Integrals

Multiple Choice Questions

Choose and write the correct option in the following questions.

1. $\int x^2 e^{x^3} dx$ is equal to **[CBSE 2020 (65/5/1)]**

   (a) $\frac{1}{3}e^{x^3} + C$ (b) $\frac{1}{3}e^{x^4} + C$ (c) $\frac{1}{2}e^{x^3} + C$ (d) $\frac{1}{2}e^{x^2} + C$

2. $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$ is equal to **[CBSE 2020 (65/3/1)]**

   (a) $\tan(xe^x) + C$ (b) $\cot(xe^x) + C$ (c) $\cot(e^x) + C$ (d) $\tan[e^x(1+x)] + C$

3. $\int e^x (\cos x - \sin x)\, dx$ is equal to

   (a) $e^x \cos x + C$ (b) $e^x \sin x + C$ (c) $-e^x \cos x + C$ (d) $-e^x \sin x + C$

4. $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to **[NCERT Exemplar]**

   (a) $\tan x + \cot x + C$ (b) $(\tan x + \cot x)^2 + C$

   (c) $\tan x - \cot x + C$ (d) $(\tan x - \cot x)^2 + C$

5. If $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} dx = ax + b\log|4e^x + 5e^{-x}| + C$ then **[NCERT Exemplar]**

   (a) $a = \frac{-1}{8}, b = \frac{7}{8}$ (b) $a = \frac{1}{8}, b = \frac{7}{8}$ (c) $a = \frac{-1}{8}, b = \frac{-7}{8}$ (d) $a = \frac{1}{8}, b = \frac{-7}{8}$

6. $\int_{-1}^{1} \frac{|x-2|}{x-2}\, dx$, $x \neq 2$ is equal to [CBSE 2023 (65/5/1)]

(a) 1 (b) –1 (c) 2 (d) –2

7. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ is equal to [NCERT Exemplar]

(a) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$ (b) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

(c) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$ (d) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

8. $\int \tan^{-1}\sqrt{x}\, dx$ is equal to [NCERT Exemplar]

(a) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$ (b) $x\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(c) $\sqrt{x} - x\tan^{-1}\sqrt{x} + C$ (d) $\sqrt{x} - (x+1)\tan^{-1}\sqrt{x} + C$

9. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to [NCERT Exemplar]

(a) $\frac{e^x}{1+x^2} + C$ (b) $\frac{-e^x}{1+x^2} + C$ (c) $\frac{e^x}{(1+x^2)^2} + C$ (d) $\frac{-e^x}{(1+x^2)^2} + C$

10. $\int 2^{x+2} dx$ is equal to [CBSE 2023 (65/3/2)]

(a) $2^{x+2} + C$ (b) $2^{x+2}\log 2 + C$ (c) $\frac{2^{x+2}}{\log 2} + C$ (d) $2 . \frac{2^x}{\log 2} + C$

11. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\, dx$ is equal to [CBSE 2020 (65/1/1)]

(a) – 1 (b) 0 (c) 1 (d) 2

12. The integral $\int \frac{x^9}{(4x^2+1)^6} dx$ is equal to [NCERT Exemplar]

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5} + C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5} + C$ (c) $\frac{1}{10x}(5)^{-5} + C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5} + C$

13. $\int_{0}^{\pi/8} \tan^2(2x)\, dx$ is equal to [CBSE 2020 (65/4/1)]

(a) $\frac{4-\pi}{8}$ (b) $\frac{4+\pi}{8}$ (c) $\frac{4-\pi}{4}$ (d) $\frac{4-\pi}{2}$

14. $\int_{a+c}^{b+c} f(x)\, dx$ is equal to [NCERT Exemplar]

(a) $\int_a^b f(x-c)\, dx$ (b) $\int_a^b f(x+c)\, dx$ (c) $\int_a^b f(x)\, dx$ (d) $\int_{a-c}^{b-c} f(x)\, dx$

15. If f and g are continuous functions in [0, 1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x).g(x)\, dx$ is equal to [NCERT Exemplar]

(a) $\frac{a}{2}$ (b) $\frac{a}{2}\int_0^a f(x)\, dx$ (c) $\int_0^a f(x)\, dx$ (d) $a\int_0^a f(x)\, dx$

16. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1} dx$ is equal to [NCERT Exemplar]

(a) log 2 (b) 2 log 2 (c) $\frac{1}{2}\log 2$ (d) 4 log 2

17. $\int_0^1 \frac{e^t}{1+t} dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2} dt$ is equal to [NCERT Exemplar]

(a) $a-1+\frac{e}{2}$ (b) $a+1-\frac{e}{2}$ (c) $a-1-\frac{e}{2}$ (d) $a+1+\frac{e}{2}$

18. $\int_{-2}^{2} |x \cos \pi x| \, dx$ is equal to [NCERT Exemplar]

(a) $\frac{8}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{2}{\pi}$ (d) $\frac{1}{\pi}$

19. If $f'(x) = x + \frac{1}{x}$, then $f(x)$ is [CBSE Sample Paper 2023]

(a) $x^2 + \log|x| + C$ (b) $\frac{x^2}{2} + \log|x| + C$ (c) $\frac{x}{2} + \log|x| + C$ (d) $\frac{x}{2} - \log|x| + C$

20. The value of $\int_2^3 \frac{x}{x^2+1} dx$ is [CBSE Sample Paper 2023]

(a) $\log 4$ (b) $\log \frac{3}{2}$ (c) $\frac{1}{2}\log 2$ (d) $\log \frac{9}{4}$

21. $\int_0^2 \sqrt{4-x^2}\, dx$ equals [CBSE 2023 (65/3/2)]

(a) $2 \log 2$ (b) $-2 \log 2$ (c) $\frac{\pi}{2}$ (d) π

22. If $\frac{d}{dx}(f(x)) = \log x$, then $f(x)$ equals : [CBSE 2023 (65/1/1)]

(a) $-\frac{1}{x} + C$ (b) $x(\log x - 1) + C$ (c) $x(\log x + x) + C$ (d) $\frac{1}{x} + C$

23. $\int_0^{\frac{\pi}{6}} \sec^2\left(x - \frac{\pi}{6}\right) dx$ is equal to : [CBSE 2023 (65/1/1)]

(a) $\frac{1}{\sqrt{3}}$ (b) $-\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

24. If $\frac{d}{dx}[f(x)] = ax + b$ and $f(0) = 0$, then $f(x)$ is equal to [CBSE 2023 (65/2/1)]

(a) $a + b$ (b) $\frac{ax^2}{2} + bx$ (c) $\frac{ax^2}{2} + bx + c$ (d) b

25. Anti-derivative of $\frac{\tan x - 1}{\tan x + 1}$ with respect to x is [CBSE 2023 (65/2/1)]

(a) $\sec^2\left(\frac{\pi}{4} - x\right) + C$ (b) $-\sec^2\left(\frac{\pi}{4} - x\right) + C$

(c) $\log\left|\sec\left(\frac{\pi}{4} - x\right)\right| + C$ (d) $-\log\left|\sec\left(\frac{\pi}{4} - x\right)\right| + C$

26. $\int \frac{\sec x}{\sec x - \tan x} dx$ equals [CBSE 2023 (65/5/1)]

(a) $\sec x - \tan x + C$ (b) $\sec x + \tan x + C$ (c) $\tan x - \sec x + C$ (d) $-(\sec x + \tan x) + C$

Answers

1. (a)	2. (a)	3. (a)	4. (c)	5. (a)	6. (d)	7. (c)
8. (a)	9. (a)	10. (c)	11. (d)	12. (d)	13. (a)	14. (b)
15. (b)	16. (b)	17. (b)	18. (a)	19. (b)	20. (c)	21. (d)
22. (b)	23. (a)	24. (b)	25. (c)	26. (b)		

Solutions of Selected Multiple Choice Questions

1. We have, $I = \int x^2 e^{x^3}\, dx$

$$x^3 = t \quad \Rightarrow 3x^2 dx = dt \quad \Rightarrow x^2 dx = \frac{dt}{3}$$

$$\therefore \quad I = \int e^t \frac{dt}{3} = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + C$$
$$= \frac{1}{3}e^{x^3} + C$$

$\therefore$ Option (*a*) is correct.

2. Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Let $xe^x = t \quad \Rightarrow \quad (xe^x + e^x)dx = dt$

$\Rightarrow \quad (x+1)e^x dx = dt$

$$\therefore \quad I = \int \frac{dt}{\cos^2 t}$$

$$\Rightarrow \quad I = \int \sec^2 t\, dt = \tan t + C = \tan(xe^x) + C$$

$\therefore$ Option (*a*) is correct.

4. $I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)dx}{\sin^2 x \cos^2 x}$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx = \tan x - \cot x + C$$

$\therefore$ Option (*c*) is correct.

6. $I = \int_{-1}^{1} \frac{|x-2|}{x-2} dx, \quad x \neq 2$

$$\because \quad |x-2| = \begin{cases} x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases}$$

$$\text{i.e.,} \quad |x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

$$\therefore \quad I = \int_{-1}^{1} -\frac{(x-2)}{(x-2)} dx = -\int_{-1}^{1} dx = -[x]_{-1}^{1}$$

$$= -[1-(-1)] = -2$$

$\therefore$ Option (*d*) is correct.

7. $I = \int \frac{dx}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} dx$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b))}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)}\int [\cot(x-b) - \cot(x-a)]\, dx$$

$$= \frac{1}{\sin(b-a)}[\log|\sin(x-b)| - \log|\sin(x-a)|] + C$$

$$= \operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$$

$\therefore$ Option (*c*) is correct.

10. $\int 2^{x+2}dx = \int 2^x . 2^2 dx = 2^2 \int 2^x dx$

$$= 2^2 \times \frac{2^x}{\log 2} + C = \frac{2^{x+2}}{\log 2} + C$$

$\therefore$ Option (*c*) is correct.

11. We have $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x\, dx = [\tan x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$$= \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right) = 1 + 1 = 2$$

$\therefore$ Option (*d*) is correct.

13. $\int_0^{\pi/8} \tan^2(2x)\,dx = \int_0^{\pi/8} \left(\sec^2(2x) - 1\right)dx$

$$= \left[\frac{\tan(2x)}{2} - x\right]_0^{\pi/8}$$

$$= \frac{\tan\frac{\pi}{4}}{2} - \frac{\pi}{8} - 0 = \frac{1}{2} - \frac{\pi}{8} = \frac{4-\pi}{8}$$

$\therefore$ Option (*a*) is correct.

15. $I = \int_0^a f(x).g(x)dx = \int_0^a f(a-x)g(a-x)dx = \int_0^a f(x)(a - g(x))dx$

$$= a\int_0^a f(x)dx - \int_0^a f(x).g(x)dx \Rightarrow I = a\int_0^a f(x)dx - I \Rightarrow I = \frac{a}{2}\int_0^a f(x)dx$$

$\therefore$ Option (*b*) is correct.

16. $I = \int_{-1}^{1} \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1}dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2 + 2|x| + 1}dx + \int_{-1}^{1} \frac{|x| + 1}{x^2 + 2|x| + 1}dx = 0 + 2\int_0^1 \frac{|x| + 1}{(|x| + 1)^2}dx$$

[odd function + even function]

$$= 2\int_0^1 \frac{x+1}{(x+1)^2}dx = 2\int_0^1 \frac{1}{x+1}dx = 2\Big|\log|x+1|\Big|_0^1 = 2\log 2.$$

$\therefore$ Option (*b*) is correct.

18. Since $I = \int_{-2}^{2} |x\cos \pi x|dx = 2\int_0^2 |x\cos \pi x|dx$

$$= 2\left\{\int_0^{\frac{1}{2}} |x\cos \pi x|\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} |x\cos \pi x|\,dx + \int_{\frac{3}{2}}^{2} |x\cos \pi x|\,dx\right\} = \frac{8}{\pi}$$

$\therefore$ Option (*a*) is correct.

19. $f'(x) = x + \frac{1}{x}$, we get

Integrating both the sides w.r.t. x, we get

$$\int f'(x) = \int\left(x + \frac{1}{x}\right)dx \Rightarrow f(x) = \frac{x^2}{2} + \log|x| + C$$

$\therefore$ Option (b) is correct.

20. $$\int_2^3 \frac{x\,dx}{x^2+1} = \frac{1}{2}\int_2^3 \frac{2x}{x^2+1}dx = \frac{1}{2}\left[\log(x^2+1)\right]_2^3 = \frac{1}{2}(\log 10 - \log 5) = \frac{1}{2}\log\left(\frac{10}{5}\right)$$

$$= \frac{1}{2}\log 2$$

$\therefore$ Option (c) is correct.

21. We have,

$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^2 \sqrt{(2)^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{(2)^2 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 2\sin^{-1}1 = 2 \times \frac{\pi}{2} = \pi$$

$\therefore$ Option (d) is correct.

22. Given, $\frac{d(f(x))}{dx} = \log x$

On integrating both sides, we have

$\Rightarrow \quad f(x) = \int \log x dx = \int 1.\log x dx$

$\Rightarrow \quad f(x) = \log x.\int 1dx - \int\left(\frac{d}{dx}\log x.\int 1dx\right)dx$

$\Rightarrow \quad f(x) = \log x \times x - \int \frac{1}{x} \times x\,dx$

$\Rightarrow \quad f(x) = x\log x - \int dx = x\log x - x + C$

$\Rightarrow \quad f(x) = x(\log x - 1) + C$

$\therefore$ Option (b) is correct.

23. $$\int_0^{\frac{\pi}{6}} \sec^2\left(x - \frac{\pi}{6}\right)dx = \left[\tan\left(x - \frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}}$$

$$= \tan\left(\frac{\pi}{6} - \frac{\pi}{6}\right) - \tan\left(0 - \frac{\pi}{6}\right) = \tan 0 - \tan\left(-\frac{\pi}{6}\right) = 0 + \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$\therefore$ Option (a) is correct.

24. Given, $\frac{d}{dx}[f(x)] = ax + b$ and $f(0) = 0$

On integrating both sides, we have

$$f(x) = \int(ax+b)dx = \frac{ax^2}{2} + bx + C$$

$$\Rightarrow \quad f(x) = \frac{ax^2}{2} + bx + C \qquad \ldots(i)$$

Also, $f(0) = 0$, we have from (i)

$$f(0) = \frac{a \times 0}{2} + b \times 0 + C \Rightarrow 0 = C$$

Putting in (i), we have

$$f(x) = \frac{ax^2}{2} + bx$$

$\therefore$ Option (b) is correct.

25. We have,

$$\int \frac{\tan x - 1}{\tan x + 1}\,dx = -\int \frac{1-\tan x}{1+\tan x}dx$$

$$= -\int \tan\left(\frac{\pi}{4} - x\right)dx = \frac{-\log\left|\sec\left(\frac{\pi}{4} - x\right)\right|}{-1} + C$$

$$= \log\left|\sec\left(\frac{\pi}{4} - x\right)\right| + C$$

$\therefore$ Option (c) is correct.

26. $$\int \frac{\sec x}{\sec x - \tan x}dx = \int \frac{\sec x(\sec x + \tan x)}{\sec^2 x - \tan^2 x}dx$$

$$= \int (\sec^2 x + \sec x \tan x)dx$$

$$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx = \tan x + \sec x + C$$

$\therefore$ Option (b) is correct.

Assertion-Reason Questions

The following questions consist of two statements—Assertion(A) and Reason(R). Answer these questions selecting the appropriate option given below:

(a) Both A and R are true and R is the correct explanation for A.

(b) Both A and R are true but R is not the correct explanation for A.

(c) A is true but R is false.

(d) A is false but R is true.

1. **Assertion (A):** $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx = 3$

   Reason (R): $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ *[CBSE 2023 (65/5/1)]*

2. **Assertion(A):** $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$, where C is an arbitrary constant.

