UNIT 6

GRAVITATION

"The most remarkable discovery in all of astronomy is that the stars are made up of atoms of same kind as those in the Earth" – Richard Feynman

Learning Objectives

In this unit, the student is exposed to

- Kepler's laws for planetary motion
- Newton's law of gravitation
- connection between Kepler's laws and law of gravitation
- calculation of gravitational field and potential
- calculation of variation of acceleration due to gravity
- calculation of escape speed and energy of satellites
- concept of weightlessness
- advantage of heliocentric system over geocentric system
- measurement of the radius of Earth using Eratosthenes method
- recent developments in gravitation and astrophysics

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6.1 INTRODUCTION

We are amazed looking at the glittering sky; we wonder how the Sun rises in the East and sets in the West, why there are comets or why stars twinkle. The sky has been an object of curiosity for human beings from time immemorial. We have always wondered about the motion of stars, the Moon, and the planets. From Aristotle to Stephen Hawking, great minds have tried to understand the movement of celestial objects in space and what causes their motion.

The 'Theory of Gravitation' was developed by Newton in the late 17th century to explain the motion of celestial objects and terrestrial objects and answer most of the queries raised. In spite of the study of gravitation and its effect on celestial objects, spanning last three centuries, "gravitation" is still one of the active areas of research in physics today. In 2017, the Nobel Prize in Physics was given for the detection of 'Gravitational waves' which was theoretically predicted by Albert Einstein in the year 1915. Understanding planetary motion, the formation of stars and galaxies, and recently massive objects like black holes and their life cycle have remained the focus of study for the past few centuries in physics.

Geocentric Model of Solar System

In the second century, Claudius Ptolemy, a famous Greco-Roman astronomer, developed a theory to explain the motion of celestial objects like the Sun, the Moon, Mars, Jupiter etc. This theory was called the geocentric model. According to the geocentric model, the Earth is at the center of the universe and all celestial objects including the Sun, the Moon, and other planets orbit the Earth. Ptolemy's model closely matched with the observations of the sky with our naked eye. But later, astronomers found that even though Ptolemy's model successfully explained the motion of the Sun and the Moon up to a certain level, the motion of Mars and Jupiter could not be explained effectively.

Heliocentric Model of Nicholas Copernicus

In the 15^{th} century, a Polish astronomer, Nicholas Copernicus (1473-1543) proposed a new model called the 'Heliocentric model' in which the Sun was considered to be at the center of the solar system and all planets including the Earth orbited the Sun in circular orbits. This model successfully explained the motion of all celestial objects.

Around the same time, Galileo, a famous Italian physicist discovered that all objects close to Earth were accelerated towards the Earth at the same rate. Meanwhile, a noble man called Tycho Brahe (1546-1601) spent his entire lifetime in recording the observations of the stellar and planetary positions with his naked eye. The data that he compiled were analyzed later by his assistant Johannes Kepler (1571–1630) and eventually the analysis led to the deduction of the laws of the planetary motion. These laws are termed as 'Kepler's laws of planetary motion'.

6.1.1 Kepler's Laws of Planetary Motion

Kepler's laws are stated as follows:

1. **Law of orbits:**
   Each planet moves around the Sun in an elliptical orbit with the Sun at one of the foci.

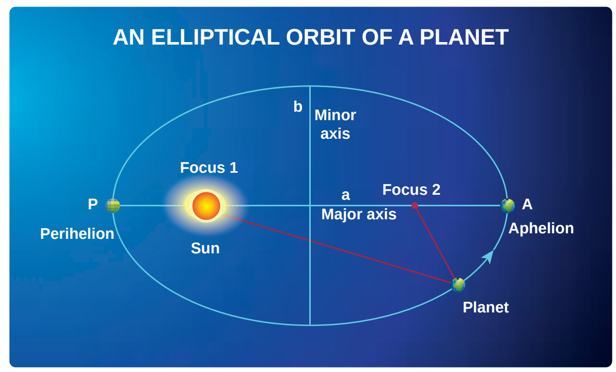


Figure 6.1 An ellipse traced out by a planet around the Sun.

The closest point of approach of the planet to the Sun 'P' is called perihelion and the farthest point 'A' is called aphelion (Figure 6.1). The semi-major axis is 'a' and semi-minor axis is 'b'. In fact, both Copernicus and Ptolemy considered planetary orbits to be circular, but Kepler discovered that the actual orbits of the planets are elliptical.

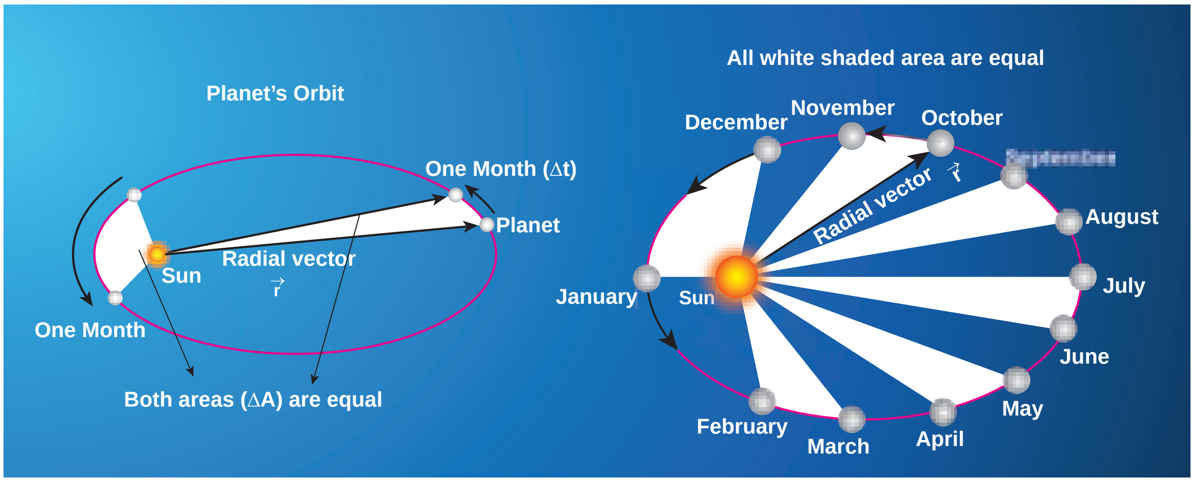


Figure 6.2 Motion of a planet around the Sun depicting 'law of area'.

2. **Law of area:**

The radial vector (line joining the Sun to a planet) sweeps equal areas in equal intervals of time.

In Figure 6.2, the white shaded portion is the area ΔA swept in a small interval of time Δt, by a planet around the Sun. Since the Sun is not at the center of the ellipse, *the planets travel faster when they are nearer to the Sun and slower when they are farther from it, to cover equal area in equal intervals of time.* Kepler discovered the law of area by carefully noting the variation in the speed of planets.

3. **Law of period:**

The square of the time period of revolution of a planet around the Sun in its elliptical orbit is directly proportional to the cube of the semi-major axis of the ellipse. It can be written as:

$$T^2 \propto a^3 \tag{6.1}$$

$$\frac{T^2}{a^3} = constant \tag{6.2}$$

where, T is the time period of revolution for a planet and *a* is the semi-major axis. Physically this law implies that as the distance of the planet from the Sun increases, the time period also increases but not at the same rate.

In Table 6.1, the time period of revolution of planets around the Sun along with their semi-major axes are given. From column four, we can realize that $\frac{T^2}{a^3}$ is nearly a constant endorsing Kepler's third law.

Table 6.1 The time period of revolution of the planets revolving around the Sun and their semi-major axes.

Planet	a ($10^{10}\,m$)	T (years)	$\frac{T^2}{a^3}$
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99

Points to Contemplate

DATA			PROBLEM
Planet	a	T	What is the law connecting *a* and T?
A	1	3	
B	2	6	
C	4	18	

Comment on the relation between *a* and T for these imaginary planets

6.1.2 Universal Law of Gravitation

Even though Kepler's laws were able to explain the planetary motion, they failed to explain the forces responsible for it. It was Isaac Newton who analyzed Kepler's laws, Galileo's observations and deduced the law of gravitation.

Newton's law of gravitation states that a particle of mass M_1 attracts any other particle of mass M_2 in the universe with an attractive force. The strength of this force of attraction was found to be directly proportional to the product of their masses and is inversely proportional to the square of the distance between them. In mathematical form, it can be written as:

$$\vec{F} = -\frac{GM_1M_2}{r^2}\hat{r} \qquad (6.3)$$

where $\hat{r}$ is the unit vector from M_1 towards M_2 as shown in Figure 6.3, and G is the Gravitational constant that has the value of $6.67\times10^{-11}\,N\,m^2kg^{-2}$, and r is the distance between the two masses M_1 and M_2. In Figure 6.3, the vector $\vec{F}$ denotes the gravitational force experienced by M_2 due to M_1. Here the negative sign indicates that the gravitational force is always attractive in nature and the direction of the force is along the line joining the two masses.

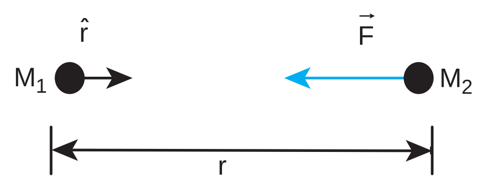
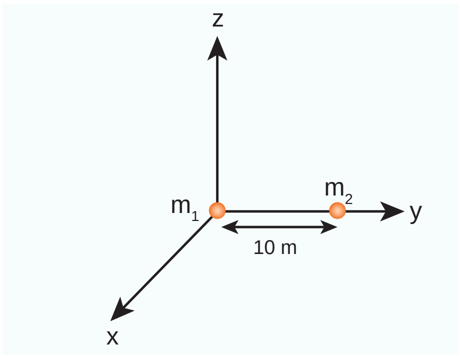


Figure 6.3 Attraction of two masses towards each other.

In cartesian coordinates, the square of the distance is expressed as $r^2 = \left(x^2 + y^2 + z^2\right)$. This is dealt in unit 2.

EXAMPLE 6.1

Consider two point masses m_1 and m_2 which are separated by a distance of 10 meter as shown in the following figure. Calculate the force of attraction between them and draw the directions of forces on each of them. Take m_1= 1 kg and m_2 = 2 kg



Solution

The force of attraction is given by

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

From the figure, r =10 m.

First, we can calculate the magnitude of the force

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{100}$$
$$= 13.34\times10^{-13}\,\text{N}$$

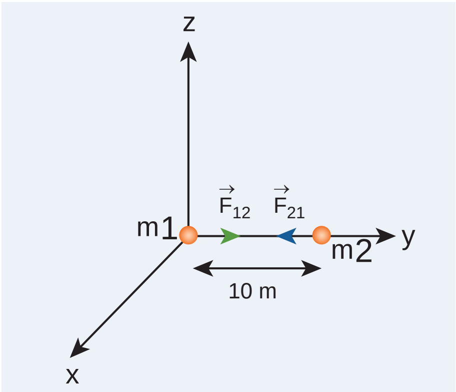
It is to be noted that this force is very small. This is the reason we do not feel the gravitational force of attraction between each other. The small value of G plays a very crucial role in deciding the strength of the force.

The force of attraction ($\vec{F}_{21}$) experienced by the mass m_2 due to m_1 is in the negative 'y' direction ie., $\hat{r} = -\hat{j}$. According to Newton's third law, the mass m_2 also exerts equal and opposite force on m_1. So the force of attraction ($\vec{F}_{12}$) experienced by m_1 due to m_2 is in the direction of positive 'y' axis ie., $\hat{r} = \hat{j}$.

$$\vec{F}_{21} = -13.34\times10^{-13}\,\hat{j}$$

$$\vec{F}_{12} = 13.34\times10^{-13}\,\hat{j}$$

The direction of the force is shown in the figure,



Gravitational force of attraction between m_1 and m_2

$\vec{F}_{12} = -\vec{F}_{21}$ which confirms Newton's third law.

Important features of gravitational force:

- As the distance between two masses increases, the strength of the force tends to decrease because of inverse dependence on r^2. Physically it implies that the planet Uranus experiences less gravitational force from the Sun than the Earth since Uranus is at larger distance from the Sun compared to the Earth.

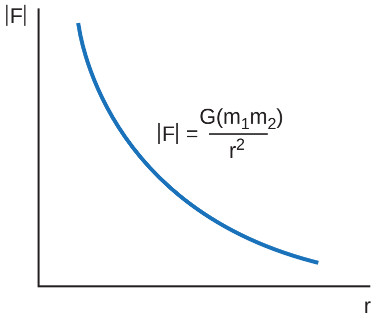


Figure 6.4 Variation of gravitational force with distance

- The gravitational forces between two particles always constitute an action-reaction pair. It implies that the gravitational force exerted by the Sun on the Earth is always towards the Sun. The reaction-force is exerted by the Earth on the Sun. The direction of this reaction force is towards Earth.

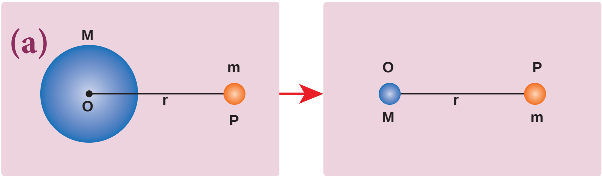
- The torque experienced by the Earth due to the gravitational force of the Sun is given by

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \left(-\frac{GM_S M_E}{r^2} \hat{r} \right) = 0$$

Since $\vec{r} = r\,\hat{r}$, $(\hat{r} \times \hat{r}) = 0$

So $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$. It implies that angular momentum $\vec{L}$ is a constant vector. The angular momentum of the Earth about the Sun is constant throughout the motion. It is true for all the planets. In fact, this constancy of angular momentum leads to the Kepler's second law.

