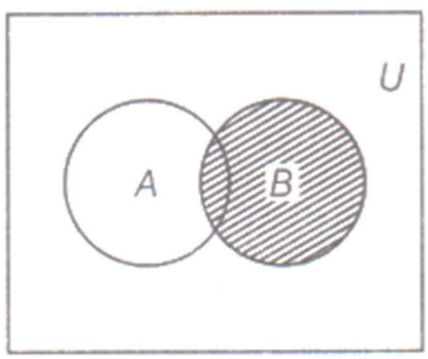
$P(A' \cup B') = \frac{2}{3}$ and $P(A \cup B) = \frac{5}{9}$ **, then P(A') + P(B') = …………**

Sol. Here, $P(A' \cup B') = \frac{2}{3}$ and $P(A \cup B) = \frac{5}{9}$

$$P(A' \cup B') = 1 - P(A \cap B)$$

$$\Rightarrow \frac{2}{3} = 1 - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = 1 - \frac{2}{3} = \frac{1}{3}$$



$$\because \quad P(A') + P(B') = 1 - P(A) + 1 - P(B)$$

$$= 2 - [P(A) + P(B)]$$

$$= 2 - [P(A \cup B) + P(A \cap B)]$$

$$= 2 - \left(\frac{5}{9} + \frac{1}{3}\right) = 2 - \left(\frac{5+3}{9}\right)$$

$$= \frac{18-8}{9} = \frac{10}{9}$$

106. If X follows binomial distribution with parameters $n = 5,\ p$ **and** $P(X=2) = 9,\ P(X=3)$**, then p = __________**

Sol. $\because P(X=2) = 9 \cdot P(X=3)$ (where, n=5 and q=1−p)

$$\Rightarrow {}^5C_2 p^2 (1-p)^3 = 9 \cdot {}^5C_3 P^3 (1-p)^2$$

$$\Rightarrow \frac{5!}{2!3!} p^2 (1-p)^3 = 9 \cdot \frac{5!}{3!2!} p^3 (1-p)^2$$

$$\Rightarrow \frac{p^2(1-p)^3}{p^3(1-p)^2} = 9$$

$$\Rightarrow \frac{(1-p)}{p} = 9 \Rightarrow 9p + p = 1$$

$$\therefore\ p = \frac{1}{10}$$

107. Let X be a random variable taking values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n,$ respectively. Then $var(X)$ = _________.

Sol. $Var(X) = (X^2) - [E(X)]^2$

$$= \sum_{i=1}^{n} X^2 P(X) - \left[\sum_{i=1}^{n} XP(X)\right]^2$$

$$= \sum P_i x_i^2 - (\sum P_i x_i)^2$$

108. Let A and B be two events. If P(A}B) = P(A), then A is _________ of B.

Sol. $\because\ P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow P(A) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A) \cdot P(B) = P(A \cap B)$$

So, A is independent of B.

Probability
Short Answer Type Questions

1. For a loaded die, the probabilities of outcomes are given as under:
$P(1)=P(2)=0.2,\ P(3)=P(5)=P(6)=0.1\ and\ P(4)=0.3$.

The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more', respectively. Determine whether or not A and B are independent.

Sol. For a loaded die, it is given that

$P(1)=P(2)=0.2,$

$P(3)=P(5)=P(6)=0.1\ and\ P(4)=0.3$

Also, die is thrown two times

Here, A = Same number each time and B = Total score is 10 or more

$\therefore\ A=\{(1,1),\ (2,2),\ (3,3),\ (4,4),\ (5,5),\ (6,6)\}$

So, $P(A)=[P(1,1)+P(2,2)+P(3,3)+P(4,4)+P(5,5)+P(6,6)]$

$=[P(1).P(1)+P(2).P(2)+P(3).P(3)+P(4).P(4)+P(5).P(5)+P(6).P(6)]$

$=[0.2\times0.2+0.2\times0.2+0.1\times0.1+0.3\times0.3+0.1\times0.1+0.1\times0.1]$

$=0.04+0.04+0.01+0.09+0.01+0.01=0.20$

$And\ B=\{(4,6),(6,4),(5,5),(5,6),(6,5),(6,6)\}$

$P(B)=P(4,6)+P(6,4)+P(5,5),P(5,6),(6,5),(6,6)\}$

$=P(4).P(6)+P(6).P(4)+P(5).P(5)+P(5).P(6)+P(6).P(5)+P(6).P(6)$

$=0.3\times0.1+0.1\times0.3+0.1\times0.1+0.1\times0.1+0.1\times0.1+0.1\times0.1$

$=0.03+0.03+0.01+0.01+0.01+0.01=0.10$

Also, $A\cap B=\{(5,5),(6,6)\}$

$\therefore\ P(A\cap B)=P(5,5)+P(6,6)=P(5).P(5)+P(6).P(6)$

$=0.1\times0.1+0.1\times0.1=0.01+0.01=0.02$

We know that, for two events A and B, if $P(A\cap B)=P(A).P(B)$, then both are independent events.

Here, $P(A\cap B)=0.02$ and $P(A).P(B)=0.20\times0.10=0.02$

Thus, $P(A\cap B)=P(A).P(B)=0.02$

Hence, A and B are independents events.

2. Refer to Exercise 1 above. If the die were fair, determine whether or not the events A and B are independent.

Sol. Referring to the above solution, we have

$A=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

$\Rightarrow n(A)=6$ and $n(S)=6^2=36$ [Where, S is sample space]

$\therefore\ P(A)=\frac{n(A)}{n(B)}=\frac{6}{36}=\frac{1}{6}$

And $B=\{(4,6),(6,4),(5,5),(6,5),(5,6),(6,6)\}$

$\Rightarrow n(B)=6$ and $n(S)=6^2=36$

$\therefore P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$

Also, $A\cap B=\{(5,5),(6,6)\}$

$\Rightarrow n(A\cap B)=2$ and $n(S)=36$

$\therefore\ (A\cap B)=\frac{2}{36}=\frac{1}{18}$

Also, $P(A).P(B)=\frac{1}{36}$

Thus, $P(A\cap B)\neq P(A).P(B)$ $\left[\because \frac{1}{18}\neq\frac{1}{36}\right]$

So, we can say that both A and B are not independent events.

3. The probability that at least one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, evaluate $P(\bar{A})+P(\bar{B})$ **.**

Sol. We know that, $A\cup B$ denotes the occurrence of at least one of A and B and $A\cap B$ denotes the occurrence of both A and B, simultaneously.

Thus, $P(A\cup B)=0.6$ and $P(A\cap B)=0.3$

Also, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow 0.6=P(A)+P(B)-0.3$

$\Rightarrow\ P(A)+P(B)=0.9$

$\Rightarrow \left[1-P(\bar{A})\right]+\left[1-P(\bar{B})\right]=0.9$ $[\because P(A)=1-P(\bar{A})$ and $P(B)=1-P(\bar{B})]$

$\Rightarrow\ P(\bar{A})+P(\bar{B})=2-0.9=1.1$

4. A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be black, if the first marble is red?

Sol. Let R = {5 Red marbles} and B = (3 black marbles}

For at least one of the three marbles drawn be black, if the first marble is red, then the following three conditions will be followed.

(i) Second ball is black and third is red (E_1).

(ii) Second ball is black and third is also black (E_2).

(iii) Second ball is red and third is black (E_3).

$\therefore P(E_1) = P(R_1).P(B_1 / R_1).P(R_2 / R_1B_1) = \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} = \frac{60}{336} = \frac{5}{28}$

$P(E_2) = P(R_1).P(B_1 / R_1).P(B_2 / R_1B_1) = \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} = \frac{30}{336} = \frac{5}{56}$

And $P(E_3) = P(R_1).P(R_2 / R_1).P(B_1 / R_1R_2) = \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{60}{336} = \frac{5}{28}$

$\therefore P(E) = P(E_1) + P(E_2) + P(E_3) = \frac{5}{28} + \frac{5}{56} + \frac{5}{28}$

$= \frac{10+5+10}{56} = \frac{25}{56}$

5. Two dice are thrown together and the total score is noted. The events E, F *and* G are 'a total of 4', 'a total of 9 or more', and 'a total divisible by 5', respectively. Calculate $P(E)$, $P(F)$ and $P(G)$ and decide which pairs of events, if any, are independent.