   Reason (R): Since $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\therefore \int \frac{1}{1+x^2}dx = \tan^{-1}x + C$

3. **Assertion(A):** $\int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + C$

   Reason (R): $\int \cos x\, dx = \sin x + C$

4. **Assertion(A):** If $f'(x) = x + \frac{1}{1+x^2}$ and $f(0) = 0$ then $f(x) = \frac{x^2}{2} + \tan^{-1}x$.

   Reason (R): $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

5. **Assertion(A):** $\int_0^{\frac{\pi}{2}} \cos x\, dx = 1$

   Reason (R): If $f(x)$ is continuous in $[a, b]$ and $\int f(x)\,dx = \phi(x)$, then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

6. **Assertion(A) :** $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}$

Reason (R) : $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

7. **Assertion(A) :** $\int_0^{\pi/2} \sqrt{1+\cos x}\, dx = 2$

Reason (R) : $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

8. **Assertion(A) :** $\int \frac{(\log x)^2}{x}dx = \frac{1}{3}(\log x)^3 + C$

Reason (R) : $\int \frac{1}{x}dx = \log x + C$

Answers

1. (*a*) **2.** (*a*) **3.** (*b*) **4.** (*b*) **5.** (*a*) **6.** (*b*) **7.** (*b*) **8.** (*b*)

Solutions of Assertion-Reason Questions

1. $I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$ ...(*i*)

Also $I = \int_2^8 \frac{\sqrt{10-(8+2-x)}}{\sqrt{8+2-x}+\sqrt{10-(8+2-x)}}dx$

$= \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$2I = \int_2^8 \frac{\sqrt{x}+\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx = \int_2^8 dx = [x]_2^8 = 8-2=6$

$\Rightarrow \quad I = 3$

So statement *A* is correct also statement *R* is true and gives correct explanation of the statement *A*.

$\therefore$ Option (*a*) is correct.

2. Clearly, both Assertion (*A*) and Reason (*R*) are true and Reason (*R*) is the correct explanation of Assertion (*A*).

$\therefore$ Option (*a*) is correct.

3. We have, $\int \frac{1}{2\sqrt{x}}dx = \int d(\sqrt{x}) = \sqrt{x} + C$

Clearly, both Assertion (*A*) and Reason (*R*) are true but Reason (*R*) is not the correct explanation of Assertion (*A*).

$\therefore$ Option (*b*) is correct.

4. We have, $f'(x) = x + \frac{1}{1+x^2}$

$\therefore \quad f(x) = \int f'(x)\,dx = \int\left\{x+\left(\frac{1}{1+x^2}\right)\right\}dx = \frac{x^2}{2} + \tan^{-1}x + C$

$\Rightarrow \quad f(x) = \frac{x^2}{2} + \tan^{-1}x + C \qquad \because \; f(0) = 0 \quad \Rightarrow \quad f(0) = 0 + \tan^{-1}0 + C = 0 + C = C$

$\Rightarrow\ 0 = C \quad \therefore f(x) = \frac{x^2}{2} + \tan^{-1} x$

Clearly, both Assertion (*A*) and Reason (*R*) are true but Reason (*R*) is not correct explanation of Assertion (*A*).

$\therefore$ Option (*b*) is correct.

5. We have, $\int_0^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$

Clearly, both Assertion (*A*) and Reason (*R*) are true and Reason (*R*) is the correct explanation of Assertion (*A*).

$\therefore$ Option (*a*) is correct.

6. We have, $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1 = \sin^{-1} 1 - \sin^{-1} 0$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Clearly, both Assertion (*A*) and Reason (*R*) are correct and Reason (*R*) does not gives correct explanation of Assertion (*A*).

$\therefore$ Option (*b*) is correct.

7. $\int_0^{\pi/2} \sqrt{1+\cos x}\, dx = \int_0^{\pi/2} \sqrt{2\cos^2\left(\frac{x}{2}\right)}\, dx = \sqrt{2} \int_0^{\pi/2} \cos\frac{x}{2}\, dx$

$$= \sqrt{2} \times \left.\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right|_0^{\pi/2} = 2\sqrt{2}[\sin x]_0^{\pi/4}$$

$$= 2\sqrt{2}\left[\sin\frac{\pi}{4} - \sin 0\right] = 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right] = 2$$

So statement *A* is correct.

Also statement *R* is correct but *R* does not gives correct explanation of *A*.

$\therefore$ Option (*b*) is correct.

8. $I = \int \frac{(\log x)^2}{x} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore \quad I = \int t^2 dt = \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$

So statement *A* is correct.

Also statement *R* is correct but *R* does not give explanation of statement *A*.

$\therefore$ Option (*b*) is correct.

CONCEPTUAL QUESTIONS

1. **Evaluate:** $\int \frac{dx}{9+4x^2}$ **[CBSE 2020 (65/3/1)]**

Sol. $\int \frac{dx}{9+4x^2} = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C$ ½ + ½

[CBSE Marking Scheme 2020 (65/3/1)]

2. **Find:** $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx$ **[*CBSE 2020 (65/4/1)*]**

Sol. $I = \int \left(2(5^{-x}) - \frac{1}{5}(2^{-x})\right) dx = -\frac{2}{5^x \log 5} + \frac{1}{5(2^x)\log 2} + C$ **1**

[*CBSE Marking Scheme 2020 (65/4/1)*]

3. $\int \frac{(x^2+2)}{x+1} dx$ **[*NCERT Exemplar*]**

Sol. Let $I = \int \frac{x^2+2}{x+1} dx$

$$= \int \left(x - 1 + \frac{3}{x+1}\right) dx$$

$$= \int (x-1)\, dx + 3\int \frac{1}{x+1} dx$$

$$= \frac{x^2}{2} - x + 3\log|x+1| + C$$

4. **Evaluate:** $\int_1^3 |2x - 1|\, dx$ **[*CBSE 2020 (65/1/1)*]**

Sol.

$\int_1^3 |2x-1| dx$

$= \int_1^3 2x-1\, dx$

$= [x^2 - x]_1^3$

$= 9-3-1+1$

$= 6$

[*Topper's Answer* (65/1/1) 2020]

5. **Find** $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$ **[*CBSE 2020 (65/1/1)*]**

Sol.

$\int \frac{2x\, dx}{\sqrt[3]{x^2+1}}$

let $x^2+1 = z$

$2x\, dx = dz$

$= \int \frac{dz}{z^{1/3}}$

$= \int z^{-1/3} dz$

$= \frac{3z^{2/3}}{2} + C$

$= \frac{3(x^2+1)^{2/3}}{2} + C$

[*Topper's Answer* (65/1/1) 2020]

6. **Find:** $\int \frac{dx}{\sqrt{9-4x^2}}$ [CBSE 2020 (65/1/1)]

Sol. $\int \frac{dx}{\sqrt{9-4x^2}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{(3/2)^2 - x^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$

[*Topper's Answer* (65/1/1) 2020]

7. **If** $\int_0^a \frac{dx}{1+4x^2} = \frac{\pi}{8}$, **then find the value of** a. [CBSE 2020 (65/4/1)]

Sol. $\int_0^a \frac{dx}{1+4x^2} = \frac{\pi}{8}$

$\Rightarrow \quad \frac{1}{2}[\tan^{-1}2x]_0^a = \frac{\pi}{8}$ ½

$\Rightarrow \quad a = \frac{1}{2}$ ½

[*CBSE Marking Scheme 2020 (65/4/1)*]

8 **Find the value of** $\int_1^4 |x-5|\, dx$ [CBSE 2020 (65/5/1)]

Sol. $\int_1^4 |x-5| dx = \int_1^4 (5-x)\, dx = \frac{15}{2}$ ½ + ½

[*CBSE Marking Scheme 2020 (65/5/1)*]

9. **Find** $\int x^4 \log x\, dx$ [CBSE 2020 (65/1/1)]

Sol. $I = \int x^4 . \log x\, dx = \log x . \frac{x^5}{5} - \int \frac{1}{x} . \frac{x^5}{5}\, dx$ ½

$= \frac{x^5 \log x}{5} - \frac{x^5}{25} + C$ ½

[*CBSE Marking Scheme 2020 (65/1/1)*]

Very Short Answer Questions

1. **Find** $\int \frac{dx}{\sqrt{4x-x^2}}$. [CBSE 2021-22 (Term-2)]

Sol. We have, $\int \frac{dx}{\sqrt{4x-x^2}}$

$= \int \frac{dx}{\sqrt{-(x^2-4x)}} = \int \frac{dx}{\sqrt{-(x^2-4x+4-4)}} = \int \frac{dx}{\sqrt{-\{(x-2)^2-(2)^2\}}}$

$= \int \frac{dx}{\sqrt{(2)^2-(x-2)^2}} = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$, where $x-2=t \quad \Rightarrow dx = dt$

$= \sin^{-1}\left(\frac{t}{2}\right) + C = \sin^{-1}\left(\frac{x-2}{2}\right) + C$

2. Find $\int \frac{dx}{x^2 - 6x + 13}$ *[CBSE 2021-22 (Term-2)]*

Sol.

$$\int \frac{dx}{x^2 - 6x + 13}$$

$$= \int \frac{dx}{x^2 - 6x + 9 + 4}$$

$$= \int \frac{dx}{(x-3)^2 + 2^2} \qquad [x^2 - 2ax + a^2 = (x-a)^2]$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x-3}{2}\right) + C$$

[where C is constant]

$$\left[\int \frac{1}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \right]$$

Answer: $\frac{1}{2} \tan^{-1}\left(\frac{x-3}{2}\right) + C$

[Topper's Answer 2022]

3. Evaluate: $\int_0^1 x^2 e^x \, dx$ *[CBSE 2021-22 (Term-2)]*

Sol. $\int_0^1 x^2 e^x \, dx = [x^2 e^x]_0^1 - \int_0^1 2xe^x \, dx$ 1

$= [x^2 e^x - 2xe^x + 2e^x]_0^1$ ½

$= e - 2$ ½

[CBSE Marking Scheme 2021-22 (Term-2)]

4. Find: $\int \frac{\tan^3 x}{\cos^3 x} dx$ *[CBSE 2020 (65/2/1)]*

Sol. Given Integral is $I = \int \frac{\sin^3 x}{\cos^6 x} dx$ ½

Let $\cos x = t$

$\Rightarrow \sin x \, dx = -dt$ ½

$= \int \left[\frac{-1}{t^6} + \frac{1}{t^4} \right] dt$

$= \frac{t^{-5}}{5} + \frac{t^{-3}}{-3} + C$ ½

$= \frac{1}{5(\cos x)^5} - \frac{1}{3(\cos x)^3} + C$ or $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$ ½

[CBSE Marking Scheme 2020 (65/2/1)]

5. **Find** $\int \frac{x}{x^2+3x+2}dx$. **[CBSE 2020 (65/5/1)]**

Sol. $\int \frac{x}{x^2+3x+2}dx = \int \frac{x}{(x+1)(x+2)}dx = \int\left(\frac{-1}{x+1}+\frac{2}{x+2}\right)dx$ 1

$= -\log|x+1| + 2\log|x+2| + C$ 1

[CBSE Marking Scheme 2020 (65/5/1)]

6. **Evaluate** $\int_1^2 \left[\frac{1}{x}-\frac{1}{2x^2}\right]e^{2x}\,dx$. **[CBSE 2020 (65/5/1)]**

Sol. Put $2x = t$, $\therefore\ dx = \frac{1}{2}dt$ ½

$\therefore\quad I = \int_1^2 \left[\frac{1}{x}-\frac{1}{2x^2}\right]e^{2x}dx = \int_2^4 \left[\frac{1}{t}-\frac{1}{t^2}\right]e^t\,dt$ ½

$= \left[\frac{1}{t}e^t\right]_2^4 = \frac{e^4}{4}-\frac{e^2}{2}$ ½ +½

[CBSE Marking Scheme 2020 (65/5/1)]

7. **Find the value of** $\int_0^1 x(1-x)^n\,dx$. **[CBSE 2020 (65/5/1)]**

Sol. $\int_0^1 x(1-x)^x dx = \int_0^1 (1-x)(1-1+x)^n dx = \int_0^1 (x^n - x^{n+1})\,dx$ 1

$= \left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1}-\frac{1}{n+2}$ or $\frac{1}{(n+1)(n+2)}$ ½ +½

[CBSE Marking Scheme 2020 (65/5/1)]

8. **Find:** $\int \frac{x-1}{(x-2)(x-3)}dx$ **[CBSE 2019 (65/5/1)]**

Sol. $\because \frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$, where $A = \left.\frac{x-1}{x-3}\right|_{x=2}$ & $B = \left.\frac{x-1}{x-2}\right|_{x=3}$

$\therefore\quad A = \frac{1}{-1} = -1$ & $B = 2$

$\Rightarrow\quad \frac{x-1}{(x-2)(x-3)} = \frac{-1}{(x-2)}+\frac{2}{(x-3)}$

$\Rightarrow\quad \int \frac{x-1}{(x-2)(x-3)}dx = -\int\frac{dx}{x-2}+2\int\frac{dx}{x-3} = -\log(x-2)+2\log(x-3)+C$

$\Rightarrow\quad \int \frac{x-1}{(x-2)(x-3)}dx = -\log(x-2)+\log(x-3)^2+C = \log\frac{(x-3)^2}{(x-2)}+C$

9. **Find:** $\int_{-\pi/4}^{0} \frac{1+\tan x}{1-\tan x}dx$. **[CBSE 2019 (65/5/1)]**

Sol. $\int_{-\pi/4}^{0} \frac{1+\tan x}{1-\tan x}\,dx = \int_{-\pi/4}^{0} \tan\left(\frac{\pi}{4}+x\right)dx$

$= \log\sec\left(\frac{\pi}{4}+x\right)\Big|_{-\pi/4}^{0} = \log\sec\left(\frac{\pi}{4}\right)-\log\sec\left(\frac{\pi}{4}-\frac{\pi}{4}\right)$

$= \log(\sqrt{2})-\log(\sec(0)) = \log(\sqrt{2})-\log 1$

$= \log\sqrt{2} = \frac{1}{2}\log 2$

10. Find: $\int \frac{dx}{\sqrt{5-4x-2x^2}}$ **[CBSE 2019 (65/4/1)]**

Sol. $\int \frac{dx}{\sqrt{5-4x-2x^2}} = \int \frac{dx}{\sqrt{7-2-4x-2x^2}}$

$$= \int \frac{dx}{\sqrt{7-2(1+2x+x^2)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2}-(x+1)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}} = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}}(x+1)\right) + C$$

11. Find: $\int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x} dx$ **[CBSE Sample Paper 2018]**

Sol. $I = \int e^x \frac{\sqrt{1+\sin 2x}}{1+\cos 2x} dx$

$$= \int e^x \frac{\sqrt{\sin^2 x + \cos^2 x + 2\sin x . \cos x}}{1+\cos 2x} dx$$

$$= \int e^x \frac{\sqrt{(\sin x + \cos x)^2}}{1+\cos 2x} dx = \int e^x \frac{\sin x + \cos x}{2\cos^2 x} dx$$

$$= \frac{1}{2} \int e^x \left(\frac{\sin x}{\cos^2 x} + \frac{\cos x}{\cos^2 x}\right) dx$$

$$= \frac{1}{2} \int e^x (\sec x + \sec x . \tan x)\, dx$$

$$= \frac{1}{2} e^x . \sec x + C \qquad [\because \int e^x (f(x)+f'(x))dx = e^x f(x) + C]$$

12. Evaluate: $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$ **[CBSE Delhi 2011]**

Sol. Let $t = \tan^{-1} x \Rightarrow dt = \frac{1}{1+x^2} dx$

Also when, $x = 0, t = 0$ and when $x = 1,\ t = \frac{\pi}{4}$

$$\therefore \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \int_0^{\frac{\pi}{4}} t\, dt$$

$$= \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

13. Evaluate: $\int_0^1 \frac{dx}{\sqrt{2x+3}}$ **[CBSE (F) 2009]**

Sol. Let $I = \int_0^1 \frac{dx}{\sqrt{2x+3}} = \int_0^1 (2x+3)^{-1/2} dx$

$$= \left[\frac{(2x+3)^{-1/2+1}}{\left(-\frac{1}{2}+1\right)\times 2}\right]_0^1 = \left[\frac{(2x+3)^{1/2}}{\frac{1}{2}\times 2}\right]_0^1 = 5^{1/2} - 3^{1/2} = \sqrt{5} - \sqrt{3}$$