- The expression $\vec{F} = -\frac{GM_1 M_2}{r^2} \hat{r}$ has one inherent assumption that both M_1 and M_2 are treated as point masses. When it is said that Earth orbits around the Sun due to Sun's gravitational force, we assumed Earth and Sun to be point masses. This assumption is a good approximation because the distance between the two bodies is very much larger than their diameters. For some irregular and extended objects separated by a small distance, we cannot directly use the equation (6.3). Instead, we have to invoke separate mathematical treatment which will be brought forth in higher classes.
- However, this assumption about point masses holds even for small distance for one special case. To calculate force of attraction between a hollow sphere of mass M with uniform density and point mass m kept outside the hollow sphere, we can replace the hollow sphere of mass M as equivalent to a point mass M located at the center of the hollow sphere. The force of attraction between the hollow sphere of mass M and point mass m can be calculated by treating the hollow sphere also as another point mass. Essentially the entire mass of the hollow sphere appears to be concentrated at the center of the hollow sphere. It is shown in the Figure 6.5(a).
- There is also another interesting result. Consider a hollow sphere of mass M. If we place another object of mass 'm' inside this hollow sphere as in Figure 6.5(b), the force experienced by this mass 'm' will be zero. This calculation will be dealt with in higher classes.



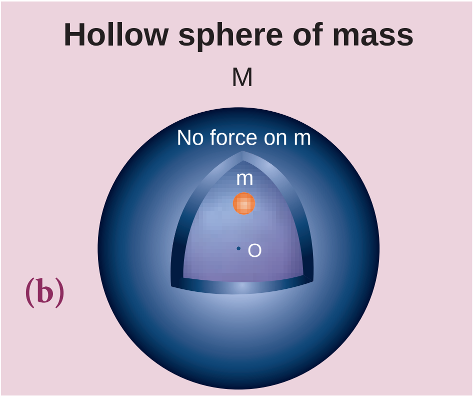


Figure 6.5 A mass placed in a hollow sphere.

- The triumph of the law of gravitation is that it concludes that the mango that is falling down and the Moon orbiting the Earth are due to the same gravitational force.

Newton's inverse square Law:

Newton considered the orbits of the planets as circular. For circular orbit of radius r, the centripetal acceleration towards the center is

$$a = -\frac{v^2}{r} \qquad (6.4)$$

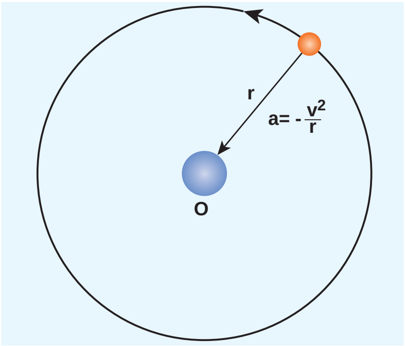


Figure 6.6 Point mass orbiting in a circular orbit.

Here v is the velocity and r, the distance of the planet from the center of the orbit (Figure 6.6).

The velocity in terms of known quantities r and T, is

$$v = \frac{2\pi r}{T} \qquad (6.5)$$

Here T is the time period of revolution of the planet. Substituting this value of v in equation (6.4) we get,

$$a = -\frac{\left(\frac{2\pi r}{T}\right)^2}{r} = -\frac{4\pi^2 r}{T^2} \qquad (6.6)$$

Substituting the value of 'a' from (6.6) in Newton's second law, $F = ma$, where 'm' is the mass of the planet.

$$F = -\frac{4\pi^2 mr}{T^2} \qquad (6.7)$$

From Kepler's third law,

$$\frac{r^3}{T^2} = k(constant) \qquad (6.8)$$

$$\frac{r}{T^2} = \frac{k}{r^2} \qquad (6.9)$$

By substituting equation 6.9 in the force expression, we can arrive at the law of gravitation.

$$F = -\frac{4\pi^2 mk}{r^2} \qquad (6.10)$$

Here negative sign implies that the force is attractive and it acts towards the center. In equation (6.10), mass of the planet 'm' comes explicitly. But Newton strongly felt that according to his third law, if Earth is attracted by the Sun, then the Sun must also be attracted by the Earth with the same magnitude of force. So he felt that the Sun's mass (M) should also occur explicitly in the expression for force (6.10). From this insight, he equated the constant $4\pi^2 k$ *to* GM which turned out to be the law of gravitation.

$$F = -\frac{GMm}{r^2}$$

Again the negative sign in the above equation implies that the gravitational force is attractive.

In the above discussion we assumed that the orbit of the planet to be circular which is not true as the orbit of the planet around the Sun is elliptical. But this circular orbit assumption is justifiable because planet's orbit is very close to being circular and there is only a very small deviation from the circular shape.

Points to Contemplate

If Kepler's third law was "r^3T^2 = constant" instead of "$\frac{r^3}{T^2}$ = constant" what would be the new law of gravitation? Would it still be an inverse square law? How would the gravitational force change with distance? In this new law of gravitation, will Neptune experience greater gravitational force or lesser gravitational force when compared to the Earth?

EXAMPLE 6.2

Moon and an apple are accelerated by the same gravitational force due to Earth. Compare the acceleration of the two.

The gravitational force experienced by the apple due to Earth

$$F = -\frac{GM_E M_A}{R^2}$$

Here M_A – Mass of the apple, M_E – Mass of the Earth and R – Radius of the Earth.

Equating the above equation with Newton's second law,

$$M_A a_A = -\frac{GM_E M_A}{R^2}.$$

Simplifying the above equation we get,

$$a_A = -\frac{GM_E}{R^2}$$

Here a_A is the acceleration of apple that is equal to 'g'.

Similarly the force experienced by Moon due to Earth is given by

$$F = -\frac{GM_E M_m}{R_m^2}.$$

Here R_m - distance of the Moon from the Earth, M_m – Mass of the Moon

The acceleration experienced by the Moon is given by

$$a_m = -\frac{GM_E}{R_m^2}.$$

The ratio between the apple's acceleration to Moon's acceleration is given by

$$\frac{a_A}{a_m} = \frac{R_m^2}{R^2}.$$

From the Hipparchrus measurement, the distance to the Moon is 60 times that of Earth radius. $R_m = 60R$.

$$a_A / a_m = \frac{(60R)^2}{R^2} = 3600.$$

The apple's acceleration is 3600 times the acceleration of the Moon.

The same result was obtained by Newton using his gravitational formula. The apple's acceleration is measured easily and it is 9.8 m s^{-2}. Moon orbits the Earth once in 27.3 days and by using the centripetal acceleration formula, (Refer unit 3).

$$\frac{a_A}{a_m} = \frac{9.8}{0.00272} = 3600$$

which is exactly what he got through his law of gravitation.

Note The above calculation depends on knowing the distance between the Earth and the Moon and the radius of the Earth. The radius of the Earth was measured by Greek librarian Eratosthenes and distance between the Earth and the Moon was measured by Greek astronomer Hipparchrus 2400 years ago. It is very interesting to note that in order to measure these distances he used only high school geometry and trigonometry. These details are discussed in the astronomy section (6.5).

6.1.3 Gravitational Constant

In the law of gravitation, the value of gravitational constant G plays a very important role. The value of G explains why the gravitational force between the Earth and the Sun is so great while the same force between two small objects (for example between two human beings) is negligible.

The force experienced by a mass 'm' which is on the surface of the Earth (Figure 6.7) is given by

$$F = -\frac{GM_E m}{R_E^2} \qquad (6.11)$$

M_E-mass of the Earth, m - mass of the object, R_E- radius of the Earth.

Equating Newton's second law, $F = -mg$, to equation (6.11) we get,

$$-mg = -\frac{GM_E m}{R_E^2}$$

$$g = \frac{GM_E}{R_E^2} \qquad (6.12)$$

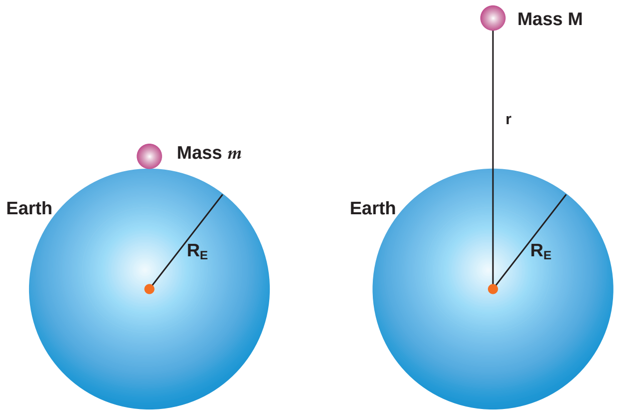


Figure 6.7 Force experienced by a mass on the (i) surface of the Earth (ii) at a distance from the centre of the Earth

Now the force experienced by some other object of mass M at a distance r from the center of the Earth is given by,

$$F = -\frac{GM_E M}{r^2}$$

Using the value of g in equation (6.12), the force F will be,

$$F = -gM\frac{R_E^2}{r^2} \qquad (6.13)$$

From this it is clear that the force can be calculated simply by knowing the value of g. It is to be noted that in the above calculation G is not required.

DO YOU KNOW? In the year 1798, Henry Cavendish experimentally determined the value of gravitational constant 'G' by using a torsion balance. He calculated the value of 'G' to be equal to $6.75 \times 10^{-11}\,N\,m^2kg^{-2}$. Using modern techniques a more accurate value of G could be measured. The currently accepted value of G is $6.67259 \times 10^{-11}\,N\,m^2kg^{-2}$.

6.2 GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

6.2.1 Gravitational field

Force is basically due to the interaction between two particles. Depending upon the type of interaction we can have two kinds of forces: Contact forces and Non-contact forces (Figure 6.8).

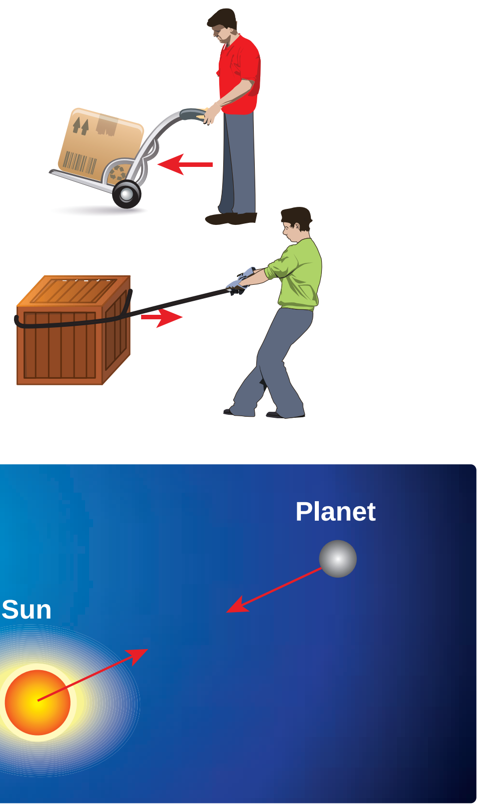


Figure 6.8 Depiction of contact and non-contact forces

Contact forces are the forces applied where one object is in physical contact with the other. The movement of the object is caused by the physical force exerted through the contact between the object and the agent which exerts force.

Consider the case of Earth orbiting around the Sun. Though the Sun and the Earth are not physically in contact with each other, there exists an interaction between them. This is because of the fact that the Earth experiences the gravitational force of the Sun. This gravitational force is a non-contact force.

It sounds mysterious that the Sun attracts the Earth despite being very far from it and without touching it. For contact forces like push or pull, we can calculate the strength of the force since we can feel or see. But how do we calculate the strength of non-contact force at different distances? To understand and calculate the strength of non-contact forces, the concept of 'field' is introduced.

The gravitational force on a particle of mass 'm_2' due to a particle of mass 'm_1' is

$$\vec{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{r} \qquad (6.14)$$

where $\hat{r}$ is a unit vector that points from m_1 to m_2 along the line joining the masses m_1 and m_2.

The gravitational field intensity $\vec{E}_1$ (here after called as gravitational field) at a point which is at a distance r from m_1 is defined as the gravitational force experienced by unit mass placed at that point. It is given by the ratio $\frac{\vec{F}_{21}}{m_2}$ (where m_2 is the mass of the object on which $\vec{F}_{21}$ acts)

Using $\vec{E}_1 = \frac{\vec{F}_{21}}{m_2}$ in equation (6.14) we get,

$$\vec{E}_1 = -\frac{Gm_1}{r^2}\hat{r} \qquad (6.15)$$

$\vec{E}_1$ is a vector quantity that points towards the mass m_1 and is independent of mass m_2, Here m_2 is taken to be of unit magnitude . The unit $\hat{r}$ is along the line between m_1 and the point in question. The field $\vec{E}_1$ is due to the mass m_1.

In general, the gravitational field intensity due to a mass M at a distance r is given by

$$\vec{E} = -\frac{GM}{r^2}\hat{r} \quad (6.16)$$

Now in the region of this gravitational field, a mass 'm' is placed at a point P (Figure 6.9). Mass 'm' interacts with the field $\vec{E}$ and experiences an attractive force due to M as shown in Figure 6.9. The gravitational force experienced by 'm' due to 'M' is given by

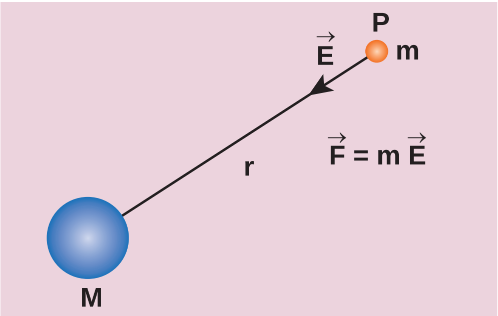


Figure 6.9 Gravitational Field intensity measured with an object of unit mass

$$\vec{F}_m = m\vec{E} \quad (6.17)$$

Now we can equate this with Newton's second law $\vec{F} = m\vec{a}$

$$m\vec{a} = m\vec{E} \quad (6.18)$$

$$\vec{a} = \vec{E} \quad (6.19)$$

In other words, equation (6.18) implies that the gravitational field at a point is equivalent to the acceleration experienced by a particle at that point. However, it is to be noted that $\vec{a}$ and $\vec{E}$ are separate physical quantities that have the same magnitude and direction. The gravitational field $\vec{E}$ is the property of the source and acceleration $\vec{a}$ is the effect experienced by the test mass (unit mass) which is placed in the gravitational field $\vec{E}$. The non-contact interaction between two masses can now be explained using the concept of "Gravitational field".