Sol. Two dice are thrown together i.e., sample space $(S) = 36 \Rightarrow n(S) = 36$

E = A total of $4 = \{(2,2),(3,1),(1,3)\}$

$\Rightarrow n(E) = 3$

F = A total of 9 or more

$= \{(3,6),(6,3),(4,5),(4,6),(5,4),(6,4),(5,5),(5,6),(6,5),(6,6)\}$

$\Rightarrow n(F) = 10$

G = a total divisible by $5 = \{(1,4),(4,1),(2,3),(3,2),(4,6),(6,4),(5,5)\}$

$\Rightarrow n(G) = 7$

Here, $(E \cap F) = \phi$ and $(E \cap G) = \phi$

Also, $(F \cap G) = \{(4,6),(6,4),(5,5)\}$

$\Rightarrow n(F \cap G) = 3$ and $(E \cap F \cap G) = \phi$

$\therefore P(E) = \frac{n(F)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

$P(F) = \frac{n(F)}{n(S)} = \frac{10}{36} = \frac{5}{18}$

$P(G) = \frac{n(G)}{n(S)} = \frac{7}{36}$

$P(F \cap G) = \frac{3}{36} = \frac{1}{12}$

And $P(F).P(G) = \frac{5}{18} \cdot \frac{7}{36} = \frac{35}{648}$

Here, we see that $P(F \cap G) \neq P(F).P(G)$

[since, only F and G have common events, so only F and G are used here]. Hence, there is no pair which is independent.

6. Explain why the experiment of tossing a coin three times is said to have binomial distribution.

Sol. We know that, a random variable X taking values $0,1,2,\ldots,n$ is said to have a binomial distribution with parameters n and P, if its probability distribution is given by

$P(X=r)={}^nC_r p^r q^{n-r}$

Where $q=1-p$

And $r=0,1,2,\ldots,n$

Similarly, in an experiment of tossing a coin three times, we have n = 3 and random variable X can take values r = 0, 1, 2 and 3 with $p=\frac{1}{2}$ and $q=\frac{1}{2}$.

X	0	1	2	3
P(X)	${}^3C_0 q^3$	${}^3C_1 Pq^2$	${}^3C_2 P^2 q$	${}^3C_3 P^3$

So, we see that in the experiment of tossing coin three times, we have random variable X which can take values 0, 1, 2 and 3 with parameters n = 3 and $p=\frac{1}{2}$.

Therefore, it is said to have a Binomial distribution.

7. If A and B are two events such that P(A) $=\frac{1}{2}$**, P(B)** $=\frac{1}{3}$ **and** $P(A\cap B)=\frac{1}{4}$**. Find:**

(i) P(A|B)
(ii) P(B|A)
(iii) P(A'|B)
(iv) P(A'|B')

Sol. Here, $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{4}$

(i) $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{1/4}{1/3}=\frac{3}{4}$

(ii) $P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{1/4}{1/2}=\frac{1}{2}$

(iii) $P(A'/B)=1-P(A/B)=1-\frac{3}{4}=\frac{1}{4}$

Or $P(A'/B)=\frac{P(A'\cap B)}{P(B)}=\frac{P(B)-P(A\cap B)}{P(B)}=\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{3}}=\frac{\frac{1}{12}}{\frac{1}{3}}=\frac{1}{4}$

(iv) $P(A'/B') = \frac{P(A'\cap B')}{P(B')} = \frac{1-P(A\cup B)}{1-P(B)} = \frac{1-[P(A)+P(B)-P(A\cap B)]}{1-P(B)}$

$= \frac{1-\left[\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right]}{1-\frac{1}{3}} = \frac{1-\left(\frac{5}{6}-\frac{1}{4}\right)}{\frac{2}{3}}$

$= \frac{1-14/24}{2/3} = \frac{10/24}{2/3} = \frac{30}{48} = \frac{5}{8}$

8. Three events A, B and C have probabilities $\frac{2}{5'}\frac{1}{3}$ and $\frac{1}{2'}$ respectively. Given that $P(A\cap B)=\frac{1}{5}$ and $P(B\cap C)=\frac{1}{4'}$ find the value of $P(C\,|\,B)$ and $P(A'\cap C')$.

Sol. Here, $P(A) = \frac{2}{5}, P(B) = \frac{1}{3}, P(C) = \frac{1}{2}.P(A\cap C) = \frac{1}{5}$ and $P(B\cap C) = \frac{1}{4}$

$\therefore P(C/B) = \frac{P(B\cap C)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$

And $P(A'\cap C') = 1-P(A\cup C) = 1-[P(A)+P(C)-P(A\cap C)]$

$= 1-\left[\frac{2}{5}+\frac{1}{2}-\frac{1}{5}\right] = 1-\left[\frac{4+5-2}{10}\right] = 1-\frac{7}{10} = \frac{3}{10}$

9. Let E_1 *and* E_2 be two independent events such that $P(E_1) = p_1$ and $P(E_2) = P_2$. Describe in words of the events whose probabilities are:

(i) P_1P_2

(ii) $(1-p_1)\,p_2$

(iii) $1-(1-p_1)(1-p_2)$

(iv) $p_1+p_2-2p_1p_2$

Sol. $P(E_1) = p_1$ and $P(E_2) = P_2$

(i) $P_1P_2 \Rightarrow P(E_1)\cdot P(E_2) = P(E_1'\cap E_2)$

So, E_1 *and* E_2 occurs.

(ii) $(1-P_1)P_2 = P(E_1)'.P(E_2) = P(E_1'\cap E_2)$

So, E_1 does not occur but E_2 occurs.

(iii) $1-(1-P_1)(1-P_2) = 1-P(E_1)'P(E_2)' = 1-P(E_1'\cap E_2')$

$= 1-[1-P(E_1\cup E_2)] = P(E_1\cup E_2)$

So, either E_1 *or* E_2 or both E_1 *and* E_2 occurs.

(iv) $P_1+P_2-2P_1P_2 = P(E_1)+P(E_2)-2P(E_1)\cdot P(E_2)$

$= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$

$= P(E_1 \cup E_2) - P(E_1 \cap E_2)$

So, either E_1 *or* E_2 occurs but not both.

10. A discrete random variable X has the probability distribution given as below:

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

(i) Find the value of k

(ii) Determine the mean of the distribution.

Sol. We have,

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

(i) We know that, $\sum_{i=1}^{n} P_i = 1$, where $P_i \geq 0$

$\Rightarrow P_1 + P_2 + P_3 + P_4 = 1$

$\Rightarrow k + k^2 + 2k^2 + k = 1$

$\Rightarrow 3k^2 + 2k - 1 = 0$

$\Rightarrow 3k^2 + 3k - k - 1 = 0$

$\Rightarrow 3k(k+1) - 1(k+1) = 0$

$\Rightarrow (3k-1)(k+1) = 0$

$\Rightarrow k = 1/3 \Rightarrow k = -1$

Since, k is $\geq 0 \Rightarrow$ k=1/3

(ii) Mean of the distribution $(\mu) = E(X) = \sum_{i=1_i}^{n} x_i p_i$

$= 0.5(k) + 1(k^2) + 1.5(2k^2) + 2k = 4k^2 + 2.5k$

$= 4 \cdot \frac{1}{9} + 2.5 \cdot \frac{1}{3} \left[\because k = \frac{1}{3} \right]$

$= \frac{4 + 7.5}{9} = \frac{23}{18}$

11. Prove that

(i) $P(A) = P(A \cap B) + \mathrm{P}(\mathrm{A} \cap \overline{\mathrm{B}})$

(ii) $P(A \cup B) = P(A \cap B) + P(A \cap \overline{B}) + P(\overline{A} \cap B)$

Sol. (i) $\because$ $P(A) = P(A \cap B) + P(A \cap \overline{B})$

$\therefore \quad RHS = P(A \cap B) + P(A \cap \overline{B})$

$= P(A) \cdot P(B) + P(A) \cdot P(\overline{B})$

$= P(A)[P(B) + P(\overline{B})]$

$= P(A)[P(B) + 1 - P(B)] \qquad [\because P(\overline{B}) = 1 - P(B)]$

$= P(A) = LHS \quad Hence\ proved.$

(ii) $\because P(A \cup B) = P(A \cap B) + P(A \cap \overline{B}) + P(\overline{A} \cap B)$

$\therefore RHS = P(A) \cdot P(B) + P(A) \cdot P(\overline{B}) + P(\overline{A}) \cdot P(B)$

$= P(A) \cdot P(B) + P(A) \cdot [1 - P(B)] + [1 - P(A)]P(B)$

$= P(A) \cdot P(B) + P(A) - P(A) \cdot P(B) + P(B) - P(A) \cdot P(B)$

$= P(A) + P(B) - P(A) \cdot P(B)$

$= P(A) + P(B) - P(A \cap B)$

$= P(A \cup B) = LHS \quad Hence\ proved.$

12. If X is the number of tails in three tosses of a coin, determine the standard deviation of X.

Sol. Given that, random variable X is the number of tails in three tosses of a coin.