14. **The acceleration of an object is given by $a(t) = \cos(\pi t)$, and its velocity at time $t = 0$ is $\frac{1}{2\pi}$. Find the net distance travelled in the first 1.5 seconds.**

Sol. We know that, velocity is given by

$$v(t) = v(0) + \int_0^t a(u)\,du$$

where $v(t)$ is velocity function, $v(0)$ is initial velocity and $a(u)$ is acceleration.

$$\therefore \quad v(t) = \frac{1}{2\pi} + \int_0^t \cos(\pi u)\,du$$

$$= \frac{1}{2\pi} + \left[\frac{\sin \pi u}{\pi}\right]_0^t = \frac{1}{2\pi} + \frac{1}{\pi}\sin(\pi t)$$

$\therefore$ Net distance travelled is

$$= S(1.5) - S(0) = S\left(\frac{3}{2}\right) - S(0) = \int_0^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin \pi t\right) dt$$

$$= \frac{1}{\pi}\left[\frac{t}{2} - \frac{1}{\pi}\cos \pi t\right]_0^{3/2} = \frac{3}{4\pi} + \frac{1}{\pi^2} \text{ metres}$$

15. **Find: $\int e^x(\cos x - \sin x)\,\text{cosec}^2 x\,dx$** *[CBSE 2019 (65/5/1)] [NCERT]*

Sol. Let $I = \int e^x(\cos x - \sin x)\,\text{cosec}^2 x\,dx = \int e^x(\cot x \,.\, \text{cosec}\, x - \text{cosec}\, x)\,dx$

$$= \int e^x \text{cosec}\, x \cot x\,dx - \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{\text{cosec}\, x}\,dx$$

$$= \int e^x \text{cosec}\, x.\cot x - \text{cosec}\, x\, e^x + C + \int -\text{cosec}\, x \cot x\, e^x\,dx$$

[Using integration by parts for 2nd integral]

$$= \int e^x \text{cosec}\, x.\cot x\,dx - e^x \text{cosec}\, x + C - \int e^x\,\text{cosec}\, x.\cot x\,dx$$

$$= -e^x \text{cosec}\, x + C.$$

16. **Find: $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}dx$** *[CBSE (AI) 2013]*

Sol. Let $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx = \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}dx = 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}dx$

$$= 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{(\cos x - \cos\alpha)}dx = 2\int (\cos x + \cos\alpha)\,dx$$

$$= 2\int \cos x\,dx + 2\cos\alpha \int 1.dx \quad = 2\sin x + 2x\cos\alpha + C$$

Short Answer Questions

1. **Find: $\int e^x.\sin 2x\,dx$** *[CBSE 2021-22 (Term-2)]*

Sol. Let $I = \int \underset{I}{e^x}.\underset{II}{\sin 2x}dx$

Using ILATE we have

$$I = \sin 2x \int e^x dx - \int\left(\frac{d \sin 2x}{dx}.\int e^x dx\right)dx$$

$$\Rightarrow I = \sin 2x\, e^x - \int 2\cos 2x\, e^x dx = \sin 2x.e^x - 2\int \underset{\text{I}}{\cos 2x}\quad \underset{\text{II}}{e^x dx}$$

$$\Rightarrow I = \sin 2x.e^x - 2\left[\cos 2x \int e^x - \int\left(\frac{d\cos 2x}{dx}.\int e^x\,dx\right)dx\right] \text{ (Again using ILATE)}$$

$$\Rightarrow \quad I = e^x \sin 2x - 2\left[\cos 2x . e^x - \int -2 \sin 2x . e^x dx\right]$$

$$\Rightarrow \quad I = e^x \sin 2x - 2e^x \cos 2x - 4\int \sin 2x \, e^x dx$$

$$\Rightarrow \quad I = e^x (\sin 2x - 2\cos 2x) - 4I + C_1$$

$$\Rightarrow \quad 5I = e^x (\sin 2x - 2\cos 2x) + C_1$$

$$\therefore \quad I = \frac{e^x}{5}(\sin 2x - 2\cos 2x) + \frac{C_1}{5}$$

$$\therefore \quad I = \frac{e^x}{5}(\sin 2x - 2\cos 2x) + C \qquad \left(\text{Where } C = \frac{C_1}{5}\right).$$

2. Find: $\int \frac{2x}{(x^2+1)(x^2+2)} dx$ **[*CBSE 2021-22 (Term-2)*]**

Sol. We have, $I = \int \frac{2x}{(x^2+1)(x^2+2)} dx$

Let $x^2 = t \Rightarrow 2x \, dx = 2dt$

$$\therefore \quad I = \int \frac{dt}{(t+1)(t+2)} = \int \frac{1}{(t+1)(t+2)} dt$$

$$I = \int \frac{(t+2)-(t+1)}{(t+1)(t+2)} dt$$

$$= \int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = \int \frac{1}{t+1} dt - \int \frac{1}{t+2} dt$$

$$\Rightarrow \quad I = \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C = \log\left|\frac{x^2+1}{x^2+2}\right| + C$$

$$\therefore \quad \int \frac{2x}{(x^2+1)(x^2+1)} dx = \log\left|\frac{x^2+1}{x^2+2}\right| + C.$$

3. Find: $\int \frac{x^4}{(x-1)(x^2+1)} dx.$ **[*CBSE 2023 (65/5/1)*]**

Sol. $I = \int \frac{x^4}{(x-1)(x^2+1)} dx$

$$\because \quad \frac{x^4}{(x-1)(x^2+1)} = \frac{x^4}{x^3 - x^2 + x - 1}$$

$$= x + 1 + \frac{1}{x^3 - x^2 + x - 1} = x + 1 + \frac{1}{(x-1)(x^2+1)}$$

$$\begin{array}{r|l} x^3 - x^2 + x - 1 \,) \; x^4 & (x+1 \\ x^4 - x^3 + x^2 - x & \\ - \quad + \quad - \quad + & \\ \hline x^3 - x^2 + x & \\ x^3 - x^2 + x - 1 & \\ - \quad + \quad - \quad + & \\ \hline 1 & \end{array}$$

$$\because I = \int \frac{x^4}{(x-1)(x^2+1)} dx$$

$$= \int \left[x + 1 + \frac{1}{(x-1)(x^2+1)}\right] dx = \int x \, dx + \int dx + \int \frac{1}{(x-1)(x^2+1)} dx$$

$$= \frac{x^2}{2} + x + I_1$$

Where $I_1 = \int \frac{1}{(x-1)(x^2+1)} dx$

$$\because \quad \frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)} = \frac{Ax^2 + A + Bx^2 + Cx - Bx - C}{(x-1)(x^2+1)}$$

$$= \frac{(A+B)x^2+(C-B)x+(A-C)}{(x-1)(x^2+1)}$$

$\Rightarrow \quad 1 = (A+B)x^2 + (C-B)x + (A-C)$

Equating both sides like powers of x, we get

$A + B = 0 \quad \Rightarrow A = -B$

$C - B = 0 \quad \Rightarrow C = B$

$A - C = 1 \quad \Rightarrow -B - B = 1 \quad \Rightarrow \quad -2B = 1 \quad \Rightarrow \quad B = \frac{-1}{2}$

$\therefore A = \frac{1}{2}, C = -\frac{1}{2}$

$$\therefore \frac{1}{(x-1)(x^2+1)} = \frac{1}{2} \times \frac{1}{x-1} + \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2+1}$$

$$= \frac{1}{2} \times \frac{1}{x-1} - \frac{1}{2} \times \frac{(x-1)}{x^2+1}$$

$$\therefore I_1 = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x-1}{x^2+1}dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

$$\therefore I = \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

4. Find: $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)}dx$ *[CBSE 2023 (65/1/1)]*

Sol. We have, $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)}dx$

Let $\sqrt{x} = t \quad \Rightarrow \frac{1}{2\sqrt{x}}dx = dt \quad \Rightarrow \frac{1}{\sqrt{x}}dx = 2dt$

$$= 2\int \frac{1}{(t+1)(t+2)}dt$$

$$= 2\int \frac{(t+2)-(t+1)}{(t+1)(t+2)}dt = 2\left(\int \frac{1}{t+1}dt - \int \frac{1}{t+2}dt\right)$$

$$= 2[\log|t+1| - \log|t+2|] + C$$

$$= 2\log\left|\frac{t+1}{t+2}\right| + C = 2\log\left|\frac{\sqrt{x}+1}{\sqrt{x}+2}\right| + C$$

5. Find: $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$ *[CBSE 2023 (65/1/1)]*

Sol. We have, $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin[(x-a)-(x-b)]}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\int \frac{\sin[(x-a).\cos(x-b) - \cos(x-a)\sin(x-b)]}{\cos(x-a).\cos(x-b)}dx$$

$$= \frac{1}{\sin(b-a)}\left[\int \tan(x-a)dx - \int \tan(x-b)dx\right]$$

$$= \frac{1}{\sin(b-a)}\left[\log\left|\sec(x-a)\right| - \log\left|\sec(x-a)\right|\right] + C$$

$$= \frac{1}{\sin(b-a)}\log\frac{\sec(x-a)}{\sec(x-b)} + C = \frac{1}{\sin(a-b)}\log\frac{\sec(x-b)}{\sec(x-a)} + C$$

$$= \frac{1}{\sin(a-b)}\log\frac{\cos(x-a)}{\cos(x-b)} + C$$

6. Find $\int \frac{dx}{\sqrt{\sin^3 x \cos(x-\alpha)}}$. *[CBSE 2023 (65/2/1)]*

Sol. We have, $\int \frac{dx}{\sqrt{\sin^3 x \cos(x-\alpha)}}$

$$= \int \frac{dx}{\sqrt{\sin^3 x\{\cos x.\cos\alpha + \sin x.\sin\alpha\}}}$$

$$= \int \frac{dx}{\sqrt{\sin^4 x\{\sin\alpha + \cos\alpha.\cot x\}}} = \int \frac{\text{cosec}^2 x}{\sqrt{\sin\alpha + \cos\alpha\cot x}}dx$$

Let $\sqrt{\sin\alpha + \cos\alpha\cot x} = t$ $\quad \Rightarrow \frac{1\times(-\cos\alpha).\text{cosec}^2 x}{2\sqrt{\sin\alpha + \cos\alpha\cot x}}dx = dt$

$$\Rightarrow \frac{\text{cosec}^2 x}{\sqrt{\sin\alpha + \cos\alpha\cot x}}dx = \frac{-2}{\cos\alpha}dt$$

$$\Rightarrow \frac{-2}{\cos\alpha}\int dt = \frac{-2}{\cos\alpha}\times t + C$$

$$\Rightarrow \frac{-2}{\cos\alpha}\sqrt{\sin\alpha + \cos\alpha\cot x} + C$$

7. Evaluate: $\int_0^{\pi/4}\log(1+\tan x)\,dx$ *[CBSE 2023 (65/2/1), (AI) 2011] [NCERT]*

Sol. Let $I = \int_0^{\pi/4}\log(1+\tan x)dx$ ...(*i*)

$\therefore I = \int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$ (By using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$)

$$= \int_0^{\pi/4}\log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}\right]dx$$

$$= \int_0^{\pi/4}\log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx = \int_0^{\pi/4}\log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]dx$$

$I = \int_0^{\pi/4}\log\frac{2}{1+\tan x}dx = \int_0^{\pi/4}[\log 2 - \log(1+\tan x)]dx$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi/4}\log 2\,dx = \log 2\int_0^{\pi/4}dx = \log 2[x]_0^{\pi/4}$$

$$2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$$

8. Find $\int e^{\cot^{-1}x}\left(\frac{1-x+x^2}{1+x^2}\right)dx$. *[CBSE 2023 (65/2/1)]*

Sol. We have, $I = \int e^{\cot^{-1}x}\left(\frac{1-x+x^2}{1+x^2}\right)dx$

Let $\cot^{-1}x = \theta \qquad \Rightarrow x = \cot\theta$

$$\Rightarrow \frac{-1}{1+x^2}dx = d\theta \quad \Rightarrow \frac{1}{1+x^2}dx = -\,d\theta$$

$$\therefore \quad I = -\int e^{\theta}(1-\cot\theta+\cot^2\theta)d\theta = -\int e^{\theta}(\text{cosec}^2\theta-\cot\theta)d\theta$$

$$= \int e^{\theta}(\cot\theta-\text{cosec}^2\theta)d\theta = \int e^{\theta}\{\underset{f(\theta)}{\underset{\downarrow}{\cot\theta}}+\underset{f'(\theta)}{\underset{\downarrow}{(-\text{cosec}^2\theta)}}\}d\theta$$

$$= e^{\theta}.\cot\theta + C \qquad (\because \int e^x\{f(x)+f'(x)\}\,dx = e^x.f(x)+C)$$

$$= e^{\cot^{-1}x}.x + C = x\,e^{\cot^{-1}x}+C$$

9. Find : $\int \frac{\cos x}{\sin 3x}dx$ *[CBSE 2023 (65/3/2)]*

Sol. We have, $I = \int \frac{\cos x}{\sin 3x}dx$

$$\Rightarrow \qquad I = \int \frac{\cos x\,dx}{3\sin x - 4\sin^3 x}$$

Let $\sin x = t \quad \Rightarrow \quad \cos x\,dx = dt$

$$\therefore \qquad I = \int \frac{dt}{3t-4t^3} = \int \frac{dt}{t(3-4t^2)} = \frac{-1}{4}\int \frac{dt}{t\left(t^2-\frac{3}{4}\right)}$$

$$\Rightarrow \qquad I = -\frac{1}{4}\int \frac{dt}{t\left(t-\frac{\sqrt{3}}{2}\right)\left(t+\frac{\sqrt{3}}{2}\right)}$$

Using partial fraction, we have

$$\frac{1}{t\left(t-\frac{\sqrt{3}}{2}\right)\left(t+\frac{\sqrt{3}}{2}\right)} = \frac{A}{t}+\frac{B}{t-\frac{\sqrt{3}}{2}}+\frac{C}{t+\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \qquad 1 = A\left(t^2-\frac{3}{4}\right)+Bt\left(t+\frac{\sqrt{3}}{2}\right)+Ct\left(t-\frac{\sqrt{3}}{2}\right)$$

For the value of A, put $t = 0$

$$\therefore \qquad 1 = A\times\left(-\frac{3}{4}\right) \Rightarrow A = -\frac{4}{3}$$

For the value of B, put $t = \frac{\sqrt{3}}{2}$

$$\therefore \qquad 1 = B\times\frac{\sqrt{3}}{2}\times\frac{2\sqrt{3}}{2} = B\times\frac{3}{2} \Rightarrow B = \frac{2}{3}$$

For the value of C, put $t = -\frac{\sqrt{3}}{2}$

$$\therefore \qquad 1 = C\times\left(-\frac{\sqrt{3}}{2}\right)\times\left(\frac{-2\sqrt{3}}{2}\right) = C\times\frac{3}{2} \Rightarrow C = \frac{2}{3}$$

$$\therefore I = -\frac{1}{4}\left[\frac{-4}{3}\int\frac{1}{t}dt+\frac{2}{3}\int\frac{1}{t-\frac{\sqrt{3}}{2}}dt+\frac{2}{3}\int\frac{1}{t+\frac{\sqrt{3}}{2}}dt\right]$$

$$= -\frac{1}{4}\left[\frac{-4}{3}\log|t|+\frac{2}{3}\log\left|t-\frac{\sqrt{3}}{2}\right|+\frac{2}{3}\log\left|t+\frac{\sqrt{3}}{2}\right|\right]+C$$

$$= -\frac{1}{4}\times\frac{2}{3}\left[-2\log|t|+\log\left|t-\frac{\sqrt{3}}{2}\right|+\log\left|t+\frac{\sqrt{3}}{2}\right|\right]+C$$

$$= -\frac{1}{6}\left[\log\left(t^2 - \frac{3}{4}\right) - \log t^2\right] + C = -\frac{1}{6}\log\left(\frac{4t^2-3}{4t^2}\right) + C$$

$$= -\frac{1}{6}\log\left|1 - \frac{3}{4t^2}\right| + C = -\frac{1}{6}\log\left|1 - \frac{3}{4\sin^2 x}\right| + C$$