Points to be noted:

i) The strength of the gravitational field decreases as we move away from the mass M as depicted in the Figure 6.10. The magnitude of $\vec{E}$ decreases as the distance r increases.

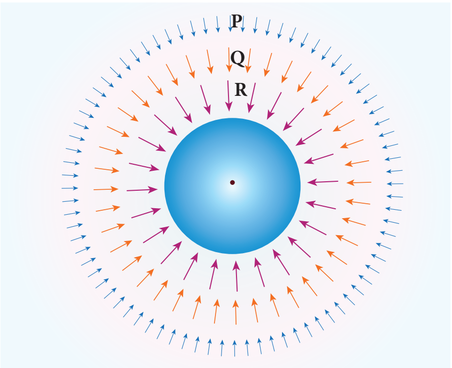


Figure 6.10 Strength of the Gravitational field lines decreases with distance

Figure 6.10 shows that the strength of the gravitational field at points P, Q, and R is given by $|\vec{E}_P| < |\vec{E}_Q| < |\vec{E}_R|$. It can be understood by comparing the length of the vectors at points P, Q, and R.

ii) The "field" concept was introduced as a mathematical tool to calculate gravitational interaction. Later it was found that field is a real physical quantity and it carries energy and momentum in space. The concept of field is inevitable in understanding the behavior of charges.

iii) The unit of gravitational field is Newton per kilogram (N/kg) or m s^{-2}.

6.2.2 Superposition principle for Gravitational field

Consider 'n' particles of masses $m_1, m_2, \ldots m_n$, distributed in space at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots$ etc, with respect to point P. The total gravitational field at a point P due to all the masses is given by the vector sum of the gravitational field due to the individual masses (Figure 6.11). This principle is known as superposition of gravitational fields.

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 + \ldots \vec{E}_n$$

$$= -\frac{Gm_1}{r_1^2}\hat{r}_1 - \frac{Gm_2}{r_2^2}\hat{r}_2 - \ldots - \frac{Gm_n}{r_n^2}\hat{r}_n$$

$$= -\sum_{i=1}^{n} \frac{Gm_i}{r_i^2}\hat{r}_i. \quad (6.20)$$

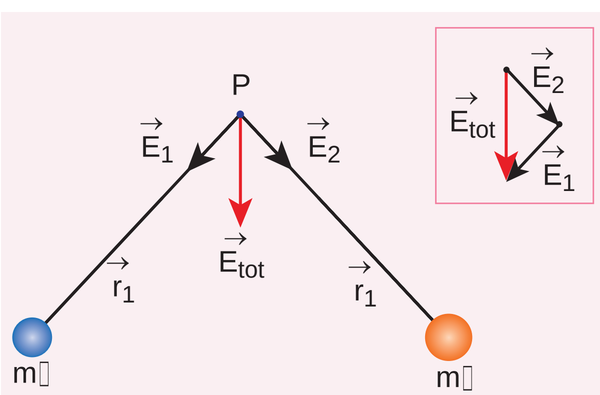
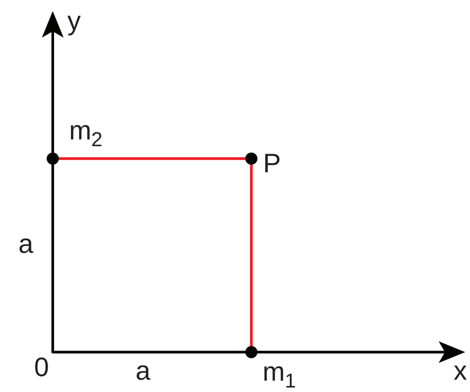


Figure 6.11 Superposition of two gravitational field intensities giving resultant field.

Instead of discrete masses, if we have continuous distribution of a total mass M, then the gravitational field at a point P is calculated using the method of integration.

EXAMPLE 6.3

(a) Two particles of masses m_1 *and* m_2 are placed along the x and y axes respectively at a distance 'a' from the origin. Calculate the gravitational field at a point P shown in figure below.



Solution

Gravitational field due to m_1 at a point P is given by,

$$\vec{E}_1 = -\frac{Gm_1}{a^2}\hat{j}$$

Gravitational field due to m_2 at the point p is given by,

$$\vec{E}_2 = -\frac{Gm_2}{a^2}\hat{i}$$

$$\vec{E}_{total} = -\frac{Gm_1}{a^2}\hat{j} - \frac{Gm_2}{a^2}\hat{i}$$

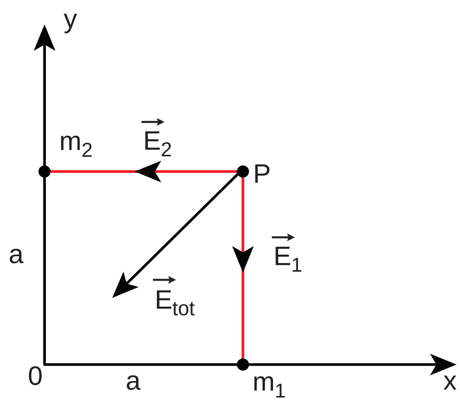
$$= -\frac{G}{a^2}\left(m_1\hat{j} + m_2\hat{i}\right)$$

The direction of the total gravitational field is determined by the relative value of m_1 and m_2.

When $m_1 = m_2 = m$

$$\vec{E}_{total} = -\frac{Gm}{a^2}(\hat{i}+\hat{j})$$

($\hat{i}+\hat{j}=\hat{j}+\hat{i}$ as vectors obeys commutation law).

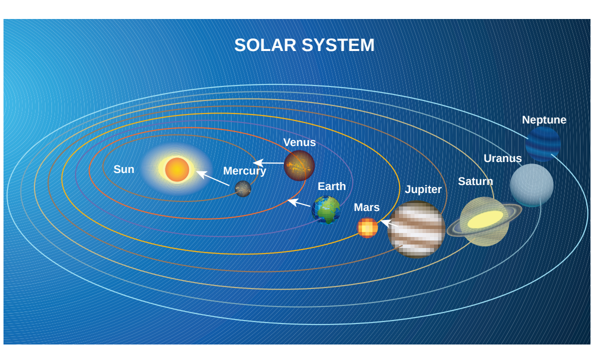


$\vec{E}_{total}$ points towards the origin of the co-ordinate system and the magnitude of $\vec{E}_{total}$ is $\sqrt{2}\,\frac{Gm}{a^2}$.

EXAMPLE 6.4

Qualitatively indicate the gravitational field of Sun on Mercury, Earth, and Jupiter shown in figure.

Since the gravitational field decreases as distance increases, Jupiter experiences a weak gravitational field due to the Sun. Since Mercury is the nearest to the Sun, it experiences the strongest gravitational field.



Solar System

6.2.3 Gravitational Potential Energy

The concept of potential energy and its physical meaning were dealt in unit 4. The gravitational force is a conservative force and hence we can define a gravitational potential energy associated with this conservative force field.

Two masses m_1 and m_2 are initially separated by a distance r'. Assuming m_1 to be fixed in its position, work must be done on m_2 to move the distance from r' to r as shown in Figure 6.12(a)

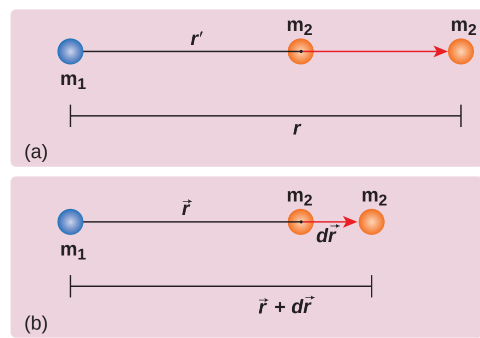


Figure 6.12 Two distant masses changing the linear distance

To move the mass m_2 through an infinitesimal displacement $d\vec{r}$ from $\vec{r}$ *to* $\vec{r}+d\vec{r}$ (shown in the Figure 6.12(b)), work has to be done externally. This infinitesimal work is given by

$$dW = \vec{F}_{ext}.d\vec{r} \qquad (6.21)$$

The work is done against the gravitational force, therefore,

$$\left|\vec{F}_{ext}\right| = \left|\vec{F}_G\right| = \frac{Gm_1m_2}{r^2} \qquad (6.22)$$

Substituting Equation (6.22) in 6.21, we get

$$dW = \frac{Gm_1m_2}{r^2}\hat{r}.d\vec{r} \qquad (6.23)$$

Also we know,

$$d\vec{r} = dr\,\hat{r} \qquad (6.24)$$

$$\Rightarrow dW = \frac{Gm_1m_2}{r^2}\hat{r}.(dr\,\hat{r}) \qquad (6.25)$$

$$\hat{r}.\hat{r} = 1(\textit{since both are unit vectors})$$

$$\therefore \quad dW = \frac{Gm_1m_2}{r^2}dr \qquad (6.26)$$

Thus the total work done for displacing the particle from r' to r is

$$W = \int_{r'}^{r} dW = \int_{r'}^{r} \frac{Gm_1m_2}{r^2}dr \qquad (6.27)$$

$$W = -\left(\frac{Gm_1m_2}{r}\right)_{r'}^{r}$$

$$W = -\frac{Gm_1m_2}{r} + \frac{Gm_1m_2}{r'} \qquad (6.28)$$

$$W = U(r) - U(r')$$

where $U(r) = \frac{-Gm_1m_2}{r}$

This work done W gives the gravitational potential energy difference of the system of masses m_1 and m_2 when the separation between them are r and r' respectively.

Case 1: If $r < r'$

Since gravitational force is attractive, m_2 is attracted by m_1.Then m_2 can move from r' to r without any external work (Figure 6.13(a)). Here work is done by the system spending its internal energy and hence the work done is said to be negative.

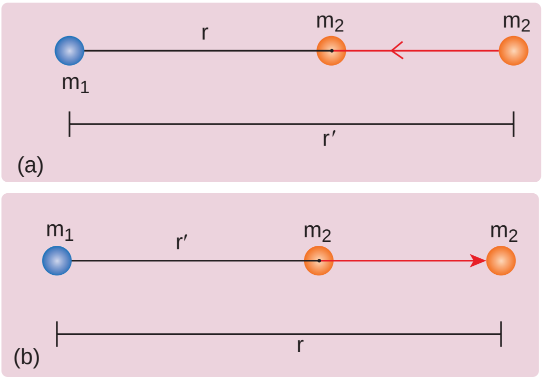


Figure 6.13 Cases for calculation of work done by gravity

Case 2: If $r > r'$

Work has to be done against gravity to move the object from r' to r (Figure 6.13(b)). Therefore work is done on the body by external force and hence work done is positive.

It is to be noted that only potential energy difference has physical significance. Now gravitational potential energy can be discussed by choosing one point as the reference point.

Let us choose $r' = \infty$. Then the second term in the equation (6.28) becomes zero.

$$W = -\frac{Gm_1m_2}{r} + 0 \qquad (6.29)$$

Now we can define gravitational potential energy of a system of two masses m_1 and m_2 separated by a distance r as the amount of work done to take the mass m_2 from a distance r to infinity assuming m_1 to be fixed in its position and is written as $U(r) = -\frac{Gm_1m_2}{r}$. It is to be noted that the gravitational potential energy of the system consisting of two masses m_1 and m_2 separated by a distance r, is the gravitational potential energy difference of the system when the masses are separated by an infinite distance and by distance r. $U(r) = U(r) - U(\infty)$. Here we choose $U(\infty) = 0$ as the reference point. The gravitational potential energy $U(r)$ is always negative because when two masses come together slowly from infinity, work is done by the system.

The unit of gravitational potential energy $U(r)$ is Joule and it is a scalar quantity. The gravitational potential energy depends upon the two masses and the distance between them.

6.2.4 Gravitational potential energy near the surface of the Earth

It is already discussed in chapter 4 that when an object of mass m is raised to a height h, the potential energy stored in the object is mgh (Figure 6.14). This can be derived using the general expression for gravitational potential energy.

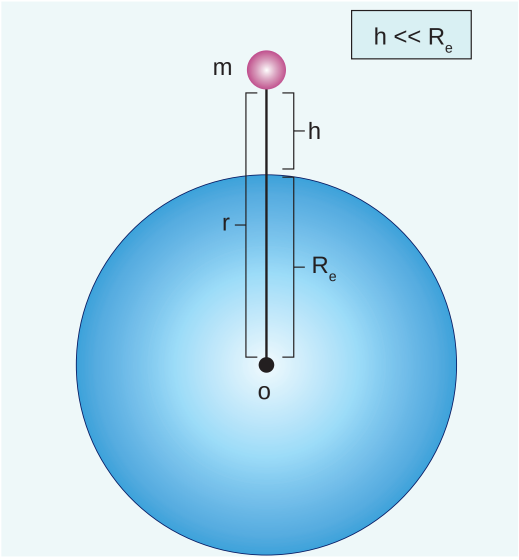


Figure 6.14 Mass placed at a distance r from the center of the Earth

Consider the Earth and mass system, with *r*, the distance between the mass m and the Earth's centre. Then the gravitational potential energy,

$$U = -\frac{GM_e m}{r} \qquad (6.30)$$

Here r = R_e+h, where R_e is the radius of the Earth. h is the height above the Earth's surface

$$U = -G\frac{M_e m}{(R_e + h)} \qquad (6.31)$$

If h << R_e, equation (6.31) can be modified as

$$U = -G\frac{M_e m}{R_e(1 + h/R_e)}$$

$$U = -G\frac{M_e m}{R_e}(1 + h/R_e)^{-1} \qquad (6.32)$$

By using Binomial expansion and neglecting the higher order terms, we get

$$U = -G\frac{M_e m}{R_e}\left(1 - \frac{h}{R_e}\right). \qquad (6.33)$$

We know that, for a mass m on the Earth's surface,

$$G\frac{M_e m}{R_e} = mgR_e \qquad (6.34)$$

Substituting equation (6.34) in (6.33) we get,

$$U = -mgR_e + mgh \qquad (6.35)$$

It is clear that the first term in the above expression is independent of the height h. For example, if the object is taken from height h_1 to h_2 then the potential energy at h_1 is

$$U(h_1) = -mgR_e + mgh_1 \qquad (6.36)$$

and the potential energy at h_2 is

$$U(h_2) = -mgR_e + mgh_2 \qquad (6.37)$$

The potential energy difference between h_1 and h_2 is

$$U(h_2) - U(h_1) = mg(h_2 - h_1). \qquad (6.38)$$

The term mgR_e in equations (6.36) and (6.37) plays no role in the result. Hence in the equation (6.35) the first term can be omitted or taken to zero. Thus it can be stated that The gravitational potential energy stored in the particle of mass m at a height h from the surface of the Earth is $U = mgh$. On the surface of the Earth, $U = 0$, since h is zero.

