

2 Straight Lines in Space

INTRODUCTION

In class XI, we studied Analytical Geometry in two dimensions and introduced 3-dimensional geometry, and restricted our study to cartesian method only. Now we shall use vector algebra to 3-dimensional geometry which makes its study simple and elegant.

In this chapter, we shall study :

- (i) the vector and cartesian equations of a line in different forms
- (ii) angle between two lines when their equations are known
- (iii) coplanar and skew lines
- (iv) shortest distance between two lines
- (v) conditions for intersection of two lines
- (vi) condition for coplanarity of two lines.

2.1 STRAIGHT LINE (IN SPACE)

We know that in space a straight line is uniquely determined if

- (i) it passes through a given point and has a given direction or
- (ii) it passes through two given points.

We shall obtain the vector equation and cartesian equation of a line (in space).

2.1.1 Equation of a straight line passing through a given point and parallel to a given vector

Let the line pass through the given point A with position vector $\vec{a}$ and let it be parallel to the given vector $\vec{b}$ (as shown in fig. 2.1).

Let P with position vector $\vec{r}$ be any point, then P lies on the line iff $\vec{AP}$ is parallel to the vector $\vec{b}$ i.e. iff $\vec{AP} = \lambda \vec{b}$, where λ is a real number (called *parameter*)

$$\text{i.e. iff } \vec{OP} - \vec{OA} = \lambda \vec{b}$$

$$\text{i.e. iff } \vec{r} - \vec{a} = \lambda \vec{b}$$

$$\text{i.e. iff } \vec{r} = \vec{a} + \lambda \vec{b}$$

which is the required *vector equation* of the line.

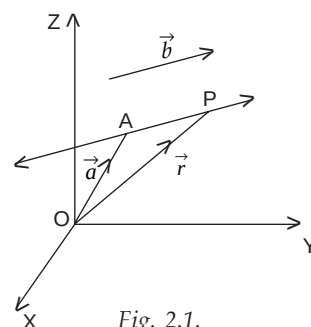


Fig. 2.1.

... (1)

Cartesian form

Let the given point be A (x_1, y_1, z_1) and $\langle a, b, c \rangle$ be the direction numbers of the line and P be (x, y, z), then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and}$$

$$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}.$$

On substituting these values of $\vec{r}$, $\vec{a}$ and $\vec{b}$ in (1), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1 + a\lambda)\hat{i} + (y_1 + b\lambda)\hat{j} + (z_1 + c\lambda)\hat{k}$$

On equating the coefficients of the base vectors $\hat{i}, \hat{j}, \hat{k}$, we get

$$\left. \begin{aligned} x &= x_1 + a\lambda \\ y &= y_1 + b\lambda \\ z &= z_1 + c\lambda \end{aligned} \right\} \quad \dots(2)$$

These are called **parametric (or one point) form** of the line.

Any point on the line is P ($x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda$). The point A is called the **base point**.

On eliminating λ from equations (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \dots(3)$$

These are called **symmetrical form** of the line.

2.1.2 Equation of a straight line passing through two given points

Let the line pass through two given points A and B with position vectors $\vec{a}$ and $\vec{b}$ respectively (as shown in fig 2.2).

Let P with position vector $\vec{r}$ be any point, then P lies on the line AB iff

$$\vec{AP} = \lambda \vec{AB}, \text{ where } \lambda \text{ is real number (called parameter)}$$

$$\text{i.e. iff } \vec{OP} - \vec{OA} = \lambda(\vec{OB} - \vec{OA})$$

$$\text{i.e. iff } \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \dots(1)$$

which is the required **vector equation** of the line.

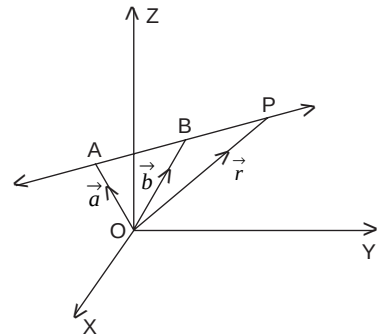


Fig. 2.2.