So, $X = 0, 1, 2, 3$

$\Rightarrow P(X - x) = {}^nC_x (p)^x q^{n-x},$

Where $n = 3,\ p = ½,\ q = ½\ and\ x = 0, 1, 2, 3$

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
XP(X)	0	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{8}$
$X^2P(X)$	0	$\frac{3}{8}$	$\frac{3}{2}$	$\frac{9}{8}$

We know that, $Var(X) = E(X^2) - [E(X)]^2$

Where, $E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$ and $E(X) = \sum_{i=1}^{n} x_i P(x_i)$

$\therefore E(X^2) = \sum_{i=1}^{n} x_i^2 P(X_i) = 0 + \frac{3}{8} + \frac{3}{2} + \frac{9}{8} = \frac{24}{8} = 3$

And $[E(X)]^2 = \left[\sum_{i=1}^{n} x_i^2 P(x_i)\right]^2 = \left[0 + \frac{3}{8} + \frac{3}{4} + \frac{3}{8}\right]^2 = \left[\frac{12}{8}\right]^2 = \frac{9}{4}$

$\therefore Var(X) = 3 - \frac{9}{4} = \frac{3}{4}$ [using Eq. (i)]

And standard deviation of $X = \sqrt{Var(X)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

13. In a dice game, a player pays a stake of Re1 for each throw of a die. She receives Rs 5 if the die shows a 3, Rs 2 if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?

Sol. Let X is the random variable of profit or throw.

X	-1	1	4
P(X)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Since, she loss Rs 1 on getting any of 2, 4 or 5.

So, at $X = -1,\ P(X) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

Similarly, at $X = 1,\ P(X) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ [if Die shows of either 1 or 6]

And at $X = 4,\ P(X) = \frac{1}{6}$ [if die shows a 3]

$\therefore$ Player's expected profit $= E(X) = \sum XP(X)$

$= -1 \times \frac{1}{2} + 1 \times \frac{1}{3} + 4 \times \frac{1}{6}$

$= \frac{-3+2+4}{6} = \frac{3}{6} = \frac{1}{2} = Rs\ 0.50$

14. Three dice are thrown at the same time. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

Sol. On a throw of three dice, we have sample space $\left[n(S)\right] = 6^3 = 216$

Let E_1 is the event when the sum of numbers on the dice was six and E_2 is the event when three two's occurs.

$\Rightarrow E_1 = \{(1,1,4),(1,2,3),(1,3,2),(1,4,1),(2,1,3),(2,2,2),(2,3,1),(3,1,2),(3,2,1),(4,1,1)\}$

$\Rightarrow n(E_1) = 100\ and\ E_2 = \{2,2,2\}$

$\Rightarrow n(E_2) = 1$

Also, $(E_1 \cap E_2) = 1$

$\therefore P(E_2 / E_1) = \frac{P \cdot (E_1 \cap E_2)}{P(E_1)} = \frac{1/216}{10/216} = \frac{1}{10}$

15. Suppose 10,000 tickets are sold in a lottery each for Re 1. First prize is of Rs 3000 and the second prize is of Rs. 2000. There are three third prizes of Rs. 500 each. If you buy one ticket, what is your expectation.

Sol. Let x is the random variable for the prize.

X	0	500	2000	3000
P(X)	$\frac{9995}{10000}$	$\frac{3}{10000}$	$\frac{1}{10000}$	$\frac{1}{10000}$

Since, $E(X)=\sum X\ P(X)$

$$\therefore E(X)=0\times\frac{9995}{10000}+\frac{1500}{10000}+\frac{2000}{10000}+\frac{3000}{10000}$$

$$=\frac{1500+2000+3000}{10000}$$

$$=\frac{6500}{10000}=\frac{13}{20}=Rs\,0.65$$

16. A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

Sol. Here, W_1 = {4 white balls} and B_1 = {5 black balls}

And W_2 = {9 white balls} and B_2 = {7 black balls}

Let E_1 is the event that ball transferred from the first bag is white and E_2 is the event that the ball transferred from the first bag is black.

Also, E is the event that the ball drawn from the second bag is white.

$$\therefore P(E/E_1)=\frac{10}{17}, P(E/E_2)=\frac{9}{17}$$

And $P(E_1)=\frac{4}{9}$ and $P(E_2)=\frac{5}{9}$

$$\therefore P(E)=P(E_1)\cdot P(E/E_1)+P(E_2)\cdot P(E/E_2)$$

$$=\frac{4}{9}\cdot\frac{10}{17}+\frac{5}{9}\cdot\frac{9}{17}$$

$$=\frac{40+45}{153}=\frac{85}{153}=\frac{5}{9}$$

17. Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

Sol. Bag I = {3B, 2W}, Bag II = {2B, $W}

Let E_1 = Event that bag I is selected

E_2 = Event that bag II is selected

And E = Event that a black ball is selected

$$\Rightarrow P(E_1) = 1/2, P(E_2) = \frac{1}{2}, P(E/E_1) = \frac{3}{5}, P(E/E_2) = \frac{2}{6} = \frac{1}{3}$$

$$\therefore P(E) = P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$$

$$= \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{6} = \frac{3}{10} + \frac{2}{12}$$

$$= \frac{18+10}{60} = \frac{28}{60} = \frac{7}{15}$$

18. A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

Sol. A box = {5 blue, 4 red}

Let E_1 is the event that first ball drawn is blue, E_2 is the event that first ball drawn is red and E is the event that second ball drawn is blue.

$$\therefore P(E) = P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2)$$

$$= \frac{5}{9} \cdot \frac{4}{8} + \frac{4}{9} \cdot \frac{5}{8} = \frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

19. Four cards are successively drawn without replacement from a deck of 52 playing cards. What is the probability that all the four cards are kings?

Sol. Let E_1, E_2, E_3 and E_4 are the events that the first, second, third and fourth card is king, respectively.

$$\therefore P(E_1 \cap E_2 \cap E_3 \cap E_4) = P(E_1) \cdot P(E_2/E_1) \cdot P(E_3/E_1 \cap E_2) \cdot P[E_4/(E_1 \cap E_2 \cap E_3 \cap E_4]$$

$$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} = \frac{24}{52 \cdot 51 \cdot 50 \cdot 49}$$

$$= \frac{1}{13 \cdot 17 \cdot 25 \cdot 49} = \frac{1}{270725}$$

20. A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.

Sol. Here, $n = 5, p = \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right) = \frac{1}{2}$ and $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Also, r = 3

$$\therefore P(X = r) = {}^nC_r (p)^r (q)^{n-r} = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3}$$

$$= \frac{5!}{3!2!} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{10}{32} = \frac{5}{16}$$

21. Ten coins are tossed. What is the probability of getting at least 8 heads?

Sol. In this case, we have to find out the probability of getting at least 8 heads. Let X is the random variable for getting a head.

Here, $n=10, r \geq 8,$

i.e., $r=8,9,10,\ p=\frac{1}{2}, q=\frac{1}{2}$

we know that, $P(X=r)={}^{n}C_{r}p^{r}q^{n-r}$

$\therefore\ P(X=r)=P(r=8)+P(r=9)+P(r=10)$

$$={}^{10}C_{8}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{10-8}+{}^{10}C_{9}\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)^{10-9}+{}^{10}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{10-10}$$

$$=\frac{10!}{8!2!}\left(\frac{1}{2}\right)^{10}+\frac{10!}{9!1!}\left(\frac{1}{2}\right)^{10}+\frac{10!}{0!10!}\left(\frac{1}{2}\right)^{10}$$

$$=\left(\frac{1}{2}\right)^{10}\left[\frac{10\times 9}{2}+10+1\right]$$

$$=\left(\frac{1}{2}\right)^{10}\cdot 56=\frac{1}{2^{7}\cdot 2^{3}}\cdot 56=\frac{7}{128}$$

22. The probability of a man hitting a target is 0.25. He shoots 7 times. What is the probability of his hitting at least twice?

Sol. Here, $n=7\ \ p=0.25=\frac{1}{4}, q=1-\frac{1}{4}=\frac{3}{4}\ r\geq 2,$

Where, $P(X)={}^{n}C_{r}(p)^{r}(q)^{n-r}$

In this case for easy approach we shall first find out the probability of his hitting at most once (i.e., r = 0, 1) and then subtract this probability from 1 to get the desired probability.

