HIGHER ALGEBRA.

CHAPTER I.

RATIO.

1. Definition. **Ratio** is the relation which one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part, or parts, one quantity is of the other.

The ratio of A to B is usually written $A : B$. The quantities A and B are called the *terms* of the ratio. The first term is called the **antecedent**, the second term the **consequent**.

2. To find what multiple or part A is of B, we divide A by B; hence the ratio $A : B$ may be measured by the fraction $\frac{A}{B}$, and we shall usually find it convenient to adopt this notation.

In order to compare two quantities they must be expressed in terms of the same unit. Thus the ratio of £2 to 15*s.* is measured by the fraction $\frac{2 \times 20}{15}$ or $\frac{8}{3}$.

Note. A ratio expresses the *number* of times that one quantity contains another, and therefore *every ratio is an abstract quantity.*

3. Since by the laws of fractions,

$$\frac{a}{b} = \frac{ma}{mb},$$

it follows that the ratio $a : b$ is equal to the ratio $ma : mb$; that is, *the value of a ratio remains unaltered if the antecedent and the consequent are multiplied or divided by the same quantity.*

4. Two or more ratios may be compared by reducing their equivalent fractions to a common denominator. Thus suppose $a : b$ and $x : y$ are two ratios. Now $\frac{a}{b} = \frac{ay}{by}$, and $\frac{x}{y} = \frac{bx}{by}$; hence the ratio $a : b$ is greater than, equal to, or less than the ratio $x : y$ according as ay is greater than, equal to, or less than bx.

5. The ratio of two fractions can be expressed as a ratio of two integers. Thus the ratio $\frac{a}{b} : \frac{c}{d}$ is measured by the fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$, or $\frac{ad}{bc}$; and is therefore equivalent to the ratio $ad : bc$.

6. If either, or both, of the terms of a ratio be a surd quantity, then no two integers can be found which will *exactly* measure their ratio. Thus the ratio $\sqrt{2} : 1$ cannot be exactly expressed by any two integers.

7. Definition. If the ratio of any two quantities can be expressed exactly by the ratio of two integers, the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable**.

Although we cannot find two integers which will exactly measure the ratio of two incommensurable quantities, we can always find two integers whose ratio differs from that required by as small a quantity as we please.

Thus $$\frac{\sqrt{5}}{4} = \frac{2{\cdot}236068\ldots}{4} = {\cdot}559017\ldots$$

and therefore $$\frac{\sqrt{5}}{4} > \frac{559017}{1000000} \text{ and } < \frac{559018}{1000000};$$

so that the difference between the ratios $559017 : 1000000$ and $\sqrt{5} : 4$ is less than $\cdot000001$. By carrying the decimals further, a closer approximation may be arrived at.

8. Definition. Ratios are *compounded* by multiplying together the fractions which denote them; or by multiplying together the antecedents for a new antecedent, and the consequents for a new consequent.

Example. Find the ratio compounded of the three ratios

$$2a : 3b,\ 6ab : 5c^2,\ c : a$$

$$\text{The required ratio} = \frac{2a}{3b} \times \frac{6ab}{5c^2} \times \frac{c}{a}$$
$$= \frac{4a}{5c}.$$

9. DEFINITION. When the ratio $a : b$ is compounded with itself the resulting ratio is $a^2 : b^2$, and is called the **duplicate ratio** of $a : b$. Similarly $a^3 : b^3$ is called the **triplicate ratio** of $a : b$. Also $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **subduplicate ratio** of $a : b$.

Examples. (1) The duplicate ratio of $2a : 3b$ is $4a^2 : 9b^2$.
(2) The subduplicate ratio of $49 : 25$ is $7 : 5$.
(3) The triplicate ratio of $2x : 1$ is $8x^3 : 1$.

10. DEFINITION. A ratio is said to be a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater than*, *less than*, or *equal to* the consequent.

11. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

Let $\frac{a}{b}$ be the ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding x to both its terms.

Now
$$\frac{a}{b} - \frac{a+x}{b+x} = \frac{ax - bx}{b(b+x)}$$
$$= \frac{x(a-b)}{b(b+x)};$$

and $a - b$ is positive or negative according as a is greater or less than b.

Hence if $a > b$,
$$\frac{a}{b} > \frac{a+x}{b+x};$$

and if $a < b$,
$$\frac{a}{b} < \frac{a+x}{b+x};$$

which proves the proposition.