Cartesian form

Let the given points be A (x_1, y_1, z_1), B (x_2, y_2, z_2) and the point P be (x, y, z), then from (1), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + t[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}].$$

On equating the coefficients of base vectors, we get

$$\left. \begin{aligned} x &= x_1 + \lambda(x_2 - x_1) \\ y &= y_1 + \lambda(y_2 - y_1) \\ z &= z_1 + \lambda(z_2 - z_1) \end{aligned} \right\} \quad \dots(2)$$

These are called **parametric (or two points) form** of the line.

On eliminating λ from the equations (2), we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \dots(3)$$

These are called **symmetrical form**.

ILLUSTRATIVE EXAMPLES

Example 1. Find the vector equation of the line which is parallel to the vector $2\hat{i} - \hat{j} + 3\hat{k}$ and which passes through the point $(5, -2, 4)$. Also find its cartesian equations.

Solution. Let $\vec{r}$ be the position vector of any point P (x, y, z) on the given line, then the vector equation of the line is

$$\vec{r} = 5\hat{i} - 2\hat{j} + 4\hat{k} + \lambda(2\hat{i} - \hat{j} + 3\hat{k}).$$

Now as $\vec{r}$ is the position vector of P (x, y, z) , we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (5 + 2\lambda)\hat{i} - (2 + \lambda)\hat{j} + (4 + 3\lambda)\hat{k}.$$

On equating coefficients of the base vectors, we get

$$x = 5 + 2\lambda, y = -2 - \lambda, z = 4 + 3\lambda.$$

On eliminating the parameter λ from these equations, we get

$$\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}, \text{ which are the required cartesian equations.}$$

Example 2. If the cartesian equations of a line are $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-5}{-1}$, find its vector equation.

Solution. The given line passes through the point $(1, -2, 5)$ and is parallel to the vector $2\hat{i} + 3\hat{j} - \hat{k}$.

Let $\vec{r}$ be the position vector of any point on the given line, then the vector equation of the line is

$$\vec{r} = \hat{i} - 2\hat{j} + 5\hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k}) \text{ where } \lambda \text{ is a parameter.}$$

Example 3. Find the vector and the cartesian equations of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

Solution. Direction ratios of the given line are $\langle 3, 5, 6 \rangle$. The required line passes through the point A $(-2, 4, -5)$ and is parallel to the vector $3\hat{i} + 5\hat{j} + 6\hat{k}$, therefore, its vector equation is

$$\vec{r} = -2\hat{i} + 4\hat{j} - 5\hat{k} + \lambda(3\hat{i} + 5\hat{j} + 6\hat{k}), \text{ where } \lambda \text{ is a parameter.}$$

Cartesian equations of the required line are

$$\frac{x - (-2)}{3} = \frac{y - 4}{5} = \frac{z - (-5)}{6} \quad \left| \quad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \right.$$

i.e. $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}.$

Example 4. Find the vector equation of a line passing through the point with position vector $2\hat{i} - \hat{j} + \hat{k}$ and parallel to the line joining the points $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$. Also find cartesian equations of the line.

Solution. Let A, B and C be the points with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

$$\begin{aligned} \therefore \quad \vec{BC} &= \text{P.V. of C} - \text{P.V. of B} \\ &= (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}. \end{aligned}$$

We have to find the vector equation of a line through A and parallel to $\vec{BC}$.

$\therefore$ The vector equation of the required line is

$$\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \text{ where } \lambda \text{ is a parameter.}$$

Now, as $\vec{r}$ is the position vector of any point P (x, y, z) on the line, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k}).$$

On equating the coefficients of the base vectors, we get

$$x = 2 + 2\lambda, y = -1 - 2\lambda, z = 1 + \lambda.$$

On eliminating the parameter λ from these equations, we get

$$\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}, \text{ which are the required cartesian equations.}$$

Example 5. Find the vector and the cartesian equations of the line that passes through the points A ($3, -2, -5$) and B ($3, -2, 6$).