$\therefore\ P(X=r)=1-[P(r=0)+P(r=1)]$

$$=1-\left[{}^{7}C_{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{7-0}+{}^{7}C_{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{7-1}\right]$$

$$=1-\left[\frac{7!}{0!7!}\left(\frac{3}{4}\right)^{7}+\frac{7!}{1!6!}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{6}\right]$$

$$=1-\left[\left(\frac{3}{4}\right)^{6}\left(\frac{3}{4}\cdot 1+\frac{1}{4}\cdot 7\right)\right]$$

$$=1-\left[\frac{3^{6}}{4^{6}}\left(\frac{10}{4}\right)\right]=1-\left[\frac{3^{6}\times 10}{4^{7}}\right]=1-\left[\frac{27\cdot 27\cdot 10}{64\cdot 256}\right]$$

$$=1-\left[\frac{7290}{16384}\right]=1-\frac{3645}{8192}=\frac{4547}{8192}$$

23. A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch?

Sol. Probability of defective watch from a lot of 100 watches $=\frac{10}{100}=\frac{1}{10}$

$\therefore\ p=1/10, q=\frac{9}{10}, n=8$ and $r\geq 1$

$\therefore\ P(r\geq 1)=1-P(r=0)=1-{}^{8}C_{0}\left(\frac{1}{10}\right)^{0}\left(\frac{9}{10}\right)^{8-0}$

$=1-\frac{8!}{0!8!}\cdot\left(\frac{9}{10}\right)^{8}=1-\left(\frac{9}{10}\right)^{8}$

24. Consider the probability distribution of a random variable X:

X	**0**	**1**	**2**	**3**	**4**
P(X)	**0.1**	**0.25**	**0.3**	**0.2**	**0.15**

Calculate (i) $V\left(\frac{X}{2}\right)$

(ii) Variance of X.

Sol. We have,

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15
XP(X)	0	0.25	0.6	0.6	0.60
$X^2P(X)$	0	0.25	1.2	1.8	2.40

$Var(X)=E(X^2)-[E(X)]^2$

Where, $E(X)=\mu=\sum_{i=1}^{n}x_iP_i(xi)$

And $E(X^2)=\sum_{i=1}^{n}x_i^2P(x_i)$

$\therefore\ E(X)=0+0.25+0.6+0.6+0.60=2.05$

$E(X^2)=0+0.25+1.2+1.8+2.40=5.65$

(i) $V\left(\frac{X}{2}\right)=\frac{1}{4}V(X)=\frac{1}{4}[5.65-(2.05)^2]$

$\frac{1}{4}[5.65-4.2025]=\frac{1}{4}\times 1.4475=0.361875$

(ii) V(X) = 1.44475

25. The probability distribution of a random variable X is given below:

X	0	1	2	3
P(X)	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

(i) Determine the value of k.

(ii) Determine $P(X \le 2)\ and\ P(X > 2)$

(iii) Find $P(X \le 2) + P(X > 2)$**.**

Sol. We have,

X	0	1	2	3
P(X)	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

(i) since, $\sum_{i=1}^{n} P_i = 1, i = 1, 2, ..., n$ and $p_i \ge 0$

$\therefore k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} = 1$

$\Rightarrow 8k + 4k + 2k + k = 8$

$\therefore k = \frac{8}{15}$

(ii) $P(X \le 2) = P(0) + P(1) + P(2) = k + \frac{k}{2} + \frac{k}{4}$

$= \frac{(4k + 2k + k)}{4} = \frac{7k}{4} = \frac{7}{4} \cdot \frac{8}{15} = \frac{14}{15}$

And $P(X > 2) = P(3) = \frac{k}{8} = \frac{1}{8} \cdot \frac{8}{15} = \frac{1}{15}$

(iii) $P(X \le 2) + P(X > 2) = \frac{14}{15} + \frac{1}{15} = 1$

26. For the following probability distribution determine standard deviation of the random variable X.

X	2	3	4
P(X)	0.2	0.5	0.3

Sol. We have,

X	2	3	4
P(X)	0.2	0.5	0.3
XP(X)	0.4	1.5	1.2
$X^2P(X)$	0.8	4.5	4.8

We know that, standard deviation of $X = \sqrt{Var\ X}$

Where, $Var\ X = E\left(X^2\right) - \left[E\left(X\right)\right]^2$

$$= \sum_{i=1}^{n} x_i^2 P(x_1) - \left[\sum_{i=1}^{n} x_i P_i\right]^2$$

$$\therefore Var\ X = [0.8 + 4.5 + 4.8] - [0.4 + 1.5 + 1.2]^2$$

$= 10.1 - (3.1)^2 = 10.1 - 9.61 = 0.49$

$\therefore$ Standard deviation of X $= \sqrt{Var\ X} = \sqrt{0.49} = 0.7$

27. A biased die is such that $P(4)=\frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of four seen', find the variance of the random variable X.

Sol. Since, X = number of four seen

On tossing two die, X = 0, 1, 2.

Also, $P_{(4)} = \frac{1}{10}$ and $P_{(not\ 4)} = \frac{9}{10}$

So, $P(X=0) = P_{(not\ 4)} \cdot P_{(not\ 4)} = \frac{9}{10} \cdot \frac{9}{10} = \frac{81}{100}$

$$P(X=1) = P_{(not\ 4)} \cdot P_{(4)} + P_{(4)} \cdot P_{(not\ 4)} = \frac{9}{10} \cdot \frac{1}{10} + \frac{1}{10} \cdot \frac{9}{10} = \frac{18}{100}$$

$$P(X=2) = P_{(4)} \cdot P_{(4)} = \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100}$$

Thus, we get following table

X	0	1	2
P(X)	$\frac{81}{100}$	$\frac{18}{100}$	$\frac{1}{100}$
XP(X)	0	18/100	2/100
$X^2P(X)$	0	18/100	4/100

$\therefore Var(X) = E(X^2) - [E(X)]^2 = \sum X^2 P(X) - [\sum XP(X)]^2$

$= \left[0 + \frac{18}{100} + \frac{4}{100}\right] - \left[0 + \frac{18}{100} + \frac{2}{100}\right]^2$

$= \frac{22}{100} - \left(\frac{20}{100}\right)^2 = \frac{11}{50} - \frac{1}{25}$

$= \frac{11-2}{50} = \frac{9}{50} = \frac{18}{100} = 0.18$

28. A die is thrown three times. Let X be 'the number of twos seen'. Find the expectation of X.

Sol. We have, X = number of twos seen

So, on throwing a die three times, we will have X = 0, 1, 2, 3.

$\therefore P(X=0) = P_{(not\ 2)} \cdot P_{(not\ 2)} \cdot P_{(not\ 2)} = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{125}{216}$

$P(X=1) = P_{(not\ 2)} \cdot P_{(not\ 2)} \cdot P_{(2)} + P_{(not\ 2)} \cdot P_{(2)} \cdot P_{(not\ 2)} + P_{(2)} \cdot P_{(not\ 2)} \cdot P_{(not\ 2)}$

$= \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36} \cdot \frac{3}{6} = \frac{25}{72}$

$P(X=2) = P_{(not\ 2)} \cdot P_{(2)} \cdot P_{(2)} + P_{(2)} \cdot P_{(2)} . P_{(not\ 2)} + P_{(2)} \cdot P_{(not\ 2)} \cdot P_{(2)}$

$= \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$

$= \frac{1}{36} \cdot \left[\frac{15}{6}\right] = \frac{15}{216}$

$P(X=3) = P_{(2)} \cdot P_{(2)} \cdot P_{(2)} = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$

We know that, $E(X) = \sum X \quad P(X) = 0 \cdot \frac{125}{216} + 1 \cdot \frac{25}{216} + 3 \cdot \frac{1}{216}$

$= \frac{75+30+3}{216} = \frac{108}{216} = \frac{1}{2}$

29. Two biased dice are thrown together. For the first die $P(6)=\frac{1}{2}$, the other scores being equally likely while for the second die, $P(1)=\frac{2}{5}$ and the other scores are equally likely. Find the probability distribution of 'the number of ones seen'.

Sol. For first die, $P(6) = \frac{1}{2}$ and $P(6') = \frac{1}{2}$

$\Rightarrow P(1)+P(2)+P(3)+P(4)+P(5)=\frac{1}{2}$

$\Rightarrow P(1)=\frac{1}{10}$ *and* $P(1')=\frac{9}{10}$ $[\because P(1)=P(2)=P(3)=P(4)=P(5)]$

For second die, $P(1)=\frac{2}{5}$ and $P(1')=1-\frac{2}{5}=\frac{3}{5}$

Let X = Number of one's seen

For $X=0$, $P(X=0)=P(1')\cdot P(1')=\frac{9}{10}\cdot\frac{3}{5}=\frac{27}{50}=0.54$

$P(X=1)=P(1')\cdot P(1')+P(1')\cdot P(1')=\frac{9}{10}\cdot\frac{2}{5}+\frac{1}{10}\cdot\frac{3}{5}$

$=\frac{18}{50}+\frac{3}{50}=\frac{21}{50}=0.42$

$P(X=2)=P(1)\cdot P(1)=\frac{1}{10}\cdot\frac{2}{5}=\frac{2}{50}=0.04$

Hence, the required probability distribution is as below.

