07

Trigonometric Indentities

7.01. Introduction

In the previous chapter we have studied trigonometric ratios and their mutual relations. In this chapter we will discuss trigonometric identities.

Trigonometric Identities

Trigonometric identities are true for all angles involved. Here we discuss proof of following identities.

According to figure, in Δ ABC , $\angle B$ is right angle. For angle θ, BC is perpendicular AB is base and AC will be hypotenuse.

$$\therefore \quad BC^2 + AB^2 = AC^2 \quad \ldots (1)$$

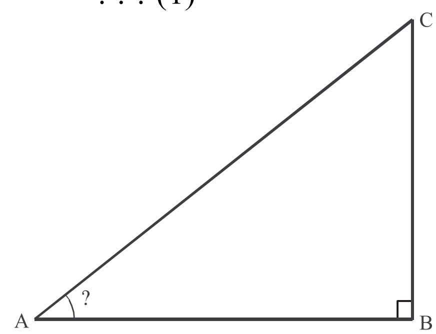


Fig. 7.01

Dividing both sides of equation (1) by AC^2

$$\frac{BC^2}{AC^2} + \frac{AB^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\Rightarrow \quad \left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

$$\Rightarrow \quad (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = 1 \quad \ldots (2)$$

For all θ where $0^o \le \theta \le 90^o$ is true value

This is an identity.

Now dividing each term of equation (1) by AB^2

$$\frac{BC^2}{AB^2} + \frac{AB^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\left(\frac{BC}{AB}\right)^2 + \left(\frac{AB}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$$(\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\Rightarrow \quad \tan^2\theta + 1 = \sec^2\theta$$

$$\Rightarrow \quad 1 + \tan^2\theta = \sec^2\theta \quad \ldots (3)$$

Now dividing each term of equation (1) by BC^2

$$\frac{BC^2}{BC^2} + \frac{AB^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\left(\frac{BC}{BC}\right)^2+\left(\frac{AB}{BC}\right)^2=\left(\frac{AC}{BC}\right)^2$$

$$1+(\cot\theta)^2=(\operatorname{cosec}\theta)^2$$

$$1+\cot^2\theta=\operatorname{cosec}^2\theta \qquad \text{. . . (iv)}$$

Above identities can be written as

1. $\sin^2\theta+\cos^2\theta=1$ or $\sin^2\theta=1-\cos^2\theta$ or $\cos^2\theta=1-\sin^2\theta$
2. $1+\tan^2\theta=\sec^2\theta$ or $\sec^2\theta-\tan^2\theta=1$ or $\tan^2\theta=\sec^2\theta-1$
3. $1+\cot^2\theta=\operatorname{cosec}^2\theta$ or $\operatorname{cosec}^2\theta-\cot^2\theta=1$ or $\cot^2\theta=\operatorname{cosec}^2-1$

Table

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

Illustrative Examples

Example 1. Prove that $\cot\theta+\tan\theta=\operatorname{cosec}\theta\sec\theta$

Solution : LHS $=\cot\theta+\tan\theta$

$$=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta} \qquad (\because \sin^2\theta+\cos^2\theta=1)$$

$$=\frac{1}{\sin\theta\cos\theta}=\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}$$

$$=\operatorname{cosec}\theta\cdot\sec\theta = \text{RHS}$$

Example 2. Prove that $\left(1+\tan^2\theta\right)\left(1+\sin\theta\right)\left(1-\sin\theta\right)=1$

Solution : LHS $=\left(1+\tan^2\theta\right)\left(1+\sin\theta\right)\left(1-\sin\theta\right)$

$$=\left(1+\tan^2\theta\right)\left(1-\sin^2\theta\right)$$

$$=\sec^2\theta\cos^2\theta$$

$$=\frac{1}{\cos^2\theta}\cdot\cos^2\theta$$

$=1=$ RHS

Example 3. Prove that $\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}=2\sec^2\theta$

Solution : LHS $=\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$

$$=\frac{1-\sin\theta+1+\sin\theta}{\left(1+\sin\theta\right)\left(1-\sin\theta\right)}$$

$$=\frac{2}{1-\sin^2\theta}=\frac{2}{\cos^2\theta}=2\cdot\frac{1}{\cos^2\theta}$$

$=2\sec^2\theta$ = RHS

Example 4. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}=\operatorname{cosec}\theta+\cot\theta$

Solution : LHS $=\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

Multiplying numerator and denominator by $\sqrt{1+\cos\theta}$

$$\sqrt{\frac{\left(1+\cos\theta\right)}{\left(1-\cos\theta\right)}\times\frac{\left(1+\cos\theta\right)}{\left(1+\cos\theta\right)}}$$

$$=\sqrt{\frac{\left(1+\cos\theta\right)^2}{1-\cos^2\theta}}=\sqrt{\frac{\left(1+\cos\theta\right)^2}{\sin^2\theta}}$$

$$=\frac{1+\cos\theta}{\sin\theta}=\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}$$

$=\operatorname{cosec}\theta+\cot\theta$ = RHS

Example 5. Prove that $(\sec\theta-\tan\theta)^2=\dfrac{1-\sin\theta}{1+\sin\theta}$

Solution : LHS $=(\sec\theta-\tan\theta)^2$

$$=\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=\left(\frac{1-\sin\theta}{\cos\theta}\right)^2$$

$$=\frac{(1-\sin\theta)^2}{\cos^2\theta}=\frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$=\frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$=\frac{1-\sin\theta}{1+\sin\theta}=\text{RHS}$$

Example 6. Prove that $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$

Solution : LHS $=\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}$

$$=\frac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\sin^2\theta+1+\cos^2\theta+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{1+\sin^2\theta+\cos^2\theta+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}=\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}=\frac{2}{\sin\theta}=2\cdot\frac{1}{\sin\theta}$$

$$=2\operatorname{cosec}\theta=\text{RHS}$$

Example 7. Prove that $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$

Solution : LHS $=\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}$

$$=\frac{\sin\theta\left(1-2\sin^2\theta\right)}{\cos\theta\left(2\cos^2\theta-1\right)}$$

$$=\frac{\sin\theta\left[\sin^2\theta+\cos^2\theta-2\sin^2\theta\right]}{\cos\theta\left[2\cos^2\theta-\left(\sin^2\theta+\cos^2\theta\right)\right]}\quad\left(\because\ \sin^2\theta+\cos^2\theta=1\right)$$

$$=\frac{\sin\theta\left[\cos^2\theta-\sin^2\theta\right]}{\cos\theta\left[\cos^2\theta-\sin^2\theta\right]}=\frac{\sin\theta}{\cos\theta}$$

$$=\tan\theta\ =\text{RHS}$$

Example 8. Prove that $\frac{1+\sec A}{\sec A}=\frac{\sin^2 A}{1-\cos A}$

Solution : LHS $=\frac{1+\sec A}{\sec A}$

$$=\frac{1+\frac{1}{\cos A}}{\frac{1}{\cos A}}=\frac{\frac{\cos A+1}{\cos A}}{\frac{1}{\cos A}}=\frac{\cos A+1}{\cos A}\times\frac{\cos A}{1}$$

$$=\frac{1+\cos A}{1}$$

$$=\frac{(1+\cos A)(1-\cos A)}{1-\cos A}$$

Multiplying numerator and denominator bya (1 – cosA)

$$=\frac{(1)^2-(\cos A)^2}{1-\cos A}=\frac{1-\cos^2 A}{1-\cos A}$$

$$=\frac{\sin^2 A}{1-\cos A}\ =\text{RHS}\quad\left(\because\ 1-\cos^2 A=\sin^2 A\right)$$

Example 9. Prove that $\cos^4\theta-\sin^4\theta=1-2\sin^2\theta$

Solution : LHS $=\cos^4\theta-\sin^4\theta$

$$=\left(\cos^2\theta\right)^2-\left(\sin^2\theta\right)^2$$

$$=\left(\cos^2\theta+\sin^2\theta\right)\left(\cos^2\theta-\sin^2\theta\right)\qquad\because\quad a^2-b^2=(a+b)(a-b)$$

$$=1\cdot\left(\cos^2\theta-\sin^2\theta\right)\qquad\because\quad\sin^2\theta+\cos^2\theta=1$$

$$=1\cdot\left(1-\sin^2\theta-\sin^2\theta\right)=1-2\sin^2\theta\ =\text{RHS}$$

Example 10 . If $\sin\theta+\cos\theta=p$ and $\sec\theta+\operatorname{cosec}\theta=q$, then prove that $q\left(p^2-1\right)=2p$.

Solution : LHS $=q\left(p^2-1\right)$

Putting value of p and q

$$=(\sec\theta+\operatorname{cosec}\theta)\left[(\sin\theta+\cos\theta)^2-1\right]$$

$$=\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)\left[\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1\right]$$

$$=\left(\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}\right)[1+2\sin\theta\cos\theta-1]$$

$$=\left[\frac{\sin\theta+\cos\theta}{\sin\theta\cdot\cos\theta}\right]\times(2\sin\theta\cos\theta)$$

$$=2[\sin\theta+\cos\theta]=2p=\text{RHS}$$

Example 11 . Prove that $\dfrac{\cot A+\operatorname{cosec}A-1}{\cot A-\operatorname{cosec}A+1}=\dfrac{1+\cos A}{\sin A}$

Solution : LHS $=\dfrac{\cot A+\operatorname{cosec}A-1}{\cot A-\operatorname{cosec}A+1}$

$$=\frac{(\cot A+\operatorname{cosec}A)-\left(\operatorname{cosec}^2A-\cot^2A\right)}{\cot A-\operatorname{cosec}A+1} \qquad \left(\because\ \operatorname{cosec}^2A-\cot^2A=1\right)$$

$$=\frac{(\operatorname{cosec}A+\cot A)-\left[(\operatorname{cosec}A+\cot A)(\operatorname{cosec}A-\cot A)\right]}{\cot A-\operatorname{cosec}A+1}$$

$$=\frac{(\operatorname{cosec}A+\cot A)\left[1-(\operatorname{cosec}A-\cot A)\right]}{\cot A-\operatorname{cosec}A+1}$$

$$=\frac{(\operatorname{cosec}A+\cot A)[\cot A-\operatorname{cosec}A+1]}{(\cot A-\operatorname{cosec}A+1)}$$

$$=\operatorname{cosec}A+\cot A$$

$$=\frac{1}{\sin A}+\frac{\cos A}{\sin A}=\frac{1+\cos A}{\sin A}=\text{RHS}$$

Example 12. Prove that $\left(\dfrac{1+\tan^2A}{1+\cot^2A}\right)=\left(\dfrac{1-\tan A}{1-\cot A}\right)^2=\tan^2A$