Solution. Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A ($3, -2, -5$) and B ($3, -2, 6$) respectively. Then

$$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k} \Rightarrow \vec{b} - \vec{a} = 11\hat{k}.$$

Let $\vec{r}$ be the position vector of any point P (x, y, z) on the line passing through the points A and B. The vector equation of the line is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \text{ i.e. } \vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda(11\hat{k})$$

$$\text{i.e. } \vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + 11\lambda\hat{k}, \text{ where } \lambda \text{ is a parameter.}$$

Now as $\vec{r}$ is the position vector of P (x, y, z), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} - 2\hat{j} - 5\hat{k} + 11\lambda\hat{k}$$

$$\text{i.e. } x\hat{i} + y\hat{j} + z\hat{k} = 3\hat{i} - 2\hat{j} + (11\lambda - 5)\hat{k}.$$

On equating the coefficients of base vectors, we get

$$x = 3, y = -2, z = 11\lambda - 5, \text{ which are the required cartesian equations (in parametric form).}$$

Example 6. The points A ($1, 2, 3$), B ($-1, -2, -1$) and C ($2, 3, 2$) are three vertices of a parallelogram ABCD. Find vector and cartesian equations of the sides AB and BC. Also find the coordinates of D.

Solution. Given A ($1, 2, 3$), B ($-1, -2, -1$) and C ($2, 3, 2$).

Direction numbers of the line AB are $\langle -1 - 1, -2 - 2, -1 - 3 \rangle$ i.e. $\langle 1, 2, 2 \rangle$.

$\therefore$ The vector equation of the side AB is

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \text{ is a parameter,}$$

$$\text{and the cartesian equations of this line are } \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{2}.$$

Direction numbers of the side BC are $\langle 2 + 1, 3 + 2, 2 + 1 \rangle$ i.e. $\langle 3, 5, 3 \rangle$.

$\therefore$ The vector equation of the side BC is

$$\vec{r} = -\hat{i} - 2\hat{j} - \hat{k} + \mu(3\hat{i} + 5\hat{j} + 3\hat{k}), \mu \text{ is a parameter,}$$

$$\text{and the cartesian equations of this line are } \frac{x+1}{3} = \frac{y+2}{5} = \frac{z+1}{3}.$$

Let D be (α, β, γ) . Since ABCD is a parallelogram, diagonals AC and BD bisect each other i.e. mid-point of segment AC is same as mid-point of segment BD

$$\Rightarrow \left(\frac{1+2}{2}, \frac{2+3}{2}, \frac{3+2}{2} \right) = \left(\frac{\alpha-1}{2}, \frac{\beta-2}{2}, \frac{\gamma-1}{2} \right)$$

$$\Rightarrow \alpha - 1 = 3, \beta - 2 = 5, \gamma - 1 = 5 \Rightarrow \alpha = 4, \beta = 7, \gamma = 6.$$

Hence, the point D is $(4, 7, 6)$.

Example 7. Find the coordinates of the point where the line through the points $(5, 1, 6)$ and $(3, 4, 1)$ crosses the yz -plane.

Solution. Direction numbers of the line passing through the points $(5, 1, 6)$ and $(3, 4, 1)$ are

$$< 3 - 5, 4 - 1, 1 - 6 > \text{ i.e. } < -2, 3, -5 > \text{ i.e. } < 2, -3, 5 >.$$

$\therefore$ The equations of the line through the given points are

$$\frac{x-5}{2} = \frac{y-1}{-3} = \frac{z-6}{5} \quad \dots(i)$$

Any point on the line (i) is $(5 + 2\lambda, 1 - 3\lambda, 6 + 5\lambda)$.

If this point lies on yz -plane, then its x -coordinate is zero

$$\Rightarrow 5 + 2\lambda = 0 \Rightarrow \lambda = -\frac{5}{2}.$$

$\therefore$ The required point is $\left(0, 1 + 3 \cdot \frac{5}{2}, 6 - 5 \cdot \frac{5}{2} \right)$ i.e. $\left(0, \frac{17}{2}, -\frac{13}{2} \right)$.

Example 8. Show that the points whose position vectors are given by $-2\hat{i} + 3\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ and $7\hat{i} - \hat{k}$ are collinear.

Solution. Let A, B and C be the points whose position vectors are $-2\hat{i} + 3\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ and $7\hat{i} - \hat{k}$ respectively.