X	0	1	2
P(X)	0.54	0.42	0.04

30. Two probability distributions of the discrete random variable X and Y are given below.

X	**0**	**1**	**2**	**3**
P(X)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Y	**0**	**1**	**2**	**3**
P(Y)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Prove that $E(Y^2)=2E(X)$**.**

Sol.

X	**0**	**1**	**2**	**3**
P(X)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Y	0	1	2	3
P(Y)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Since, we have to prove that, $E\left(Y^2\right)=2E\left(X\right)$

$$\therefore\ E(X)=\sum X\ \ P(X)$$

$$=0\cdot\frac{1}{5}+1\cdot\frac{2}{5}+2\cdot\frac{1}{5}+3\cdot\frac{1}{5}=\frac{7}{5}$$

$$\Rightarrow\ 2E(X)=\frac{14}{5}\ \ ...(i)$$

$$E(Y)^2=\sum Y^2P(Y)$$

$$=0\cdot\frac{1}{5}+1\cdot\frac{3}{10}+4\cdot\frac{2}{5}+9\cdot\frac{1}{10}$$

$$=\frac{3}{10}+\frac{8}{5}+\frac{9}{10}=\frac{28}{10}=\frac{14}{5}$$

$$\Rightarrow\ E(Y)^2=\frac{14}{5}\ \ ...(ii)$$

From Eqs. (i) and (ii),

$E\left(Y\right)^2=2E\left(X\right)$ *Hence proved.*

31. A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that
(i) none of the bulbs is defective
(ii) exactly two bulbs are defective
(iii) more than 8 bulbs work properly

Sol. Let X is the random variable which denotes that a bulb is defective.

Also, $n=10, p=\frac{1}{50}$ and $q=\frac{49}{50}$ and $P(X=r)={}^nC_r\ \ p^r\ q^{n-r}$

(i) None of the bulbs are defective i.e., r = 0

$$\therefore\ P(X=r)=P_{(0)}={}^{10}C_0\left(\frac{1}{50}\right)^0\left(\frac{49}{50}\right)^{10-0}=\left(\frac{49}{50}\right)^{10}$$

(ii) Exactly two bulbs are defective i.e., r = 2

$$\therefore\ P(X=r)=P_{(2)}={}^{10}C_2\left(\frac{1}{50}\right)^2\left(\frac{49}{50}\right)^8$$

$$=\frac{10!}{8!2!}\left(\frac{1}{50}\right)^2\cdot\left(\frac{49}{50}\right)^8=45\times\left(\frac{1}{50}\right)^{10}\times(49)^8$$

(iii) More than 8 bulbs work properly i.e., there is less than 2 bulbs which are defective.

So, $r < 2 \Rightarrow r = 0,1$

$\therefore P(X = r) = P(r < 2) = P(0) + P(1)$

$= {}^{10}C_0\left(\frac{1}{50}\right)^0\left(\frac{49}{50}\right)^{10-0} + {}^{10}C_1\left(\frac{1}{50}\right)^1\left(\frac{49}{50}\right)^{10-1}$

$= \left(\frac{49}{50}\right)^{10} + \frac{10!}{1!9!}\cdot\frac{1}{50}\cdot\left(\frac{49}{50}\right)^9$

$= \left(\frac{49}{50}\right)^{10} + \frac{1}{5}\cdot\left(\frac{49}{50}\right)^9 = \left(\frac{49}{50}\right)^9\left(\frac{49}{50} + \frac{1}{5}\right)$

$= \left(\frac{49}{50}\right)^9\left(\frac{59}{50}\right) = \frac{59(49)^9}{(50)^{10}}$

32. Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

Sol. Let E_1 = Event that 5 fair coin is drawn

E_2 = Event that 2 headed coin in drawn

E = Event that tossed coin get a head

$\therefore P(E_1) = 1/2, P(E_2) = 1/2, P(E/E_1) = 1/2$ and $P(E/E_2) = 1$

Now using Baye's theorem $P(E_1/E) = \frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2)}$

$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

33. Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?

Sol.

	Blood group 'O'	Other than blood group 'O'
I. Number of people II. Percentage of left handed people	30 % 6 %	70 % 10 %

E_1 = Event that the person selected is of blood group O

E_2 = Event that the person selected is of other than blood group O

(E_3) = Event that selected person is left handed

$\therefore\ P(E_1) = 0.30,\ P(E_2) = 0.70$

$P(E_3/E_1) = 0.06\ and\ P(E_3/E_2) = 0.10$

By using Baye's theorem,

$$P(E_1/E_3) = \frac{P(E_1)\cdot P(E_3/E_1)}{P(E_1)\cdot P(E_3/E_1) + P(E_2)\cdot P(E_3/E_2)}$$

$$= \frac{0.30\times 0.06}{0.30\cdot 0.06 + 0.70\cdot 0.10}$$

$$= \frac{0.0180}{0.0180 + 0.0700}$$

$$= \frac{0.0180}{0.0880} = \frac{180}{880} = \frac{9}{44}$$

34. Two natural numbers r, s are drawn one at a time, without replacement from the set $S = \{1, 2, 3, ..., n\}$. Find P[r ≤ p|s ≤ p], where $p \in S$.

Sol. $\because$ Set $S = \{1, 2, 3, ..., n\}$

$$\therefore P(r \le p/s \le p) = \frac{P(p \cap S)}{P(S)}$$

$$= \frac{p-1}{n} \times \frac{n}{n-1} = \frac{p-1}{n-1}$$

35. Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

Sol. Let X is the random variable score obtained when a die is thrown twice.

$\therefore$ X = 1, 2, 3, 4, 5, 6

Here, $S = \{(1,1),(1,2),(2,1),(2,2),(1,3),(2,3),(3,1),(3,2),(3,3),....,(6,6)\}$

$$\therefore P(X = 1) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$$

$$P(X = 2) = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{3}{36}$$

$$P(X = 3) = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{5}{36}$$

Similarly,

$$P(X = 4) = \frac{7}{36}$$

$$P(X = 5) = \frac{9}{36}$$

$$P(X = 6) = \frac{11}{36}$$

So, the required distribution is,

X	1	2	3	4	5	6
P(X)	1/36	3/36	5/36	7/36	9/36	11/36

Also, we know that, Mean $\{E(X)\} = \sum X \ P(X)$

$$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} = \frac{161}{36}$$

36. The random variable X can take only the values of 0, 1, 2. Give that $P(X = 0) = P(X = 1) = p$ and that $E(X^2) = E[X]$, find the value of p.

Sol. Since, X = 0, 1, 2 and P(X) at X = 0 and 1 is p, let at X = 2, P(X) is x.

$\Rightarrow p + p + x = 1$

$\Rightarrow x = 1 - 2p$

We get, the following distribution.

X	0	1	2
P(X)	p	p	1 – 2p

$\therefore E(X) = \sum X \ P(X)$

$= 0 \cdot p + 1 \cdot p + 2(1 - 2p)$

$= p + 2 - 4p = 2 - 3p$

And $E(X^2) = \sum X^2 \ P(X)$

$= 0 \cdot p + 1 \cdot p + 4 \cdot (1 - 2p)$

$= p + 4 - 8p = 4 - 7p$

Also, given that $E(X^2) = E[X]$

$\Rightarrow 4 - 7p = 2 - 3p$

$\Rightarrow 4p = 2 \ \Rightarrow p = \frac{1}{2}$

37. Find the variance of the distribution:

x	**0**	**1**	**2**	**3**	**4**	**5**
P(x)	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

Sol. We have,

X	0	1	2	3	4	5
P(X)	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$
XP(X)	0	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{5}{18}$
$X^2P(X)$	0	$\frac{5}{18}$	$\frac{8}{9}$	$\frac{3}{2}$	$\frac{16}{9}$	$\frac{25}{18}$

$\therefore$ Variance $= E(X^2) - [E(X)]^2 = \sum X^2 P(X) - [\sum X\, P(X)]^2$

$$= \left[0 + \frac{5}{18} + \frac{8}{9} + \frac{3}{2} + \frac{16}{9} + \frac{25}{18}\right] - \left[0 + \frac{5}{18} + \frac{4}{9} + \frac{1}{2} + \frac{4}{9} + \frac{5}{18}\right]^2$$

$$= \left[\frac{5+16+27+32+25}{18}\right] - \left[\frac{5+8+9+8+5}{18}\right]^2$$

$$= \frac{105}{18} - \frac{35 \cdot 35}{18 \cdot 18} = \frac{18 \cdot 105 - 35 \cdot 35}{18 \cdot 18}$$

$$= \frac{35}{18 \cdot 18}[54 - 35] = \frac{19 \cdot 35}{324} = \frac{665}{324}$$