Similarly it can be proved that *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking the same quantity from both its terms.*

12. When two or more ratios are equal many useful propositions may be proved by introducing a single symbol to denote each of the equal ratios.

The proof of the following important theorem will illustrate the method of procedure.

If $$\frac{\mathrm{a}}{\mathrm{b}}=\frac{\mathrm{c}}{\mathrm{d}}=\frac{\mathrm{e}}{\mathrm{f}}=\ldots\ldots,$$

each of these ratios $=\left(\dfrac{\mathrm{pa}^{\mathrm{n}}+\mathrm{qc}^{\mathrm{n}}+\mathrm{re}^{\mathrm{n}}+\ldots}{\mathrm{pb}^{\mathrm{n}}+\mathrm{qd}^{\mathrm{n}}+\mathrm{rf}^{\mathrm{n}}+\ldots}\right)^{\frac{1}{\mathrm{n}}}$,

where p, q, r, n *are any quantities whatever.*

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k;$$

then $$a=bk,\ c=dk,\ e=fk,\ldots;$$

whence $$pa^n=pb^nk^n,\ qc^n=qd^nk^n,\ re^n=rf^nk^n,\ldots;$$

$$\therefore\ \frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}=\frac{pb^nk^n+qd^nk^n+rf^nk^n+\ldots}{pb^n+qd^n+rf^n+\ldots}$$
$$=k^n;$$

$$\therefore\ \left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\ldots$$

By giving different values to p, q, r, n many particular cases of this general proposition may be deduced; or they may be proved independently by using the same method. For instance,

if $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots,$$

each of these ratios $=\dfrac{a+c+e+\ldots}{b+d+f+\ldots}$;

a result of such frequent utility that the following verbal equivalent should be noticed: *When a series of fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

Example 1. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, shew that

$$\frac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\frac{ace}{bdf}.$$

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k;$$

then $$a=bk,\ c=dk,\ e=fk;$$

$$\therefore \frac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\frac{b^4k^3+2d^2fk^3-3bf^3k^3}{b^4+2d^2f-3bf^3}$$

$$=k^3=\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}$$

$$=\frac{ace}{bdf}.$$

Example 2. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}.$$

Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$, so that $x=ak$, $y=bk$, $z=ck$;

then $$\frac{x^2+a^2}{x+a}=\frac{a^2k^2+a^2}{ak+a}=\frac{(k^2+1)\,a}{k+1};$$

$$\therefore \frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(k^2+1)\,a}{k+1}+\frac{(k^2+1)\,b}{k+1}+\frac{(k^2+1)\,c}{k+1}$$

$$=\frac{(k^2+1)\,(a+b+c)}{k+1}$$

$$=\frac{k^2\,(a+b+c)^2+(a+b+c)^2}{k\,(a+b+c)+a+b+c}$$

$$=\frac{(ka+kb+kc)^2+(a+b+c)^2}{(ka+kb+kc)+a+b+c}$$

$$=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}.$$

13. If an equation is homogeneous with respect to certain quantities, we may for these quantities substitute in the equation any others proportional to them. For instance, the equation

$$lx^3y+mxy^2z+ny^2z^2=0$$

is homogeneous in x, y, z. Let α, β, γ be three quantities proportional to x, y, z respectively.

Put $k=\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$, so that $x=\alpha k$, $y=\beta k$, $z=\gamma k$;

then $$l\alpha^3\beta k^4+m\alpha\beta^2\gamma k^4+n\beta^2\gamma^2k^4=0,$$

that is, $$l\alpha^3\beta+m\alpha\beta^2\gamma+n\beta^2\gamma^2=0;$$

an equation of the same form as the original one, but with α, β, γ in the places of x, y, z respectively.

14. The following theorem is important.

If $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}, \dots \dfrac{a_n}{b_n}$ *be unequal fractions, of which the denominators are all of the same sign, then the fraction*

$$\frac{a_1 + a_2 + a_3 + \dots + a_n}{b_1 + b_2 + b_3 + \dots + b_n}$$

lies in magnitude between the greatest and least of them.