Equation of the line passing through points A and B is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

$$\text{i.e. } \vec{r} = -2\hat{i} + 3\hat{j} + 5\hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k} - (-2\hat{i} + 3\hat{j} + 5\hat{k}))$$

$$\text{i.e. } \vec{r} = -2\hat{i} + 3\hat{j} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} - 2\hat{k}) \quad \dots(ii)$$

The points A, B and C are collinear if the point C lies on (i)

$$\text{i.e. if } 7\hat{i} - \hat{k} = -2\hat{i} + 3\hat{j} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} - 2\hat{k}) \text{ is consistent}$$

$$\text{i.e. if } 7 = -2 + 3\lambda, 0 = 3 - \lambda \text{ and } -1 = 5 - 2\lambda \text{ are consistent}$$

$$\text{i.e. if } \lambda = 3, \lambda = 3 \text{ and } \lambda = 3, \text{ which are true.}$$

Hence, the given points are collinear.

Alternative method (cartesian approach)

The coordinates of the points A, B and C are

$$(-2, 3, 5), (1, 2, 3) \text{ and } (7, 0, -1) \text{ respectively.}$$

Direction numbers of the line AB are $< 1 + 2, 2 - 3, 3 - 5 >$ i.e. $< 3, -1, -2 >$.

The equations of the line AB are

$$\frac{x+2}{3} = \frac{y-3}{-1} = \frac{z-5}{-2} \quad \dots(ii)$$

Now the points A, B and C are collinear if the point C $(7, 0, -1)$ lies on (ii)

$$\text{i.e. if } \frac{7+2}{3} = \frac{0-3}{-1} = \frac{-1-5}{-2} \text{ are consistent}$$

$$\text{i.e. if } 3 = 3 = 3, \text{ which is true.}$$

Hence, the given points are collinear.

Example 9. Find the values of p and q by using vector method such that the points $A(5, 0, 5)$, $B(2, 1, 3)$ and $C(-4, p, q)$ are collinear.

Solution. Direction numbers of the line AB are $\langle 2-5, 1-0, 3-5 \rangle$ i.e. $\langle -3, 1, -2 \rangle$ i.e. $\langle 3, -1, 2 \rangle$.

$\therefore$ The vector equation of the line AB is

$$\vec{r} = 5\hat{i} + 0\hat{j} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} + 2\hat{k}).$$

The position vector of the point C is $-4\hat{i} + p\hat{j} + q\hat{k}$.

Since the points A, B and C are collinear, C lies on the line AB

$$\Rightarrow -4\hat{i} + p\hat{j} + q\hat{k} = 5\hat{i} + 5\hat{k} + \lambda(3\hat{i} - \hat{j} + 2\hat{k}).$$

Equating the coefficients of the base vectors, we get

$$-4 = 5 + 3\lambda, p = -\lambda \text{ and } q = 5 + 2\lambda$$

$$\Rightarrow \lambda = -3, p = -(-3) = 3, q = 5 + 2(-3) = -1.$$

Hence, $p = 3$ and $q = -1$.

Example 10. The cartesian equations of a line are $2x - 3 = 3y + 1 = 5 - 6z$. Find the direction ratios of the line and write down the vector and cartesian equations of the line through $(7, -5, 0)$ which is parallel to the given line.

Solution. The equations of the line are

$$2x - 3 = 3y + 1 = 5 - 6z \text{ i.e. } 2\left(x - \frac{3}{2}\right) = 3\left(y + \frac{1}{3}\right) = -6\left(z - \frac{5}{6}\right)$$

$$\text{i.e. } \frac{x - \frac{3}{2}}{\frac{1}{2}} = \frac{y + \frac{1}{3}}{\frac{1}{3}} = \frac{z - \frac{5}{6}}{-\frac{1}{6}} \text{ i.e. } \frac{x - \frac{3}{2}}{3} = \frac{y + \frac{1}{3}}{2} = \frac{z - \frac{5}{6}}{-1}.$$

Therefore, direction ratios of the given line are $\langle 3, 2, -1 \rangle$. The vector equation of the line through $(7, -5, 0)$ and parallel to the given line is

$\vec{r} = 7\hat{i} - 5\hat{j} + \lambda(3\hat{i} + 2\hat{j} - \hat{k})$, where λ is a parameter
and the cartesian equations of this line are

$$\frac{x-7}{3} = \frac{y+5}{2} = \frac{z}{-1}.$$

Example 11. Find the points on the line through the points $A(1, 2, 3)$ and $B(3, 5, 9)$ at a distance of 14 units from the mid-point of segment AB.