It is to be noted that mgh is the work done on the particle when we take the mass m from the surface of the Earth to a height h. This work done is stored as a gravitational potential energy in the mass m. Even though mgh is gravitational potential energy of the system (Earth and mass m), we can take mgh as the gravitational potential energy of the mass m since Earth is stationary when the mass moves to height h.

6.2.5 Gravitational potential V(r)

It is explained in the previous sections that the gravitational field $\vec{E}$ depends only on the source mass which creates the field. It is a vector quantity. We can also define a scalar quantity called "gravitational potential" which depends only on the source mass.

The gravitational potential at a distance r due to a mass is defined as the amount of work required to take unit mass from the distance r to infinity and it is denoted as V(r). In other words, the gravitational potential at distance r is equivalent to gravitational potential energy per unit mass at the same distance r. It is a scalar quantity and its unit is J kg^{-1}

We can determine gravitational potential from gravitational potential energy. Consider two masses m_1 and m_2 separated by a distance r which has gravitational potential energy $U(r)$ (Figure 6.15). The gravitational potential due to mass m_1 at a point P which is at a distance r from m_1 is obtained by making m_2 equal to unity (m_2 = 1kg). Thus the gravitational potential $V(r)$ due to mass m_1 at a distance r is

$$V(r) = -\frac{Gm_1}{r} \qquad (6.39)$$

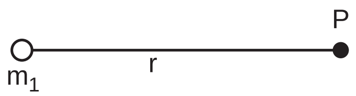


Figure 6.15 Point mass placed at a distance

Gravitational field and gravitational force are vector quantities whereas the gravitational potential and gravitational potential energy are scalar quantities. The motion of particles can be easily analyzed using scalar quantities than vector quantities. Consider the example of a falling apple:

Figure 6.16 shows an apple which falls on Earth due to Earth's gravitational force. This can be explained using the concept of gravitational potential $V(r)$ as follows.

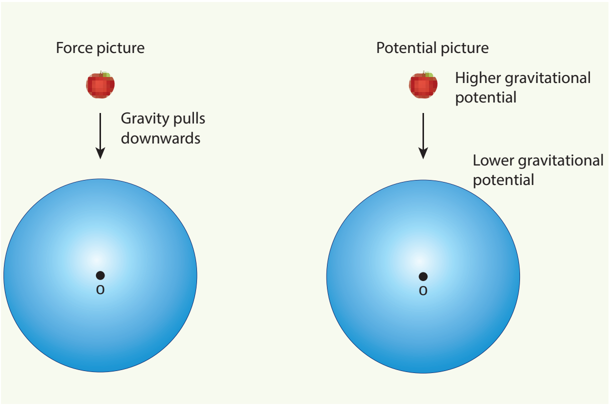


Figure 6.16 Apple falling freely under gravity

The gravitational potential $V(r)$ at a point of height h from the surface of the Earth is given by,

$$V(r = R + h) = -\frac{GM_e}{(R+h)} \qquad (6.40)$$

The gravitational potential $V(r)$ on the surface of Earth is given by,

$$V(r = R) = -\frac{GM_e}{R} \qquad (6.41)$$

Thus we see that

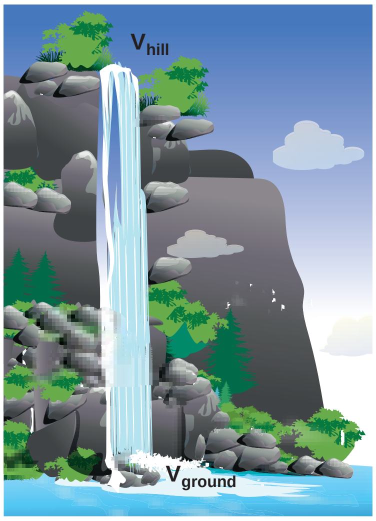
$$V(r = R) < V(r = R + h). \qquad (6.42)$$

It is already discussed in the previous section that the gravitational potential energy near the surface of the Earth at height h is mgh. The gravitational potential at this point is simply $V(h) = U(h)/m = gh$. In fact, the gravitational potential on the surface of the Earth is zero since h is zero. So the apple falls from a region of a higher gravitational potential to a region of lower gravitational potential. In general, the mass will move from a region of higher gravitational potential to a region of lower gravitational potential.

EXAMPLE 6.5

Water falls from the top of a hill to the ground. Why?

This is because the top of the hill is a point of higher gravitational potential than the surface of the Earth i.e. $V_{hill} > V_{ground}$

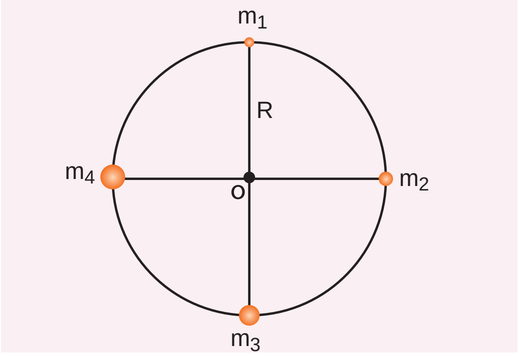


Water falling from hill top

The motion of particles can be analyzed more easily using scalars like $U(r)$ or $V(r)$ than vector quantities like $\vec{F}$ or $\vec{E}$. In modern theories of physics, the concept of potential plays a vital role.

EXAMPLE 6.6

Consider four masses m_1, m_2, m_3, and m_4 arranged on the circumference of a circle as shown in figure below



Calculate

(a) The gravitational potential energy of the system of 4 masses shown in figure.

(b) The gravitational potential at the point O due to all the 4 masses.

Solution

The gravitational potential energy $U(r)$ can be calculated by finding the sum of gravitational potential energy of each pair of particles.

$$U = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_1m_4}{r_{14}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_2m_4}{r_{24}} - \frac{Gm_3m_4}{r_{34}}$$

Here r_{12}, r_{13} ... are distance between pair of particles

$$r_{14}^{\ 2} = R^2 + R^2 = 2R^2$$

$$r_{14} = \sqrt{2}R = r_{12} = r_{23} = r_{34}$$

$$r_{13} = r_{24} = 2R$$

$$U = -\frac{Gm_1m_2}{\sqrt{2}R} - \frac{Gm_1m_3}{2R} - \frac{Gm_1m_4}{\sqrt{2}R} - \frac{Gm_2m_3}{\sqrt{2}R} - \frac{Gm_2m_4}{2R} - \frac{Gm_3m_4}{\sqrt{2}R}$$

$$U = -\frac{G}{R}\left[\frac{m_1m_2}{\sqrt{2}} + \frac{m_1m_3}{2} + \frac{m_1m_4}{\sqrt{2}} + \frac{m_2m_3}{\sqrt{2}} + \frac{m_2m_4}{2} + \frac{m_3m_4}{\sqrt{2}}\right]$$

If all the masses are equal, then $m_1 = m_2 = m_3 = m_4 = M$

$$U = -\frac{GM^2}{R}\left[\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}}\right]$$

$$U = -\frac{GM^2}{R}\left[1 + \frac{4}{\sqrt{2}}\right]$$

$$U = -\frac{GM^2}{R}\left[1 + 2\sqrt{2}\right]$$

The gravitational potential V(r) at a point O is equal to the sum of the gravitational potentials due to individual mass. Since potential is a scalar, the net potential at point O is the algebraic sum of potentials due to each mass.

$$V_O(r) = -\frac{Gm_1}{R} - \frac{Gm_2}{R} - \frac{Gm_3}{R} - \frac{Gm_4}{R}$$

If $m_1 = m_2 = m_3 = m_4 = M$

$$V_O(r) = -\frac{4GM}{R}$$

6.3 ACCELERATION DUE TO GRAVITY OF THE EARTH

When objects fall on the Earth, the acceleration of the object is towards the Earth. From Newton's second law, an object is accelerated only under the action of a force. In the case of Earth, this force is the gravitational pull of Earth. This force produces a constant acceleration near the Earth's surface in all bodies, irrespective of their masses. The gravitational force exerted by Earth on the mass m near the surface of the Earth is given by

$$\vec{F} = -\frac{GmM_e}{R_e^{\ 2}}\hat{r}$$

Now equating Gravitational force to Newton's second law,

$$m\vec{a} = -\frac{GmM_e}{R_e^2}\hat{r}$$

hence, acceleration is,

$$\vec{a} = -\frac{GM_e}{R_e^2}\hat{r} \quad (6.43)$$

The acceleration experienced by the object near the surface of the Earth due to its gravity is called acceleration due to gravity. It is denoted by the symbol g. The magnitude of acceleration due to gravity is

$$|g| = \frac{GM_e}{R_e^2}. \quad (6.44)$$

It is to be noted that the acceleration experienced by any object is independent of its mass. The value of g depends only on the mass and radius of the Earth. Infact, Galileo arrived at the same conclusion 400 years ago that *all objects fall towards the Earth with the same acceleration* through various quantitative experiments. The acceleration due to gravity g is found to be 9.8 m s^{-2} on the surface of the Earth near the equator.

6.3.1 Variation of g with altitude, depth and latitude

Consider an object of mass m at a height h from the surface of the Earth. Acceleration experienced by the object due to Earth is

$$g' = \frac{GM}{(R_e + h)^2} \quad (6.45).$$

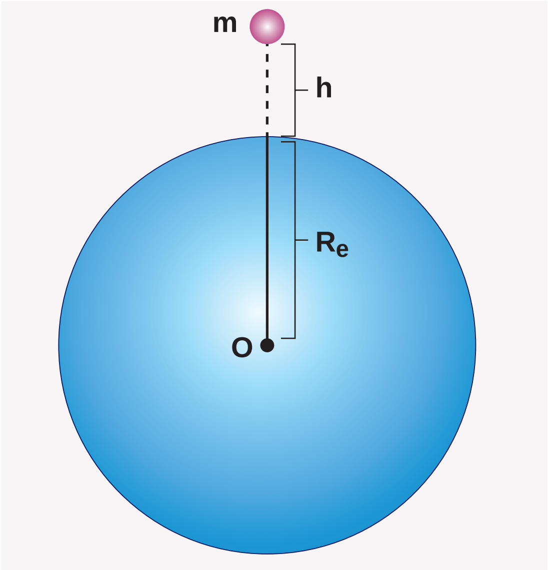


Figure 6.17(a) Mass at a height h from the center of the Earth

$$g' = \frac{GM}{R_e^2\left(1 + \frac{h}{R_e}\right)^2}$$

$$g' = \frac{GM}{R_e^2}\left(1 + \frac{h}{R_e}\right)^{-2}$$

If $h \ll R_e$

We can use Binomial expansion. Taking the terms upto first order

$$g' = \frac{GM}{R_e^2}\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - 2\frac{h}{R_e}\right) \quad (6.46)$$

We find that $g' < g$. This means that as altitude h increases the acceleration due to gravity g decreases.

EXAMPLE 6.7

1. Calculate the value of g in the following two cases:

   (a) If a mango of mass ½ kg falls from a tree from a height of 15 meters, what is the acceleration due to gravity when it begins to fall?

Solution

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = 9.8\left(1 - \frac{2 \times 15}{6400 \times 10^3}\right)$$

$$g' = 9.8\left(1 - 0.469 \times 10^{-5}\right)$$

But $1 - 0.00000469 \cong 1$

Therefore $g' = g$

   (b) Consider a satellite orbiting the Earth in a circular orbit of radius 1600 km above the surface of the Earth. What is the acceleration experienced by the satellite due to Earth's gravitational force?

Solution

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - \frac{2 \times 1600 \times 10^3}{6400 \times 10^3}\right)$$

$$g' = g\left(1 - \frac{2}{4}\right)$$

$$g' = g\left(1 - \frac{1}{2}\right) = g/2$$

The above two examples show that the acceleration due to gravity is a constant near the surface of the Earth.

Note Can we substitute $h = R_e$ in the equation 6.46? No. To get equation 6.46 we assumed that $h << R_e$. However $h = R_e$ can be substituted in equation 6.45.

Variation of g with depth:

Consider a particle of mass m which is in a deep mine on the Earth. (Example: coal mines in Neyveli). Assume the depth of the mine as d. To calculate g' at a depth d, consider the following points.