Solution : LHS $=\dfrac{1+\tan^2A}{1+\cot^2A}=\dfrac{\sec^2A}{\operatorname{cosec}^2A}$

$$= \left[\frac{\sec A}{\operatorname{cosec} A}\right]^2 = \left[\frac{1/\cos A}{1/\sin A}\right]^2 = \left[\frac{1}{\cos A} \times \frac{\sin A}{1}\right]^2$$

$$= \left[\frac{\sin A}{\cos A}\right]^2 = [\tan A]^2 = \tan^2 A = \text{RHS}$$

Now, $$\left[\frac{1-\tan A}{1-\cot A}\right]^2 = \left[\frac{1-\frac{\sin A}{\cos A}}{1-\frac{\cos A}{\sin A}}\right]^2 = \left[\frac{\frac{\cos A - \sin A}{\cos A}}{\frac{\sin A - \cos A}{\sin A}}\right]^2$$

$$= \left[\frac{\cos A - \sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A}\right]^2 = \left[-\frac{(\sin A - \cos A)}{\cos A} \times \frac{\sin A}{(\sin A - \cos A)}\right]^2$$

$$= \left[-\frac{\sin A}{\cos A}\right]^2 = [-\tan A]^2 = \tan^2 A = \text{RHS}$$

Exercise 7.1

1. Express all trigonometric ratios in term of $\sec\theta$, for angle θ.
2. Express trigonometric ratios $\sin\theta, \sec\theta, \tan\theta$ in terms of $\cot\theta$.

Prove the following with the help of identities.

3. $\cos^2\theta + \cos^2\theta\cot^2\theta = \cot^2\theta$
4. $\sec\theta(1-\sin\theta)(\sec\theta+\tan\theta) = 1$

7. $\operatorname{cosec}^2\theta + \sec^2\theta = \operatorname{cosec}^2\theta\sec^2\theta$

6. $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

7. $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

8. $\frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\tan\beta$

9. $\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta} = 2\sec\theta$

10. $\frac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta} = 1$

11. $\cot\theta - \tan\theta = \frac{1-2\sin^2\theta}{\sin\theta\cos\theta}$

12. $\cos^4\theta + \sin^4\theta = 1 - 2\cos^2\theta\sin^2\theta$

13. $(\sec\theta - \cos\theta)(\cot\theta + \tan\theta) = \tan\theta\sec\theta$

14. $\dfrac{1-\tan^2\alpha}{\cot^2\alpha - 1} = \tan^2\alpha$

15. $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$

16. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

17. $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = 1 + \tan\theta + \cot\theta$

18. $\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta) = \operatorname{cosec}\theta + \sec\theta$

19. $\sin^2\theta\cos\theta + \tan\theta\sin\theta + \cos^3\theta = \sec\theta$

20. $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\operatorname{cosec}\theta$

21. $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

22. $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

23. $\sqrt{\dfrac{\sec\theta + 1}{\sec\theta - 1}} = \cot\theta + \operatorname{cosec}\theta$

24. $\dfrac{(1+\cot\theta+\tan\theta)(\sin\theta - \cos\theta)}{\sec^3\theta - \operatorname{cosec}^3\theta} = \sin^2\theta\cos^2\theta$

27. $\dfrac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta} + \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \dfrac{2}{1-2\cos^2\theta} = \dfrac{2}{2\sin^2\theta - 1}$

26. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$

27. $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

28. $\dfrac{\cos^2\theta}{1-\tan\theta} + \dfrac{\sin^3\theta}{\sin\theta - \cos\theta} = 1 + \sin\theta\cos\theta$

29. If $\sec\theta + \tan\theta = P$ then, prove that $\dfrac{P^2-1}{P^2+1} = \sin\theta$

30. If $\dfrac{\cos A}{\cos B} = m$ and $\dfrac{\cos A}{\sin B} = n$ then, prove that $(m^2+n^2)\cos^2 B = n^2$

7.02. Trigonometric Ratios of Complementary Angles

Complementary Angles

If sum of two angle is 90° then two angles are called complementary angles. Complementary angle of any acute angle θ will be (90° – θ). If in a right angled Δ ABC, $\angle B$ = 90° then sum of $\angle A$ and $\angle C$ will be 90°.

$$\angle A + \angle C = 90^\circ$$

If $\angle A = \theta$ then

$$\angle C = 90^\circ - \theta$$

Thus θ and 90° – θ are complementary angles of each other.

In right angled triangle ABC for angle θ, side BC and AB will be Perpendicular and base respectively. Thus in ΔABC, trigonometric ratios for angle θ and (90° – θ).

$$\sin(90^\circ - \theta) = \frac{AB}{AC} \qquad \sin\theta = \frac{BC}{AC}$$

$$\cos(90^\circ - \theta) = \frac{BC}{AC} \qquad \cos\theta = \frac{AB}{AC}$$

$$\tan(90^\circ - \theta) = \frac{AB}{BC} \qquad \tan\theta = \frac{BC}{AB}$$

$$\cot(90^\circ - \theta) = \frac{BC}{AB} \qquad \cot\theta = \frac{AB}{BC}$$

$$\sec(90^\circ - \theta) = \frac{AC}{BC} \qquad \sec\theta = \frac{AC}{AB}$$

$$\text{cosec}(90^\circ - \theta) = \frac{AC}{AB} \qquad \text{cosec}\,\theta = \frac{AC}{BC}$$

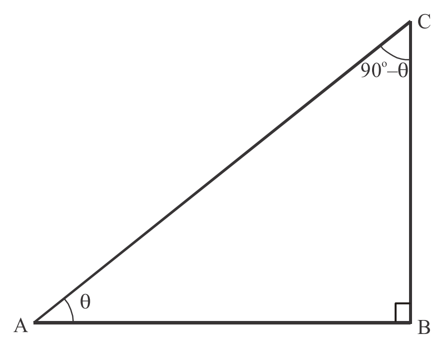


Fig. 7.02

Comparing above equation, we get

$$\sin(90^\circ - \theta) = \cos\theta \qquad \cos(90^\circ - \theta) = \sin\theta$$

$$\tan(90^\circ - \theta) = \cot\theta \qquad \cot(90^\circ - \theta) = \tan\theta$$

$$\sec(90^\circ - \theta) = \text{cosec}\,\theta \qquad \text{cosec}(90^\circ - \theta) = \sec\theta$$

$$0^\circ \le \theta \le 90^\circ$$

Note : We can say that

sin of any angle = cos of its complementary angle
tan of any angle = cot of its complementary angle
sec of any angle = cosec of its complementary angle

Its converse is also true

Illustrative Examples

Example 13. Find the value of $\frac{\tan 49^\circ}{\cot 41^\circ}$

Solution : $\tan 49^\circ = \cot(90^\circ - 49^\circ) = \cot 41^\circ$ $\quad \{\tan\theta = \cot(90^\circ - \theta)\}$

$\therefore \quad \frac{\tan 49^\circ}{\cot 41^\circ} = \frac{\cot 41^\circ}{\cot 41^\circ} = 1$

Example 14. Find the value of $\sin^2 50^\circ + \sin^2 40^\circ$

Solution : $\because \; 40^\circ = 90^\circ - 50^\circ$

$\therefore \quad \sin 40^\circ = \sin(90^\circ - 50^\circ) = \cos 50^\circ$

Thus, $\sin^2 50^\circ + \sin^2 40^\circ = \sin^2 50^\circ + \cos^2 50^\circ = 1$ $\quad (\because \; \sin^2\theta + \cos^2\theta = 1)$

Example 15. Find the value of $\tan 39^\circ - \cot 51^\circ$

Solution : $\tan 39^\circ = \cot(90 - 39^\circ) = \cot 51^\circ$

Thus, $\tan 39^\circ - \cot 51^\circ = \cot 51^\circ - \cot 51^\circ = 0$

Example 16. Find the value of $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

Solution : $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec}(90^\circ - 40^\circ)$

$= \operatorname{cosec} 40^\circ \sin 40^\circ + \cos 40^\circ \sec 40^\circ$

$= \frac{1}{\sin 40^\circ} \cdot \sin 40^\circ + \cos 40^\circ \cdot \frac{1}{\cos 40^\circ} = 1 + 1 = 2$

Example 17. Prove that $\tan 15^\circ \tan 20^\circ \tan 70^\circ \tan 75^\circ = 1$

Solution : LHS $= \tan 15^\circ \tan 20^\circ \tan 70^\circ \tan 75^\circ$

$= \tan 15^\circ \tan 20^\circ \tan(90^\circ - 20) \tan(90^\circ - 15^\circ)$

$= \tan 15^\circ \tan 20^\circ \cdot \cot 20^\circ \cdot \cot 15^\circ$

$= \tan 15^\circ \tan 20^\circ \cdot \frac{1}{\tan 20^\circ \tan 15^\circ} = 1$ (RHS)

Example 18. Prove that $\tan 2A = \cot(A - 18^\circ)$ then find the value of *A*.