38. A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. It A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

Sol. Let A_1 = A total of 6 = {(2, 4), (1, 5), (5, 1), (4, 2), (3, 3)}

And B_1 = A total of 7 = {(2, 5), (1, 6), (6, 1), (5, 2), (3, 4), (4, 3)}

Let P(A) is the probability, if A wins in a throw $\Rightarrow P(A) = \frac{5}{36}$

And P(B) is the probability, if B wins in a throw $\Rightarrow P(B) = \frac{1}{36}$

$\therefore$ Required probability $= P(\overline{A}) \cdot P(\overline{B}) \cdot P(A) = \frac{31}{36} \cdot \frac{5}{6} \cdot \frac{5}{36} = \frac{775}{216 \cdot 36} = \frac{775}{7776}$

39. Two dice are tossed. Find whether the following two events A and B are independent:

$A = \{(x, y) : x + y = 11\}$ *and* $B = \{(x, y) : x \neq 5\}$, Where (x, y) denotes a typical sample point.

Sol. We have, $A = \{(x, y) : x + y = 11\}$ *and* $B = \{(x, y) : x \neq 5\}$,

$\therefore$ A = {(5, 6), (6, 5)}, B = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$\Rightarrow n(a) = 2, n(B) = 30$ and $n(A \cap B) = 1$

$\therefore P(A)=\frac{2}{36}=\frac{1}{18}$ and $P(B)=\frac{30}{36}=\frac{5}{6}$

$\Rightarrow P(A)\cdot P(B)=\frac{5}{108}$ and $P(A\cap B)=\frac{1}{36}\neq P(A)\cdot P(B)$

So, A and B are not independent.

40. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.

Sol. Let U = {m white, n black balls}

E_1 = {First ball drawn of white colour}

E_2 = {First ball drawn of black colour}

And E_3 = {Second ball drawn of white colour}

$\therefore P(E_1)=\frac{m}{m+n}$ and $P(E_2)=\frac{n}{m+n}$

Also, $P(E_3/E_1)=\frac{m+k}{m+n+k}$ and $P(E_3/E_2)=\frac{m+k}{m+n+k}$

$\therefore P(E_3)=P(E_1)\cdot P(E_3/E_1)+P(E_2)\cdot P(E_3/E_2)$

$=\frac{m}{m+n}\cdot\frac{m+k}{m+n+k}+\frac{n}{m+n}\cdot\frac{m}{m+n+k}$

$=\frac{m(m+k)+nm}{(m+n+k)(m+n)}=\frac{m^2+mk+nm}{(m+n+k)(m+n)}$

$=\frac{m(m+k+n)}{(m+n+k)(m+n)}=\frac{m}{m+n}$

Hence, the probability of drawing a white ball does not depend on k.

Long Answer Type Questions

41. Three bags contain a number of red and white balls as follows:
Bag 1 : 3 red balls, Bag 2 : 2 red balls and 1 white ball and Bag 3 : 3 white balls.
The probability that bag i will be chosen and a ball is selected from it is $\frac{i}{6},\ i=1,2,3$. What is the probability that
(i) a red ball will be selected?
(ii) a white ball is selected?

Sol. Bag I: 3 red balls and 0 white ball.
Bag II: 2 red balls and 1 white ball.
Bag III: 0 red ball and 3 white balls.
Let E_1, E_2 and E_3 be the events that bag I, bag II and bag III is selected and a ball is chosen from it.

$P(E_1)=\frac{1}{6}, P(E_2)=\frac{2}{6}$ and $P(E_3)=\frac{3}{6}$

(i) Let E be the event that a red ball is selected. Then, Probability that red ball will be selected.

$$P(E)=P(E_1)\cdot P(E/E_1)+P(E_2)\cdot P(E/E_2)+P(E_3)\cdot P(E/E_3)$$
$$=\frac{1}{6}\cdot\frac{3}{3}+\frac{2}{6}\cdot\frac{2}{3}+\frac{3}{6}\cdot 0$$
$$=\frac{1}{6}+\frac{2}{9}+0$$
$$=\frac{3+4}{18}=\frac{7}{18}$$

(ii) Let F be the event that a white ball is selected.

$$\therefore P(F)=P(E_1)\cdot P(F/E_1)+P(E_2)\cdot P(F/E_2)+P(E_3)\cdot P(F/E_3)$$
$$=\frac{1}{6}\cdot 0+\frac{2}{6}\cdot\frac{1}{3}+\frac{3}{6}\cdot 1=\frac{1}{9}+\frac{3}{6}=\frac{11}{18}$$

Note: $P(F)=1-P(E)=1-\frac{7}{18}=\frac{11}{18}$ [since, we know that $P(E)+P(F)=1$]

42. Refer to Question 41 above. If a white ball is selected, what is the probability that it came from
(i) Bag 2?
(ii) Bag 3?

Sol. Referring to the previous solution, using Bay's theorem, we have

(i) $$P(E_2/F)=\frac{P(E_2)\cdot P(F/E_2)}{P(E_1)\cdot P(F/E_1)+P(E_2)\cdot P(F/E_2)+P(E_3)\cdot P(F/E_3)}$$

$$= \frac{\frac{2}{6} \cdot \frac{1}{3}}{\frac{1}{6} \cdot 0 + \frac{2}{6} \cdot \frac{1}{3} + \frac{3}{6} \cdot 1} = \frac{\frac{2}{18}}{\frac{2}{18} + \frac{3}{6}}$$

$$= \frac{2/18}{\frac{2+9}{18}} = \frac{2}{11}$$

(ii) $P(E_3/F) = \dfrac{P(E_3) \cdot P(F/E_3)}{P(E_1) \cdot P(F/E_1) + P(E_2) \cdot P(F/E_2) + P(E_3) \cdot P(F/E_3)}$

$$= \frac{\frac{3}{6} \cdot 1}{\frac{1}{6} \cdot 0 + \frac{2}{6} \cdot \frac{1}{3} + \frac{3}{6} \cdot 1}$$

$$= \frac{\frac{3}{6}}{\frac{2}{18} + \frac{3}{6}} = \frac{3/6}{\frac{2}{18} + \frac{9}{18}} = \frac{9}{11}$$

43. A shopkeeper sells three types of flower seeds A_1, A_2, and A_3. They are sold as a mixture where the proportions are 4:4:2 respectively. The germination rates of the three types of seeds are 45%, 60% and 35%. Calculate the probability

(i) of a randomly chosen seed to germinate

(ii) that it will not germinate given that the seed is of type A_3,

(iii) that it is of the type A_2 given that a randomly chosen seed does not germinate.

Sol. We have, $A_1 : A_2 : A_3 = 4 : 4 : 2$

$P(A_1) = \dfrac{4}{10}, P(A_2) = \dfrac{4}{10}$ and $P(A_3) = \dfrac{2}{10}$

Where A_1, A_2 and A_3 denote the three types of flower seeds.

Let E be the event that a seed germinates and $\overline{E}$ be the event that a seed does not germinate.

$\therefore P(E/A_1) = \dfrac{45}{100}$, $P(E/A_2) = \dfrac{60}{100}$ and $P(E/A_3) = \dfrac{35}{100}$

And $P(\overline{E}/A_1) = \dfrac{55}{100}, P(\overline{E}/A_2) = \dfrac{40}{100}$ and $P(\overline{E}/A_3) = \dfrac{65}{100}$

(i) $\therefore P(E) = P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2) + P(A_3) \cdot P(E/A_3)$

$$= \frac{4}{10} \cdot \frac{45}{100} + \frac{4}{10} \cdot \frac{60}{100} + \frac{2}{10} \cdot \frac{35}{100}$$

$$= \frac{180}{1000} + \frac{240}{1000} + \frac{70}{1000} = \frac{490}{1000} = 0.49$$

(ii) $P(\overline{E}/A_3) = 1 - P(E/A_3) = 1 - \frac{35}{100} = \frac{65}{100}$ [as given above]

(iii) $P(A_2/\overline{E}) = \frac{P(A_2)\cdot P(\overline{E}/A_2)}{P(A_1)\cdot P(\overline{E}/A_1) + P(A_2) + P(\overline{E}/A_2) + P(A_3)\cdot P(\overline{E}/A_3)}$

$$= \frac{\frac{4}{10}\cdot\frac{40}{100}}{\frac{4}{10}\cdot\frac{55}{100} + \frac{4}{10}\cdot\frac{40}{100} + \frac{2}{10}\cdot\frac{65}{100}} = \frac{\frac{160}{1000}}{\frac{220}{1000} + \frac{160}{1000} + \frac{130}{1000}}$$

$$= \frac{160/1000}{510/1000} = \frac{16}{51} = 0.313725 = 0.314$$

44. A letter is known to have come either from TATA NAGAR or from CALCUTTA. On the envelope, just two consecutive letter TA are visible. What is the probability that the letter came from TATA NAGAR.