Solution. Direction numbers of the line AB are $\langle 3-1, 5-2, 9-3 \rangle$ i.e. $\langle 2, 3, 6 \rangle$.

The equations of the line AB are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$.

Any point on the line AB is $P(1 + 2\lambda, 2 + 3\lambda, 3 + 6\lambda)$.

Mid-point of AB is $M\left(\frac{1+3}{2}, \frac{2+5}{2}, \frac{3+9}{2}\right)$ i.e. $M\left(2, \frac{7}{2}, 6\right)$.

According to given $|MP| = 14$

$$\Rightarrow \sqrt{(2\lambda + 1 - 2)^2 + \left(3\lambda + 2 - \frac{7}{2}\right)^2 + (6\lambda + 3 - 6)^2} = 14$$

$$\Rightarrow (2\lambda - 1)^2 + \left(3\lambda - \frac{3}{2}\right)^2 + (6\lambda - 3)^2 = 196$$

$$\Rightarrow 4\lambda^2 - 4\lambda + 1 + 9\lambda^2 - 9\lambda + \frac{9}{4} + 36\lambda^2 - 36\lambda + 9 = 196$$

$$\Rightarrow 49\lambda^2 - 49\lambda - \frac{735}{4} = 0$$

$$\Rightarrow 4\lambda^2 - 4\lambda - 15 = 0 \Rightarrow (2\lambda - 5)(2\lambda + 3) = 0$$

$$\Rightarrow \lambda = \frac{5}{2}, -\frac{3}{2}.$$

$\therefore$ The required points are $\left(1 + 2 \cdot \frac{5}{2}, 2 + 3 \cdot \frac{5}{2}, 3 + 6 \cdot \frac{5}{2}\right)$ and $\left(1 - 2 \cdot \frac{3}{2}, 2 - 3 \cdot \frac{3}{2}, 3 - 6 \cdot \frac{3}{2}\right)$

i.e. $\left(6, \frac{19}{2}, 18\right)$ and $\left(-2, -\frac{5}{2}, -6\right)$.

EXERCISE 2.1

- Find the vector equation of the line passing through the point $(2, -3, 5)$ and parallel to the vector $3\hat{i} + 2\hat{j} - 7\hat{k}$.
- Find the cartesian equation of the line which passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + 2\hat{j} - \hat{k}$.
- Find the vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$.
- Find the cartesian equations of the line which passes through the point $(-2, 4, -5)$ and parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.
- Find the vector equation of a line passing through $(1, -2, 3)$ and parallel to the line $\frac{x-1}{3} = \frac{y+2}{-5} = \frac{z+5}{2}$.
- Find the vector equation of the line passing through the points $(-1, 0, 2)$ and $(3, 4, 6)$.
- Find the vector and the cartesian equations of the line that passes through origin and the point $(5, -2, 3)$.
- The cartesian equations of a line AB are $\frac{2x-1}{2} = \frac{4-y}{7} = \frac{z+1}{2}$. Write the direction ratios of a line parallel to AB.
- The cartesian equations of a line are $2x - 3 = 3y + 1 = 5 - 6z$. Write the direction ratios of the line.
- Write the vector equation of the x -axis.
- Find the vector and the cartesian equations of the line through the point $(5, 2, -4)$ and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.
- Find the vector equation of the line passing through the point A $(2, -1, 1)$ and parallel to the line joining the points B $(-1, 4, 1)$ and C $(1, 2, 2)$. Also find the cartesian equation of the line.
- Find the vector equation of a line passing through the point with position vector $\hat{i} - 2\hat{j} - 3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i} - \hat{j} + 4\hat{k}$ and $2\hat{i} + \hat{j} + 2\hat{k}$. Also find the cartesian equations of the line.
- Find the vector equation of the line joining the points whose position vectors are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - 3\hat{k}$. Also find its cartesian equations.
- The cartesian equations of a line are $6x - 2 = 3y + 1 = 2z - 2$. Find direction ratios of the line and write down the vector equation of the line through $(2, -1, -1)$ which is parallel to the given line.