Solution : $\tan 2A = \tan[90^\circ - (A - 18^\circ)]$

$\tan 2A = \tan(108^\circ - A)$

$\therefore \quad 2A = 108^\circ - A$

$3A = 108^\circ \Rightarrow A = 36^\circ$

Example 19. Find the value of x form the following equation.

$$\operatorname{cosec}(90^\circ-\theta)+x\cos\theta\cot(90^\circ-\theta)=\sin(90^\circ-\theta)$$

Solution : $\operatorname{cosec}(90^\circ-\theta)+x\cos\theta\cot(90^\circ-\theta)=\sin(90^\circ-\theta)$

$\sec\theta+x\cos\theta\tan\theta=\cos\theta$

$x\sin\theta=\cos\theta-\sec\theta$ $\qquad\left(\because \tan\theta=\dfrac{\sin\theta}{\cos\theta}\right)$

$x=\dfrac{\cos\theta-\sec\theta}{\sin\theta}=\dfrac{\cos^2\theta-1}{\sin\theta\cos\theta}$ $\qquad\left(\because \sec\theta=\dfrac{1}{\cos\theta}\right)$

$=-\left[\dfrac{1-\cos^2\theta}{\sin\theta\cos\theta}\right]$ $\qquad(\because 1-\cos^2\theta=\sin^2\theta)$

$=-\dfrac{\sin^2\theta}{\sin\theta\cos\theta}$ $\qquad x=-\tan\theta$

Exercise 7.2

Find the value of the following :

1. (i) $\dfrac{\cos 37^\circ}{\sin 53^\circ}$ (ii) $\dfrac{\cos ec32^\circ}{\sec 58^\circ}$ (iii) $\dfrac{\tan 10^\circ}{\cot 80^\circ}$ (iv) $\dfrac{\cos 19^\circ}{\sin 71^\circ}$

2. (i) $\cos ec25^\circ-\sec 65^\circ$ (ii) $\cot 34^\circ-\tan 56^\circ$

   (iii) $\dfrac{\sin 36^\circ}{\cos 54^\circ}-\dfrac{\sin 54^\circ}{\cos 36^\circ}$ (iv) $\sin\theta\cos(90^\circ-\theta)+\cos\theta\sin(90^\circ-\theta)$

3. (i) $\sin 70^\circ\sin 20^\circ-\cos 20^\circ\cos ec70^\circ$ (ii) $\dfrac{2\cos 67^\circ}{\sin 23^\circ}-\dfrac{\tan 40^\circ}{\cot 50^\circ}-\cos 60^\circ$

4. (i) $\left(\dfrac{\sin 35^\circ}{\cos 55^\circ}\right)^2+\left(\dfrac{\cos 55^\circ}{\sin 35^\circ}\right)^2-2\cos 60^\circ$ (ii) $\left(\dfrac{\sin 27^\circ}{\cos 63^\circ}\right)^2+\left(\dfrac{\cos 63^\circ}{\sin 27^\circ}\right)^2$

5. (i) $\cot 12^\circ\cot 38^\circ\cot 52^\circ\cot 60^\circ\cot 78^\circ$ (ii) $\tan 5^\circ\tan 25^\circ\tan 30^\circ\tan 45^\circ\tan 65^\circ\tan 85^\circ$

6. Express the following in terms of trigonometric ratios of angles between 0° and 45°

   (i) $\sin 81^\circ+\sin 71^\circ$ (ii) $\tan 68^\circ+\sec 68^\circ$

   Prove the following :

7. $\sin 65^\circ+\cos 25^\circ=2\cos 25^\circ$

8. $\sin 35^\circ\sin 55^\circ-\cos 35^\circ\cos 55^\circ=0$

9. $\dfrac{\cos 70^\circ}{\sin 20^\circ}+\dfrac{\cos 59^\circ}{\sin 31^\circ}-8\sin^2 30^\circ=0$

10. $\sin(90^\circ-\theta)\cos(90^\circ-\theta)=\dfrac{\tan\theta}{1+\tan^2\theta}$

11. $\dfrac{\cos(90^\circ-\theta)\cos\theta}{\tan\theta}+\cos^2(90^\circ-\theta)=1$

12. $\dfrac{\tan(90^\circ-\theta)\cot\theta}{\operatorname{cosec}^2\theta}-\cos^2\theta=0$

13. $\dfrac{\cos(90^\circ-\theta)\sin(90^\circ-\theta)}{\tan(90^\circ-\theta)}=\sin^2\theta$

14. $\dfrac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sec(90^\circ-\theta)}+\dfrac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\operatorname{cosec}(90^\circ-\theta)}=\sin\theta\cos\theta$

15. If $\sin 3\theta=\cos(\theta-6^\circ)$, where 3θ and $(\theta-6^\circ)$ are acute angles, then find the value of θ.

16. If $\sec 5\theta=\operatorname{cosec}(\theta-36^\circ)$, where 5θ is an acute angle, then find the value of θ.

17. If *A, B* and *C* are interior angles of a ΔABC, then prove that $\tan\left(\dfrac{B+C}{2}\right)=\cot\dfrac{A}{2}$.

18. If $\cos 2\theta=\sin 4\theta$ where 2θ and 4θ are acute angles,then find the value of θ.

Answer

Exercise 7.1

1. $\sin\theta=\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$, $\cos\theta=\dfrac{1}{\sec\theta}$, $\tan\theta=\sqrt{\sec^2\theta-1}$, $\cot\theta=\dfrac{1}{\sqrt{\sec^2\theta-1}}$, $\operatorname{cosec}\theta=\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$

2. (i) $\sin\theta=\dfrac{1}{\sqrt{1+\cot^2\theta}}$, $\tan\theta=\dfrac{1}{\cot\theta}$, $\sec\theta=\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$

Exercise 7.2

1. (1) 1 (ii) 1 (iii) 1 (iv) 1

2. (i) 0 (ii) 0 (iii) 0 (iv) 1

3. (i) 0 (ii) 1 / 2

4. (i) 1 (ii) 2

5. (i) $\dfrac{1}{\sqrt{3}}$ (ii) $\dfrac{1}{\sqrt{3}}$

6. (i) $\cos 9^\circ+\cos 19^\circ$ (ii) $\cot 22^\circ+\operatorname{cosec} 22^\circ$

15. $\theta=24^\circ$

16. $\theta=21^\circ$

18. $\theta=15^\circ$

08

Height and Distance

8.01. Introduction

In the preceding chapter, we have studied trigonometric identities and trigonometric ratios for complementary angles. In this chapter we will study problems based on height and distace by using trigonometric ratios, we will see how trigonometry is used for finding the heights and distances of various objects without actually measuring them. Before this, we will study some definitations.

8.02. Important Definitions

Line of sight : The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.

In figure 8.01, if eye is at point O and object is at point P then OP is the line of sight.

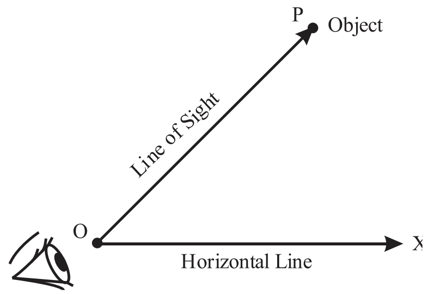


Fig. 8.01

Angle of Elevation

If an oject is above the eye then we raise our head to look at the object. The angle formed by the line of sight with the horizontal line is called angle of elevation.

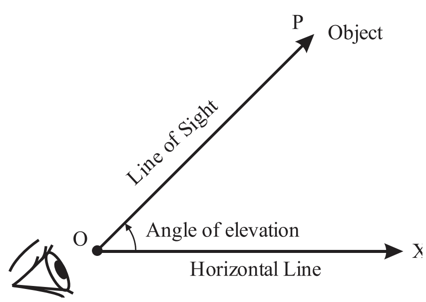


Fig. 8.02

In fig. 8.02, eye is at point O and object is at point P then line of sight OP makes an angle $\angle XOP$ with horizontal line OX then

Angle of elevation $= \angle XOP$

Note : Angle of elevation is also called as angular height of the object.

Angle of depression

The angle of depression of a point on the object being viewed is the angle formed by the line when the point is below the horizontal line.

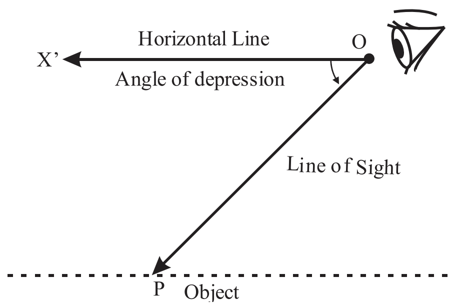


Fig. 8.03

In figure 8.03, eye is at point O and object is at point P then OP is line of sight which makes an angle $X'OP$ with horizontal line OX' then angle of depession $= \angle XO'P$

In solving the problems related to height and distance, following points are to be kept in mind.

(i) First read the question carefully then draw figure and prepare right angled triangle.

(ii) In right angled triangle, express trignometric ratios (sin, cos, tan etc.) of given angle in the terms of given sides.

Note : Complemetary angles : If sum of two angles is 90^o then they are called complelmentary angles.

Following are the Examples with figure of angle of depression subtended at the eye of the observer by the objects.

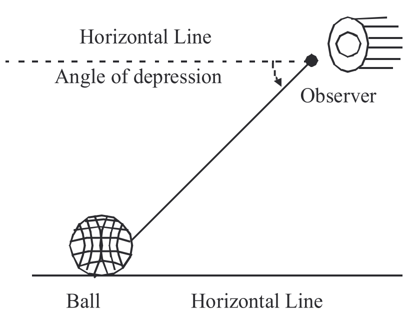


Fig. 8.04

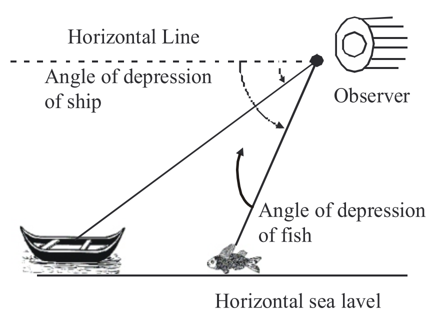


Fig. 8.05

Illustrative Examples

Example 1. The angle of elevation of the top of a tower form a point on the ground which is $10\sqrt{3}$ m away from the foot of the tower is 60°. Find the height of the tower.

Solution : Let AB is a tower. The angle of elevation of the top of a tower from point C on the ground, which is $10\sqrt{3}$ m away from the foot of the tower is 60°. Let h is the height of the tower AB.

In right angled ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{10\sqrt{3}}$$

or $\quad h = 10\sqrt{3} \times \sqrt{3}$

or $\quad h = 10 \times 3 = 30$

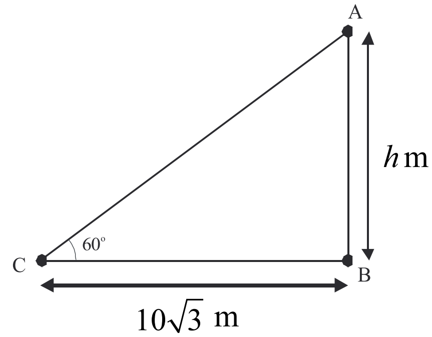


Fig. 8.06

Thus, height of the tower AB is 30 m.

Example 2. The angle of drepression of any boat from a 50 m high bridge is 30°. Find the horizontal distance between boat and bridge.

Solution : Let horizontal distance between boat and bridge is x m.