10. Find : $\int x^2 \log(x^2+1)\,dx$ *[CBSE 2023 (65/3/2)]*

Sol. We have, $I = \int x^2 \log(x^2+1)dx$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$\therefore I = \int \tan^2\theta \log(1+\tan^2\theta).\sec^2\theta\, d\theta = \int 2\log\sec\theta.\tan^2\theta\sec^2\theta\, d\theta$

$$= 2\int \log\sec\theta.(\tan^2\theta\sec^2\theta)d\theta$$

$$= 2\left[\log\sec\theta\int\tan^2\theta.\sec^2\theta\,d\theta - \int\frac{d\log\sec\theta}{d\theta}\times\int\tan^2\theta\sec^2\theta\,d\theta\right]$$

$$= 2\left[\log\sec\theta\times\frac{\tan^3\theta}{3} - \int\frac{1}{\sec\theta}\times\sec\theta\tan\theta\times\frac{\tan^3\theta}{3}d\theta\right]$$

$$= 2\left[\frac{\tan^3\theta}{3}\log\sec\theta - \frac{1}{3}\int\tan^4\theta\,d\theta\right] = 2\left[\frac{\tan^3\theta}{3}\log\sec\theta - \frac{1}{3}\int\tan^2\theta.\tan^2\theta\,d\theta\right]$$

$$= \frac{2}{3}\tan^3\theta\log\sec\theta - \frac{2}{3}\int(\sec^2 - 1)\tan^2\theta\,d\theta$$

$$= \frac{2}{3}\tan^3\theta\log\sec\theta - \frac{2}{3}\int\sec^2\theta\tan^2\theta\,d\theta + \frac{2}{3}\int\tan^2\theta\,d\theta$$

$$= \frac{2}{3}\tan^3\theta\log\sec\theta - \frac{2}{3}\times\frac{\tan^3\theta}{3} + \frac{2}{3}\int(\sec^2\theta - 1)d\theta$$

$$= \frac{2}{3}\tan^3\theta\log\sec\theta - \frac{2}{9}\tan^3\theta + \frac{2}{3}\tan\theta - \frac{2}{3}\theta + C$$

$$= \frac{2}{3}x^3\log\sqrt{1+x^2} - \frac{2}{9}x^3 + \frac{2}{3}x - \frac{2}{3}\tan^{-1}x + C$$

11. Evaluate: $\int\frac{x^2}{1-x^4}dx$ *[NCERT Exemplar]*

Sol. Let $I = \int\frac{x^2}{1-x^4}dx = \int\frac{\frac{1}{2}+\frac{x^2}{2}-\frac{1}{2}+\frac{x^2}{2}}{(1-x^2)(1+x^2)}dx$ $[\because a^2-b^2=(a+b)(a-b)]$

$$= \int\frac{\frac{1}{2}(1+x^2)-\frac{1}{2}(1-x^2)}{(1-x^2)(1+x^2)}dx = \int\frac{\frac{1}{2}(1+x^2)}{(1-x)^2(1+x^2)}dx - \frac{1}{2}\int\frac{(1-x)^2}{(1-x)^2(1+x)^2}dx$$

$$= \frac{1}{2}\int\frac{1}{1-x^2}dx - \frac{1}{2}\int\frac{1}{1+x^2}dx = \frac{1}{2}.\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C_1 - \frac{1}{2}\tan^{-1}x + C_2$$

$$= \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C \qquad [\because C = C_1 + C_2]$$

12. Evaluate: $\int\sin x\sin 2x\sin 3x\,dx$ *[CBSE (F) 2010, Delhi 2012, 2019 (65/5/3)]*

Sol. Let $I = \int\sin x\sin 2x\sin 3x\,dx$

$$= \frac{1}{2}\int 2\sin x.\sin 2x.\sin 3x dx = \frac{1}{2}\int\sin x.(2\sin 2x.\sin 3x)dx$$

$$= \frac{1}{2}\int\sin x.(\cos x - \cos 5x)dx \qquad [\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2\times2}\int 2\sin x.\cos x dx - \frac{1}{2\times2}\int 2\sin x.\cos 5x dx \qquad [\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{4}\int \sin 2x\,dx - \frac{1}{4}\int (\sin 6x - \sin 4x)\,dx$$

$$= -\frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$

13. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx$ *[CBSE Delhi 2014]*

Sol. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x.\cos^2 x}dx$

$\Rightarrow \quad I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x.\cos^2 x}dx$

$\Rightarrow \quad I = \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x)}{\sin^2 x.\cos^2 x}dx$

$\Rightarrow \quad I = \int \frac{\sin^4 x - \sin^2 x.\cos^2 x + \cos^4 x}{\sin^2 x.\cos^2 x}dx = \int \tan^2 x\,dx - \int dx + \int \cot^2 x\,dx$

$\Rightarrow \quad I = \int (\sec^2 x - 1)\,dx - x + \int (\text{cosec}^2 x - 1)\,dx$

$\Rightarrow \quad I = \int \sec^2 x\,dx + \int \text{cosec}^2 x\,dx - x - x - x + C = \tan x - \cot x - 3x + C$

14. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}dx$ *[CBSE Delhi 2013; (F) 2015]*

Sol. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$

Let $x + a = t \quad \Rightarrow \quad x = t - a \quad \Rightarrow \quad dx = dt$

$\therefore \quad I = \int \frac{\sin(t-2a)}{\sin t}dt = \int \frac{\sin t.\cos 2a - \cos t.\sin 2a}{\sin t}dt$

$= \cos 2a \int dt - \int \sin 2a.\cot t\,dt = \cos 2a.t - \sin 2a.\log|\sin t| + C$

$= \cos 2a\,.\,(x+a) - \sin 2a\,.\,\log|\sin(x+a)| + C$

$= x\cos 2a + a\cos 2a - (\sin 2a)\log|\sin(x+a)| + C$

15. Evaluate: $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}dx$ *[CBSE Delhi 2009, 2019 (65/5/1)]*

Sol. Let $I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$, we get

$\therefore \quad I = \int \frac{dt}{\sqrt{5 - 4t - t^2}} = \int \frac{dt}{\sqrt{-(t^2 + 4t - 5)}} = \int \frac{dt}{\sqrt{-(t^2 + 2.t.2 + 2^2 - 9)}}$

$= \int \frac{dt}{\sqrt{3^2 - (t+2)^2}} = \sin^{-1}\frac{t+2}{3} + C = \sin^{-1}\left(\frac{e^x + 2}{3}\right) + C$

16. Evaluate: $\int e^x\left(\frac{\sin 4x - 4}{1 - \cos 4x}\right)dx$ *[CBSE Delhi 2010]*

Sol. Let $I = \int e^x\left(\frac{\sin 4x - 4}{1 - \cos 4x}\right)dx$

$= \int e^x\left(\frac{2\sin 2x\,.\cos 2x - 4}{2\sin^2 2x}\right)dx \quad [\because \sin 2x = 2\sin x\,.\cos x \text{ and } \cos 2x = 1 - 2\sin^2 x]$

$= \int e^x(\cot 2x - 2\text{cosec}^2 2x)\,dx$

Let $f(x) = \cot 2x \;\therefore\; f'(x) = -2\text{cosec}^2 2x$

$\therefore \quad I = \int e^x (f(x) + f'(x))\, dx$

$\Rightarrow \quad I = e^x . f(x) + C = e^x . \cot 2x + C \qquad [\because \int e^x (f(x) + f'(x))\, dx = e^x f(x) + C]$

17. Evaluate: $\int \frac{(x^2 - 3x)}{(x-1)(x-2)} dx$ **[*CBSE (F) 2010*]**

Sol. Let $I = \int \frac{(x^2 - 3x)}{(x-1)(x-2)} dx = \int \frac{(x^2 - 3x)}{x^2 - 3x + 2} dx$

$$= \int \frac{x^2 - 3x + 2 - 2}{x^2 - 3x + 2} dx = \int dx - \int \frac{2dx}{x^2 - 3x + 2}$$

$$= x - 2\int \frac{dx}{x^2 - 2.x.\frac{3}{2} + \frac{9}{4} - \frac{9}{4} + 2} = x - 2\int \frac{dx}{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= x - 2\log\left|\frac{x - \frac{3}{2} - \frac{1}{2}}{x - \frac{3}{2} + \frac{1}{2}}\right| + C \qquad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= x - 2\log\left|\frac{x-2}{x-1}\right| + C$$

18. Find: $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$ **[*NCERT Exemplar*]**

Sol. Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$

Put $x = a\tan^2\theta \quad \Rightarrow \quad dx = 2a \tan\theta \sec^2\theta\, d\theta$

$\therefore \quad I = \int \sin^{-1}\left(\sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}}\right)(2a\tan\theta.\sec^2\theta)\, d\theta$

$$= 2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta.\sec^2\theta d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta.\sec^2\theta d\theta = 2a\int \underset{\text{I}}{\theta}.\tan\theta \underset{\text{II}}{\sec^2\theta} d\theta$$

$$= 2a\left[\theta.\int \tan\theta.\sec^2\theta d\theta - \int\left(\frac{d}{d\theta}\theta.\int \tan\theta.\sec^2\theta\, d\theta\right)d\theta\right]$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2} d\theta\right]$$

$$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\, d\theta = a\theta.\tan^2\theta - a\tan\theta + a\theta + C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

19. Find: $\int \frac{dx}{\sin x + \sin 2x}$ **[*CBSE Delhi 2012*]**

Sol. Here, $I = \int \frac{1}{\sin x + \sin 2x} dx$

$\Rightarrow I = \int \frac{1}{\sin x + 2\sin x\cos x} dx \quad \Rightarrow \quad I = \int \frac{1}{\sin x(1 + 2\cos x)} dx$

$\Rightarrow I = \int \frac{\sin x}{\sin^2 x(1 + 2\cos x)} dx \quad \Rightarrow \quad I = \int \frac{\sin x}{(1 - \cos^2 x)(1 + 2\cos x)} dx$

Let $\cos x = z \quad \Rightarrow -\sin x\, dx = dz$

$\Rightarrow I = \int \frac{-dz}{(1 - z^2)(1 + 2z)} \quad \Rightarrow \quad I = -\int \frac{dz}{(1+z)(1-z)(1+2z)}$

Here, integrand is proper rational function. Therefore, by the form of partial fraction, we can write

$$\frac{1}{(1+z)(1-z)(1+2z)} = \frac{A}{1+z} + \frac{B}{1-z} + \frac{C}{1+2z} \quad ...(i)$$

$$\Rightarrow \quad \frac{1}{(1+z)(1-z)(1+2z)} = \frac{A(1-z)(1+2z) + B(1+z)(1+2z) + C(1+z)(1-z)}{(1+z)(1-z)(1+2z)}$$

$$\Rightarrow \quad 1 = A(1-z)(1+2z) + B(1+z)(1+2z) + C(1+z)(1-z) \quad ....(ii)$$

Putting the value of $z = -1$ in (*ii*), we get

$$\Rightarrow \quad 1 = -2A + 0 + 0 \quad \Rightarrow \quad A = -1/2$$

Again, putting the value of $z = 1$ in (*ii*), we get

$$\Rightarrow \quad 1 = 0 + B.2.(1+2) + 0 \quad \Rightarrow \quad 1 = 6B \quad \Rightarrow \quad B = \frac{1}{6}$$

Similarly, putting the value of $z = -\frac{1}{2}$ in (*ii*), we get

$$\Rightarrow \quad 1 = 0 + 0 + C\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \quad \Rightarrow \quad 1 = \frac{3}{4}C \quad \Rightarrow \quad C = \frac{4}{3}$$

Putting the value of A, B, C in (*i*), we get

$$\frac{1}{(1+z)(1-z)(1-2z)} = \frac{-1}{2(1+z)} + \frac{1}{6(1-z)} + \frac{4}{3(1+2z)}$$

$$\therefore \quad I = -\int\left[-\frac{1}{2(1+z)} + \frac{1}{6(1-z)} + \frac{4}{3(1+2z)}\right]dz = \int\left[\frac{1}{2(1+z)} - \frac{1}{6(1-z)} - \frac{4}{3(1+2z)}\right]dz$$

$$\Rightarrow \quad I = \frac{1}{2}\log|1+z| + \frac{1}{6}\log|1-z| - \frac{4}{3\times 2}\log|1+2z| + C$$

Putting the value of z, we get

$$\Rightarrow \quad I = \frac{1}{2}\log|1+\cos x| + \frac{1}{6}\log|1-\cos x| - \frac{2}{3}\log|1+2\cos x| + C$$

20. Find: $\int \frac{x^2}{x^4 - x^2 - 12}dx$ **[*NCERT Exemplar*]**

Sol. Let $I = \int \frac{x^2}{x^4 - x^2 - 12}dx = \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12}dx$

$$= \int \frac{x^2 dx}{x^2(x^2-4) + 3(x^2-4)} = \int \frac{x^2}{(x^2-4)(x^2+3)}dx \quad [\text{let } x^2 = t]$$

$$\Rightarrow \quad \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$$

$$\Rightarrow \quad t = A(t+3) + B(t-4) = (A+B)t + (3A - 4B)$$

On comparing the coefficient of t on both sides, we get

$$A + B = 1$$

and $3A - 4B = 0 \Rightarrow 3(1-B) - 4B = 0$

$$\Rightarrow 3 - 3B - 4B = 0 \Rightarrow 7B = 3 \quad \Rightarrow \quad B = \frac{3}{7}$$

If $B = \frac{3}{7}$, then $A + \frac{3}{7} = 1 \quad \Rightarrow \quad A = 1 - \frac{3}{7} = \frac{4}{7}$

Now, $\int \frac{x^2 dx}{(x^2-4)(x^2+3)} = \int\left[\frac{4}{7(x^2-4)} + \frac{3}{7(x^2+3)}\right]dx$

$$= \frac{4}{7}\int \frac{1}{x^2-(2)^2}dx + \frac{3}{7}\int \frac{1}{x^2+(\sqrt{3})^2}dx$$

$$= \frac{4}{7}\cdot\frac{1}{2.2}\log\left|\frac{x-2}{x+2}\right| + \frac{3}{7}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

21. Evaluate: $\int \frac{x^2}{(x^2+4)(x^2+9)}dx$ *[CBSE Delhi 2013; (F) 2015]*

Sol. Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)}dx$

Put $x^2 = t$, we get

$$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)}$$

Now, $\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9} = \frac{A(t+9)+B(t+4)}{(t+4)(t+9)}$

$\Rightarrow \quad t = (A+B)t + (9A+4B)$

Equating the coefficients, we get

$A + B = 1$ and $9A + 4B = 0$

Solving above two equations, we get

$A = -\frac{4}{5},\ B = \frac{9}{5}$

$$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$\Rightarrow \int \frac{x^2\,dx}{(x^2+4)\,(x^2+9)} = -\frac{4}{5}\int\frac{dx}{x^2+2^2} + \frac{9}{5}\int\frac{dx}{x^2+3^2} = -\frac{4}{5}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5}\times\frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

$$= -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C$$

22. Find: $\int \frac{(3\sin\theta - 2)\cos\theta}{5-\cos^2\theta - 4\sin\theta}d\theta$ *[CBSE Delhi 2016]*

Sol. We have

$$I = \int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$$

Let $\sin\theta = z \Rightarrow \cos\theta\, d\theta = dz$

$$\therefore \quad I = \int \frac{(3z-2)dz}{5-(1-z^2)-4z}$$

$$= \int \frac{(3z-2)dz}{5-1+z^2-4z} = \int \frac{(3z-2)}{4-4z+z^2}dz$$

$$= \int \frac{3z-2}{(z-2)^2}dz = \int \frac{3z}{(z-2)^2}dz - 2\int\frac{dz}{(z-2)^2}$$