Sol. Let E_1 be the event that letter is from TATA NAGAR and E_2 be the event that letter is from CALCUTTA.

Also, let E_3 be the event on the letter, two consecutive letters TA are visible.

$\therefore P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$

And $P(E_3/E_1) = \frac{2}{8}$ and $P(E_3/E_2) = \frac{1}{7}$

[Since, if letter is from TATA NAGAR, we see that the events of two consecutive letters visible are {TA, AT, TA, AN, NA, AG, GA, AR}. So, $P(E_3/E_1) = \frac{2}{8}$ and if letter is from CALCUTTA, we see that the events of two consecutive letters to visible are {CA, AL, LC, CU, UT, TT, TA}. So, $P(E_3/E_2) = \frac{1}{7}$]

$\therefore P(E_1/E_3) = \frac{P(E_1)\cdot P(E_3/E_1)}{P(E_1)\cdot P(E_3/E_1) + P(E_2)\cdot P(E_3/E_2)}$

$$= \frac{\frac{1}{2}\cdot\frac{2}{8}}{\frac{1}{2}\cdot\frac{2}{8} + \frac{1}{2}\cdot\frac{1}{7}} = \frac{\frac{1}{8}}{\frac{1}{8} + \frac{1}{14}} = \frac{1/8}{\frac{22}{8\times 14}} = \frac{\frac{1}{8}}{\frac{11}{56}} = \frac{7}{11}$$

45. There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the 1st bag; but it shows up any other number, a ball is chosen from the second bag. Find the probability of choosing a black ball.

Sol. Since, Bag I = {3 black, 4 white balls}, Bag II = {4 black, 3 white balls}

Let E_1 be the event that bag I is selected and E_2 be the event that bag II is selected.

Let E_3 be the event that black ball is chosen.

$\therefore P(E_1)=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$ and $P(E_2)=1-\frac{1}{3}=\frac{2}{3}$

And $P(E_3/E_1)=\frac{3}{7}$ and $P(E_3/E_2)=\frac{4}{7}$

$\therefore P(E_3)=P(E_1)\cdot P(E_3/E_1)+P(E_2)\cdot P(E_3/E_2)$

$=\frac{1}{3}\cdot\frac{3}{7}+\frac{2}{3}\cdot\frac{4}{7}=\frac{11}{21}$

46. There are three urns containing 2 white and 3 black balls, 3 white and 2 black balls, and 4 white and 1 black balls, respectively. There is an equal probability of each urn being chosen. A ball is drawn at random from the chosen urn and it is found to be white. Find the probability that the ball drawn was from the second urn.

Sol. Let U_1 = {2 white, 3 black balls}

U_2 = {3 white, 2 black balls}

And U_3 = {4 white, 1 black balls}

$\therefore P(U_1)=P(U_2)=P(U_3)=\frac{1}{3}$

Let E1 be the event that a ball is chosen from urn U_1, E_2 be the event that a ball is chosen from urn U_1 and E_3 be the event that a ball is chosen from urn U_3.

Also, $P(E_1) = P(E_2) = P(E_3) = 1/3$

Now, let E be the event that white ball is drawn.

$\therefore P(E/E_1)=\frac{2}{5}, P(E/E_2)=\frac{3}{5}, P(E/E_3)=\frac{4}{5}$

Now, $P(E_2/E)=\frac{P(E_2)\cdot P(E/E_2)}{P(E_1)\cdot P(E/E_1)+P(E_2)\cdot P(E/E_2)+P(E_3)\cdot P(E/E_3)}$

$$=\frac{\frac{1}{3}\cdot\frac{3}{5}}{\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{3}{5}+\frac{1}{3}\cdot\frac{4}{5}}$$

$$=\frac{\frac{3}{15}}{\frac{2}{15}+\frac{3}{15}+\frac{4}{15}}=\frac{3}{9}=\frac{1}{3}$$

47. By examining the chest X ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of an healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?

Sol. Let E_1 = Event that person that TB

E_2 = Event that person does not have TB

E = Event that the person is diagnosed to have TB

$$\therefore P(E_1) = \frac{1}{1000} = 0.001, P(E_2) = \frac{999}{1000} = 0.999$$

And $P(E/E_1) = 0.99$ and $P(E/E_2) = 0.001$

$$\therefore P(E_1/E) = \frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2)}$$

$$= \frac{0.001\times 0.99}{0.001\times 0.99 + 0.999\times 0.001}$$

$$= \frac{0.000990}{0.000990 + 0.000999}$$

$$= \frac{990}{1989} = \frac{110}{221}$$

48. An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on A, 30% on B and 20% on C. 2% of the items produced on A and 2% of items produced on B are defective, and 3% of these produced on C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A?

Sol. Let E_1 = Event that item is manufactured on A.

E_2 = Event that an item is manufactured on B.

E_3 = Event that an item is manufactured on C.

Let E be the event that an item is defective.

$$\therefore P(E_1) = \frac{50}{100} = \frac{1}{2}, P(E_2) = \frac{30}{100} = \frac{3}{10} \text{ and } P(E_3) = \frac{20}{100} = \frac{1}{5}$$

$$P\left(\frac{E}{E_1}\right) = \frac{2}{100} = \frac{1}{50}, P\left(\frac{E}{E_2}\right) = \frac{2}{100} = \frac{1}{50} \text{ and } P\left(\frac{E}{E_3}\right) = \frac{3}{100}$$

$$\therefore P\left(\frac{E_1}{E}\right) = \frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right) + P(E_3)\cdot P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{50}}{\frac{1}{2}\cdot\frac{1}{50} + \frac{3}{10}\cdot\frac{1}{50} + \frac{1}{5}\cdot\frac{3}{100}}$$

$$= \frac{\frac{1}{100}}{\frac{1}{100} + \frac{3}{500} + \frac{3}{500}} = \frac{\frac{1}{100}}{\frac{5+3+3}{500}} = \frac{5}{11}$$

49. Let X be a discrete random variable whose probability distribution is defined as follows:

$$P(X = x) = \begin{cases} k(x+1) & for\ x = 1,2,3,4 \\ 2kx & for\ x = 5,6,7 \\ 0 & otherwise \end{cases}$$

where k is a constant. Calculate

(i) the value of k

(ii) E (X)

(iii) Standard deviation of X.

Sol. $$P(X = x) = \begin{cases} k(x+1) & for\ x = 1,2,3,4 \\ 2kx & for\ x = 5,6,7 \\ 0 & otherwise \end{cases}$$

Thus, we have following table

X	1	2	3	4	5	6	7	Otherwise
P(X)	2k	3k	4k	5k	10k	12k	14k	0
XP(X)	2k	6k	12k	20k	50k	72k	98k	0
X_2P(X)	2k	12k	36k	80k	250k	432k	686k	0

(i) Since, $\sum P_i = 1$

$$\Rightarrow K(2+3+4+5+10+12+14) = 1 \Rightarrow k = \frac{1}{50}$$

(ii) $\because E(X) = \sum XP(X)$

$\therefore E(X) = 2k + 6k + 12k + 20k + 50k + 72k + 98k + 0 = 260k$

$$= 260 \times \frac{1}{50} = \frac{26}{5} = 5.2 \left[\because k = \frac{1}{50}\right] \quad ...(i)$$

(iii) We know that,

$Var(X) = [E(X^2)] - [E(X)]^2 = \sum X^2 P(X) - [\sum\{XP(X)\}]^2$

$= [2k + 12k + 36k + 80k + 250k + 432k + 686k + 0] - [5.2]^2$ [using Eq. (i)]

$$= [1498k] - 27.04 = \left[1498 \times \frac{1}{50}\right] - 27.04 \left[\because k = \frac{1}{50}\right]$$

$= 29.96 - 27.04 = 2.92$

We know that, standard deviation of $X = \sqrt{Var(X)} = \sqrt{2.92} = 1.7088 = 1.7(\text{approx})$

50. The probability distribution of a discrete random variable X is given as under:

X	1	2	4	2A	3A	5A
P(X)	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

Calculate:

(i) The value of A if E(X) = 2.94

(ii) Variance of X.