Given, Angle of depression is 30°

Here $\quad PQ = 50$ m

$$\angle XPO = \angle POQ = 30^\circ \text{ (Alternate angles)}$$

In right angled ΔPQO

$$\because \quad \tan 30^\circ = \frac{PQ}{OQ}$$

$$\therefore \quad \frac{1}{\sqrt{3}} = \frac{50}{x}$$

or $\quad x = 50\sqrt{3} = 50 \times 1.732 \quad (\because \sqrt{3} = 1.732)$

or $\quad x = 86.60$

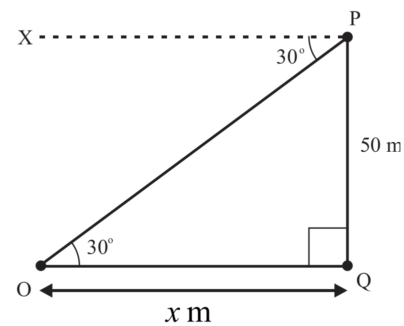


Fig. 8.07

Thus, horizontal distance between boat and bridge is 86.60 m.

Example 3. The shadow of a 1.5 m tall student standing on a plane ground in found to be 1 m and at the same time shadow of a tower on ground is 5 m, then find height of the tower.

Solution : Given, Length of student AC = 1.5 m

Shadow of student BC = 1 m

In right angled ΔACB

$$\tan\theta = \frac{AC}{BC} \Rightarrow \tan\theta = \frac{1.5}{1}$$

or $\quad \tan\theta = 1.5 \quad \ldots (1)$

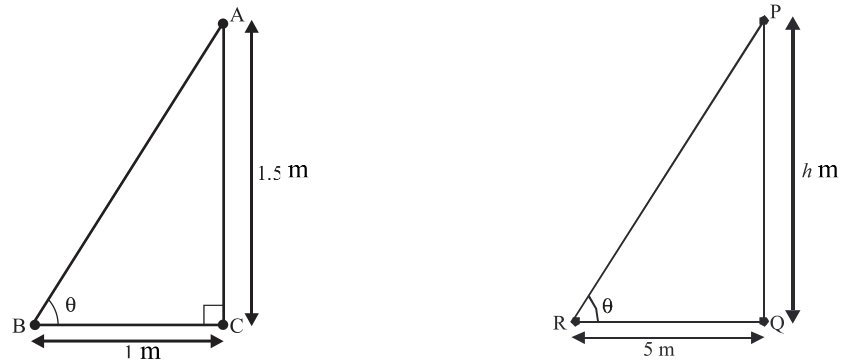


Fig 8.08

Now it is given that

Length of shadow of the tower $QR = 5$ m

Let height of the tower $= PQ = h$ m

In right angled ΔPQR

or $\tan\theta = \dfrac{PQ}{QR}$

or $\dfrac{h}{5} = 1.5$ $[\because \tan\theta = 1.5$ from equation (i)$]$

or $h = 5 \times 1.5$

or $h = 7.5$

Thus, height of tower is 7.5 m.

Example 4. There is a small island in 100 m broad river and there is a tall tree on this island. P and Q lie on the opposite banks of the river such that P, Q and tree are in the same line. If angle of elevation form P and Q at top of the tree are 30° and 45° respectivaly then find height of the tree.

Solution : Let OA is tree whose height is h m.

In figure, $PQ = 100$ m

$\angle APO = 30^\circ$ and $\angle AQO = 45^\circ$

Now, in right angled ΔPOA and ΔQOA

$\tan 30^\circ = \dfrac{OA}{OP}$ and $\tan 45^\circ = \dfrac{OA}{OQ}$

$\dfrac{1}{\sqrt{3}} = \dfrac{h}{OP}$ and $1 = \dfrac{h}{OQ}$

$OP = h\sqrt{3}$ and $OQ = h$

$\therefore$ From figure $PQ = OP + OQ$

$100 = h\sqrt{3} + h$

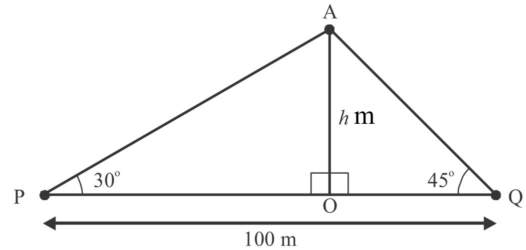


Fig 8.09

$$100 = h\left(\sqrt{3}+1\right)$$

$$\therefore \quad h = \frac{100}{\sqrt{3}+1} = \frac{100}{\left(\sqrt{3}+1\right)} \times \left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)$$

$$h = \frac{100\left(\sqrt{3}-1\right)}{2}$$

$$h = 50\left(\sqrt{3}-1\right) = 36.6 \text{ m} \qquad \left(\because \sqrt{3} = 1.732\right)$$

Thus, height of the tree is 36.6 m.

Example 5. A car is moving on a straight road which goes towards a tower. At a distance of 500 m from tower, driver of car observe that angle of elevation of top of tower is 30° after driving the car for 10 sec. towards tower then he observe that the angle of elevation of top of tower became 60°. Find the speed of the car.

Solution : Let height of tower $AB = h$ m and distance covered by car in 10 sec. (DC) = x m.

$$BD = 500 \text{ m}$$

$$\therefore \quad BC = (500 - x) \text{ m}$$

$$\angle ADB = 30^\circ, \angle ACB = 60^\circ$$

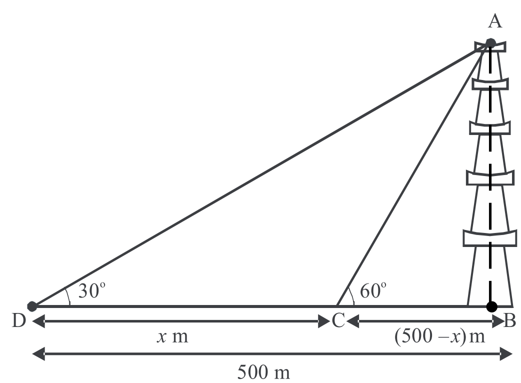


Fig 8.10

Now, in right angled ? ABD

$$\frac{AB}{BD} = \tan 30$$

$$\frac{h}{500} = \frac{1}{\sqrt{3}} \Rightarrow h = \frac{500}{\sqrt{3}} \ldots(1)$$

Now, in right angled ΔABC

$$\frac{AB}{BC} = \tan 60$$

or $$\frac{h}{500-x} = \sqrt{3} \Rightarrow h = (500-x)\sqrt{3} \ldots(2)$$

From equation (1) and (2)

$$\frac{500}{\sqrt{3}} = (500-x)\sqrt{3} \Rightarrow 500 = (500-x)\sqrt{3}.\sqrt{3}$$

or $$500 = (500-x).3$$

or $$500 = 1500 - 3x$$

or $$3x = 1500 - 500 = 1000$$

or $$x = \frac{1000}{3}$$

Distance covered by car in 10 sec $= \frac{1000}{3}$ m

? Distance covered by car in 1 min. $= \frac{1000 \times 60}{3 \times 10} = 2000$ m $= 2$ km

Thus speed of car = 2 km/min.

Example 6. The angle of elevations of the top of a tower form two pointrs C and D from base of tower and in the same straight line at a distance a and b respectively , are complement to each other. Prove that height of tower is $\sqrt{ab}$.

Solution : Let the height of tower $AB = h$ meter and points C and D are in such a way that $BC = a$, $BD = b$.

If $\angle ACB = \theta$ then $\angle ADB = 90^\circ - \theta$

Again, in right angled ΔABC

$$\tan\theta = \frac{AB}{BC} = \frac{h}{a} \quad \text{... (1)}$$

Again, in right angled ΔABD

$$\tan(90^\circ - \theta) = \frac{AB}{BD}$$

or $$\cot\theta = \frac{h}{b} \quad \text{... (2)}$$

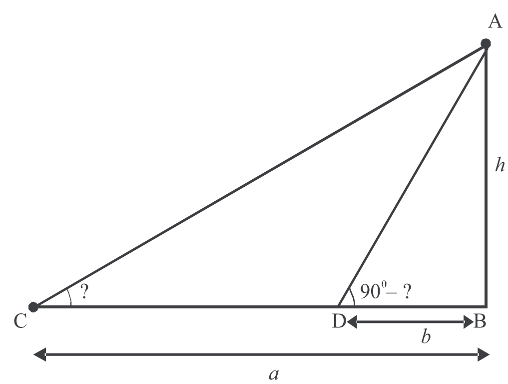


Fig. 8.11

On multiplying equation (1) and (2), we get

$$\tan\theta \times \cot\theta = \frac{h}{a} \times \frac{h}{b}$$

or $$1 = \frac{h^2}{ab} \Rightarrow h^2 = ab$$

or $$h = \sqrt{ab}$$

Example 7. Two poles of equal height are standing opposite each other on either side of the road which is 80 m wide. From a point between them on the road, the angle of elevation of the top of the poles are 60° and 30° respectively. Find the height of the poles and the distances of the point from the poles.

Solution: Let BC and DE are two poles of same height (in meter). From a point between the poles,on the road BD,the angles of elevations of the top of the poles are 60° and 30°

Thus, $\angle CAB = 60^\circ$ and $\angle EAD = 30$ $BC = DE = h$ m. $BD = 80$ m.

Let $AD = x$ m

$\therefore$ $AB = BD - AD = (80 - x)$ m

In right angle ΔADE

$$\tan 30^\circ = \frac{DE}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\therefore \qquad h = \frac{x}{\sqrt{3}} \qquad \text{... (1)}$$

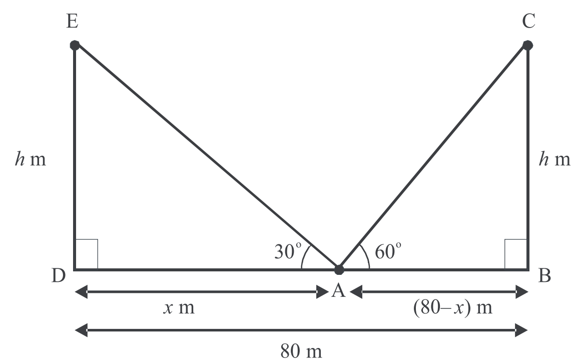


Fig. 8.12

Again, in right angled ΔABC

$$\tan 60^\circ = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{h}{(80-x)}$$

$$h = (80-x)\sqrt{3} \text{ m} \qquad \text{... (2)}$$

From equation (1) and (2)

$$\frac{x}{\sqrt{3}} = \sqrt{3}(80-x)$$

$$x = \sqrt{3}.\sqrt{3}(80-x)$$

$$x = 3(80-x)$$

$$x ? 240 ? 3x$$

$$\Rightarrow \quad x + 3x = 240$$

$$4x = 240$$

$$x = \frac{240}{4} = 60$$

From equation (1)

$$h = \frac{60}{\sqrt{3}} = \frac{60}{\sqrt{3}}.\frac{\sqrt{3}}{\sqrt{3}} = \frac{60\sqrt{3}}{3}$$

$$h = 20\sqrt{3}$$

Thus height of poles $(h) = 20\sqrt{3}$ m and distance of the point form the poles is 20 m and 60 m.