Let $z - 2 = t \Rightarrow dz = dt$

$$= \int \frac{3(t+2)dt}{t^2} - 2\int\frac{dt}{t^2} = 3\int\frac{t.dt}{t^2} + 6\int\frac{dt}{t^2} - 2\int\frac{dt}{t^2} = 3\int\frac{dt}{t} + 4\int\frac{dt}{t^2} = 3\log|t| + 4\frac{t^{-2+1}}{-2+1} + C$$

$$= 3\log|t| - 4.\frac{1}{t} + C$$

Putting value of t in terms of z then z in terms of θ, we get

$$= 3\log|\sin\theta - 2| - \frac{4}{\sin\theta - 2} + C$$

23. Find: $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$ *[CBSE Delhi 2016]*

Sol. We have $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{x^{1/2} dx}{\sqrt{a^3 - x^3}} = \int \frac{x^{1/2} dx}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$

Let $x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt \Rightarrow x^{1/2} dx = \frac{2}{3} dt$

$$I = \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} \qquad [\because x^{3/2} = t \Rightarrow x^3 = t^2]$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C = \frac{2}{3} \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C = \frac{2}{3} \sin^{-1}\sqrt{\frac{x^3}{a^3}} + C$$

24. Find: $\int \frac{(2x-5)e^{2x}}{(2x-3)^3} dx$ *[CBSE (North) 2016]*

Sol. We have,

$$\int \frac{(2x-5)e^{2x}}{(2x-3)^3} dx = \int e^{2x}\left[\frac{(2x-3)-2}{(2x-3)^3}\right] dx$$

$$= \int e^3 . e^{2x-3}\left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3}\right] dx$$

$$= e^3 \int e^{2x-3}\left[\frac{1}{(2x-3)^2} - \frac{2}{(2x-3)^3}\right] dx$$

Let $2x - 3 = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{dt}{2}$

$$\Rightarrow I = \frac{e^3}{2} \int e^t \left[\frac{1}{t^2} - \frac{2}{t^3}\right] dt \qquad \Rightarrow \qquad I = \frac{e^3}{2} e^t . \frac{1}{t^2} + C$$

Putting $t = 2x - 3$

$$I = \frac{e^3}{2} e^{2x-3} \frac{1}{(2x-3)^2} + C \qquad \Rightarrow \qquad I = \frac{e^{2x}}{2(2x-3)^2} + C$$

25. Evaluate: $\int \frac{dx}{\sin(x-a)\cos(x-b)}$ *[CBSE 2019 (65/4/2)]*

Sol. Let $I = \int \frac{dx}{\sin(x-a)\cos(x-b)} = \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)\, dx}{\sin(x-a)\cos(x-b)}$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(a - x + x - b)}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos((x-b)-(x-a))}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(x-b)\cos(x-a) + \sin(x-b)\sin(x-a)}{\sin(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \left\{\frac{\cos(x-a)}{\sin(x-a)} + \frac{\sin(x-b)}{\cos(x-b)}\right\} dx$$

$$= \frac{1}{\cos(a-b)}\left[\int \frac{\cos(x-a)}{\sin(x-a)}dx + \int \frac{\sin(x-b)}{\cos(x-b)}dx\right]$$

$$= \frac{1}{\cos(a-b)}[\log|\sin(x-a)| - \log|\cos(x-b)|] + C$$

$$= \frac{1}{\cos(a-b)}\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$$

26. Find: $\int [\log(\log x) + \frac{1}{(\log x)^2}]dx$ *[CBSE Bhubaneswar 2015, (South) 2016]*

Sol. Let $I = \int [\log(\log x) + \frac{1}{(\log x)^2}]dx$

Let $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$\therefore \quad I = \int \left\{\log t + \frac{1}{t^2}\right\}e^t\,dt$$

$$= \int \left\{\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right\}e^t\,dt = \int \left[\left(\log t + \frac{1}{t}\right)e^t + \left(-\frac{1}{t} + \frac{1}{t^2}\right)e^t\right]dt$$

$$= e^t.\log t - \frac{1}{t}.e^t + C \qquad [\because \int (f(x) + f'(x))e^x\,dx = f(x)e^x + C]$$

$$= e^{\log x}\log(\log x) - \frac{1}{\log x}e^{\log x} + C \qquad [\text{Put } t = \log x]$$

$$= x.\log(\log x) - \frac{x}{\log x} + C$$

27. Evaluate: $\int_{-1}^{2} |x^3 - x|\,dx$ *[CBSE 2021-22 (Term-2)]*

Sol.

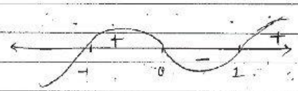
$I = \int_{-1}^{2} |x^3 - x|\,dx$

$I = \int_{-1}^{2} |x(x-1)(x+1)|$

It changes sign at 0, 1, -1. So, we have to break limits here

$I = \int_{-1}^{0} |x^3-x|\,dx + \int_{0}^{1} |x^3-x|\,dx + \int_{1}^{2} |x^3-x|\,dx$

$\left[\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right]$ $a < c < b$



in $(-1, 0)$ $\quad x^3 - x$ is +ve

in $(0, 1)$ $\quad x^3 - x$ is −ve

in $(1, 2)$ $\quad x^3 - x$ is +ve

$$I = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$I = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{2}^{2}$$

$$I = \left(0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right) - \left(\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right) + \left(\left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)\right)$$

$$I = \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4}$$

$$I = 2 + \frac{3}{4}$$

$$\therefore I = \frac{11}{4}$$

Answer: $\int_{-1}^{2} |x^3 - x|\,dx = \frac{11}{4}$

[*Topper's Answer 2022*]

28. Evaluate: $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$. **[CBSE 2023 (65/5/1)]**

Sol. Let $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$ ...(*i*)

Also $I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi - x)}}\,dx = \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}\,dx$

$\Rightarrow I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\,dx$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{2\pi} \frac{1+e^{\sin x}}{1+e^{\sin x}}\,dx = \int_0^{2\pi} dx = [x]_0^{2\pi} = 2\pi - 0 = 2\pi$$

$\Rightarrow I = \pi$

29. Evaluate: $\int_0^{\frac{\pi}{2}} [\log(\sin x) - \log(2\cos x)]\,dx$ **[CBSE 2023 (65/1/1)]**

Sol. Let $I = \int_0^{\frac{\pi}{2}} [\log \sin x - \log(2\cos x)]\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x - \log 2 - \log \cos x)\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \tan x\,dx - \log 2 \int_0^{\frac{\pi}{2}} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \tan x\,dx - \log 2 [x]_0^{\pi/2}$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \tan x\,dx - \frac{\pi}{2}\log 2$ ...(*i*)

Let $I_1 = \int_0^{\frac{\pi}{2}} \log \tan x$ ...(ii)

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I_1 = \int_0^{\frac{\pi}{2}} \log \tan\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cot x \, dx \quad ...(iii)$$

Adding (ii) and (iii), we get

$$2I_1 = \int_0^{\frac{\pi}{2}} (\log \tan x + \log \cot x)dx = \int_0^{\frac{\pi}{2}} \log (\tan x . \cot x)dx$$

$$2I_1 = \int_0^{\frac{\pi}{2}} \log 1 \, dx = 0 \qquad \Rightarrow I_1 = 0$$

Putting $I_1 = 0$ in (i), we get

$$I = 0 - \frac{\pi}{2} \log 2 = \frac{-\pi}{2} \log 2$$

30. Evaluate: $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$ **[CBSE 2023 (65/1/1)]**

Sol. Let $I = \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$

Using integration by parts, we have

$$I = \sin x \int e^x \, dx - \int\left(\frac{d \sin x}{dx} . \int e^x dx\right)dx$$

$\Rightarrow \quad I = \sin x . e^x - \int \cos x . e^x dx$

$\Rightarrow \quad I = \sin x . e^x - [\cos x \, e^x - \int -\sin x \, e^x \, dx]$

$\Rightarrow \quad I = \sin x e^x - \cos x \, e^x - \int e^x \sin x \, dx$

$\Rightarrow \quad I = (\sin x - \cos x)e^x - I$

$\Rightarrow \quad 2I = (\sin x - \cos x)e^x \qquad \Rightarrow I = \frac{1}{2}\left[(\sin x - \cos x)e^x\right]_0^{\pi/2}$

$\therefore \quad I = \frac{1}{2}\left[\left(\sin \frac{\pi}{2} - \cos \frac{\pi}{2}\right)e^{\pi/2} - (\sin 0 - \cos 0)e^0\right] = \frac{1}{2}\left[(1-0)e^{\pi/2} - (0-1) \times 1\right] = \frac{1}{2}\left(e^{\pi/2} + 1\right)$

31. Evaluate: $\int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x + e^{-x})(e^x - e^{-x})} dx$ **[CBSE 2023 (65/2/1)]**

Sol. Let $I = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x + e^{-x})(e^x - e^{-x})} dx$

$$= \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1 \times e^{2x}}{(e^{2x}+1)(e^{2x}-1)} dx$$

Let $e^{2x} = t \qquad \Rightarrow 2e^{2x} \, dx = dt$

When $x = \log\sqrt{2} \Rightarrow t = e^{2\log\sqrt{2}} = e^{\log(\sqrt{2})^2} = 2$

When $x = \log\sqrt{3} \Rightarrow t = e^{2\log\sqrt{3}} = e^{\log(\sqrt{3})^2} = 3$

$\therefore \quad I = \frac{1}{2}\int_2^3 \frac{dt}{(t+1)(t-1)} = \frac{1}{2} \times \frac{1}{2}\int_2^3 \frac{(t+1)-(t-1)}{(t+1)(t-1)} dt$

$$= \frac{1}{4}\int_2^3 \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt = \frac{1}{4}\left[\log|t-1| - \log|t+1|\right]_2^3$$

$$= \frac{1}{4}\left[\log \frac{t-1}{t+1}\right]_2^3 = \frac{1}{4}\left(\log \frac{3-1}{3-1} - \log \frac{2-1}{2+1}\right)$$

$$= \frac{1}{4}\left(\log \frac{1}{2} - \log \frac{1}{3}\right) = \frac{1}{4} \log \frac{3}{2}$$

32. Evaluate : $\int_{-\pi/2}^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx$ [*CBSE 2023 (65/3/2)*]

Sol. Let $I = \int_{-\frac{\pi}{2}}^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx$

Using property $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, when $f(x)$ is even.

$$\therefore \quad I = 2\int_0^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx \qquad ...(i)$$

Using properties $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = 2\int_0^{\pi/2} \frac{\sin^{100}\left(\frac{\pi}{2} - x\right)}{\sin^{100}\left(\frac{\pi}{2} - x\right) + \cos^{100}\left(\frac{\pi}{2} - x\right)} dx$$

$$I = 2\int_0^{\pi/2} \frac{\cos^{100} x}{\cos^{100} x + \sin^{100} x} dx \qquad ...(ii)$$

Adding (i) and (ii), we have

$$2I = 2\int_0^{\pi/2} \frac{\sin^{100} x + \cos^{100} x}{\sin^{100} x + \cos^{100} x} dx = 2\int_0^{\pi/2} dx = 2[x]_0^{\pi/2}$$

$$\Rightarrow \quad 2I = 2\left[\frac{\pi}{2} - 0\right] = \pi$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

33. Evaluate: $\int_0^1 x \log(1+2x)\, dx$ [*NCERT Exemplar*]

Sol. Let $I = \int_0^1 \underset{\text{II}}{x} \log(1+\underset{\text{I}}{2x})\, dx$

$$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int \frac{1}{1+2x}.2.\frac{x^2}{2} dx = \frac{1}{2}[x^2 \log(1+2x)]_0^1 - \int_0^1 \frac{x^2}{1+2x} dx$$

$$= \frac{1}{2}[1\log 3 - 0] - \left[\int_0^1 \left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x}\right) dx\right] = \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\, dx + \frac{1}{2}\int_0^1 \frac{x}{1+2x} dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\frac{1}{2}(2x+1-1)}{(2x+1)} dx = \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2} - 0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1 \frac{1}{1+2x} dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\log|(1+2x)|]_0^1 = \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3 = \frac{3}{8}\log 3$$

34. Evaluate: $\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx$ **[CBSE (AI) 2014]**

Sol. Let $I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x} dx$ ...(i)

$$= \int_0^{\pi} \frac{4(\pi - x).\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$$

$$I = \int_0^{\pi} \frac{4(\pi - x).\sin x}{1+\cos^2 x} dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{4(x+\pi - x)\sin x}{1+\cos^2 x} dx \Rightarrow \quad 2I = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$$

$$I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Let $\cos x = z \Rightarrow -\sin x\, dx = dz \Rightarrow \sin x\, dx = -dz$

The limits are, $x = 0 \Rightarrow z = 1$

$x = \pi \Rightarrow z = -1$

$$\therefore \quad I = 2\pi \int_1^{-1} \frac{-dz}{1+z^2} = 2\pi [\tan^{-1} z]_{-1}^{1}$$

$$= 2\pi[\tan^{-1} 1 - \tan^{-1}(-1)] = 2\pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = 2\pi \times \frac{\pi}{2}$$

$\Rightarrow \quad I = \pi^2.$

35. Evaluate: $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$ **[CBSE Delhi 2015]**

Sol. Here, $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

$\Rightarrow \quad I = \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx) dx$

$\Rightarrow \quad I = \int_{-\pi}^{\pi} \cos^2 ax\, dx + \int_{-\pi}^{\pi} \sin^2 bx\, dx - \int_{-\pi}^{\pi} 2\cos ax \sin bx\, dx$

$\Rightarrow \quad I = 2\int_0^{\pi} \cos^2 ax\, dx + 2\int_0^{\pi} \sin^2 bx\, dx - 0$ [First two integrands are even function while third is odd function.]

$\Rightarrow \quad I = \int_0^{\pi} 2\cos^2 ax\, dx + \int_0^{\pi} 2\sin^2 bx\, dx$

$\Rightarrow \quad I = \int_0^{\pi} (1 + \cos 2ax)\, dx + \int_0^{\pi} (1 - \cos 2bx)\, dx$

$\Rightarrow \quad I = \int_0^{\pi} dx + \int_0^{\pi} \cos 2ax\, dx + \int_0^{\pi} dx - \int_0^{\pi} \cos 2bx\, dx$

$\Rightarrow \quad I = 2\int_0^{\pi} dx + \int_0^{\pi} \cos 2ax\, dx - \int_0^{\pi} \cos 2bx\, dx$

$\Rightarrow \quad I = 2[x]_0^{\pi} + \left[\frac{\sin 2ax}{2a}\right]_0^{\pi} - \left[\frac{\sin 2bx}{2b}\right]_0^{\pi}$

$\Rightarrow \quad I = 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$

36. Evaluate: $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ **[CBSE Delhi 2017; (AI) 2012, CBSE 2020 (65/4/1)]**

Sol. Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\, dx}{1 + \cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\, dx}{1 + \cos^2 x} = \pi \int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x} - I$$

or $\quad 2I = \pi \int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x} \quad$ or $\quad I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x\, dx}{1 + \cos^2 x}$

Put $\cos x = t$ so that $-\sin x\, dx = dt$.