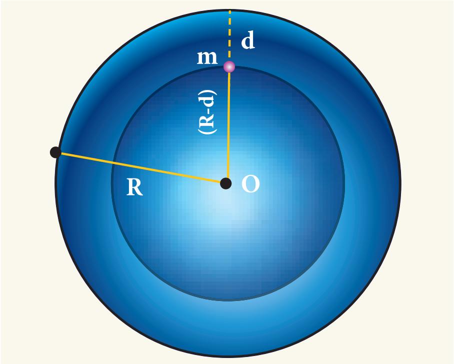


Figure 6.17(b) Particle in a mine

The part of the Earth which is above the radius $(R_e - d)$ do not contribute to the acceleration. The result is proved earlier and is given as

$$g' = \frac{GM'}{(R_e - d)^2} \qquad (6.47)$$

Here M' is the mass of the Earth of radius $(R_e - d)$

Assuming the density of Earth ρ to be constant,

$$\rho = \frac{M}{V} \qquad (6.48)$$

where M is the mass of the Earth and V its volume, Thus,

$$\rho = \frac{M'}{V'}$$

$$\frac{M'}{V'} = \frac{M}{V} \text{ and } M' = \frac{M}{V}V'$$

$$M' = \left(\frac{M}{\frac{4}{3}\pi R_e^3}\right)\left(\frac{4}{3}\pi(R_e - d)^3\right)$$

$$M' = \frac{M}{R_e^3}(R_e - d)^3 \qquad (6.49)$$

Substituting equation (6.49) in equation (6.47)

$$g' = G\frac{M}{R_e^3}(R_e - d)^3 . \frac{1}{(R_e - d)^2}$$

$$g' = GM\frac{R_e\left(1 - \frac{d}{R_e}\right)}{R_e^3}$$

$$g' = GM\frac{\left(1 - \frac{d}{R_e}\right)}{R_e^2}$$

Thus

$$g' = g\left(1 - \frac{d}{R_e}\right) \qquad (6.50)$$

Here also $g' < g$. As depth increases, g' decreases. It is very interesting to know that acceleration due to gravity is maximum on the surface of the Earth but decreases when we go either upward or downward.

Variation of g with latitude:

Whenever we analyze the motion of objects in rotating frames [explained in chapter 3] we must take into account the centrifugal force. Even though we treat the Earth as an inertial frame, it is not exactly correct because the Earth spins about its own axis. So when an object is on the surface of the Earth, it experiences a centrifugal force that depends on the latitude of the object on Earth. If the Earth were not spinning, the force on the object would have been mg. However, the object experiences an additional centrifugal force due to spinning of the Earth.

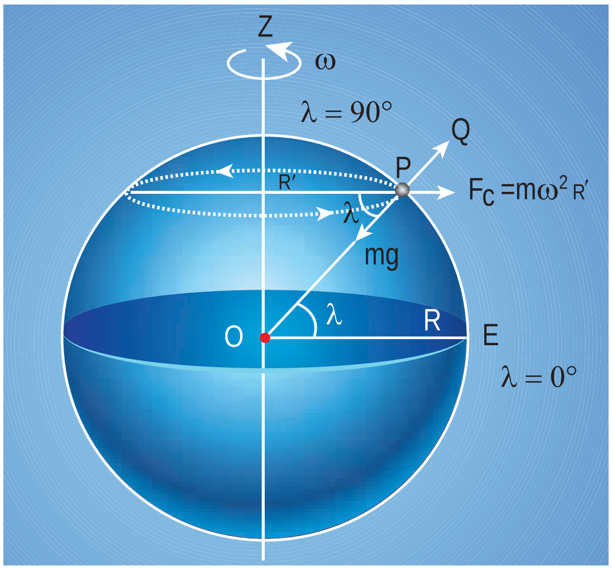


Figure 6.18 Variation of g with latitude

This centrifugal force is given by $m\omega^2 R'$.

$$R' = R\cos\lambda \qquad (6.51)$$

where λ is the latitude. The component of centrifugal acceleration experienced by the object in the direction opposite to g is

$$a_{PQ} = \omega^2 R' \cos\lambda = \omega^2 R \cos^2\lambda$$

$$\text{since } R' = R\cos\lambda$$

Therefore,

$$g' = g - \omega^2 R\cos^2\lambda \qquad (6.52)$$

From the expression (6.52), we can infer that at equator, $\lambda = 0$; $g' = g - \omega^2 R$. The acceleration due to gravity is minimum. At poles $\lambda = 90°$; $g' = g$, it is maximum. At the equator, g' is minimum.

EXAMPLE 6.8

Find out the value of g' in your school laboratory?

Solution

Calculate the latitude of the city or village where the school is located. The information is available in Google search. For example, the latitude of Chennai is approximately 13 degree.

$$g' = g - \omega^2 R\cos^2\lambda$$

Here $\omega^2 R$ = $(2x3.14/86400)^2$ x $(6400x10^3)$ = $3.4x10^{-2}$ m s^{-2}.

It is to be noted that the value of λ should be in radian and not in degree. 13 degree is equivalent to 0.2268 rad.

$$g' = 9.8 - \left(3.4\times10^{-2}\right)\times\left(\cos 0.2268\right)^2$$

$$g' = 9.7677 \text{ m s}^{-2}$$

Points to Contemplate

Suppose you move towards east-west along the same latitude. Will the value of g' change?

6.4 ESCAPE SPEED AND ORBITAL SPEED

Hydrogen and helium are the most abundant elements in the universe but Earth's atmosphere consists mainly of nitrogen and oxygen. The following discussion brings forth the reason why hydrogen and helium are not found in abundance on the Earth's atmosphere. When an object is thrown up with some initial speed it will reach a certain height after which it will fall back to Earth. If the same object is thrown again with a higher speed, it reaches a greater height than the previous one and falls back to Earth. This leads to the question of what should be the speed of an object thrown vertically up such that it escapes the Earth's gravity and would never come back. This speed is called "Escape speed" and it is defined as "the minimum speed required for a body to escape from the earth's gravitational pull".

Consider an object of mass M on the surface of the Earth. When it is thrown up with an initial speed v_i, the initial total energy of the object is

$$E_i = \frac{1}{2}Mv_i^2 - \frac{GMM_E}{R_E} \qquad (6.53)$$

where, M_E is the mass of the Earth and R_E- the radius of the Earth. The term $-\frac{GMM_E}{R_E}$ is the potential energy of the mass M.

When the object reaches a height far away from Earth and hence treated as approaching infinity, the gravitational potential energy becomes zero $[U(\infty) = 0]$ and the kinetic energy becomes zero as well. Therefore the final total energy of the object becomes zero. This is for minimum energy and for minimum speed to escape. Otherwise Kinetic energy can be nonzero.

$$E_f = 0$$

According to the law of energy conservation,

$$E_i = E_f \qquad (6.54)$$

Substituting (6.53) in (6.54) we get,

$$\frac{1}{2}Mv_i^2 - \frac{GMM_E}{R_E} = 0$$
$$\frac{1}{2}Mv_i^2 = \frac{GMM_E}{R_E} \qquad (6.55)$$

Consider the escape speed, the minimum speed required by an object to escape Earth's gravitational field, hence replace v_i with v_e. i.e,

$$\frac{1}{2}Mv_e^2 = \frac{GMM_E}{R_E}$$
$$v_e^2 = \frac{GMM_E}{R_E} \cdot \frac{2}{M}$$
$$v_e^2 = \frac{2GM_E}{R_E}$$

Using $g = \frac{GM_E}{R_E^2}$,

$$v_e^2 = 2gR_E$$
$$v_e = \sqrt{2gR_E} \qquad (6.56)$$

From equation (6.56) the escape speed depends on two factors: acceleration due to gravity and radius of the Earth. It is completely independent of the mass of the object. By substituting the values of g ($9.8\ m\,s^{-2}$) and $R_e = 6400\,km$, the escape speed of the Earth is $v_e = 11.2\,km\,s^{-1}$. The escape speed is independent of the direction in which the object is thrown. Irrespective of whether the object is thrown vertically up, radially outwards or tangentially it requires the same initial speed to escape Earth's gravity. It is shown in Figure 6.19

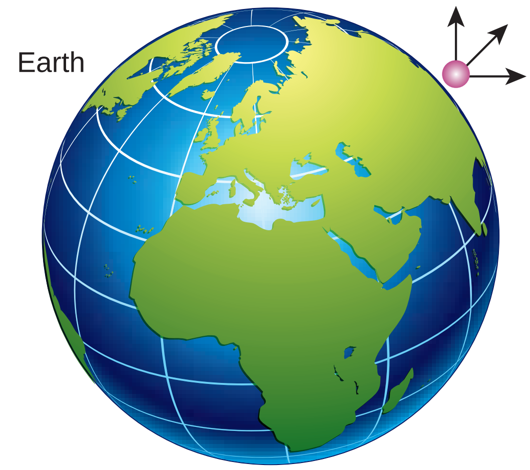


Figure 6.19 Escape speed independent of angle

Lighter molecules such as hydrogen and helium have enough speed to escape from the Earth, unlike the heavier ones such as nitrogen and oxygen. (The average speed of hydrogen and helium atoms compaired with the escape speed of the Earth,is presented in the kinetic theory of gases, unit 9).

6.4.1 Satellites, orbital speed and time period

We are living in a modern world with sophisticated technological gadgets and are able to communicate to any place on Earth. This advancement was made possible because of our understanding of solar system. Communication mainly depends on the satellites that orbit the Earth (Figure 6.20). Satellites revolve around the Earth just like the planets revolve around the Sun. Kepler's laws are applicable to man-made satellites also.

For a satellite of mass M to move in a circular orbit, centripetal force must be acting on the satellite. This centripetal force is provided by the Earth's gravitational force.

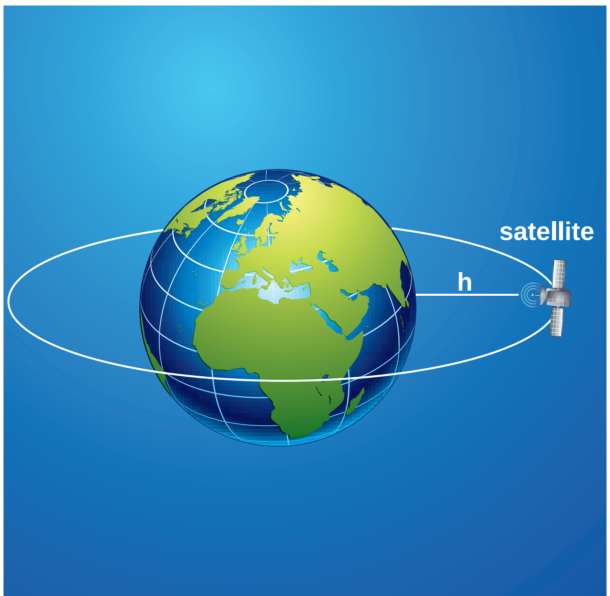


Figure 6.20 Satellite revolving around the Earth.

$$\frac{Mv^2}{(R_E+h)} = \frac{GMM_E}{(R_E+h)^2} \qquad (6.57)$$

$$v^2 = \frac{GM_E}{(R_E+h)}$$

$$v^2 = \sqrt{\frac{GM_E}{(R_E+h)}} \qquad (6.58)$$

As h increases, the speed of the satellite decreases.

Time period of the satellite:

The distance covered by the satellite during one rotation in its orbit is equal to $2\pi(R_E+h)$ and time taken for it is the time period, T. Then

$$Speed\ v = \frac{Distance\ travelled}{Time\ taken} = \frac{2\pi(R_E+h)}{T}$$

From equation (6.58)

$$\sqrt{\frac{GM_E}{(R_E+h)}} = \frac{2\pi(R_E+h)}{T} \qquad (6.59)$$

$$T = \frac{2\pi}{\sqrt{GM_E}}(R_E+h)^{3/2} \qquad (6.60)$$

Squaring both sides of the equation (6.60), we get

$$T^2 = \frac{4\pi^2}{GM_E}(R_E+h)^3$$

$$\frac{4\pi^2}{GM_E} = \text{constant say } c$$

$$T^2 = c(R_E+h)^3 \qquad (6.61)$$

Equation (6.61) implies that a satellite orbiting the Earth has the same relation between time and distance as that of Kepler's law of planetary motion. For a satellite orbiting near the surface of the Earth, h is negligible compared to the radius of the Earth R_E. Then,

$$T^2 = \frac{4\pi^2}{GM_E}R_E^{\ 3}$$

$$T^2 = \frac{4\pi^2}{GM_E/R_E^{\ 2}}R_E$$

$$T^2 = \frac{4\pi^2}{g}R_E$$

$$since\ GM_E/R_E^{\ 2} = g$$

$$T = 2\pi\sqrt{\frac{R_E}{g}} \qquad (6.62)$$

By substituting the values of $R_E = 6.4 \times 10^6$ m and g = 9.8 m s^{-2}, the orbital time period is obtained as T ≅ 85 minutes.

EXAMPLE 6.9

Moon is the natural satellite of Earth and it takes 27 days to go once around its orbit. Calculate the distance of the Moon from the surface of the Earth assuming the orbit of the Moon as circular.

Solution

We can use Kepler's third law,

$$T^2 = c(R_E + h)^3$$

$$T^{2/3} = c^{1/3}(R_E + h)$$

$$\left(\frac{T^2}{c}\right)^{1/3} = (R_E + h)$$

$$\left(\frac{T^2 GM_E}{4\pi^2}\right)^{\frac{1}{3}} = (R_E + h);$$

$$c = \frac{4\pi^2}{GM_E}$$

$$h = \left(\frac{T^2 GM_E}{4\pi^2}\right)^{1/3} - R_E$$

Here h is the distance of the Moon from the surface of the Earth. Here,

R_E – radius of the Earth = 6.4 $\times 10^6$ m

M_E – mass of the Earth = 6.02 $\times 10^{24}$ kg

G – Universal gravitational constant = 6.67 $\times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$

By substituting these values, the distance to the Moon from the surface of the Earth is calculated to be 3.77 $\times 10^5$ km.

6.4.2 Energy of an Orbiting Satellite

The total energy of a satellite orbiting the Earth at a distance h from the surface of Earth is calculated as follows; The total energy of the satellite is the sum of its kinetic energy and the gravitational potential energy. The potential energy of the satellite is,

$$U = -\frac{GM_s M_E}{(R_E + h)} \qquad (6.63)$$

Here M_s - mass of the satellite, M_E - mass of the Earth, R_E - radius of the Earth.