Example 8. The angle of elevation of a cloud form a point *'h'* m above a lake is and the angle of depression of its reflection in the lake is β. Prove that height of the cloud from surface of water is $\dfrac{h(\tan\beta + \tan \propto)}{\tan\beta - \tan \propto}$.

Solution : Let AB is surface of lake and P is point of observation. Given $AP = h$ m. Let position of cloud is c and c' is its shadow in lake $\therefore CB = C'B$. Let PM is perpendicular from P to CB, it is given that

$$\angle CPM = \alpha \text{ and } \angle MPC' = \beta \text{ Let } CM = x$$

It is clear that $CB = CM + MB = CM + PA = x + h$

In ΔCMP $\tan\alpha = \frac{CM}{PM}$

or $\tan\alpha = \frac{x}{AB}$ $(\because PM = AB)$

$\therefore$ $AB = x\cot\alpha$ . . . (1)

In $\Delta PMC'$, $\tan\beta = \frac{C'M}{PM} = \frac{x+2h}{AB}$

$\therefore$ $AB = (x+2h)\cot\beta$ . . . (2)

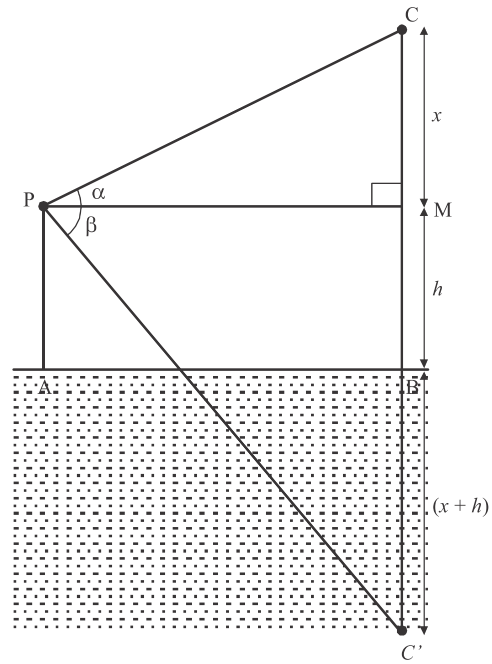


Fig. 8.13

From equations (1) and (2), we get

$$x\cot\alpha = (x+2h)\cot\beta$$

$$x(\cot\alpha - \cot\beta) = 2h\cot\beta$$

or $x\left(\frac{1}{\tan\alpha} - \frac{1}{\tan\beta}\right) = \frac{2h}{\tan\beta}$

or $x\left[\frac{\tan\beta - \tan\alpha}{\tan\alpha\tan\beta}\right] = \frac{2h}{\tan\beta}$

or $x = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha}$

Thus height of cloud form surface of water

$$CB = x + h = \frac{2h\tan\alpha}{\tan\beta - \tan\alpha} + h = \frac{h(\tan\alpha + \tan\beta)}{\tan\beta - \tan\alpha}$$

Exercise 8

1. The shadow of a verticle pillar is same the height of pillar, then angle of elevation of Sun will be :
   (a) 45° (b) 30° (c) 60° (d) 50°
2. From a point on the ground which is 100 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°, then height of tower is :
   (a) $100\sqrt{3}$ m (b) $\frac{100}{\sqrt{3}}$ m (c) $50\sqrt{3}$ m (d) $\frac{200}{\sqrt{3}}$ m

3. A 15 m long ladder touches the top of a vertical wall. If this ladder makes an angle of 60° with the wall then height of the wall is :

(a) $15\sqrt{3}$ m (b) $\frac{15\sqrt{3}}{2}$ m (c) $\frac{15}{2}$ m (d) 15 m

4. From the top of 10 m height tower, angle of depression at a point on earth is 30°. Distance of point from base of tower is

(a) $10\sqrt{3}$ m (b) $\frac{10}{\sqrt{3}}$ m (c) 10 m (d) $5\sqrt{3}$ m

5. A bridge above the river makes an angle of 45° with the bank of river. If length of bridge above the river is 150 m then breadth of river will be

(a) 75 m (b) $50\sqrt{2}$ m (c) 150 m (d) $75\sqrt{2}$ m

6. Top of two towers of height 20 m and 14 m are joined by a wire. If wire makes an angle of 30° with horizontal line then length of wire is :

(a) 12 m (b) 10 m (c) 8 m (d) 6 m

7. The angle of elevation of the top of the tower form two points distance a and b from the base of tower $(a > b)$ are 30° and 60° then height of tower is :

(a) $\sqrt{a+b}$ (b) $\sqrt{a-b}$ (c) $\sqrt{ab}$ (d) $\sqrt{\frac{a}{b}}$

8. From the top of a 25 m high pillar the angle of elevation of top of the tower is same as the angle of depression of foot of tower then height of tower is :

(a) 25 m (b) 100 m (c) 75 m (d) 50 m

9. If ratio of length of a verticle rod and length of its shadow is $1 : \sqrt{3}$ then angle of elevation of sun is :

(a) 30° (b) 45° (c) 60° (d) 90°

10. The slope of a hill makes an angle of 60° with horizontal . If to reach at top, 500 m distance have to covered then height of the hill is :

(a) $500\sqrt{3}$ m (b) $\frac{500}{\sqrt{3}}$ m (c) $250\sqrt{3}$ m (d) $\frac{250}{\sqrt{3}}$ m

11. A tower is vertically placed on a horizontal plane. If angle of elevation of sun is 30° and length of shadow of tower is 45 m the find height of the tower.

12. The upper part of a tree is broken by windstrom and it makes an angle of 60° with the ground. The distance form the bottom of the tree to the point where the top touches the ground is 10 m. Find the original height of the tree $\left(\sqrt{3} = 1.732\right)$

13. From a point on the ground which is 120 m away form the foot of the unfinished tower, the angle of elevation of the top of the tower is found to be 30°. Find how much height of tower have to increased so that its angle of elevation at same point become 60° ?

14. The angle of elevation of the top of a tower form a point situated at 100 m far form the foot of tower is 30°. Find the height of the tower.

15. The angle of elevation of the top of a pillar form a point on the ground is 15° on walking 100 m towards the tower, the angle of elevation is found to be 30°. Find the height of the tower (where $\tan 15^0 = 2 - \sqrt{3}$)
16. The shadow of a verticle tower on level ground is increased by 40 m, when the altitude of the sun changes form 60° to 30°. Find the height of the tower.
17. The angle of depression of two ships from the top of light house situated at 60 m heigtht from sea level, are 30° and 45° if two ships are on the same side of the light house then find the distance between two ships.
18. A 1.5 m tall boy is standing at some distance away form a 30 m high building when he moves towards the building then angle of elevation from his eye become 60° to 30°. Find how much distance he covered towards the building ?
19. Angle of elevation of top of a tower form a 7 m high building is 60° and angle of depression of its foot is 45°. Find the height of the tower.
20. From the top of a hill, in east side at two points of angle of depression are 30° and 45°. If distance between two points is 1 km, then find height of the hill.
21. The angle of elevation of a cloud from a point 20 m above a lake (point A) is 30°. If the angle of depression of its reflection form point A is 60° then find the distance of cloud form point A.
22. From a point on a bridge across a river, the angles of depression of the banks on opposite side of the river are 30° and 45° respectively. If the bridge is at height of 4 m form the bank, find the width of the river.
23. A man on the deck of the ship is 10 m above water -level. He observes that the angle of elevation of the top of hill is 60° and the angle of depression of the base is 30° then find the distance of the hill form the ship and height of the hill.
24. A verticle straight tree 12 m high is broken by strong wind in such a way that its top touches the ground and makes an angle of 60° with the ground. Find at what height from the ground did the tree break ? $\left(\sqrt{3} = 1.732\right)$
25. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later,the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.
26. The angles of elevation of the top of the tower form two points at a distance of 4 m and 9 m form the base of the tower in the same straight line are complementary. Prove that the height of tower is 6 m.
27. A tower and a building on the opposite side of road are situated. The angles of depression form the top of tower at the roof and base of building are 45° and 60° respectively. If height of building is 12 m then find the height of the tower $\left(\sqrt{3} = 1.732\right)$
28. If angle of elevation of sun changes from 30° to 60°. Then at these angles of elevation find the difference in the length of shadow of 15 m high pillar.

Important Points

1. The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.
2. Angle subtended by an eye with the horizontal to see an object in the upward direction is called angle of elevation.
3. Angle subtended by an eye with the horizontal to see an object in the downward direction is called angle of depression.
4. $\sin 30^\circ = 0.5774 = \cos 60^\circ$

   $\sin 45^\circ = 0.7071 = \cos 45^\circ$

   $\sin 60^\circ = 0.8660 = \cos 30^\circ$

   $\sqrt{2} = 1.4141, \ \sqrt{3} = 1.732$

Answers
Exercise 8

1. (a) 45^o	2. (a) $100\sqrt{3}$ m	3. (c) $\frac{15}{2}$ m	4. (a) $10\sqrt{3}$ m
5. (d) $75\sqrt{2}$	6. (a) 12 m	7. (c) $\sqrt{ab}$	8. (d) 50 m
9. (a) 30^o	10. (c) $250\sqrt{3}$ m	11. $15\sqrt{3}$ m	12. 37.32 m
13. 138.56 m	14. 57.73 m	15. 50 m	16. 34.64 m
17. 43.92 m	18. $19\sqrt{3}$ m	19. $7\left(\sqrt{3}+1\right)$m	20. 1.366 km
21. 40 m	22. 10.92 m	23. $10\sqrt{3}$ m, 40 m	24. 5.569 m
25. 3 m	27. 28.392 m	28. 17.32 m	

09

Co-ordinate Geometry

9.01 Introduction

In earlier classes we have studied geometry which is called Euclidean geometry. Now we will study the analytic geometry. Where position of the point is expressed by specific numbers which are called co-ordinates, lines and curves so formed are represented by algebraic equations. **Due to the use of co-ordinates in Analytic geometry, this is called as co-ordinate geometry.**

9.02 Cartesian co–ordinates

Let $X'OX$ and $Y'OY$ be two perpendicular lines in any plane, which intersects each other at point O. These lines are called coordinate axes and O is called origin. $X'OX$ and $Y'OY$ are perpendicular to each other. Thus $X'OX$ and $Y'OY$ are called rectangular axes.