The limits are, when $x = 0$, $t = 1$ and $x = \pi$, $t = -1$, we get

$$I = \frac{-\pi}{2} \int_1^{-1} \frac{dt}{1 + t^2} = \pi \int_0^1 \frac{dt}{1 + t^2} \quad \left[\because \int_a^{-a} f(x)\, dx = -\int_{-a}^{a} f(x)\, dx \text{ and } \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx\right]$$

$$= \pi [\tan^{-1} t]_0^1 = \pi [\tan^{-1} 1 - \tan^{-1} 0] = \pi \left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

37. Find: $\int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2 \sin 2x}}$ **[*CBSE (Allahabad) 2015*]**

Sol. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2 \sin 2x}} = \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2.2 \sin x . \cos x}}$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{\frac{\sin x}{\cos x} \cdot \cos^2 x}} = \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^4 x \sqrt{\tan x}}$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{\sec^4 x\, dx}{\sqrt{\tan x}} = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2 x . \sec^2 x\, dx}{\sqrt{\tan x}}$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt, \quad x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{4} \Rightarrow t = 1$

$$\therefore \quad I = \frac{1}{2} \int_0^1 \frac{(1 + t^2)\, dt}{\sqrt{t}}$$

$$= \frac{1}{2} \int_0^1 (t^{-1/2} + t^{3/2})\, dt = \frac{1}{2} \left[\frac{t^{-1/2+1}}{-1/2+1}\right]_0^1 + \frac{1}{2} \left[\frac{t^{3/2+1}}{3/2+1}\right]_0^1$$

$$= \frac{1}{2} \times \frac{2}{1} [\sqrt{t}]_0^1 + \frac{1}{2} \times \frac{2}{5} [t^{5/2}]_0^1 = 1 + \frac{1}{5} = \frac{6}{5}$$

38. Evaluate: $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx$ **[*CBSE (F) 2015*]**

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx$

$$= \int_{-\pi/2}^{0} \frac{\cos x}{1 + e^x} dx + \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx$$

$$= \int_{\pi/2}^{0} \frac{\cos t}{1 + e^{-t}} (-dt) + \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx$$

In 1st Integrand
Let $x = -t$
$dx = -dt$
$x = -\pi/2 \Rightarrow t = \pi/2$
$x = 0 \Rightarrow t = 0$

$$= \int_0^{\pi/2} \frac{\cos t}{1 + \frac{1}{e^t}} dt + \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx = \int_0^{\pi/2} \frac{e^t . \cos t}{1 + e^t} dt + \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx$$

$$= \int_0^{\pi/2} \frac{e^x . \cos x}{1 + e^x} dx + \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx \quad \left[\text{By property } \int_a^b f(x)\, dx = \int_a^b f(t)\, dt\right]$$

$$= \int_0^{\pi/2} \frac{(e^x + 1) . \cos x}{1 + e^x} dx = \int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\frac{\pi}{2}} = \sin \pi/2 - \sin 0 = 1.$$

39. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ *[CBSE (AI) 2011]*

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ ...(*i*)

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}$$ [By using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$]

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{1}{\sqrt{\tan x}}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx$$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{(1+\sqrt{\tan x})}{(1+\sqrt{\tan x})}dx = \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$\Rightarrow \quad 2I = \frac{\pi}{6} \quad$ or $\quad I = \frac{\pi}{12}$

40. Evaluate: $\int_1^3 [|x-1|+|x-2|+|x-3|]dx$ *[CBSE Delhi 2013]*

Sol. Let $I = \int_1^3 [|x-1|+|x-2|+|x-3|]dx = \int_1^3 |x-1|\,dx + \int_1^3 |x-2|\,dx + \int_1^3 |x-3|\,dx$

$$= \int_1^3 |x-1|\,dx + \int_1^2 |x-2|\,dx + \int_2^3 |x-2|\,dx + \int_1^3 |x-3|\,dx$$

[By property of definite integral]

$$= \int_1^3 (x-1)\,dx + \int_1^2 -(x-2)dx + \int_2^3 (x-2)dx + \int_1^3 -(x-3)dx \qquad \begin{cases} x-1 \ge 0, & \text{if } 1 \le x \le 3 \\ x-2 \le 0, & \text{if } 1 \le x \le 2 \\ x-2 \ge 0, & \text{if } 2 \le x \le 3 \\ x-3 \le 0, & \text{if } 1 \le x \le 3 \end{cases}$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^3 - \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3 - \left[\frac{(x-3)^2}{2}\right]_1^3$$

$$= \left(\frac{4}{2}-0\right) - \left(0-\frac{1}{2}\right) + \left(\frac{1}{2}-0\right) - \left(-0-\frac{4}{2}\right) = 2+\frac{1}{2}+\frac{1}{2}+2 = 5$$

41. Evaluate: $\int_0^\pi \frac{x\tan x}{\sec x.\text{cosec}\,x}dx$ *[CBSE Delhi 2008, 2014; Chennai 2015]*

Sol. Let $I = \int_0^\pi \frac{x\tan x}{\sec x.\text{cosec}\,x}dx = \int_0^\pi \frac{x.\frac{\sin x}{\cos x}}{\frac{1}{\cos x}.\frac{1}{\sin x}}dx$

$$I = \int_0^\pi x\sin^2 x dx = \int_0^\pi (\pi-x)\sin^2(\pi-x)dx \qquad [\because \int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$I = \int_0^\pi \pi\sin^2 x dx - \int_0^\pi x\sin^2 x\,dx \Rightarrow 2I = \frac{\pi}{2}\int_0^\pi 2\sin^2 x\,dx$$

$$= \frac{\pi}{2}\int_0^\pi (1-\cos 2x)dx = \frac{\pi}{2}\int_0^\pi dx - \frac{\pi}{2}\int_0^\pi \cos 2x dx = \frac{\pi}{2}[x]_0^\pi - \frac{\pi}{2}\left[\frac{\sin 2x}{2}\right]_0^\pi$$

$\Rightarrow \quad 2I = \frac{\pi}{2}(\pi-0) - \frac{\pi}{4}(\sin 2\pi - \sin 0)$

$\Rightarrow \quad 2I = \frac{\pi^2}{2} - 0 \Rightarrow I = \frac{\pi^2}{4}$

42. Evaluate: $\int_0^1 \cot^{-1}(1-x+x^2)dx$ **[CBSE Delhi 2008; (AI) 2008; (South) 2016]**

Sol. Let $I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$= \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}dx \qquad \left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx$$

$$= \int_0^1 \{\tan^{-1}x + \tan^{-1}(1-x)\}\,dx \qquad \left[\because \tan^{-1}(x+y) = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-(1-x))\,dx \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx = 2\int_0^1 \tan^{-1}x\,dx = 2\int_0^1 1.\tan^{-1}x\,dx$$

$$= 2\left\{[\tan^{-1}x.x]_0^1 - \int_0^1 \frac{1}{1+x^2}.x.dx\right\}$$

$$= 2\frac{\pi}{4} - \int_0^1 \frac{2x\,dx}{1+x^2} = \frac{\pi}{2} - [\log|1+x^2|]_0^1$$

$$= \frac{\pi}{2} - [\log 2 - \log 1] = \frac{\pi}{2} - \log 2$$

43. Evaluate: $\int_0^1 x^2(1-x)^n dx$ **[CBSE (F) 2010, 2019 (65/4/3)]**

Sol. Let $I = \int_0^1 x^2(1-x)^n dx$

$$\Rightarrow \quad I = \int_0^1 (1-x)^2[1-(1-x)]^n dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-2x+x^2)x^n dx = \int_0^1 (x^n - 2x^{n+1} + x^{n+2})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - 2.\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}$$

$$= \frac{n^2+5n+6-2n^2-8n-6+n^2+3n+2}{(n+1)(n+2)(n+3)} = \frac{2}{(n+1)(n+2)(n+3)}$$

44. Evaluate: $\int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$ **[CBSE Delhi 2016]**

Sol. We have $I = \int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$

Integrating by part, we get

$$I = \left[\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^{\pi} - \int_0^{\pi} \cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}dx$$

$$= \frac{1}{2}\left[\sin\frac{5\pi}{4}.e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{1}{2}\int_0^{\pi} e^{2x}.\cos\left(\frac{\pi}{4}+x\right)dx$$

$$= \frac{1}{2}\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left\{\cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^{\pi} + \int_0^{\pi}\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}dx\right]$$

$$= -\frac{e^{2\pi}+1}{2\sqrt{2}} - \frac{1}{2}\left[\cos\frac{5\pi}{4}.\frac{e^{2\pi}}{2} - \frac{1}{2}\cos\frac{\pi}{4}\right] - \frac{1}{4}\int e^{2x}.\sin\left(\frac{\pi}{4}+x\right)dx$$

$$I = -\frac{e^{2\pi}+1}{2\sqrt{2}} - \frac{1}{4}.e^{2\pi}.\cos\frac{5\pi}{4} + \frac{1}{4\sqrt{2}} - \frac{1}{4}I$$

$$\frac{5I}{4} = -\frac{e^{2\pi}+1}{2\sqrt{2}} + \frac{e^{2\pi}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} = -\frac{e^{2\pi}+1}{2\sqrt{2}} + \frac{e^{2\pi}+1}{4\sqrt{2}} = \frac{e^{2\pi}+1}{4\sqrt{2}}(-2+1) = -\frac{e^{2\pi}+1}{4\sqrt{2}}$$

$$I = -\frac{e^{2\pi}+1}{5\sqrt{2}}$$

45. Evaluate: $\int_{-2}^{2} \frac{x^2}{1+5^x}dx$ *[CBSE (North) 2016]*

Sol. Let $I = \int_{-2}^{2} \frac{x^2}{1+5^x}dx$ ...(*i*)

$$= \int_{-2}^{2} \frac{(2+(-2)-x)^2}{1+5^{(2+(-2)-x)}}dx \qquad \left[\int_a^b f(x)\,dx = \int f(a+b-x)\,dx\right]$$

$$= \int_{-2}^{2} \frac{(-x)^2}{1+5^{-x}}dx = \int_{-2}^{2} \frac{x^2}{1+\frac{1}{5^x}}dx$$

$$I = \int_{-2}^{2} \frac{5^x x^2}{1+5^x}dx \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_{-2}^{2} \frac{(1+5^x)x^2}{1+5^x}dx = \int_{-2}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$\Rightarrow \quad 2I = \frac{1}{3}[8-(-8)] \qquad \Rightarrow \qquad I = \frac{16}{3\times 2} = \frac{8}{3}$$

46. Evaluate: $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$ *[CBSE Delhi 2008, 2010, (AI) 2008, 2017, (F) 2010, 2013, 2014]*

Sol. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$ ...(*i*)

$$= \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}dx \qquad ...(ii)$$

By adding equations (*i*) and (*ii*), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x}dx$$

Multiplying and dividing by (sec x – tan x), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x}dx = \pi\int_0^{\pi}(\sec x\tan x - \tan^2 x)\,dx$$

$$= \pi\int_0^{\pi}\sec x\tan x\,dx - \pi\int_0^{\pi}\sec^2 x\,dx + \int_0^{\pi}dx$$

$$= \pi[\sec x]_0^{\pi} - \pi[\tan x]_0^{\pi} + \pi[x]_0^{\pi}$$

$$= \pi(-1-1) - 0 + \pi(\pi - 0) = \pi(\pi - 2)$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow \quad I = \frac{\pi}{2}(\pi - 2)$$

47. **Evaluate:** $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$ [*CBSE 2018*]

Sol.

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$$

Put $\sin x - \cos x = t$

$(\cos x + \sin x) dx = dt$

Also $(\sin x - \cos x)^2 = t^2$

$\sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$

$1 - \sin 2x = t^2$

$\sin 2x = 1 - t^2$

limits, when, $x = 0$, $t = -1$

$x = \pi/4$, $t = 0$

$$\int_{-1}^{0} \frac{dt}{16 + 9(1 - t^2)}$$

$$\int_{-1}^{0} \frac{dt}{16 + 9 - 9t^2}$$

$$\int_{-1}^{0} \frac{dt}{25 - 9t^2}$$

$$= \int_{-1}^{0} \frac{dt}{-9(t^2 - 25/9)}$$

$$= \frac{-1}{9} \int_{-1}^{0} \frac{dt}{t^2 - (5/3)^2}$$

$$= \frac{-1}{9} \left[\frac{1}{2 \times \frac{5}{3}} \log \left| \frac{t - 5/3}{t + 5/3} \right| \right]_{-1}^{0}$$

$$= \frac{-1}{9} \left[\frac{3}{10} \log \left| \frac{3t - 5}{3t + 5} \right| \right]_{-1}^{0}$$

$$= \frac{-1}{9_3} \times \frac{3}{10} \left[\log \left| \frac{3(0) - 5}{3(0) + 5} \right| - \log \left| \frac{3(-1) - 5}{3(-1) + 5} \right| \right]$$

$$= \frac{-1}{30} \left[\log \left| \frac{-5}{5} \right| - \log \left| \frac{-8}{2} \right| \right]$$

$$= \frac{-1}{30} \left[\log|-1| - \log|-4| \right] = \frac{-1}{30} \left[\log \left| \frac{-1}{-4} \right| \right] = \frac{-1}{30} \log\left(\frac{1}{4}\right)$$

or $\frac{1}{15} \log 2$ Ans

[*Topper's Answer 2018*]

Long Answer Questions

1. **Evaluate:** $\int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$ [*CBSE Delhi 2011; (AI) 2010*]

Sol. We can express the N^r as $5x + 3 = A\frac{d}{dx}(x^2 + 4x + 10) + B$

$\Rightarrow \quad 5x + 3 = A(2x + 4) + B \qquad \Rightarrow \qquad 5x + 3 = 2Ax + (4A + B)$

Equating the coefficients, we get

$$2A = 5 \quad \text{and} \quad 4A + B = 3$$

$$A = \frac{5}{2} \Rightarrow 4 \times \frac{5}{2} + B = 3 \Rightarrow B = 3 - 10 = -7$$

$$\therefore\ 5x + 3 = \frac{5}{2}(2x+4) + (-7)$$

$$\therefore\ I = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\int \frac{(2x+4)}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$$

$$I = \frac{5}{2}I_1 - 7I_2 \qquad \ldots(i)$$

where $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{dx}{\sqrt{x^2+4x+10}}$

Now, $I_1 = \int \frac{(2x+4)}{\sqrt{x^2+4x+10}}dx$

Let $x^2 + 4x + 10 = t \Rightarrow (2x+4)dx = dt$

$$\therefore\ I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = \frac{t^{-1/2+1}}{-\frac{1}{2}+1} + C_1 = 2\sqrt{t} + C_1$$

$$I_1 = 2\sqrt{x^2+4x+10} + C_1$$

Again, $I_2 = \int \frac{dx}{\sqrt{x^2+2.x.2+2^2-4+10}} = \int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$

$$= \log\left| (x+2) + \sqrt{x^2+4x+10} \right| + C_2$$

Putting the value of I_1 and I_2 in (i), we get

$$I = \frac{5}{2} \times 2\sqrt{x^2+4x+10} - 7\log\left| (x+2) + \sqrt{x^2+4x+10} \right| + \left(\frac{5}{2}C_1 - 7C_2\right)$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left| (x+2) + \sqrt{x^2+4x+10} \right| + C, \text{ where } C = \left(\frac{5}{2}C_1 - 7C_2\right)$$

2. Evaluate: $\int \frac{2x}{(x^2+1)(x^2+3)}dx$ **[*CBSE Delhi 2011*]**

Sol. Let $x^2 = z \Rightarrow 2x\,dx = dz$

$$\therefore\ \int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \frac{dz}{(z+1)(z+3)}$$

Using partial fraction

Let $\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$ $\qquad \ldots(i)$

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3) + B(z+1)}{(z+1)(z+3)}$$

$\Rightarrow \quad 1 = A(z+3) + B(z+1)$

$\Rightarrow \quad 1 = (A+B)z + (3A+B)$

Equating the coefficient of z and constant, we get

$A + B = 0 \qquad \ldots(ii)$

and $\quad 3A + B = 1 \qquad \ldots(iii)$

Subtracting (*ii*) from (*iii*), we get

$$2A = 1 \quad \Rightarrow \quad A = \frac{1}{2} \quad \text{and} \quad B = -\frac{1}{2}$$