The Kinetic energy of the satellite is

$$K.E = \frac{1}{2} M_s v^2 \qquad (6.64)$$

Here v is the orbital speed of the satellite and is equal to

$$v = \sqrt{\frac{GM_E}{(R_E + h)}} \qquad (6.65)$$

Substituting the value of v in (6.64), the kinetic energy of the satellite becomes,

$$K.E = \frac{1}{2}\frac{GM_E M_s}{(R_E + h)}$$

Therefore the total energy of the satellite is

$$E = \frac{1}{2}\frac{GM_E M_s}{(R_E + h)} - \frac{GM_s M_E}{(R_E + h)}$$

$$E = -\frac{GM_s M_E}{2(R_E + h)} \qquad (6.66)$$

The negative sign in the total energy implies that the satellite is bound to the Earth and it cannot escape from the Earth.

As h approaches ∞, the total energy tends to zero. Its physical meaning is that the satellite is completely free from the influence of Earth's gravity and is not bound to Earth at large distances.

EXAMPLE 6.10

Calculate the energy of the (i) Moon orbiting the Earth and (ii) Earth orbiting the Sun.

Solution

Assuming the orbit of the Moon to be circular, the energy of Moon is given by,

$$E_m = -\frac{GM_E M_m}{2R_m}$$

where M_E is the mass of Earth 6.02×10^{24} kg; M_m is the mass of Moon 7.35×10^{22} kg; and R_m is the distance between the Moon and the center of the Earth $3.84 \times 10^5 \, km$

$$G = 6.67 \times 10^{-11} \text{N m}^2 \text{kg}^{-2}.$$

$$E_m = -\frac{6.67 \times 10^{-11} \times 6.02 \times 10^{24} \times 7.35 \times 10^{22}}{2 \times 3.84 \times 10^5 \times 10^3}$$

$$E_m = -38.42 \times 10^{-19} \times 10^{46}$$

$$E_m = -38.42 \times 10^{27} \text{ Joule}$$

The negative energy implies that the Moon is bound to the Earth.

Same method can be used to prove that the energy of the Earth is also negative.

6.4.3 Geo-stationary and polar satellite

The satellites orbiting the Earth have different time periods corresponding to different orbital radii. Can we calculate the orbital radius of a satellite if its time period is 24 hours?

Kepler's third law is used to find the radius of the orbit.

$$T^2 = \frac{4\pi^2}{GM_E}(R_E + h)^3$$

$$(R_E + h)^3 = \frac{GM_E T^2}{4\pi^2}$$

$$R_E + h = \left(\frac{GM_E T^2}{4\pi^2}\right)^{1/3}$$

Substituting for the time period (24 hrs = 86400 seconds), mass, and radius of the Earth, h turns out to be 36,000 km. Such satellites are called "geo-stationary satellites", since they appear to be stationary when seen from Earth.

India uses the INSAT group of satellites that are basically geo-stationary satellites for the purpose of telecommunication. Another type of satellite which is placed at a distance

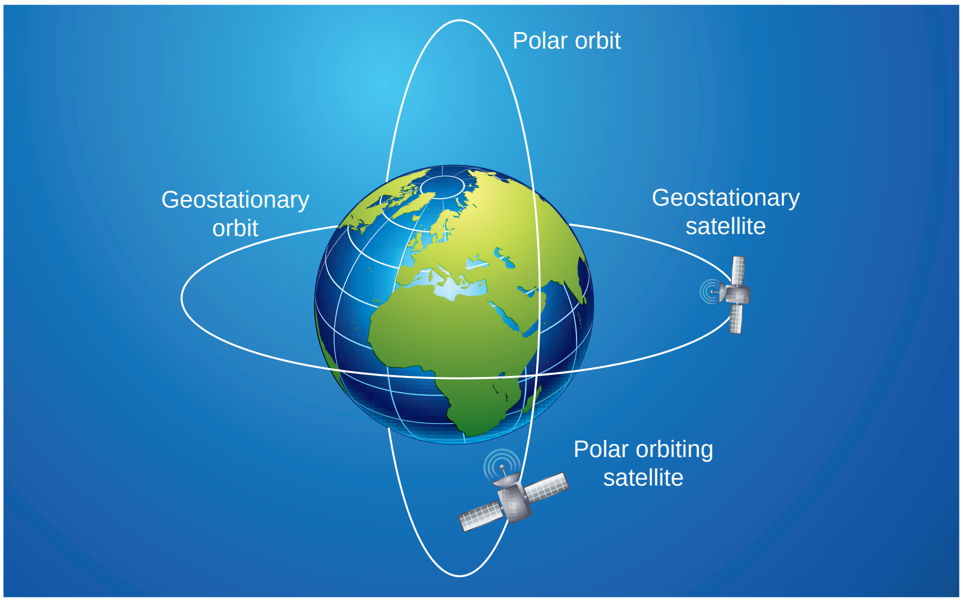


Figure 6.21 Polar orbit and geostationary satellite

of 500 to 800 km from the surface of the Earth orbits the Earth from north to south direction. This type of satellite that orbits Earth from North Pole to South Pole is called a polar satellite. The time period of a polar satellite is nearly 100 minutes and the satellite completes many revolutions in a day. A Polar satellite covers a small strip of area from pole to pole during one revolution. In the next revolution it covers a different strip of area since the Earth would have moved by a small angle. In this way polar satellites cover the entire surface area of the Earth.

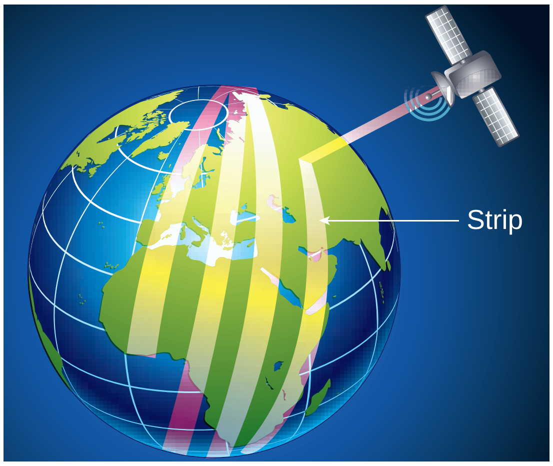


Figure 6.22 Strip of communication region, covered by a polar satellite.

6.4.4 Weightlessness

Weight of an object

Objects on Earth experience the gravitational force of Earth. The gravitational force acting on an object of mass m is mg. This force always acts downwards towards the center of the Earth. When we stand on the floor, there are two forces acting on us. One is the gravitational force, acting downwards and the other is the normal force exerted by the floor upwards on us to keep us at rest. The weight of an object $\vec{W}$ is defined as the downward force whose magnitude W is equal to that of upward force that must be applied to the object to hold it at rest or at constant velocity relative to the earth. The direction of weight is in the direction of gravitational force. So the magnitude of weight of an object is denoted as, W=N=mg. Note that even though magnitude of weight is equal to mg, it is not same as gravitational force acting on the object.

Apparent weight in elevators

Everyone who used an elevator would have felt a jerk when the elevator takes off or stops. Why does it happen? Understanding the concept of weight is crucial for explaining this effect. Let us consider a man inside an elevator in the following scenarios.

When a man is standing in the elevator, there are two forces acting on him.

1. Gravitational force which acts downward. If we take the vertical direction as positive y direction, the gravitational force acting on the man is $\vec{F}_G = -mg\hat{j}$
2. The normal force exerted by floor on the man which acts vertically upward, $\vec{N} = N\hat{j}$

Case (i) When the elevator is at rest

The acceleration of the man is zero. Therefore the net force acting on the man is zero. With respect to inertial frame (ground), applying Newton's second law on the man,

$$\vec{F}_G + \vec{N} = 0$$
$$-mg\hat{j} + N\hat{j} = 0$$

By comparing the components, we can write

$$N - mg = 0 \text{ (or) } N = mg \qquad (6.67)$$

Since weight, W =N, the apparent weight of the man is equal to his actual weight.

Case (ii) When the elevator is moving uniformly in the upward or downward direction

In uniform motion (constant velocity), the net force acting on the man is still zero. Hence, in this case also the apparent weight of the man is equal to his actual weight. It is shown in Figure 6.23(a)

Case (iii) When the elevator is accelerating upwards

If an elevator is moving with upward acceleration ($\vec{a} = a\hat{j}$) with respect to inertial frame (ground), applying Newton's second law on the man,

$$\vec{F}_G + \vec{N} = m\vec{a}$$

Writing the above equation in terms of unit vector in the vertical direction,

$$-mg\hat{j} + N\hat{j} = ma\hat{j}$$

By comparing the components,

$$N = m(g + a) \qquad (6.68)$$

Therefore, apparent weight of the man is greater than his actual weight. It is shown in Figure 6.23(b)

Case (iv) When the elevator is accelerating downwards

If the elevator is moving with downward acceleration ($\vec{a} = -a\hat{j}$), by applying Newton's second law on the man, we can write

$$\vec{F}_G + \vec{N} = m\vec{a}$$

Writing the above equation in terms of unit vector in the vertical direction,

$$-mg\hat{j} + N\hat{j} = -ma\hat{j}$$

By comparing the components,

$$N = m(g - a) \qquad (6.69)$$

Therefore, apparent weight W = N = m(g-a) of the man is lesser than his actual weight. It is shown in Figure 6.23(c)

Weightlessness of freely falling bodies

Freely falling objects experience only gravitational force. As they fall freely, they are not in contact with any surface (by neglecting air friction). The normal force acting on the object is zero. The downward acceleration is equal to the acceleration due to the gravity of the Earth. i.e (a = g). From equation (6.69) we get.

$$a = g \quad \therefore N = m\,(g - g) = 0.$$

This is called the state of weightlessness. When the lift falls (when the lift wire cuts) with downward acceleration a = g, the person inside the elevator is in the state of weightlessness or free fall. It is shown in Figure 6.23(d)

When the apple was falling from the tree it was weightless.As soon as it hit Newton's head, it gained weight! and Newton gained physics!

Weightlessness in satellites:

There is a wrong notion that the astronauts in satellites experience no gravitational force because they are far away from the Earth. Actually the Earth satellites that orbit very close to Earth experience only gravitational force. The astronauts inside the satellite also experience the same gravitational force. Because of this, they cannot exert any force on the floor of the satellite. Thus, the floor of the satellite also cannot exert any normal force on the astronaut. Therefore, the astronauts inside a satellite are in the state of weightlessness. Not only the astronauts, but all the objects in the satellite will be in the state of weightlessness which is similar to that of a free fall. It is shown in the Figure 6.24.

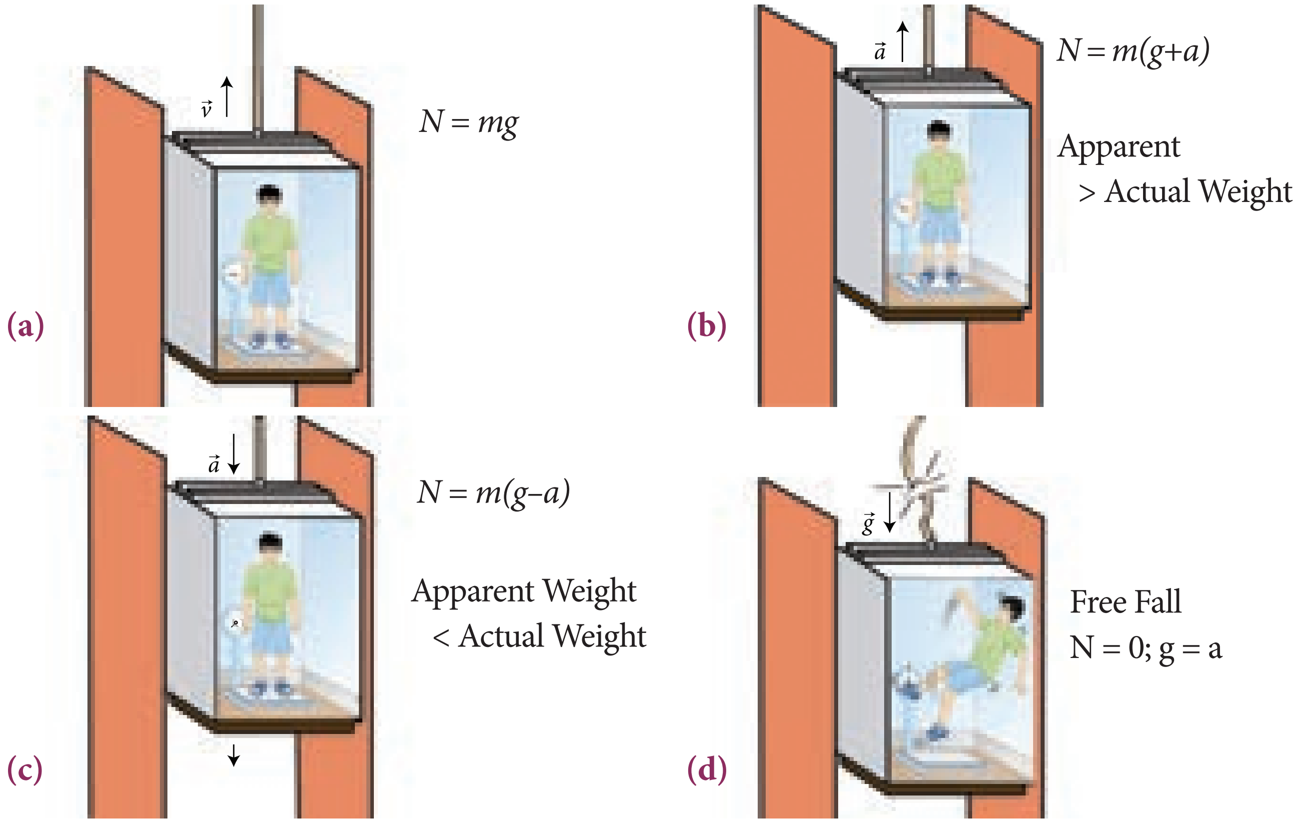


Figure 6.23 Apparent weight in the lift

Figure 6.24 The well known scientist Stephen Hawking in the state of weightlessness.
https://www.youtube.com/watch?v=OCsuHvv_D0s

6.5 ELEMENTARY IDEAS OF ASTRONOMY

Astronomy is one of the oldest sciences in the history of mankind. In the olden days, astronomy was an inseparable part of physical science. It contributed a lot to the development of physics in the 16^{th} century. In fact Kepler's laws and Newton's theory of gravitation were formulated and verified using astronomical observations and data accumulated over the centuries by famous astronomers like Hippachrus, Aristachrus, Ptolemy, Copernicus and Tycho Brahe. Without Tycho Brahe's astronomical observations, Kepler's laws would not have emerged. Without Kepler's laws, Newton's theory of gravitation would not have been formulated.