Sol. (i) We have, $\sum XP(X)=\frac{1}{2}+\frac{2}{5}+\frac{12}{25}+\frac{2A}{10}+\frac{3A}{25}+\frac{5A}{25}$

$=\frac{25+20+24+10A+6A+10A}{50}=\frac{69+26A}{50}$

Since, $E(X)=\sum XP(X)$

$\Rightarrow 2.94=\frac{69+26A}{50}$

$\Rightarrow 26A=50\times 2.94-69$

$\Rightarrow A=\frac{147-69}{26}=\frac{78}{26}=3$

(ii) We know that,

$Var(X)=E(X)^2-\left[E(X)\right]^2$

$=\sum X^2P(X)-[\sum XP(X)]^2$

$=\frac{1}{2}+\frac{4}{5}+\frac{48}{25}+\frac{4A^2}{10}+\frac{9A^2}{25}+\frac{25A^2}{25}-[E(X)]^2$

$=\frac{25+40+96+20A^2+18A^2+50A^2}{50}-[E(X)]^2$

$=\frac{161+88A^2}{50}-[E(X)]^2=\frac{161+88\times(3)^2}{50}-[E(X)]^2$ $[\because A=3]$

$=\frac{953}{50}-[2.94]^2$ $[\because E(X)=2.94]$

$=19.0600-8.6436=10.4164$

51. The probability distribution of a random variable x is given as under:

$$P(X=x)=\begin{cases} kx^2 & for\ x=1,2,3 \\ 2kx & for\ x=4,5,6 \\ 0 & otherwise \end{cases}$$

Where k is a constant. Calculate

(i) $E(X)$

(ii) $E\left(3X^2\right)$

(iii) $P(X \geq 4)$

Sol.

X	1	2	3	4	5	6	Otherwise
P(X)	k	4k	9k	8k	10k	12k	0

We know that, $\sum P_i = 1$

$\Rightarrow 44k = 1 \Rightarrow k = \frac{1}{44}$

$\therefore \sum XP(X) = k + 8k + 27k + 32k + 50k + 72k + 0$

$= 190k = 190 \times \frac{1}{44} = \frac{95}{22}$

(i) So, $E(X) = \sum XP(X) = \frac{95}{22} = 4.32$

(ii) Also, $E(X^2) = \sum X^2 P(X) = k + 16k + 81k + 128k + 250k + 432k$

$= 908k = 908 \times \frac{1}{44}$ $\left[\because k = \frac{1}{44}\right]$

$= 20.636 = 20.64 \text{(approx)}$

$\therefore E(3X^2) = 3E(X^2) = 3 \times 20.64 = 61.92 = 61.9$

(iii) $P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6)$

$= 8k + 10k + 12k = 30k = 30 \cdot \frac{1}{44} = \frac{15}{22}$

52. **A bag contains (2n + 1) coins. It is known that n of these coins have a head on both sides where as the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.**

Sol. Given, n coin have head on both sides and (n+1) coins are fair coins.

Let E_1 = Event that an unfair coin is selected.

E_2 = Event that a fair coin is selected.

E = Event that the toss results in a head.

$\therefore P(E_1) = \frac{n}{2n+1}$ and $P(E_2) = \frac{n+1}{2n+1}$

Also, $P\left(\frac{E}{E_1}\right) = 1$ and $P\left(\frac{E}{E_2}\right) = \frac{1}{2}$

$$\therefore P(E)=P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)=\frac{n}{2n+1}\cdot 1+\frac{n+1}{2n+1}\cdot\frac{1}{2}$$

$$\Rightarrow \frac{31}{42}=\frac{2n+n+1}{2(2n+1)}\Rightarrow\frac{31}{42}=\frac{3n+1}{4n+2}$$

$$\Rightarrow 124n+62=126n+42$$

$$\Rightarrow 2n=20\Rightarrow n=10$$

53. Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard variation of the random variable X where X is the number of aces.

Sol. Let X denotes a random variable of number of aces.

$\therefore$ X = 0, 1, 2

Now, $P(X=0)=\frac{48}{52}\cdot\frac{47}{51}=\frac{2256}{2652}$

$$P(X=1)=\frac{48}{52}\cdot\frac{4}{51}+\frac{4}{52}\cdot\frac{48}{51}=\frac{384}{2652}$$

$$P(X=2)=\frac{4}{52}\cdot\frac{3}{51}=\frac{12}{2652}$$

X	0	1	2
P(X)	$\frac{2256}{2652}$	$\frac{384}{2652}$	$\frac{12}{2652}$
XP(X)	0	$\frac{384}{2652}$	$\frac{24}{2652}$
$X^2P(X)$	0	$\frac{384}{2652}$	$\frac{48}{2652}$

We know that, Mean $(\mu)=E(X)=\sum XP(X)$

$$=0+\frac{384}{2652}+\frac{24}{2652}$$

$$=\frac{408}{2652}=\frac{2}{13}$$

Also, $Var(X)=E(X^2)-[E(X)]^2=\sum X^2P(X)-[E(X)]^2$

$$=\left[0+\frac{384}{2652}+\frac{48}{2652}\right]-\left(\frac{2}{13}\right)^2\quad\left[\because E(X)=\frac{2}{13}\right]$$

$$=\frac{432}{2652}-\frac{4}{169}=0.1628-0.0236=0.1391$$

$\therefore$ Standard deviation $=\sqrt{Var(X)}=\sqrt{0.139}=0.373$ (approx)

54. A die is tossed twice. A 'success' is getting an even number on a toss. Find the variance of the number of successes.

Sol. Let X be the random variable for a 'success' for getting an even number on a toss.

$\therefore\ X = 0, 1, 2,\ n = 2,\ p = \frac{3}{6} = \frac{1}{2}$ and $q = \frac{1}{2}$

At $X = 0,\ \ P(X = 0) = {}^2C_0 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{2-0} = \frac{1}{4}$

At $X = 1,\ \ P(X = 1) = {}^2C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{2-1} = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

At $X = 2,\ \ P(X = 2) = {}^2C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{2-2} = \frac{1}{4}$

Thus,

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
XP(X)	0	$\frac{1}{2}$	$\frac{1}{2}$
$X^2P(X)$	0	$\frac{1}{2}$	1

$\therefore \sum XP(X) = 0 + \frac{1}{2} + \frac{1}{2} = 1\ \ ...(i)$

And $\sum X^2 P(X) = 0 + \frac{1}{2} + 1 = \frac{3}{2}\ \ ...(ii)$

$\because Var(X) = E(X^2) - [E(X)]^2$

$= \sum X^2 P(X) - [\sum XP(X)]^2 = \frac{3}{2} - (1)^2 = \frac{1}{2}$ [using Eqs. (i) and (ii)]

55. There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

Sol. Here, S = {(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (1, 4), (4, 1), (1, 5), (5, 1), (2, 4), (4, 2), (2, 5), (5, 2), (3, 4), (4, 3), (3, 5), (5, 3), (5, 4), (4, 5)}

$\Rightarrow n(S) = 20$

Let random variable be X which denotes the sum of the numbers on two cards drawn.

$\therefore\ X = 3, 4, 5, 6, 7, 8, 9$

$$At\ X=3, P(X)=\frac{2}{20}=\frac{1}{10}$$

$$At\ X=4, P(X)=\frac{2}{20}=\frac{1}{10}$$

$$At\ X=5, P(X)=\frac{4}{20}=\frac{1}{5}$$

$$At\ X=6, P(X)=\frac{4}{20}=\frac{1}{5}$$

$$At\ X=7, P(X)=\frac{4}{20}=\frac{1}{5}$$

$$At\ X=8, P(X)=\frac{2}{20}=\frac{1}{5}$$

$$At\ X=9, P(X)=\frac{2}{20}=\frac{1}{10}$$

$$\therefore \qquad Mean, E(X)=\sum X\ P(X)=\frac{3}{10}+\frac{4}{10}+\frac{5}{5}+\frac{6}{5}+\frac{7}{5}+\frac{8}{10}+\frac{9}{10}$$

$$=\frac{3+4+10+12+14+8+9}{10}=6$$

Also, $\sum X^2 P(X)=\frac{9}{10}+\frac{16}{10}+\frac{25}{5}+\frac{36}{5}+\frac{49}{5}+\frac{64}{10}+\frac{81}{10}$

$$=\frac{9+16+50+72+98+64+81}{10}=39$$

$$\therefore Var(X)=\sum X^2 P(X)-[\sum XP(X)]^2$$

$$=39-(6)^2=39-36=3$$