Putting the values of A and B in (*i*), we get

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$\therefore \quad \int \frac{2x\,dx}{(x^2+1)(x^2+3)}$$

$$= \int\left(\frac{1}{2(z+1)} - \frac{1}{2(z+3)}\right)dz = \frac{1}{2}\int\frac{dz}{z+1} - \frac{1}{2}\int\frac{dz}{z+3}$$

$$= \frac{1}{2}\log|z+1| - \frac{1}{2}\log|z+3| + C = \frac{1}{2}\log|x^2+1| - \frac{1}{2}\log|x^2+3| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C \qquad \begin{bmatrix}\text{Note: } \log m + \log n = \log m \,.\, n \\ \text{and} \quad \log m - \log n = \log m/n\end{bmatrix}$$

$$= \log\sqrt{\frac{x^2+1}{x^2+3}} + C$$

3. Evaluate: $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$ **[*CBSE (AI) 2014*]**

Sol. Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

Now, we can express as

$$x+2 = A\frac{d}{dx}(x^2+5x+6) + B$$

$$\Rightarrow \quad x+2 = A(2x+5) + B \quad \Rightarrow \quad x+2 = 2Ax + (5A+B)$$

Equating coefficients both sides, we get

$$2A = 1,\ 5A + B = 2 \quad \Rightarrow \quad A = \frac{1}{2}, B = 2 - \frac{5}{2} = -\frac{1}{2}$$

$$\therefore \quad x+2 = \frac{1}{2}(2x+5) - \frac{1}{2}$$

Hence, $I = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}dx = \frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int\frac{dx}{\sqrt{x^2+5x+6}}$

$$I = \frac{1}{2}I_1 - \frac{1}{2}I_2 \qquad \ldots (i)$$

where, $I_1 = \int\frac{2x+5}{\sqrt{x^2+5x+6}}dx$, $I_2 = \int\frac{dx}{\sqrt{x^2+5x+6}}$

Now, $I_1 = \int\frac{2x+5}{\sqrt{x^2+5x+6}}dx$

Let $x^2+5x+6 = z \Rightarrow (2x+5)dx = dz$

$$\therefore \quad I_1 = \int\frac{dz}{\sqrt{z}} = \int z^{\frac{-1}{2}}dz = \frac{z^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + C_1 = 2\sqrt{z} + C_1 = 2\sqrt{x^2+5x+6} + C_1$$

Again $I_2 = \int\frac{dx}{\sqrt{x^2+5x+6}} = \int\frac{dx}{\sqrt{x^2 + 2\times x\times\frac{5}{2} + \left(\frac{5}{2}\right)^2 - \frac{25}{4} + 6}}$

$$= \int \frac{dx}{\left(x+\frac{5}{2}\right)^2 - \frac{1}{4}} = \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \log \left| \left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6} \right| + C_2$$

Putting the value of I_1 and I_2 in (*i*), we get

$$I = \frac{1}{2}\{2\sqrt{x^2+5x+6} + C_1\} - \frac{1}{2}\left\{\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C_2\right\}$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + \frac{1}{2}C_1 - \frac{1}{2}C_2$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C \qquad \left[\text{Here, } C = \frac{1}{2}C_1 - \frac{1}{2}C_2\right]$$

4. Evaluate: $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$ **[*CBSE (AI) 2014*]**

Sol. Let $I = \int \frac{1}{\sin^4 x + \sin^2 x.\cos^2 x + \cos^4 x} dx$

Dividing N^r and D^r by $\cos^4 x$, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

Put $z = \tan x \Rightarrow dz = \sec^2 x\, dx$

$$\therefore \quad I = \int \frac{(1+z^2)dz}{z^4+z^2+1} = \int \frac{z^2\left(1+\frac{1}{z^2}\right)}{z^2\left\{z^2+\frac{1}{z^2}+1\right\}} dz = \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+3} dz = \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+(\sqrt{3})^2} dz$$

Again, let $z - \frac{1}{z} = t \Rightarrow \left(1+\frac{1}{z^2}\right)dz = dt$

$$\Rightarrow \quad I = \int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{t}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{3}}\right) + C \qquad \left[\because z - \frac{1}{z} = t\right]$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z^2-1}{\sqrt{3}z}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3}\tan x}\right) + C$$

5. Solve: $\int \sqrt{\frac{(a+x)}{(a-x)}} dx$ **[*NCERT Exemplar*]**

Sol. Let $I = \int \sqrt{\frac{a+x}{a-x}} dx$

Put $x = a\cos 2\theta \Rightarrow dx = -a \,.\, \sin 2\theta \,.\, 2 \,.\, d\theta$

$$\therefore \quad I = -2\int \sqrt{\frac{a+a\cos 2\theta}{a-a\cos 2\theta}} \,.\, a\sin 2\theta d\theta$$

$$= -2a\int \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} \sin 2\theta d\theta = -2a\int \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}} \sin 2\theta d\theta$$

$$= -2a\int \frac{\cos\theta}{\sin\theta} 2\sin\theta.\cos\theta d\theta = -4a\int \cos^2\theta d\theta = -2a\int(1+\cos 2\theta)d\theta$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right] + C = -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\sqrt{1-\frac{x^2}{a^2}}\right] + C$$

$$= -a\left[\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{1-\frac{x^2}{a^2}}\right] + C \qquad \left[\because \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}\right]$$

6. Find: $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Sol. Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Putting $\sqrt{x} = \cos\theta$, *i.e.*, $x = \cos^2\theta \Rightarrow \theta = \cos^{-1}\sqrt{x}$ and $dx = -2\cos\theta\sin\theta d\theta$, we get

$$I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$$

$$= -2\int \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}(\sin\theta\cos\theta)\,d\theta = -2\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\right)d\theta$$

$$= -2\int 2\sin^2\frac{\theta}{2}\cos\theta d\theta$$

$$= -2\int (1-\cos\theta)\cos\theta\,.\,d\theta = -2\int (\cos\theta - \cos^2\theta)\,.\,d\theta$$

$$= -2\int \cos\theta\,d\theta + \int 2\cos^2\theta\,.\,d\theta = -2\sin\theta + \int (1+\cos 2\theta)\,.d\theta$$

$$= -2\sin\theta + \int 1.d\theta + \int \cos 2\theta.\,d\theta = -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$= -2\sqrt{1-\cos^2\theta} + \theta + \frac{2\sqrt{1-\cos^2\theta}.\cos\theta}{2} + C = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

7. Find: $\int \frac{x^2}{(x\sin x + \cos x)^2}dx$

Sol. Let $I = \int \frac{x^2 dx}{(x\sin x+\cos x)^2}dx = \int \frac{x\cos x}{(x\sin x+\cos x)^2}.\frac{x}{\cos x}dx$

Integrating by parts, taking $\frac{x}{\cos x}$ as the first function and $\frac{x\cos x}{(x\sin x+\cos x)^2}$ as the second function, we get

$$I = \frac{x}{\cos x}\int \frac{x\cos x}{(x\sin x+\cos x)^2}.dx - \int\left[\frac{d}{dx}\left(\frac{x}{\cos x}\right)\int\left(\frac{x\cos x}{(x\sin x+\cos x)^2}\right).dx\right]dx$$

Now, let us first evaluate $\int \frac{x\cos x dx}{(x\sin x+\cos x)^2}$

Putting $(x\sin x + \cos x) = t$, then $(\sin x + x\cos x - \sin x)dx = dt$ *i.e.*, $x\cos x\,dx = dt$, we get

$$\int \frac{x\cos x}{(x\sin x+\cos x)^2}dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{(x\sin x+\cos x)}$$

Hence, $I = \frac{x}{\cos x}.\frac{-1}{(x\sin x+\cos x)} - \int \frac{\cos x + x\sin x}{\cos^2 x}\times\frac{-1}{(x\sin x+\cos x)}.dx$

$$= \frac{x}{\cos x}\times\frac{-1}{(x\sin x+\cos x)} + \int \sec^2 x.\,dx = \frac{-x}{\cos x(x\sin x+\cos x)} + \tan x + C$$

$$= \frac{-x + x\sin^2 x + \sin x\cos x}{\cos x(x\sin x+\cos x)} + C = \frac{-x(1-\sin^2 x) + \sin x\cos x}{\cos x(x\sin x+\cos x)} + C$$

$$= \frac{\cos x(\sin x - x\cos x)}{\cos x(x\sin x+\cos x)} + C$$

$$\int \frac{x^2 dx}{(x\sin x+\cos x)^2} = \frac{(\sin x - x\cos x)}{(x\sin x+\cos x)} + C$$

8. Evaluate: $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ **[*CBSE Panchkula 2015; (South) 2016*]**

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ ..(*i*)

$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos x.\cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right)dx \quad [\because \cos(A-B) = \cos A \cos B + \sin A \sin B]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left\{\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right\}\right]_0^{\pi/2} \quad [\because \int \sec x dx = \log(\sec x + \tan x)]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log\left\{\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right\}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1) - \log\left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}[\log(\sqrt{2}+1) - \log(\sqrt{2}-1)] = \frac{1}{\sqrt{2}}\log\left\{\frac{\sqrt{2}+1}{\sqrt{2}-1}\right\}$$

$$= \frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{2-1}\right\} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

Hence, $I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

9. Evaluate: $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ **[*CBSE Delhi 2011, 2014; Sample Paper 2017*]**

Sol. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right).\sin\left(\frac{\pi}{2} - x\right).\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx \quad \left[\text{By Property } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\cos x.\sin x}{\cos^4 x + \sin^4 x} dx \quad \left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x.\sin x}{\sin^4 x + \cos^4 x}dx - \int_0^{\pi/2} \frac{x\sin x.\cos x}{\sin^4 x + \cos^4 x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x.\sin x}{\sin^4 x + \cos^4 x}dx - I \Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x.\cos x}{\sin^4 x + \cos^4 x}dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\frac{\sin x\,.\cos x}{\cos^4 x}dx}{\tan^4 x + 1}$$ [Dividing numerator and denominator by $\cos^4 x$]

$$= \frac{\pi}{2\times 2}\int_0^{\pi/2} \frac{2\tan x.\sec^2 x}{1 + (\tan^2 x)^2}dx$$

Let $\tan^2 x = z; \;\; 2\tan x\,.\sec^2 x\,dx = dz$

The limits are, when $x = 0, z = 0; \; x = \frac{\pi}{2}, z = \infty$

$$\therefore \quad 2I = \frac{\pi}{4}\int_0^{\infty} \frac{dz}{1+z^2} = \frac{\pi}{4}[\tan^{-1} z]_0^{\infty} = \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0)$$

$$\therefore \quad 2I = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) \quad \Rightarrow \quad I = \frac{\pi^2}{16}$$

10. Evaluate: $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}\,dx$ **[*CBSE (AI) 2009; (F) 2014*]**

Sol. Let $I = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx$ ...(*i*)

$$I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}dx \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$2I = \int_0^{\pi} \frac{\pi}{a^2\cos^2 x + b^2\sin^2 x}dx \quad \Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}dx$$ [Divide numerator and denominator by $\cos^2 x$]

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}dx \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx\right]$$

Put $b\tan x = t \;\; \Rightarrow b\sec^2 x\,dx = dt$

The limits are, when $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$I = \frac{\pi}{b}\int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b}.\frac{1}{a}\tan^{-1}\frac{t}{a}\Big]_0^{\infty}$$

$$I = \frac{\pi}{ab}(\tan^{-1}\infty - \tan^{-1} 0) = \frac{\pi}{ab}.\frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi^2}{2ab}$$

11. Evaluate the following: $\int_0^{3/2} |\,x\cos\pi x\,|\,dx$ **[*CBSE (F) 2010; Patna 2015; (Central) 2016*]**

Sol. $\int_0^{3/2} |\,x\cos\pi x\,|\,dx$

As we know, $\cos x = 0 \quad \Rightarrow \quad x = (2n-1)\frac{\pi}{2}, n \in Z$

$\therefore \quad \cos \pi x = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \frac{3}{2}$

For $0 < x < \frac{1}{2}$, $x > 0$ then $\quad \cos \pi x > 0 \quad \Rightarrow \quad x \cos \pi x > 0$

For $\frac{1}{2} < x < \frac{3}{2}$, $x > 0$ then $\quad \cos \pi x < 0 \quad \Rightarrow \quad x \cos \pi x < 0$

$$\therefore \int_0^{3/2} | x\cos\pi x | \, dx = \int_0^{1/2} x\cos\pi x dx + \int_{1/2}^{3/2} (-x\cos\pi x) dx$$

$$= \left[x \frac{\sin \pi x}{\pi}\right]_0^{1/2} - \int_0^{1/2} 1. \frac{\sin\pi x}{\pi} dx - \left[\frac{x\sin\pi x}{\pi}\right]_{1/2}^{3/2} + \int_{1/2}^{3/2} \frac{\sin\pi x}{\pi} dx$$

$$= \left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_0^{1/2} - \left[\frac{x}{\pi}\sin\pi x - \frac{1}{\pi^2}\cos\pi x\right]_{1/2}^{3/2}$$

$$= \left(\frac{1}{2\pi} + 0 - \frac{1}{\pi^2}\right) - \left(-\frac{3}{2\pi} - \frac{1}{2\pi}\right) = \frac{5}{2\pi} - \frac{1}{\pi^2}$$

12. Evaluate: $\int_0^{\pi} \frac{x}{1 + \sin\alpha \sin x} dx$ **[*CBSE (F) 2016*]**

Sol. Let $\quad I = \int_0^{\pi} \frac{x}{1 + \sin\alpha \sin x} dx = \int_0^{\pi} \frac{\pi - x}{1 + \sin\alpha . \sin(\pi - x)} dx$

$$= \int_0^{\pi} \frac{\pi}{1 + \sin\alpha \sin x} dx - \int_0^{\pi} \frac{x}{1 + \sin\alpha . \sin x} dx$$

$$I = \pi \int_0^{\pi} \frac{dx}{1 + \sin\alpha . \sin x} - I$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{dx}{1 + \sin\alpha . \sin x} = \pi \int_0^{\pi} \frac{dx}{1 + \sin\alpha . \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}}$$

$$= \pi \int_0^{\pi} \frac{\left(1 + \tan^2\frac{x}{2}\right)}{1 + \tan^2\frac{x}{2} + 2\sin\alpha . \tan\frac{x}{2}} dx = \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 2\sin\alpha . \tan\frac{x}{2} + 1} dx$$

Let $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2} dx = 2dt$; $x = 0 \Rightarrow t = 0$ and $x = \pi \Rightarrow t = \infty$

$$\therefore \quad 2I = 2\pi \int_0^{\infty} \frac{dt}{t^2 + 2\sin\alpha t + 1}$$

$$I = \pi \int_0^{\infty} \frac{dt}{t^2 + 2\sin\alpha t + \sin^2\alpha - \sin^2\alpha + 1}$$

$$= \pi \int_0^{\infty} \frac{dt}{(t + \sin\alpha)^2 + (1 - \sin^2\alpha)} = \pi \int_0^{\infty} \frac{dt}{(t + \sin\alpha)^2 + \cos^2\alpha}$$

$$= \frac{\pi}{\cos\alpha}\left[\tan^{-1}\frac{t + \sin\alpha}{\cos\alpha}\right]_0^{\infty} = \frac{\pi}{\cos\alpha}\left[\tan^{-1}\frac{\tan\frac{x}{2} + \sin\alpha}{\cos\alpha}\right]_0^{\infty}$$

$$= \frac{\pi}{\cos\alpha}\left[\frac{\pi}{2} - \tan^{-1}(\tan\alpha)\right] = \frac{\pi}{\cos\alpha}\left(\frac{\pi}{2} - \alpha\right)$$

$$= \frac{\pi(\pi - 2\alpha)}{2\cos\alpha}$$

13. Evaluate: $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)dx$ **[CBSE 2023 (65/5/1)]**

Sol. $I = \int_0^{\pi/2} \sin(2x)\tan^{-1}(\sin x)dx = \int_0^{\pi/2} \tan^{-1}(\sin x)\sin(2x)dx$

$\because \int \tan^{-1}(\sin x)\sin(2x)dx$

$$= \tan^{-1}(\sin x)\int \sin(2x)dx - \int\left\{\frac{d}{dx}(\tan^{-1}(\sin x))\int \sin(2x)dx\right\}dx$$

$$= \tan^{-1}(\sin x)\times\left(\frac{-\cos 2x}{2}\right) - \frac{1}{1+\sin^2 x}\times \cos x \times\left(\frac{-\cos 2x}{2}\right)dx$$

$$= -\frac{1}{2}\tan^{-1}(\sin x)\cos(2x) + \frac{1}{2}\int\frac{(1-2\sin^2 x)}{1+\sin^2 x}\times \cos x\, dx \qquad [\because \cos 2x = 1-2\sin^2 x]$$