It was mentioned in the beginning of this chapter that Ptolemy's geocentric model was replaced by Copernicus' heliocentric model. It is important to analyze and explain the shortcoming of the geocentric model over heliocentric model.

6.5.1 Heliocentric system over geocentric system

When the motion of the planets are observed in the night sky by naked eyes over a period of a few months, it can be seen that the planets move eastwards and reverse their motion for a while and return to eastward motion again. This is called "retrograde motion" of planets.

Figure 6.25 shows the retrograde motion of the planet Mars. Careful observation for a period of a year clearly shows that Mars initially moves eastwards (February to June), then reverses its path and moves backwards (July, August, September). It changes its direction of motion once again and continues its forward motion (October onwards). In olden days, astronomers recorded the retrograde motion of all

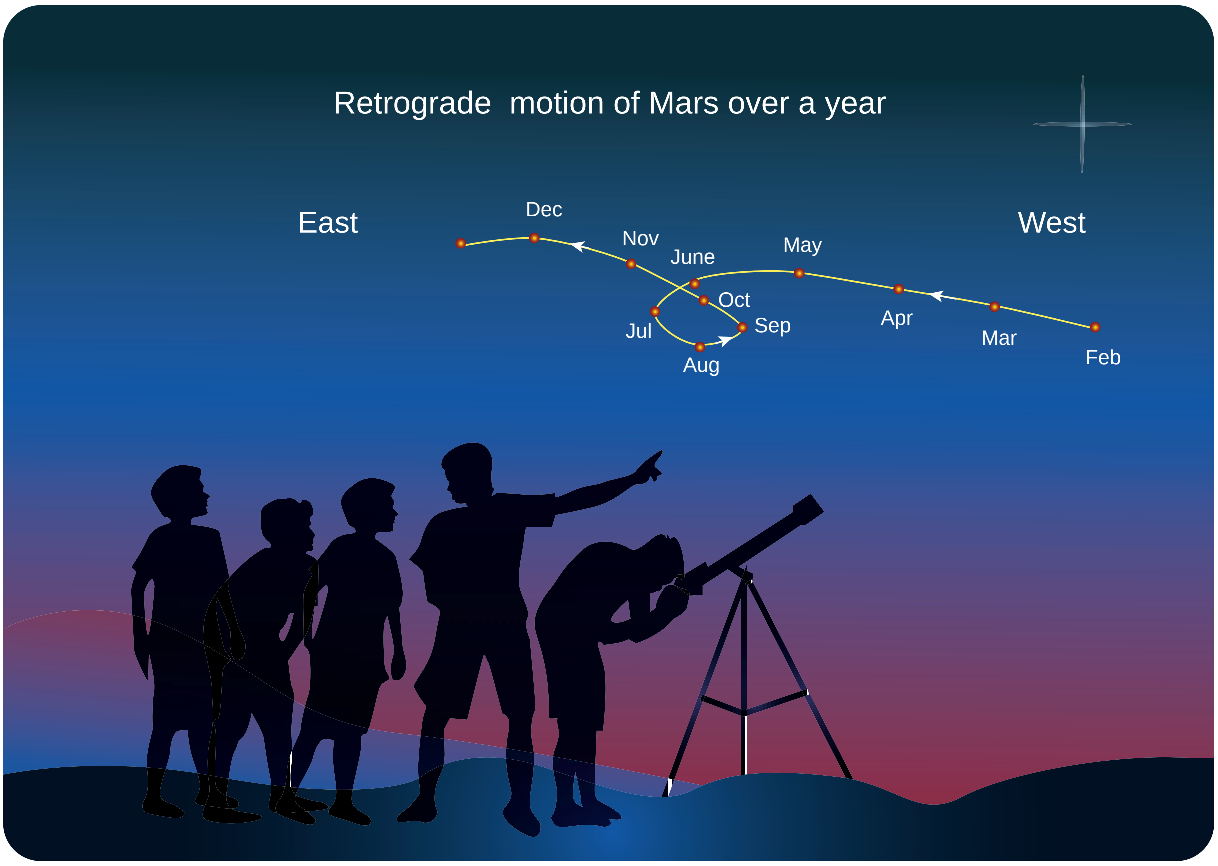


Figure 6.25 Retrograde motion of planets

visible planets and tried to explain the motion. According to Aristotle, the other planets and the Sun move around the Earth in the circular orbits. If it was really a circular orbit it was not known how the planet could reverse its motion for a brief interval. To explain this retrograde motion, Ptolemy introduced the concept of "epicycle" in his geocentric model. According to this theory, while the planet orbited the Earth, it also underwent another circular motion termed as "epicycle". A combination of epicycle and circular motion around the Earth gave rise to retrograde motion of the planets with respect to Earth (Figure 6.26). Essentially Ptolemy retained the Earth centric idea of Aristotle and added the epicycle motion to it.

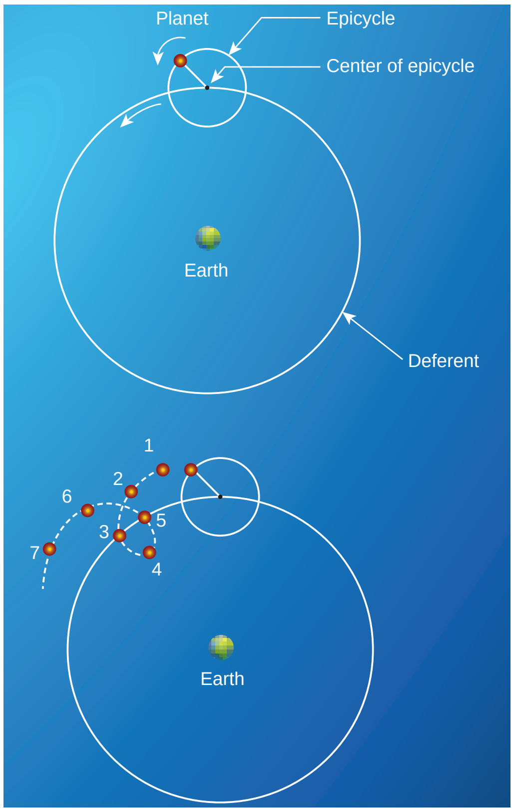


Figure 6.26 "Epicycle" motion of planetary objects around Earth, depicted with respect to months of observation.

But Ptolemy's model became more and more complex as every planet was found to undergo retrograde motion. In the 15th century, the Polish astronomer Copernicus proposed the heliocentric model to explain this problem in a simpler manner. According to this model, the Sun is at the center of the solar system and all planets orbited the Sun. The retrograde motion of planets with respect to Earth is because of the relative motion of the planet with respect to Earth. The retrograde motion from the heliocentric point of view is shown in Figure 6.27.

Figure 6.27 shows that the Earth orbits around the Sun faster than Mars. Because of the relative motion between Mars and Earth, Mars appears to move backwards from July to October. In the same way the retrograde motion of all other planets was explained successfully by the Copernicus model. It was because of its simplicity, the heliocentric model slowly replaced the geocentric model. Historically, if any natural phenomenon has one or more explanations, the simplest one is usually accepted. Though this was not the only reason to disqualify the geocentric model, a detailed discussion on correctness of the Copernicus model over to Ptolemy's model can be found in astronomy books.

Activity

Students are encouraged to observe the motion of the planet Mars by naked eye and identify its retrograde motion. As mentioned above, to observe the retrograde motion six to seven months are required. So students may start their observation of Mars from the month of June and continue till April next year. Mars is the little bright planet with reddish color. The position of the planet Mars in the sky can be easily taken from 'Google'.

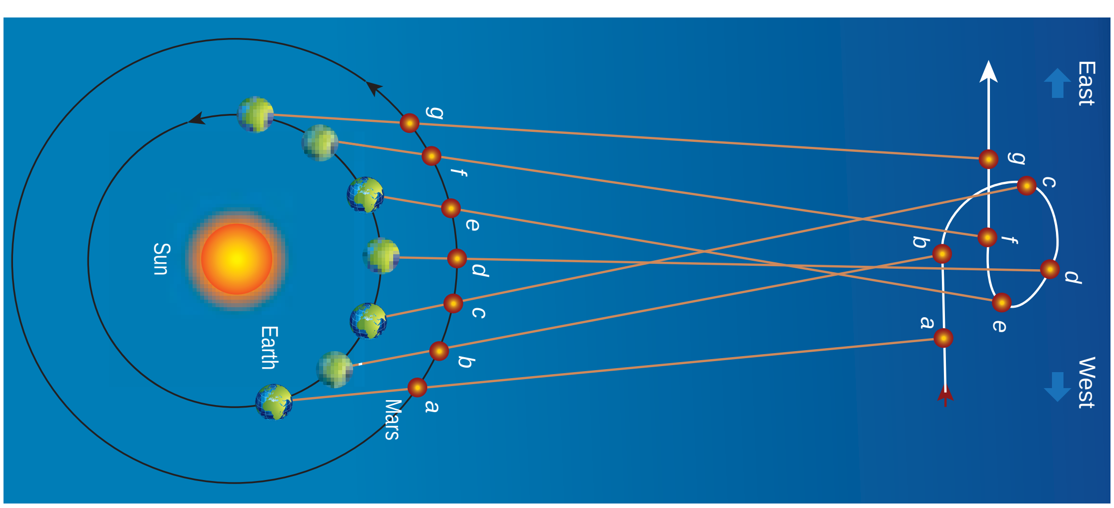


Figure 6.27 'Retrograde motion' in heliocentric model

6.5.2 Kepler's Third Law and The Astronomical Distance

When Kepler derived his three laws, he strongly relied on Tycho Brahe's astronomical observation. In his third law, he formulated the relation between the distance of a planet from the Sun to the time period of revolution of the planet. Astronomers cleverly used geometry and trigonometry to calculate the distance of a planet from the Sun in terms of the distance between Earth and Sun. Here we can see how the distance of Mercury and Venus from the Sun were measured. The Venus and Mercury, being inner planets with respect to Earth, the maximum angular distance they can subtend at a point on Earth with respect to the Sun is 46 degree for Venus and 22.5 degree for Mercury. It is shown in the Figure 6.28

Figure 6.29 shows that when Venus is at maximum elongation (i.e., 46 degree) with respect to Earth, Venus makes 90 degree to Sun. This allows us to find the distance between Venus and Sun. The distance between Earth and Sun is taken as one Astronomical unit (1 AU).

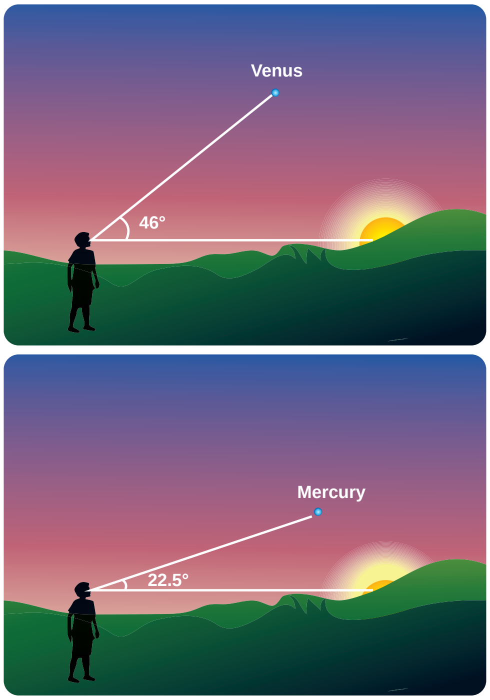


Figure 6.28 Angle of elevation for Venus and Mercury from horizon

The trigonometric relation satisfied by this right angled triangle is shown in Figure 6.29.

$$\sin\theta = \frac{r}{R}$$

where R = 1 AU.

$$r = R\sin\theta = (1AU)(\sin 46°)$$

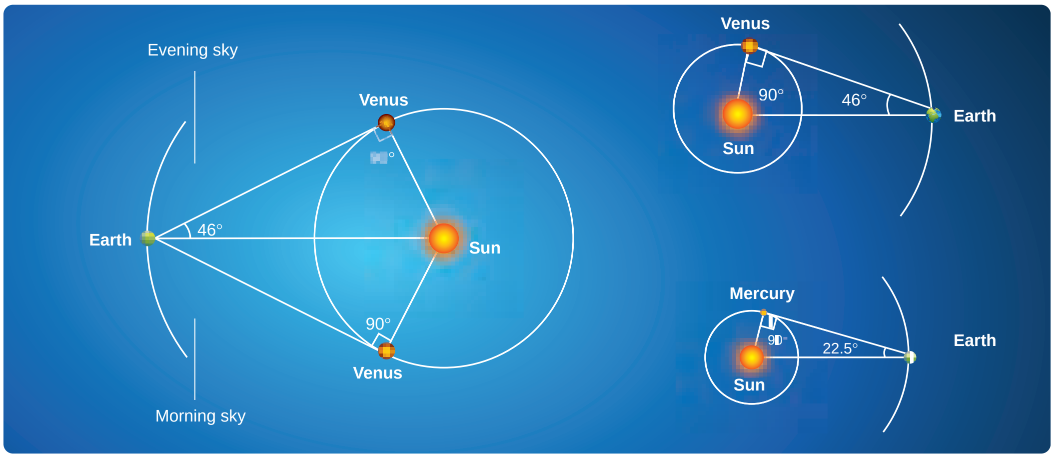


Figure 6.29 Angle of elevation for Mercury from horizon

Here $\sin 46° = 0.72$. Hence Venus is at a distance of 0.72 AU from Sun. Similarly, the distance between Mercury (θ is 22.5 degree) and Sun is calculated as 0.38 AU. To find the distance of exterior planets like Mars and Jupiter, a slightly different method is used. The distances of planets from the Sun is given in the table below.