Suppose that all the denominators are positive. Let $\dfrac{a_r}{b_r}$ be the least fraction, and denote it by k; then

$$\frac{a_r}{b_r} = k; \qquad \therefore\ a_r = kb_r;$$

$$\frac{a_1}{b_1} > k; \qquad \therefore\ a_1 > kb_1;$$

$$\frac{a_2}{b_2} > k; \qquad \therefore\ a_2 > kb_2;$$

and so on;

$\therefore$ by addition,

$$a_1 + a_2 + a_3 + \dots\dots + a_n > (b_1 + b_2 + b_3 + \dots\dots + b_n)\,k;$$

$$\therefore\ \frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b_n} > k; \text{ that is, } > \frac{a_r}{b_r}.$$

Similarly we may prove that

$$\frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b_n} < \frac{a_s}{b_s},$$

where $\dfrac{a_s}{b_s}$ is the greatest of the given fractions.

In like manner the theorem may be proved when all the denominators are negative.

15. The ready application of the *general principle* involved in Art. 12 is of such great value in all branches of mathematics, that the student should be able to use it with some freedom in any particular case that may arise, without necessarily introducing an auxiliary symbol.

Example 1. If $\dfrac{x}{b+c-a} = \dfrac{y}{c+a-b} = \dfrac{z}{a+b-c}$,

prove that $$\frac{x+y+z}{a+b+c} = \frac{x(y+z)+y(z+x)+z(x+y)}{2(ax+by+cz)}.$$

Each of the given fractions $= \dfrac{\text{sum of numerators}}{\text{sum of denominators}}$

$$= \frac{x+y+z}{a+b+c} \text{ ....(1).}$$

Again, if we multiply both numerator and denominator of the three given fractions by $y+z$, $z+x$, $x+y$ respectively,

$$\text{each fraction} = \frac{x(y+z)}{(y+z)(b+c-a)} = \frac{y(z+x)}{(z+x)(c+a-b)} = \frac{z(x+y)}{(x+y)(a+b-c)}$$

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$= \frac{x(y+z)+y(z+x)+z(x+y)}{2ax+2by+2cz} \text{ ...... (2).}$$

$\therefore$ from (1) and (2),

$$\frac{x+y+z}{a+b+c} = \frac{x(y+z)+y(z+x)+z(x+y)}{2(ax+by+cz)}.$$

Example 2. If $\dfrac{x}{l(mb+nc-la)} = \dfrac{y}{m(nc+la-mb)} = \dfrac{z}{n(la+mb-nc)}$,

prove that $\dfrac{l}{x(by+cz-ax)} = \dfrac{m}{y(cz+ax-by)} = \dfrac{n}{z(ax+by-cz)}$.

We have $\dfrac{\dfrac{x}{l}}{mb+nc-la} = \dfrac{\dfrac{y}{m}}{nc+la-mb} = \dfrac{\dfrac{z}{n}}{la+mb-nc}$

$$= \frac{\dfrac{y}{m}+\dfrac{z}{n}}{2la}$$

$=$ two similar expressions;

$$\therefore \quad \frac{ny+mz}{a} = \frac{lz+nx}{b} = \frac{mx+ly}{c}.$$

Multiply the first of these fractions above and below by x, the second by y, and the third by z; then

$$\frac{nxy+mxz}{ax} = \frac{lyz+nxy}{by} = \frac{mxz+lyz}{cz}$$

$$= \frac{2lyz}{by+cz-ax}$$

$=$ two similar expressions;

$$\therefore \quad \frac{l}{x(by+cz-ax)} = \frac{m}{y(cz+ax-by)} = \frac{n}{z(ax+by-cz)}.$$

16. If we have *two* equations containing *three* unknown quantities in the first degree, such as

$$a_1x + b_1y + c_1z = 0 \text{ ....(1)},$$

$$a_2x + b_2y + c_2z = 0 \text{ ....(2)},$$

we cannot solve these completely; but by writing them in the form

$$a_1\left(\frac{x}{z}\right) + b_1\left(\frac{y}{z}\right) + c_1 = 0,$$

$$a_2\left(\frac{x}{z}\right) + b_2\left(\frac{y}{z}\right) + c_2 = 0,$$

we can, by regarding $\frac{x}{z}$ and $\frac{y}{z}$ as the unknowns, solve in the ordinary way and obtain

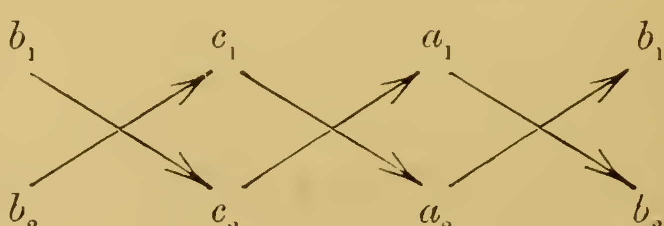
$$\frac{x}{z} = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \qquad \frac{y}{z} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1};$$

or, more symmetrically,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \text{ ....(3)}.$$

It thus appears that when we have two equations of the type represented by (1) and (2) we may always by the above formula write down the *ratios* $x : y : z$ in terms of the coefficients of the equations by the following rule:

Write down the coefficients of x, y, z in order, beginning with those of y; and repeat these as in the diagram.