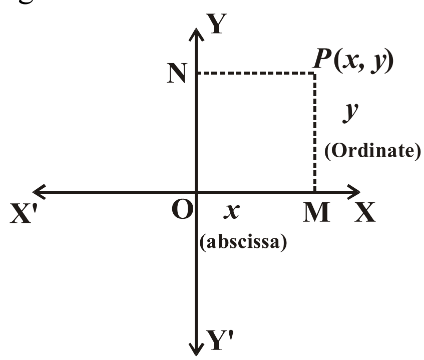


Fig. 9.01

Now to find co-ordinates of P, draw perpendiculars PM and PN from P on x and y axes respectively. The length of the segment OM ($OM=x$) is called the x-coordinate or abscissa of point P. Similarly, the length, of line segment ON is called the y-coordinate or ordinate of point. These coordinates are written in ordered pair (x,y) *i.e.,* while writing the co-ordinates of a point, write co-ordinate first and then y co-ordinate in parentheses.

9.03 Sign of Co–ordinates in quadrants

In figure 9.02, two axes $X'OX$ and $Y'OY$ divide the plane into four equal parts which are called quadrants. XOY, YOX', $X'OY'$ and $Y'OX$ are called respectively I,II,III, and IV quadrants. We always take OX and OY as + ve and OX' and OY' as - ve directions.

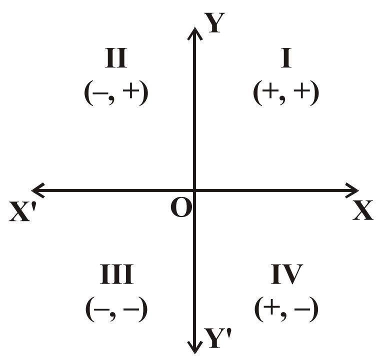


Fig.9.02

If (x,y) be coordinates of any point P in plane, then in

I Quadrant $x > 0, y > 0$; Coordinates(+, +)

II Quadrant $x < 0, y > 0$; Coordinates (–, +)

III Quadrant $x < 0, y < 0$; Coordinates (–, –)

IV Quadrant $x > 0, y < 0$; Coordinates (+, –)

Note:(i) If coordinate of any point P is (x,y) then we can write as P (x,y).

(ii) The abscissa of any point is at a perpendicular distance from y - axis.

(iii) The ordinate of any point is at a perpendicular distance form x-axis.

(iv) The abscissa of any point is positive at R.H.S of y axis and negative at LHS of y axis.

(v) The ordinate of any point is positive above the x-axis and negative below the x-axis.

(vi) If $y = 0$, then point lies on x-axis.

(vii) If $x = 0$, then point lies on y-axis.

(viii) If $x = 0, y = 0$ then point is origin.

9.04 Distance between two points

Let XOX' and YOY' are co-ordinate axes and two points in the plane are $P(x_1, y_1)$ and $Q(x_2, y_2)$ we have to find distance between these two points. From point P and Q draw perpendicular PM and QN on x-axis, respectively and draw perpendicular PR form P to QN.

$\therefore$ OM = Abscissa of $P = x_1$

Similarly $ON = x_2$, $PM = y_1$

and $QN = y_2$

According to figure $PR = MN = ON - OM = x_2 - x_1$

and $QR = QN - RN = QN - PM = y_2 - y_1$

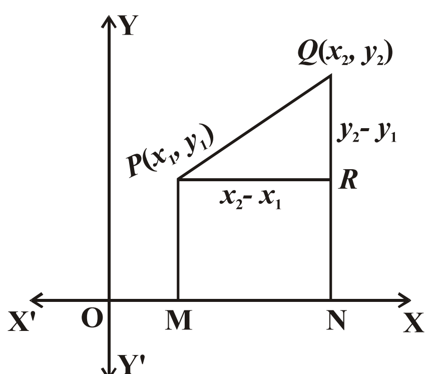


Fig. 9.03

Thus, by Bodhayan formula in right angled ΔPRQ

$$PQ^2 = PR^2 + QR^2$$

or $$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$\therefore$ $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\text{difference of } x - \text{coordinates})^2 + (\text{difference of } y - \text{coordinate})^2}$$

which is a formula to find distance between two points.

Special case : Distance of point P(x,y) from origin O (0, 0)

$$OP = \sqrt{x^2 + y^2}$$

Illustrative Examples

Example 1. Plot the points (2, 4), (–2, 3), (–4, –3) and (5, –2) in the rectangular co-ordinate system.

Solution :

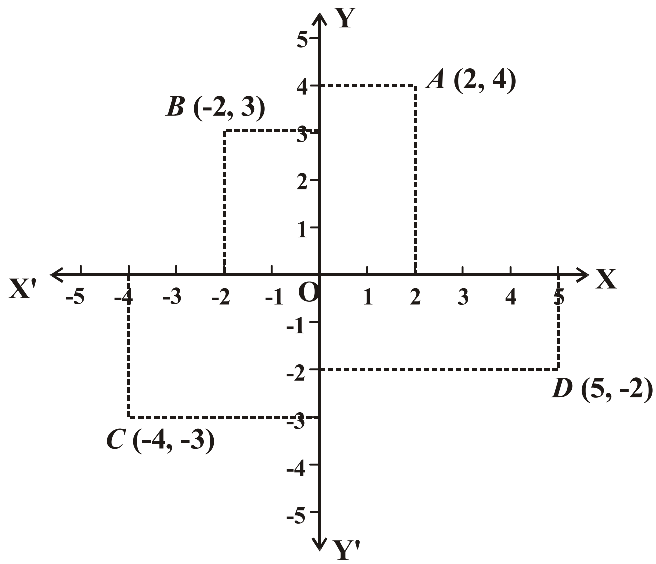


Fig. 9.04

Example 2. If $2a$ is a side of an equilateral triangle, then find the co-ordinates of its vertices.

Solution : According to figure 9.05

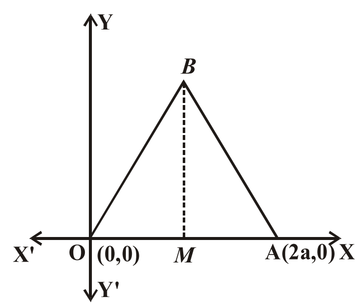


Fig. 9.05

$\because$ OAB is an equilateral triangle of side $2a$

$\therefore$ $OA = AB = OB = 2a$

Draw perpendicular BM from point B to OA

$\therefore$ $OM = MA = a$

In rigtht angle ΔOMB,

$$OB^2 = OM^2 + MB^2$$

or $$(2a)^2 = (a)^2 + MB^2$$

or $$MB^2 = 3a^2$$

$\therefore$ $$MB = \sqrt{3}a$$

Thus, co-ordinates of vertices of an equilateral triangle are O (0, 0), $A(2a, 0)$ and $B(a, \sqrt{3}a)$ since $OM = a$ and $MB = \sqrt{3}a$.

Example 3. Find the distance between the points (2,3) and (5,6).

Solution : Let points (2, 3) and (5, 6) are P and Q respectively so distance between them

$$PQ = \sqrt{(5-2)^2 + (6-3)^2}$$

$$= \sqrt{(3)^2 + (3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{18} = 3\sqrt{2}$$

Example 4. If distance between points $(x, 3)$ and $(5, 7)$ is 5, then find the value of x.

Solution : Let $P(x, 3)$ and $Q(5, 7)$ are given points then according to question.

$$PQ = 5$$

$$\sqrt{(x-5)^2+(3-7)^2} = 5$$

Squaring both sides,

$$(x-5)^2+(-4)^2 = 25$$

or $\quad x^2-10x+25+16 = 25$

or $\quad x^2-10x+16 = 0$

or $\quad (x-2)(x-8) = 0$

$\therefore \quad x = 2, 8$

Example 5. Prove that points $(-2, -1)$, $(-1, 1)$, $(5, -2)$ and $(4, -4)$ taken in order, are vertices of a rectangle.

Solution : Let given points are $P(-2, -1)$, $Q(-1, 1)$, R $(5, -2)$ and $S(4, -4)$

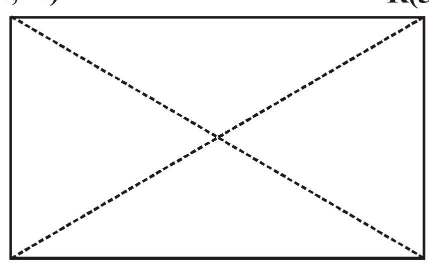


Fig. 9.06

$$PQ = \sqrt{[-2-(-1)]^2+[-1-1]^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$$

$$QR = \sqrt{[5-(-1)]^2+[-2-1]^2} = \sqrt{(6)^2+(-3)^2} = \sqrt{45}$$

$$RS = \sqrt{[4-5]^2+[-4-(-2)]^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$$

$$SP = \sqrt{[4-(-2)]^2+[-4-(-1)]^2} = \sqrt{(6)^2+(-3)^2} = \sqrt{45}$$

$\therefore \quad PQ = RS$ and $QR = SP$

So opposite sides are equal

Again, diagonal $PR = \sqrt{[5-(-2)]^2+[-2-(-1)]^2} = \sqrt{(7)^2+(-1)^2} = \sqrt{50}$

$$QS = \sqrt{[4-(-1)]^2+[-4-1)]^2} = \sqrt{(5)^2+(-5)^2} = \sqrt{50}$$

Thus, diagonals are equal. So given points are vertices of rectangle $PQRS$.

Example 6. If points (x, y) lies at equal distance form points $(a+b, b-a)$ and $(a-b, a+b)$ then prove that $bx = ay$.