16. Find the equations of a line through A(1, -1, 5) and parallel to the line $\frac{x-2}{3} = \frac{y-5}{-2}, z = -1$.
17. The cartesian equations of a line are $x = ay + b, z = cy + d$. Find direction numbers of the line, also find its equation in vector form.
18. The points A (4, 5, 10), B (2, 3, 4) and C (1, 2, -1) are three vertices of a parallelogram ABCD. Find vector and cartesian equations of the sides AB and BC. Also find the coordinates of D.
19. (i) Find the coordinates of the point where the line through A (3, 4, 1) and B (5, 1, 6) crosses the xy -plane.
(ii) Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the zx -plane.
20. (i) Find the equation of the line passing through the points A (-2, 4, 7), B (3, -6, -8). Hence show that the points A, B and C (1, -2, -2) are collinear.
(ii) If the points (-1, 3, 2), (-4, 2, -2) and (5, 5, λ) are collinear, then find the value of λ .
21. By using vector method, find the values of p and q so that the points $(p, q, 1)$, $(-1, 4, -2)$ and $(0, 2, -1)$ may be collinear.
22. Find the coordinates of the points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which are at a distance of 3 units from the point (1, -2, 3).
23. Find the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3).
24. Find points on the line through the points A(3, 4, -1) and B(5, 1, 5) at a distance of 7 units from the mid-point of [AB].

2.2 ANGLE BETWEEN TWO LINES

Angle between two lines is the same as the angle between the two vectors along their directions.

Observe that angle between a pair of straight lines depends only on their directions and not on their positions.

In this section, we shall find angle between two lines when their equations are known.

2.2.1 To find angle between two lines

$$\text{Let } \vec{r} = \vec{a} + \lambda \vec{b} \quad \dots(1) \quad \text{and} \quad \vec{r} = \vec{a'} + \mu \vec{c} \quad \dots(2)$$

be the vector equations of the two lines.

These lines are in the directions of $\vec{b}$ and $\vec{c}$.

If θ is the angle between these lines, then θ is the angle between the vectors $\vec{b}$ and $\vec{c}$.

$$\therefore \cos \theta = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}| |\vec{c}|} \quad \dots(3)$$

Cartesian form

Let the equations of the given lines in cartesian form be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \quad \dots(4)$$

6. Find the value of λ so that the line joining the points A(3, 2, 1) and B(4, λ , 5) may intersect the line joining the points C(4, 2, -2) and D(6, 5, -1).
7. Find the equations of the line joining the points (-2, 1, 3) and (5, 1, -1) and show that it is perpendicular to y -axis.
8. A(1, 0, 4), B(0, -11, 3), C(2, -3, 1) are three points and D is the foot of perpendicular from A on BC. Find the co-ordinates of D.
9. Find the equations of the perpendicular drawn from the point A(2, 4, -1) to the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$. Also obtain the length of perpendicular.
10. Find the image of the point (5, 9, 3) in the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.
11. A line with direction numbers $\langle 2, 7, -5 \rangle$ is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$.
Find the coordinates of the points of intersection and the length intercepted on it.
12. Find whether or not the two lines given below intersect :
 $\vec{r} = (2\lambda + 1)\hat{i} - (\lambda + 1)\hat{j} + (\lambda + 1)\hat{k}$, $\vec{r} = (3\mu + 2)\hat{i} - (5\mu + 5)\hat{j} + (2\mu + 1)\hat{k}$.
13. Find the shortest distance between the lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$.
Also find the equations of the line of shortest distance.

ANSWERS

EXERCISE 2.1

1. $\vec{r} = 2\hat{i} - 3\hat{j} + 5\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 7\hat{k})$, where λ is a real number.
2. $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$.
3. $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.
4. $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$.
5. $\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} - 5\hat{j} + 2\hat{k})$.
6. $\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$.
7. $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$; $\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$.
8. $\langle 1, -7, 2 \rangle$.
9. $\langle 3, 2, -1 \rangle$.
10. $\vec{r} = \lambda\hat{i}$.
11. $\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$; $\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$.
12. $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$; $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$.
13. $\vec{r} = \hat{i} - 2\hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$; $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z+3}{-2}$.
14. $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(-\hat{i} + 3\hat{j} - 4\hat{k})$; $\frac{x-2}{-1} = \frac{y+1}{3} = \frac{z-1}{-4}$.
15. $\langle 1, 2, 3 \rangle$; $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$.
16. $\frac{x-1}{3} = \frac{y+1}{-2}$, $z = 5$.
17. $\langle a, 1, c \rangle$; $\vec{r} = (b\hat{i} + d\hat{k}) + \lambda(a\hat{i} + \hat{j} + c\hat{k})$.
18. $\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$, $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-10}{3}$;
 $\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu(\hat{i} + \hat{j} + 5\hat{k})$, $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$; D(3, 4, 5).