Put $\sin x = u \Rightarrow \cos x\, dx = du$

$$= -\frac{1}{2}\tan^{-1}(\sin x)\cos(2x) + \frac{1}{2}\int\frac{1-2u^2}{1+u^2}du$$

$$\because \quad \frac{1-2u^2}{1+u^2} = -2 + \frac{3}{1+u^2}$$

$$\therefore \quad \int\frac{1-2u^2}{1+u^2}du = -2\int du + 3\int\frac{du}{1+u^2}$$

$$= -2u + 3\tan^{-1} u$$

$$= -2\sin x + 3\tan^{-1}(\sin x) + C$$

Now, $\int \tan^{-1}(\sin x)\sin(2x)dx$

$$= -\frac{1}{2}\tan^{-1}(\sin x)\cos(2x) + \frac{1}{2}[-2\sin x + 3\tan^{-1}(\sin x) + C]$$

$$\therefore \quad I = \left[-\frac{1}{2}\tan^{-1}(\sin x)\cos(2x) - \sin x + \frac{3}{2}\tan^{-1}(\sin x)\right]_0^{\pi/2}$$

$$= \left[\left\{-\frac{1}{2}\tan^{-1}\left(\sin\frac{\pi}{2}\right)\cos\left(2\times\frac{\pi}{2}\right) - \sin\frac{\pi}{2} + \frac{3}{2}\tan^{-1}\left(\sin\frac{\pi}{2}\right)\right\}\right.$$

$$\left. -\left\{-\frac{1}{2}\tan^{-1}(\sin 0)\cos(0) - \sin(0) + \frac{3}{2}\tan^{-1}(\sin 0)\right\}\right]$$

$$= \left[\left\{-\frac{1}{2}\tan^{-1}(1)\cos(\pi) - 1 + \frac{3}{2}\tan^{-1}(1)\right\} - 0\right] \qquad [\because \tan^{-1}(0) = 0 = \sin 0]$$

$$= \frac{1}{2}\times\frac{\pi}{4} - 1 + \frac{3}{2}\times\frac{\pi}{4} \qquad \left[\because \tan^{-1}(1) = \frac{\pi}{4} \text{ and } \cos\pi = -1\right]$$

$$= \frac{\pi}{8} - 1 + \frac{3\pi}{8} = \frac{4\pi}{8} - 1$$

$$= \frac{\pi}{2} - 1$$

14. **Find:** $\int \frac{2x+1}{\sqrt{3+2x-x^2}}dx$ [*CBSE 2020 (65/1/2)*]

Sol.

$\int \frac{2x+1\ dx}{\sqrt{3+2x-x^2}}$.

Now $2x+1 = A(2-2x)+B$

$2x+1 = x(-2A) + B+2A$.

$A = -1 \qquad B = 3$

$I = -1\int \frac{2-2x}{\sqrt{3+2x-x^2}}dx + 3\int \frac{dx}{\sqrt{3+2x-x^2}}$

Now let $3+2x-x^2 = z^2$ and

$(2-2x)dx = 2z\,dz$

$3+2x-x^2$

$= -(x^2-2x-3)$

$= -(x^2-2.x+1-4)$

$= 2^2-(x-1)^2$

$\therefore I = -\int \frac{2z\,dz}{z} + 3\int \frac{dx}{\sqrt{2^2-(x-1)^2}}$

$= -2z + 3\left[\sin^{-1}\left(\frac{x-1}{2}\right)\right] + C$

$= -2\sqrt{3+2x-x^2} + 3\sin^{-1}\left(\frac{x-1}{2}\right) + C$, 'c' is integration constant

[*Topper's Answer 2020*]

Questions for Practice

■ Objective Type Questions

1. **Choose and write the correct option in each of the following questions.**

(*i*) $\int \frac{x^9}{(4x^2+1)^6}dx$ is equal to [*NCERT Exemplar*]

(*a*) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (*b*) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(*c*) $\frac{1}{10x}\left(\frac{1}{x^2}+4\right)^{-5}+C$ (*d*) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

(*ii*) The integral of $\int \frac{x}{\sqrt{x}+1}dx$ is equal to

(*a*) $2\left[\frac{x\sqrt{x}}{3}-\frac{x}{2}+\sqrt{x}-\log\left|(\sqrt{x}+1)\right|\right]+C$ (*b*) $\frac{x\sqrt{x}}{3}+\frac{x}{2}-\sqrt{x}+\log(\sqrt{x}+1)+C$

(*c*) $\sqrt{x}-\log(\sqrt{x}+1)+C$ (*d*) None of these

(*iii*) $\int e^x[f(x)+f'(x)]dx = e^x\sin x + C$ then $f(x)$ is equal to

(*a*) $\sin x$ (*b*) $-\sin x$ (*c*) $\cos x - \sin x$ (*d*) $\sin x + \cos x$

(*iv*) $\int \frac{a^{\sqrt{x}}}{\sqrt{x}}dx$ is

(*a*) $a^{\sqrt{x}}\log a + C$ (*b*) $2a^{\sqrt{x}}\log_e a + C$ (*c*) $2a^{\sqrt{x}}\log_{10} a + C$ (*d*) $\frac{2a^{\sqrt{x}}}{\log_e a}+C$

(*v*) $\int_1^3 \frac{3\cos(\log x)}{x}dx$ is equal to

(*a*) sin (log 3) (*b*) cos (log 3) (*c*) 1 (*d*) $\frac{\pi}{4}$

(*vi*) $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$ is equal to

(*a*) 1 (*b*) $\frac{\pi^2}{4}$ (*c*) $\frac{\pi^2}{32}$ (*d*) none of these

Conceptual Questions

2. Write the antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$. [*CBSE Delhi 2014*]
3. If $\int (ax+b)^2 dx = f(x)+C$, find $f(x)$. [*CBSE (F) 2010*]
4. If $\int (e^{ax}+bx)dx = 4e^{4x}+\frac{3x^2}{2}$, find the values of a and b. [*CBSE (AI) 2008*]
5. Evaluate: $\int \frac{dx}{\sin^2 x\cos^2 x}$ [*CBSE (F) 2014*]
6. Evaluate: $\int \cos^{-1}(\sin x)dx$ [*CBSE Delhi 2014*]
7. Evaluate: $\int \frac{dx}{\sqrt{1-x^2}}$ [*CBSE (AI) 2011*]
8. Write the value of $\int \frac{dx}{x^2+16}$. [*CBSE Delhi 2011*]
9. If $\int_0^a 3x^2 dx = 8$, write the value of 'a'. [*CBSE (F) 2012, 2014*]
10. If $f(x)=\int_0^x t\sin t\,dt$, then write the value of $f'(x)$. [*CBSE (AI) 2014*]
11. Evaluate: $\int_0^{\pi/4} \tan x\,dx$ [*CBSE (F) 2014*]
12. Evaluate: $\int_e^{e^2} \frac{dx}{x\log x}$ [*CBSE (AI) 2014*]

Very Short Answer Questions

13. Given $\int e^x(\tan x+1)\sec x\,dx = e^x f(x)+C$ [*CBSE (AI) 2012*]
    Write $f(x)$ satisfying the above.
14. If $\int \left(\frac{x-1}{x^2}\right)e^x dx = f(x)e^x + C$, find the value of $f(x)$. [*CBSE (F) 2012*]
15. Find: $\int \frac{x\,dx}{1+x\tan x}$ [*CBSE Bhubneshwar 2015*]
16. Evaluate: $\int \frac{(x+3)e^x}{(x+5)^3}dx$ [*CBSE Punchkula 2015*]
17. Evaluate: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
18. Evaluate: $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [*CBSE (AI) 2012, (F) 2016*]

19. Show that $\int_0^{\pi/2}(\sqrt{\tan x}+\sqrt{\cot x}\,dx)=\sqrt{2}\pi$. [CBSE (AI) 2008]

20. Evaluate: $\int_0^{\pi/4}\left(\frac{\sin x+\cos x}{3+\sin 2x}\right)dx$ [CBSE (Ajmer) 2014]

21. Evaluate: $\int_0^{\pi/2}\left(\frac{5\sin x+3\cos x}{\sin x+\cos x}\right)dx$ [CBSE Bhubneshwar 2015]

22. Evaluate: $\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx$ [CBSE Patna 2015]

23. Evaluate: $\int_0^{\pi/2}e^x(\sin x-\cos x)\,dx$ [CBSE Delhi 2014]

24. Write the value of the following integral $\int_{-\pi/2}^{\pi/2}\sin^5 x\,dx$. [CBSE (AI) 2010]

■ Short Answer Questions

25. Find : $\int\frac{e^x}{\sqrt{e^{2x}-4e^x-5}}dx$ [CBSE 2023 (65/3/2)]

26. Evaluate: $\int x\sin^{-1}x\,dx$ [CBSE (AI) 2009; Chennai 2015]

27. Evaluate: $\int\frac{dx}{x(x^5+3)}$ [CBSE (AI) 2013]

28. Evaluate: $\int x^2.\cos^{-1}x\,dx$ [CBSE (F) 2009]

29. Evaluate: $\int\frac{dx}{x(x^3+8)}$ [CBSE (AI) 2013]

30. Evaluate: $\int\frac{3x+5}{\sqrt{x^2-8x+7}}dx$ [CBSE (F) 2011]

31. Evaluate: $\int\frac{1-x^2}{x(1-2x)}dx$ [CBSE Delhi 2010, (F) 2013]

32. Evaluate: $\int\frac{(x+2)}{\sqrt{(x-2)(x-3)}}dx$ [CBSE (AI) 2010]

33. Evaluate the following indefinite integral: $\int\frac{\sin\phi}{\sqrt{\sin^2\phi+2\cos\phi+3}}d\phi$ [CBSE Sample Paper 2016]

34. Find: $\int\frac{x^2}{x^4+x^2-2}dx$ [CBSE (Central) 2016]

35. Find: $\int\frac{x^2+x+1}{(x^2+1)(x+2)}dx$ [CBSE (North) 2016]

36. Find: $\int\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$ [CBSE (F) 2016]

37. Evaluate: $\int\frac{2x^2+3}{x^2+5x+6}dx$ [CBSE (F) 2013]

38. Evaluate: $\int e^{2x}.\sin(3x+1)\,dx$ [CBSE (F) 2015]

39. Evaluate: $\int\frac{1-\cos x}{\cos x(1+\cos x)}dx$ [CBSE Chennai 2015]

40. Find: $\int\frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}dx$ [CBSE 2019 (65/3/1)]

41. Evaluate: $\int(\sqrt{\cot x}+\sqrt{\tan x})dx$ [CBSE (AI) 2014; Patna 2015]

42. Evaluate: $\int_0^{\pi}\frac{x}{1+\sin x}dx$ [CBSE Delhi 2010]

43. Evaluate: $\int_0^4(|x|+|x-2|+|x-4|)dx$ [CBSE Delhi 2013]

44. Evaluate: $\int_0^{\pi}\frac{x\sin x}{1+3\cos^2 x}dx$ [CBSE (East) 2016]

45. Evaluate the following definite integral: [CBSE Sample Paper 2016]

$$\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx$$

46. Evaluate: $\int_{-\pi/2}^{\pi/2}e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ [CBSE Guwahati 2015]

47. Evaluate: $\int_0^{\pi/2}\log\sin x\,dx$

Long Answer Questions

48. Evaluate: $\int\frac{\sin x-x\cos x}{x(x+\sin x)}dx$ [CBSE Ajmer 2015]

49. Find: $\int\frac{1}{x(x^4-1)}dx$

50. Evaluate: $\int_0^{\pi/2}\log(\sin x)\,dx$

51. Evaluate: $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$ [CBSE Delhi 2014]

52. Find: $\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$ [CBSE Allahabad 2015]

Answers

1. (*i*) (*d*) (*ii*) (*a*) (*iii*) (*a*) (*iv*) (*d*) (*v*) (*a*) (*vi*) (*c*)

2. $2x^{3/2}+2\sqrt{x}+C$ 3. $\frac{(ax+b)^3}{3a}$ 4. a can't be determined, $b=3$

5. $\tan x-1/\tan x+C$ 6. $\frac{\pi x}{2}-\frac{x^2}{2}+C$ 7. $\sin^{-1}x+C$ 8. $\frac{1}{4}\tan^{-1}\frac{x}{4}+C$

9. $a=2$ 10. $x\sin x$ 11. $\frac{1}{2}\log 2$ 12. $\log 2$ 13. $f(x)=\sec x$

14. $f(x)=\frac{1}{x}$ 15. $\log|\cos x+x\sin x|+C$ 16. $e^x.\frac{1}{(x+5)^2}+C$ 17. $2\sin\sqrt{x}+C$

18. $x-\sqrt{1-x^2}\sin^{-1}x+C$ 20. $\frac{1}{4}\log 3$ 21. 2π 22. $\frac{\pi}{4}$ 23. 1

24. 0 25. $\log|e^x-2+\sqrt{e^{2x}-4e^x-5}|+C$

26. $\frac{x^2}{2}\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+C$ 27. $\frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right|+C$

28. $\frac{x^3}{3}\cos^{-1}x-\frac{1}{3}\sqrt{1-x^2}+\frac{1}{9}(1-x^2)^{3/2}+C$ 29. $\frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right|+C$

30. $3\sqrt{x^2-8x+7}+17\log|(x-4)+\sqrt{x^2-8x+7}|+C$

31. $\frac{1}{2}x+\log|x|-\frac{3}{4}\log|2x-1|+C$

32. $\sqrt{x^2-5x+6}+\frac{9}{2}\log\left|\left(x-\frac{5}{2}\right)+\sqrt{x^2-5x+6}\right|+C$

33. $-\sin^{-1}\left(\frac{\cos\phi-1}{\sqrt{5}}\right)+C$

34. $\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{1}{6}\log\left|\frac{x-1}{x+1}\right|+C$

35. $\frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$

36. $\frac{2}{9}(10-4x-3x^2)^{3/2}+\frac{11}{2\sqrt{3}}\left(x+\frac{2}{3}\right)\sqrt{10-4x-3x^2}+\frac{17}{9}\sin^{-1}\left[\frac{3}{\sqrt{34}}\left(x+\frac{2}{3}\right)\right]+C$

37. $2x+11\log|x+2|-21\log|x+3|+C$

38. $\frac{2e^{2x}.\sin(3x+1)}{13}-\frac{3e^{2x}.\cos(3x+1)}{13}+C$

39. $\log|\sec x+\tan x|-2\tan\frac{x}{2}+C$

40. $\frac{1}{2}\log(\sin^2x+1)-\frac{1}{2}\log(\sin^2x+3)+C$

41. $\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$

42. π

43. 20

44. $\frac{\sqrt{3}\pi^2}{9}$

45. π^2

46. $\frac{e^{\pi/2}}{2}$

47. $-\frac{\pi}{2}\log 2$

48. $\log|x|-\log|x+\sin x|+C$ **[Hint:** Write $\int\frac{\sin x-x\cos x}{x(x+\sin x)}dx=\int\frac{(x+\sin x)-x(1+\cos x)}{x(x+\sin x)}dx$]

49. $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$

50. $-\frac{\pi}{2}\log 2$

51. $\frac{\pi}{12}$

52. $\frac{6}{5}$; Hint: $\int_0^{\pi/4}\frac{dx}{\cos^3x\sqrt{2\sin 2x}}=\frac{1}{2}\int_0^{\pi/4}\frac{dx}{\cos^3x\sqrt{\frac{\sin x}{\cos x}.\cos^2x}}=\frac{1}{2}\int_0^{\pi/2}\frac{dx}{\cos^4x\sqrt{\tan x}}$. Then put $\tan x=t$