Solution : Let $P(x, y)$, $Q(a+b, b-a)$ and $R(a-b, a+b)$ are given points, so according to question

$$PQ = PR$$

or $\quad PQ^2 = PR^2$

or $$[x-(a+b)]^2+[y-(b-a)]^2=[x-(a-b)]^2+[y-(a+b)]^2$$

or $$x^2-2(a+b)x+(a+b)^2+y^2-2(b-a)y+(b-a)^2$$

$$=x^2-2(a-b)x+(a-b)^2+y^2-2(a+b)y+(a+b)^2$$

or $$-2(a+b)x-2(b-a)y=-2(a-b)x-2(a+b)y$$

or $$ax+bx+by-ay=ax-bx-ay-by$$

or $$2bx=2ay \Rightarrow bx=ay$$

Exercise 9.1

1. Find the co-ordinates of points P, Q, R and S from given figure.

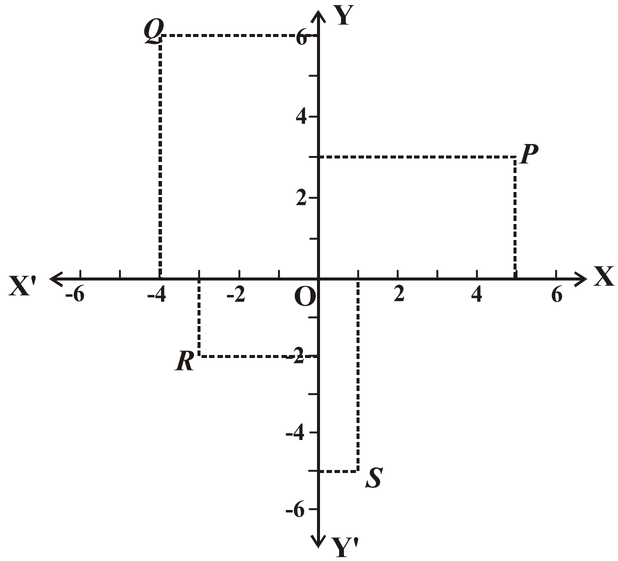


Fig. 9.07

2. Plot the points of the following co-ordinates.
   (1, 2), (–1, 3), (–2, –4), (3, –2), (2, 0), (0, 3)
3. By taking rectangular coordinate axis plot the points O (0,0), P (3, 0) and R(0, 4). If $OPQR$ is rectangle then find coordinates of Q.
4. Plot the points (–1, 0), (1, 0), (1, 1), (0, 2), (–1, 1). Which figure is obtained, by joining them serially?
5. Draw quadrilateral, if its vertices are following :
   (i) (1,1), (2, 4), (8, 4) and (10, 1)
   (ii) (–2, –2), (–4, 2), (–6, –2) and (–4, –6)
   Also, mention type of obtained quadrilateral.
6. Find the distance between the following points :
   (i) (–6, 7) and (–1, –5)
   (ii) (–1, –1) and (8, –2)
   (iii) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$
7. Prove that the points (2, –2), (–2, 1) and (5, 2) are vertics of a right angled triangle.
8. Prove that points (1, –2), (3, 0), (1, 2) and (–1, 0) are vertices of a square.
9. Prove that points (a, a), $(-a, -a)$ and $(-\sqrt{3}a, \sqrt{3}a)$ are vertices of a equilateral triangle.

10. Prove that points (1, 1), (–2, 7) and (3, –3) are collinear.
11. Find that point on x-axis which is equidistant form points (–2, –5) and (2, –3).
12. Find that point on y-axis which is equidistant form points (–5, –2) and (3, 2).
13. If points (3, K) and (K, 5) are equidistant from a point (0, 2), then find the value of K.
14. If co-ordinates of P and Q are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ respectively, then show that $OP^2 + OQ^2 = a^2 + b^2$, where O is origin.
15. If (0, 0) and $(3, \sqrt{3})$ are two vertices of an equilateral triangle then find third vertex.

9.05 Internal and external division of distance between two points

Let A and B are two points in plane. If point P lies in the middle of line AB then this type of division is called internal division.If point P is not in the middle of A and B, but it lies either left of A or right of B then such division is called external division.

(i) Internal division :

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in plane and point $P(x, y)$ divides line segment AB in the ratio $m_1 : m_2$, internally. AL, PM and BN are perpendicular drawn form A, P and B on x-axis, respectively. Draw perpendicular AQ and PR form A to PM and from P to BN. Then

$$OL = x_1,\ OM = x,\ ON = x_2$$

$$AL = y_1,\ PM = y \text{ and } BN = y_2$$

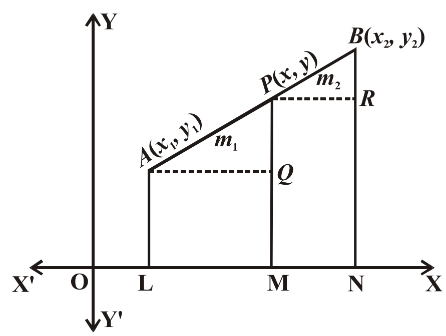


Fig. 9.8

$$\therefore \quad AQ = LM = OM - OL = x - x_1$$

$$PR = MN = ON - OM = x_2 - x$$

$$PQ = PM - QM = PM - AL = y - y_1$$

$$BR = BN - RN = BN - PM = y_2 - y$$

In Fig. 9.8, ΔAQP and ΔPRB are similar triangles.

$$\therefore \quad \frac{AP}{BP} = \frac{AQ}{PR} = \frac{PQ}{BR}$$

or $$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y_1}$$

Now $$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}$$

or $$m_1x_2 - m_1x = m_2x - m_2x_1$$

or $$(m_1 + m_2)x = m_1x_2 + m_2x_1$$

$$\therefore \quad x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

Again $$\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}$$

or $\qquad m_1 y_2 - m_1 y = m_2 y - m_2 y_1$

or $\qquad (m_1 + m_2) y = m_1 y_2 + m_2 y_1$

$\therefore \qquad y = \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$

Thus, required coordinate of $P\left(\dfrac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right)$

(ii) External division :

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ are points lie in plane. Point P externally divides line segment AB it the ratio $m_1 : m_2$. AL, BN and PM are perpendiculars drawn form A, B and P respectively. AQ and BR are perpendicular from point A and B on PM and BN respectively. Then $OL = x_1$, $ON = x_2$, $OM = x$, $AL = y_1$, $BN = y_2$ and $PM = y$

$\therefore \qquad AQ = LM = OM - OL = x - x_1$

$BR = NM = OM - ON = x - x_2$

$PQ = PM - QM = PM - AL = y - y_1$

and $\qquad PR = PM - RM = PM - BN = y - y_2$

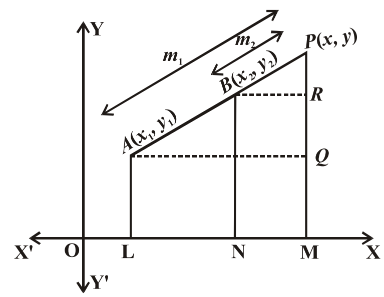


Fig. 9.9

In fig. 9.09, ΔAPQ and ? BPR are smilar triangles

$\therefore \qquad \dfrac{AP}{BP} = \dfrac{AQ}{BR} = \dfrac{PQ}{PR}$

or $\qquad \dfrac{m_1}{m_2} = \dfrac{x - x_1}{x - x_2} = \dfrac{y - y_1}{y - y_2}$

Now $\qquad \dfrac{m_1}{m_2} = \dfrac{x - x_1}{x - x_2}$

or $\qquad m_1 x - m_1 x_2 = m_2 x - m_2 x_1$

or $\qquad (m_1 - m_2) x = m_1 x_2 - m_2 x_1$

$\therefore \qquad x = \dfrac{m_1 x_2 - m_2 x_1}{m_1 - m_2}$

Again $\qquad \dfrac{m_1}{m_2} = \dfrac{y - y_1}{y - y_2}$

or $\qquad m_1 y - m_1 y_2 = m_2 y - m_2 y_1$

or $\qquad (m_1 - m_2) y = m_1 y_2 - m_2 y_1$

$\therefore \qquad y = \dfrac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$

Thus, required coordinates of P

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

Special case : If point P lie in the mid of line segment AB *i.e.,* P divides AB in the ratio 1 : 1, then co-ordinates of P are $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$

Note :

(i) Internal division formula is replaced by external division formula by putting –ve sign of m_1 or m_2.

(ii) In external division, if $|m_1| > |m_2|$ then division point is in right of B and if $|m_1| < |m_2|$ then, division point is in left of A.

(iii) If point $P(x_1, y_1)$ divides line segment AB in ratio λ : 1 then co-ordinates of P are $\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}\right)$.

So co-ordinates of any point of the line joining the points (x_1, y_1) and (x_2, y_2) can be expressed as above.

Illustrative Examples

Example 1. Find the co-ordinates of that point which divides the line joining the points (–2, 1) and (5, 4) internally in the ratio2 : 3.

Solution : Let required point is (x, y), then by formula

$$x = \frac{2\times5 + 3\times(-2)}{2+3} = \frac{10-6}{5} = \frac{4}{5}$$

and $$y = \frac{2\times4 + 3\times1}{2+3} = \frac{8+3}{5} = \frac{11}{5}$$

Therefore, co-ordinates of required point are $\left(\frac{4}{5}, \frac{11}{5}\right)$

Example 2. Find the co-oridinates of that point which externally divides the line joining the points (–4, 4) and (7, 2) in the ratio 4 : 7

Solution : Let co-ordinates of requiredpoint is (x, y), then

$$x = \frac{4\times7 - 7\times(-4)}{4-7} = \frac{28+28}{-3} = -\frac{56}{3} = -18\frac{2}{3}$$

and $$y = \frac{4\times2 - 7\times4}{4-7} = \frac{8-28}{-3} = \frac{20}{3} = 6\frac{2}{3}$$

Thus, coordinates of required point are $\left(-18\frac{2}{3}, 6\frac{2}{3}\right)$

Example 3. In which ratio x–axis divides the line joining the points $A(3, -5)$ and $B(-4, 7)$?

Solution : Ordinate of each point on x–axis is zero.So, let point $P(x, 0)$ internally divides the given line segment in the ratio $m_1 : m_2$.

$$\therefore \qquad 0 = \frac{m_1 \times 7 + m_2 \times (-5)}{m_1 + m_2}$$

$$\text{or} \qquad 7m_1 - 5m_2 = 0$$

$$\text{or} \qquad \frac{m_1}{m_2} = \frac{5}{7}$$

Therefore, x-axis internally divides the line joiniung the given points in the ratio 5: 7.