19. (i) $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ (ii) $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$.
20. (i) $\frac{x+2}{1} = \frac{y-4}{-2} = \frac{z-7}{-3}$ (ii) 10.
21. $p = 2, q = -2$.
22. $\left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right), \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$.
23. $(-2, -1, 3), \left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$.
24. $\left(6, -\frac{1}{2}, 8\right), \left(2, \frac{11}{2}, -4\right)$.

EXERCISE 2.2

1. (i) 0 (ii) $\cos^{-1}\left(\frac{19}{21}\right)$.
2. (i) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ (ii) $\cos^{-1}\left(\frac{2}{3}\right)$.
4. (i) $-\frac{10}{7}$ (ii) 1.
6. (i) $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$ (ii) $\cos^{-1}\left(\frac{3}{\sqrt{182}}\right)$ (iii) $\cos^{-1}\left(\frac{4}{\sqrt{150}}\right)$ (iv) $\cos^{-1}\left(\frac{1}{5}\right)$.
7. (i) $(2, 3, -1)$ (ii) $(3, 5, 9); \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}, 7$ units.
8. $\frac{x}{1} = \frac{y-6}{4} = \frac{z+9}{-4}; (-1, 2, -5)$.
9. $(0, 5, 1)$.
10. $(0, 5, 1)$.
11. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$.
12. $\frac{x-1}{4} = \frac{y-1}{-5} = \frac{z+3}{1}$.
13. $\vec{r} = (\hat{i} + \hat{j} - 3\hat{k}) + \lambda(4\hat{i} - 5\hat{j} + \hat{k})$.
14. $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$.

EXERCISE 2.3

1. (i) $\frac{3\sqrt{2}}{2}$ units (ii) $\frac{2}{\sqrt{5}}$ units (iii) $\frac{10}{\sqrt{59}}$ units (iv) $\frac{8}{\sqrt{29}}$ units.
2. $3\sqrt{30}$ units; $\frac{x-3}{2} = \frac{y-8}{6} = \frac{z-3}{-1}$.
3. $\frac{1}{\sqrt{6}}$ units; $\frac{x-\frac{5}{3}}{1} = \frac{y-3}{-2} = \frac{z-\frac{13}{3}}{1}$.
4. (i) No (ii) Yes (iii) Yes.
6. $\sqrt{26}$ units.
7. $\vec{r} = -\hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - 2\hat{j} + \hat{k}); \sqrt{\frac{83}{6}}$ units.
8. $\sqrt{21}$ units.
9. $\vec{r} = 3\hat{i} - 4\hat{k} + \lambda(5\hat{i} - 2\hat{j} + 4\hat{k}); \sqrt{\frac{974}{45}}$ units.
10. $(1, 3, 2)$.
11. $(10, 14, 4)$.
13. (i) Yes (ii) No.

CHAPTER TEST

1. $D(1, 10, -15); \frac{x-2}{1} = \frac{y+3}{-13} = \frac{z-4}{19}$.
2. $\lambda = 1, \mu = -13$.
3. $< 2, -3, 6 >; \vec{r} = -\hat{i} - 2\hat{j} + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$.
4. $(1, 8, 4), (1, -4, -12)$.
5. $(0, 0, -4)$.
6. 5.
7. $\frac{x+2}{7} = \frac{z-3}{-4}, y = 0$.
8. $\left(\frac{22}{9}, -\frac{11}{9}, \frac{5}{9}\right)$.
9. $\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}; 7$ units.
10. $(1, 1, 11)$.
11. $(2, 8, -3)$ and $(0, 1, 2); \sqrt{78}$ units.
12. Do not intersect.
13. $2\sqrt{29}$ units; $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$.