Multiply the coefficients across in the way indicated by the arrows, remembering that in forming the products any one obtained by descending is positive, and any one obtained by ascending is negative. The three results

$$b_1c_2 - b_2c_1, \quad c_1a_2 - c_2a_1, \quad a_1b_2 - a_2b_1$$

are proportional to x, y, z respectively.

This is called the **Rule of Cross Multiplication.**

Example 1. Find the ratios of $x : y : z$ from the equations

$$7x = 4y + 8z, \quad 3z = 12x + 11y.$$

By transposition we have $7x - 4y - 8z = 0,$

$$12x + 11y - 3z = 0.$$

Write down the coefficients, thus

$$\begin{array}{cccc} -4 & -8 & 7 & -4 \\ 11 & -3 & 12 & 11, \end{array}$$

whence we obtain the products

$$(-4) \times (-3) - 11 \times (-8), \quad (-8) \times 12 - (-3) \times 7, \quad 7 \times 11 - 12 \times (-4),$$

or $$100, \quad -75, \quad 125;$$

$$\therefore \quad \frac{x}{100} = \frac{y}{-75} = \frac{z}{125},$$

that is, $$\frac{x}{4} = \frac{y}{-3} = \frac{z}{5}.$$

Example 2. Eliminate x, y, z from the equations

$$a_1x + b_1y + c_1z = 0 \quad \text{.......(1)},$$

$$a_2x + b_2y + c_2z = 0 \quad \text{.......(2)},$$

$$a_3x + b_3y + c_3z = 0 \quad \text{.......(3)}.$$

From (2) and (3), by cross multiplication,

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2};$$

denoting each of these ratios by k, by multiplying up, substituting in (1), and dividing out by k, we obtain

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0.$$

This relation is called the **eliminant** of the given equations.

Example 3. Solve the equations

$$ax + by + cz = 0 \quad \text{.......(1)},$$

$$x + y + z = 0 \quad \text{.......(2)},$$

$$bcx + cay + abz = (b - c)(c - a)(a - b) \quad \text{.......(3)}.$$

From (1) and (2), by cross multiplication,

$$\frac{x}{b - c} = \frac{y}{c - a} = \frac{z}{a - b} = k, \text{ suppose};$$

$$\therefore \quad x = k(b - c), \; y = k(c - a), \; z = k(a - b).$$

Substituting in (3),

$$k\{bc(b - c) + ca(c - a) + ab(a - b)\} = (b - c)(c - a)(a - b),$$

$$k\{-(b - c)(c - a)(a - b)\} = (b - c)(c - a)(a - b);$$

$$\therefore \quad k = -1;$$

whence $$x = c - b, \; y = a - c, \; z = b - a.$$

17. If in Art. 16 we put $z=1$, equations (1) and (2) become

$$a_1x+b_1y+c_1=0,$$
$$a_2x+b_2y+c_2=0;$$

and (3) becomes

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

or

$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\quad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Hence any two simultaneous equations involving two unknowns in the first degree may be solved by the rule of cross multiplication.

Example. Solve $5x-3y-1=0,\ x+2y=12.$

By transposition,

$$5x-3y-\ 1=0,$$
$$x+2y-12=0;$$

$$\therefore\quad \frac{x}{36+2}=\frac{y}{-1+60}=\frac{1}{10+3};$$

whence

$$x=\frac{38}{13},\quad y=\frac{59}{13}.$$

EXAMPLES. I.

1. Find the ratio compounded of

(1) the ratio $2a : 3b$, and the duplicate ratio of $9b^2 : ab$.

(2) the subduplicate ratio of $64 : 9$, and the ratio $27 : 56$.

(3) the duplicate ratio of $\frac{2a}{b} : \frac{\sqrt{6a^2}}{b^2}$, and the ratio $3ax : 2by$.