Example 4. In which ratio, point (–2,3) divides the line joining the points (–3, 5) and (4, –9).

Solution : Let point (–2, 3) inetrnally divides the line joining the given points in the ratio $\lambda : 1$. So by internal division formula

$$-2 = \frac{\lambda \times 4 + 1 \times (-3)}{\lambda + 1}$$

$$\text{or} \qquad -2 = \frac{4\lambda - 3}{\lambda + 1}$$

$$\text{or} \qquad -2\lambda - 2 = 4\lambda - 3$$

$$\text{or} \qquad 6\lambda = 1 \Rightarrow \lambda = \frac{1}{6}$$

Thus, required ratio is 1 : 6

Note : By ordinate same ratio will be obtained.

Example 5. If point P (–1, 2) divides the line joining the points A(2, 5) and B in the ratio 3: 4 internally, then find co-ordinates of B.

Solution : Let co-ordinates of B are (x_1, y_1) and given that $AP : BP = 3 :4$

By internal division formula

$$-1 = \frac{3 \times x_1 + 4 \times 2}{3 + 4} = \frac{3x_1 + 8}{7}$$

$$\text{or} \qquad -7 = 3x_1 + 8 \Rightarrow x_1 = -\frac{15}{3} = -5$$

$$\text{and} \qquad 2 = \frac{3 \times y_1 + 4 \times 5}{3 + 4} = \frac{3y_1 + 20}{7}$$

$$\text{or} \qquad 14 = 3y_1 + 20$$

$$\Rightarrow \qquad y_1 = -\frac{6}{3} = -2$$

Therefore, co-ordinates of B are (–5, –2)

Example 6. Find in which ratio line $x + y = 4$ divides the line joining the points (–1, 1) and (5, 7)?

Solution : Let given line divides line joining the points A(–1, 1) and B(5, 7) in the ratio $\lambda : 1$. So co-ordinates of P will be

$$\left(\frac{5\lambda-1}{\lambda+1}, \frac{7\lambda+1}{\lambda+1}\right)$$

But point P lies on line $x + y = 4$

$\therefore \quad \frac{5\lambda-1}{\lambda+1}+\frac{7\lambda+1}{\lambda+1}=4$

or $\quad 5\lambda-1+7\lambda+1=4\lambda+4$

or $\quad 8\lambda=4$

$\Rightarrow \quad \lambda=\frac{1}{2}$

or $\quad \lambda:1=1:2$

Exercise 9.2

1. Find the coordinates of the point which divides the line segment joining the points (3,5) and (7,9) in the ratio 2 : 3 internally.
2. Find the coordinates of the point which divides the line segment joining the points (5, –2) and $\left(-1\frac{1}{2},4\right)$ in the ratio 7 : 9 externally.
3. Prove that origin O divides the line joining the points $A(1, -3)$ and $B(-3, 9)$ in the ratio 1 : 3 internally. Find the coordinartes of the points which devides the line AB externally in the ratio 1 : 3.
4. Find the mid point of line joining the points (22, 20) and (0, 16).
5. In which ratio, x-axis divides the line segment which joins points (5, 3) and (–3, –2)?
6. In which ratio, y-axis divides the line segment which joins points (2, –3) and (5, 6)?
7. In which ratio, point (11, 15) divides the line segment which joins (15, 5) and (9, 20)?
8. If point P (3, 5) divides line segment which joins $A(-2, 3)$ and B (x,y) in the ratio 4 : 7 intermally, then find the co-ordinates of B.
9. Find the co-ordinates of point which trisects the line joining point (11, 9) and (1, 2).
10. Find the co-ordinates of point which quartersects the line joining point (–4, 0) and (0, 6).
11. Find the ratio in which line $3x + y = 9$ divides the line segment which joins points (1, 3) and (2, 7)
12. Find the ratio where point $(-3, p)$, divides intermally the line segment which joins points (–5, –4) and (–2, 3). Also find p.

Miscellaneous Exercise-9

Objective Question [1 to 10]

1. Distance of point (3, 4) form y-axis will be :

   (a) 1 (b) 4 (c) 2 (d) 3
2. Distance of point (5, –2) from x-axis will be :

   (a) 5 (b) 2 (c) 3 (d) 4
3. Distance between points (0, 3) and (–2, 0) will be :

   (a) $\sqrt{14}$ (b) $\sqrt{15}$ (c) $\sqrt{13}$ (d) $\sqrt{5}$

4. Triangle having vertices (–2, 1), (2, –2) and (5, 2) is :

(a) Right triangle (b) Equilateral (c) Isoceles (d) None of these

5. Quadrilateral having vertices (–1, 1), (0, –3), (5, 2) and (4, 6) will be :

(a) Square (b) Rectangle (c) Rhombus (d) Parallelogram

6. Point equidistance from (0, 0), (2, 0) and (0, 2) is :

(a) (1, 2) (b) (2, 1) (c) (2, 2) (d) (1, 1)

7. *P* divides the line segment which joins points (5, 0) and (0, 4) in the ratio of 2 : 3 internally. Co-ordinates of *P* are :

(a) $\left(3, \frac{8}{5}\right)$ (b) $\left(1, \frac{4}{5}\right)$ (c) $\left(\frac{5}{2}, \frac{3}{4}\right)$ (d) $\left(2, \frac{12}{5}\right)$

8. If points (1, 2), (–1, x) and (2, 3) are collinear, then x will be :

(a) 2 (b) 0 (c) −1 (d) 1

9. If distance between point (3, a) and (4, 1) is $\sqrt{10}$, then a will be :

(a) 3, −1 (b) 2, −2 (c) 4, −2 (d) 5, −3

10. If point (x, y) is at equidistant from (2, 1) and (1, –2), then the true statement is :

(a) $x+3y=0$ (b) $3x+y=0$ (c) $x+2y=0$ (d) $2y+3x=0$

11. Find the type of quadrilateral, if its vertices are (1, 4), (–5, 4), (–5, –3) and (1,–3).

12. Which shape will be formed on joining (–2, 0), (2, 0), (2, 2), (0, 4), (–2, 2) in the given order?

13. Find the ratio in which point (3, 4) divides the line segment which joins points (1, 2) and (6, 7).

14. Opposite vetrices of any square are (5, –4) and (–3, 2), then find the length of diagonal.

15. If co-ordinate of one end and mid point of a line segment are (4, 0) and (4, 1) respectively, then find the co-ordinates of other end of line segment.

16. Find the distance between of point (1, 2) from mid point of line segment which joints the points (6, 8) and (2, 4).

17. If in any plane, there are four points *P*(2, –1), *Q*(3, 4), *R*(–2, 3) and *S*(–3, –2), then prove that *PQRS* is not a square but a rhombus.

18. Prove that mid point (*C*) of hypotaneous in a right angled triangle *AOB* is situated at equal distance form vertics *O*, *A* and *B* of triangle.

19. Find the length of median of a triangle whose vertices are (1,–1), (0, 4) and (–5, 3).

20. Prove that mid point of a line segment which joins points (5, 7) and (3, 9) is the same as mid point of line segment which joins points (5,7) and (3, 9).

21. If mid points of sides of a triangle is (1, 2), (0, –1) and (2, –1), then find its vertices.

Important Points

1. Formula of distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or

$$PQ = \sqrt{(\text{difference of abscissas})^2 + (\text{difference of ordinates})^2}$$

2. The coordinates of the point P(x,y) which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

and
$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

3. The coordinates of the point P (x, y) which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m_1 : m_2$ are

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$$

and
$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

4. The mid point of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

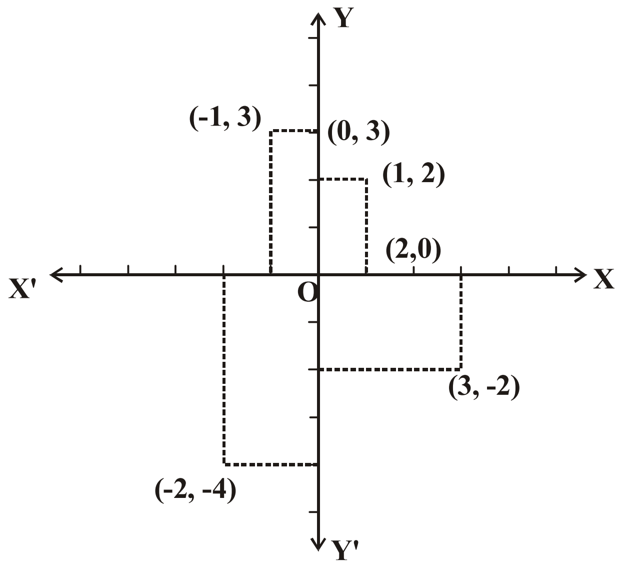
or
$$\left(\frac{\text{Sum of } x \text{ co-ordinates}}{2}, \frac{\text{Sum of } y \text{ co-ordinates}}{2}\right)$$

Answer Sheet

Exercise 9.1

1. $P(5,3), Q(-4,6), R(-3,-2), S(1,-5)$

2.



3. (3, 4) 4. Pentagon 5. (i) Trapezium (ii) Rhombus

6. (i) 13 (ii) $\sqrt{82}$ (iii) $a(t_2 - t_1)\sqrt{(t_2 + t_1)^2 + 4}$ 11. (–2, 0) 12. (0, –2)

13. 1 15. $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

Exercise 9.2

1. $\left(\frac{23}{5}, \frac{33}{5}\right)$ 2. $\left(27\frac{3}{4}, -23\right)$ 3. (3, –9) 4. (11, 18)

5. 3 : 2 6. 2 :5 external division 7. 2 :1 8. $\left(\frac{47}{4}, \frac{17}{2}\right)$

9. $\left(\frac{13}{3}, \frac{13}{3}\right), \left(\frac{23}{3}, \frac{20}{3}\right)$ 10. $\left(-3, \frac{3}{2}\right), (-2, 3), \left(-1, \frac{9}{2}\right)$ 11. 3 : 4 12. $2:1, p = \frac{2}{3}$

Miscellaneous Exercise-9

1. (d) 2. (b) 3. (c) 4. (a) 5. (d) 6. (d) 7. (a)
8. (b) 9. (c) 10. (a) 11. Rectangle 12. Pentagon 13. 2 :3. 14. 10

15. (4, 2) 16. 5 19. $\frac{\sqrt{130}}{2}, \frac{\sqrt{130}}{2}, \sqrt{13}$ 21. (1, –4), (3, 2), (–1, 2)