2. If $x+7 : 2(x+14)$ in the duplicate ratio of $5 : 8$, find x.

3. Find two numbers in the ratio of $7 : 12$ so that the greater exceeds the less by 275.

4. What number must be added to each term of the ratio $5 : 37$ to make it equal to $1 : 3$?

5. If $x : y=3 : 4$, find the ratio of $7x-4y : 3x+y$.

6. If $15(2x^2-y^2)=7xy$, find the ratio of $x : y$.

7. If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},$$
prove that
$$\frac{2a^4b^2+3a^2e^2-5e^4f}{2b^6+3b^2f^2-5f^5}=\frac{a^4}{b^4}.$$

8. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a}{d}$ is equal to
$$\sqrt{\frac{a^5+b^2c^2+a^3c^2}{b^4c+d^4+b^2cd^2}}.$$

9. If
$$\frac{x}{q+r-p}=\frac{y}{r+p-q}=\frac{z}{p+q-r},$$
shew that
$$(q-r)x+(r-p)y+(p-q)z=0.$$

10. If $\frac{y}{x-z}=\frac{y+x}{z}=\frac{x}{y}$, find the ratios of $x:y:z$.

11. If
$$\frac{y+z}{pb+qc}=\frac{z+x}{pc+qa}=\frac{x+y}{pa+qb},$$
shew that
$$\frac{2(x+y+z)}{a+b+c}=\frac{(b+c)x+(c+a)y+(a+b)z}{bc+ca+ab}.$$

12. If
$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c},$$
shew that
$$\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}.$$

13. If
$$\frac{2y+2z-x}{a}=\frac{2z+2x-y}{b}=\frac{2x+2y-z}{c},$$
shew that
$$\frac{x}{2b+2c-a}=\frac{y}{2c+2a-b}=\frac{z}{2a+2b-c}.$$

14. If
$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2,$$
shew that
$$x:a=y:b=z:c.$$

15. If $l(my+nz-lx)=m(nz+lx-my)=n(lx+my-nz)$,

prove
$$\frac{y+z-x}{l}=\frac{z+x-y}{m}=\frac{x+y-z}{n}.$$

16. Shew that the eliminant of
$$ax+cy+bz=0,\ cx+by+az=0,\ bx+ay+cz=0,$$
is
$$a^3+b^3+c^3-3abc=0.$$

17. Eliminate x, y, z from the equations
$$ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0.$$

18. If $$x=cy+bz,\quad y=az+cx,\quad z=bx+ay,$$
shew that $$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2}.$$

19. Given that $a(y+z)=x,\ b(z+x)=y,\ c(x+y)=z,$
prove that $$bc+ca+ab+2abc=1.$$

Solve the following equations:

20. $$3x-4y+7z=0,\quad 2x-y-2z=0,\quad 3x^3-y^3+z^3=18.$$

21. $$x+y=z,\quad 3x-2y+17z=0,\quad x^3+3y^3+2z^3=167.$$

22. $$7yz+3zx=4xy,\quad 21yz-3zx=4xy,\quad x+2y+3z=19.$$

23. $$3x^2-2y^2+5z^2=0,\quad 7x^2-3y^2-15z^2=0,\quad 5x-4y+7z=6.$$

24. If $$\frac{l}{\sqrt{a}-\sqrt{b}}+\frac{m}{\sqrt{b}-\sqrt{c}}+\frac{n}{\sqrt{c}-\sqrt{a}}=0,$$
$$\frac{l}{\sqrt{a}+\sqrt{b}}+\frac{m}{\sqrt{b}+\sqrt{c}}+\frac{n}{\sqrt{c}+\sqrt{a}}=0,$$
shew that $$\frac{l}{(a-b)(c-\sqrt{ab})}=\frac{m}{(b-c)(a-\sqrt{bc})}=\frac{n}{(c-a)(b-\sqrt{ac})}.$$

Solve the equations:

25. $$ax+by+cz=0,$$
$$bcx+cay+abz=0,$$
$$xyz+abc(a^3x+b^3y+c^3z)=0.$$

26. $$ax+by+cz=a^2x+b^2y+c^2z=0,$$
$$x+y+z+(b-c)(c-a)(a-b)=0.$$

27. If $$a(y+z)=x,\quad b(z+x)=y,\quad c(x+y)=z,$$
prove that $$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

28. If $ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0,$
prove that

(1) $$\frac{x^2}{bc-f^2}=\frac{y^2}{ca-g^2}=\frac{z^2}{ab-h^2}.$$

(2) $$(bc-f^2)(ca-g^2)(ab-h^2)=(fg-ch)(gh-af)(hf-bg).$$