Table 6.2 a^3/T^2 for different planets

Planet	semi major axis of the orbit(a)	Period T (days)	a^3/T^2
Mercury	0.389 AU	87.77	7.64
Venus	0.724 AU	224.70	7.52
Earth	1.000 AU	365.25	7.50
Mars	1.524 AU	686.98	7.50
Jupiter	5.200 AU	4332.62	7.49
Saturn	9.510 AU	10,759.20	7.40

It is to be noted that to verify the Kepler's law we need only high school level geometry and trigonometry.

ACTIVITY

Venus can be observed with the naked eye. We can see Venus during sunrise or sunset. Students are encouraged to observe the motion of Venus and verify that the maximum elevation is at 46 degree and calculate the distance of Venus from the Sun. As pointed out already Google or Stellarium will be helpful in locating the position of Venus in the sky.

6.5.3 Measurement of radius of the Earth

Around 225 B.C a Greek librarian "Eratosthenes" who lived at Alexandria measured the radius of the Earth with a small error when compared with results using modern measurements. The technique he used involves lower school geometry and

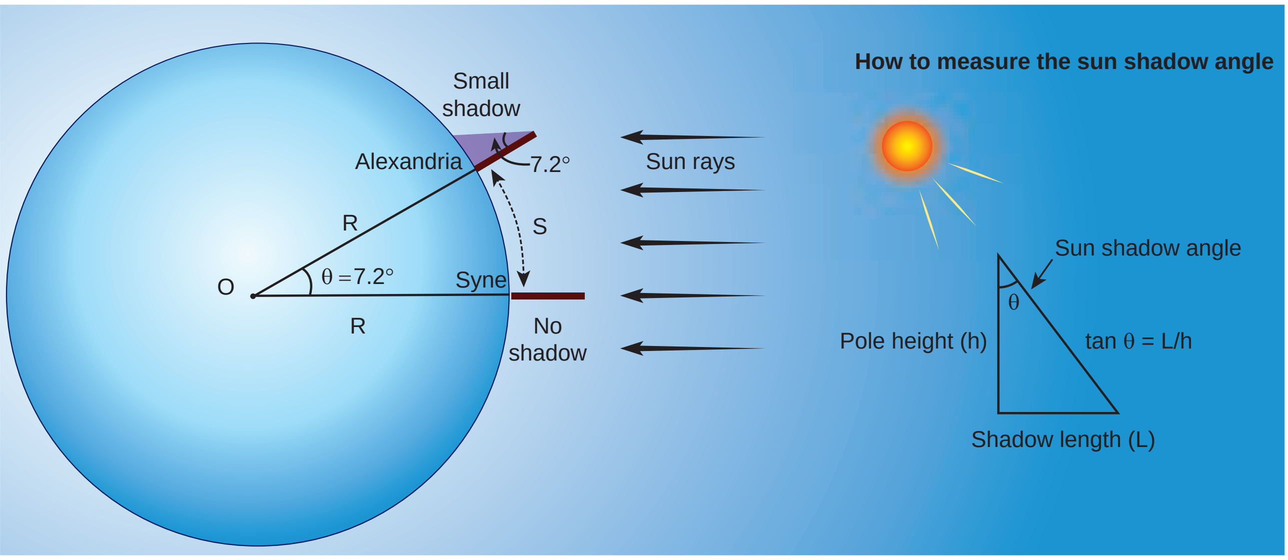


Figure 6.30 Measuring radius of The Earth

brilliant insight. He observed that during noon time of summer solstice the Sun's rays cast no shadow in the city Syne which was located 500 miles away from Alexandria. At the same day and same time he found that in Alexandria the Sun's rays made 7.2 degree with local vertical as shown in the Figure 6.30. He realized that this difference of 7.2 degree was due to the curvature of the Earth.

The angle 7.2 degree is equivalent to $\frac{1}{8}$ radian. So $\theta = \frac{1}{8}$ rad;

If S is the length of the arc between the cities of Syne and Alexandria, and if R is radius of Earth, then

$$S = R\theta = 500\,\text{miles},$$

so radius of the Earth

$$R = \frac{500}{\theta}\ \text{miles}$$

$$R = \frac{500}{\left(\frac{1}{8}\right)}\,\text{miles}$$

R = 4000 miles

1 mile is equal to 1.609 km. So, he measured the radius of the Earth to be equal to R = 6436 km, which is amazingly close to the correct value of 6378 km.

The distance of the Moon from Earth was measured by a famous Greek astronomer Hipparchus in the 3^{rd} century BC.

ACTIVITY

To measure the radius of the Earth, choose two different places (schools) that are separated by at least 500 km. It is important to note that these two places have to be along the same longitude of the Earth (For example Hosur and Kanyakumari lie along the same longitude of 77.82° E). Take poles of known length (h) and fix them vertically in the ground (it may be in the school playgrounds) at both the places. At exactly noon in both the places the length of the shadow (L) cast by each pole has to be noted down. Draw the picture like in Figure 6.30. By using the equation $\tan\theta = \frac{L}{h}$, the angle in radian can be found at each place. The difference in angle (θ') is due to the curvature of the Earth. Now the distance between the two schools can be obtained from 'Google maps'. Divide the distance with the angle (θ' in radians) which will give the radius of the Earth.

6.5.4 Interesting Astronomical Facts

1. **Lunar eclipse and measurement of shadow of Earth**

   On January 31, 2018 there was a total lunar eclipse which was observed from various places including Tamil Nadu. It is possible to measure the radius of shadow of the Earth at the point where the Moon crosses. Figure 6.31 illustrates this.

   When the Moon is inside the umbra shadow, it appears red in color. As soon as the Moon exits from the umbra

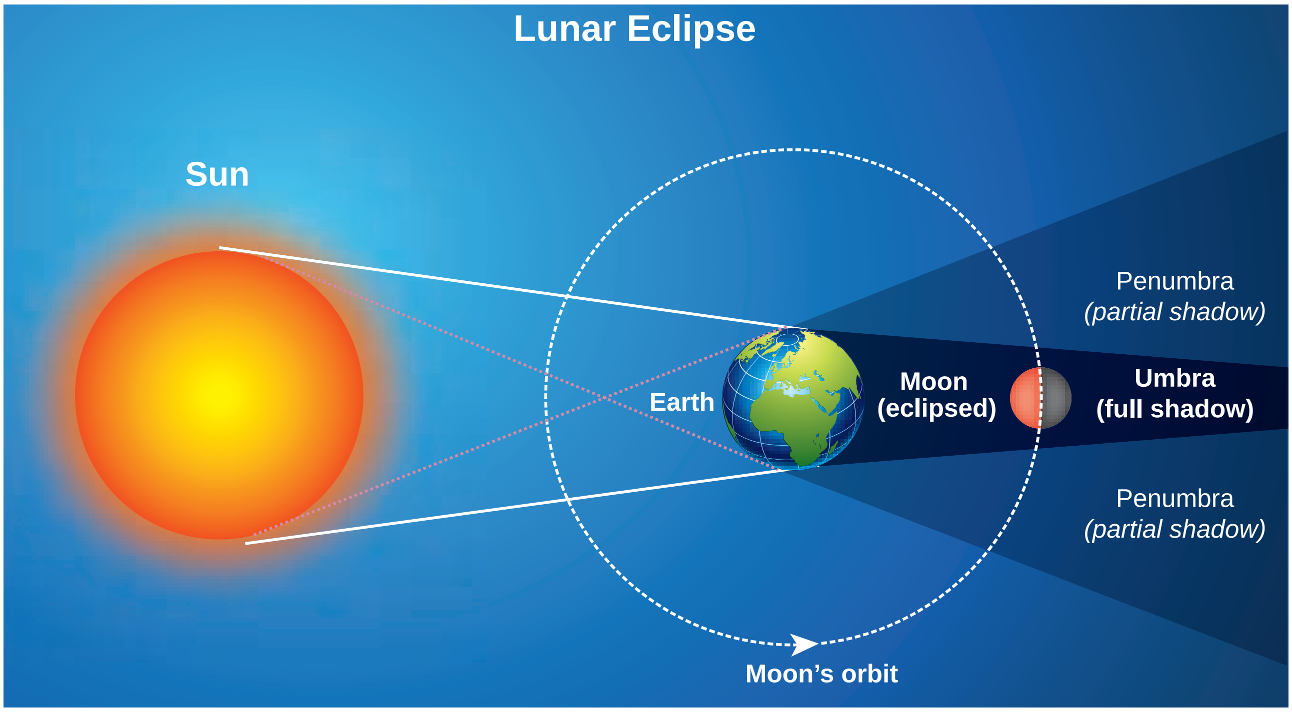


Figure 6.31 Total lunar eclipse

shadow, it appears in crescent shape. Figure 6.32 is the photograph taken by digital camera during Moon's exit from the umbra shadow.

Figure 6.32 Image of the Moon when it exits from umbra shadow

By finding the apparent radii of the Earth's umbra shadow and the Moon, the ratio of the these radii can be calculated. This is shown in Figures 6.33 and 6.34.

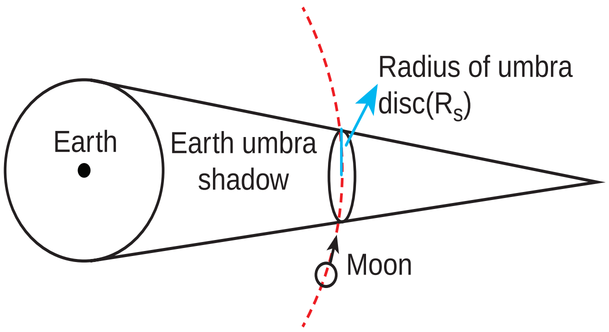


Figure 6.33 Schematic diagram of umbra disk radius

The apparent radius of Earth's umbra shadow = R_s = 13.2 cm

The apparent radius of the Moon = R_m= 5.15 cm

The ratio $\frac{R_s}{R_m} \approx 2.56$

The radius of the Earth's umbra shadow is $R_s = 2.56 \times R_m$

The radius of Moon $R_m = 1737\,\text{km}$

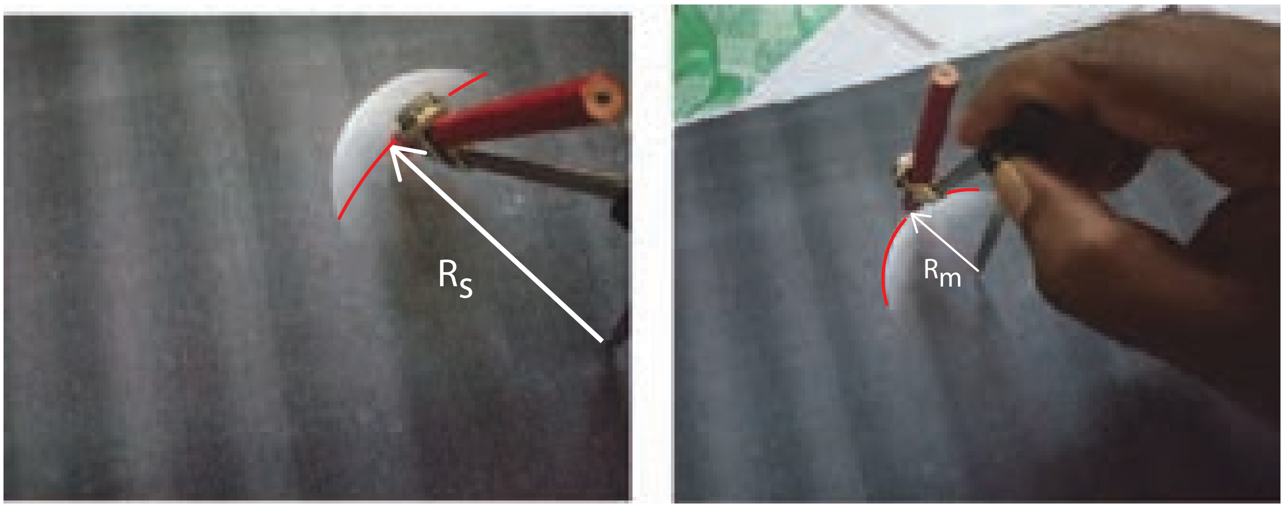


Figure 6.34 Calculation of umbra radius

The radius of the Earth's umbra shadow is $R_s = 2.56 \times 1737\,\text{km} \cong 4446\,\text{km}$.

The correct radius is 4610 km.

The percentage of error in the calculation

$$= \frac{4610 - 4446}{4610} \times 100 = 3.5\%.$$

The error will reduce if the pictures taken using a high quality telescope are used. It is to be noted that this calculation is done using very simple mathematics.

Early astronomers proved that Earth is spherical in shape by looking at the shape of the shadow cast by Earth on the Moon during lunar eclipse.

2. **Why there are no lunar eclipse and solar eclipse every month?**
If the orbits of the Moon and Earth lie on the same plane, during full Moon of every month, we can observe lunar eclipse. If this is so during new Moon we can observe solar eclipse. But Moon's orbit is tilted 5° with respect to Earth's orbit. Due to this 5° tilt, only during certain periods of the year, the Sun, Earth and Moon align in straight line leading to either lunar eclipse or solar eclipse depending on the alignment. This is shown in Figure 6.35

3. **Why do we have seasons on Earth?**
The common misconception is that 'Earth revolves around the Sun, so when the Earth is very far away, it is winter and when the Earth is nearer, it is summer'. Actually, the seasons in the Earth arise due to the rotation of Earth around the Sun with 23.5° tilt. This is shown in Figure 6.36

Due to this 23.5° tilt, when the northern part of Earth is farther to the Sun, the southern part is nearer to the Sun. So when it is summer in the northern hemisphere, the southern hemisphere experience winter.

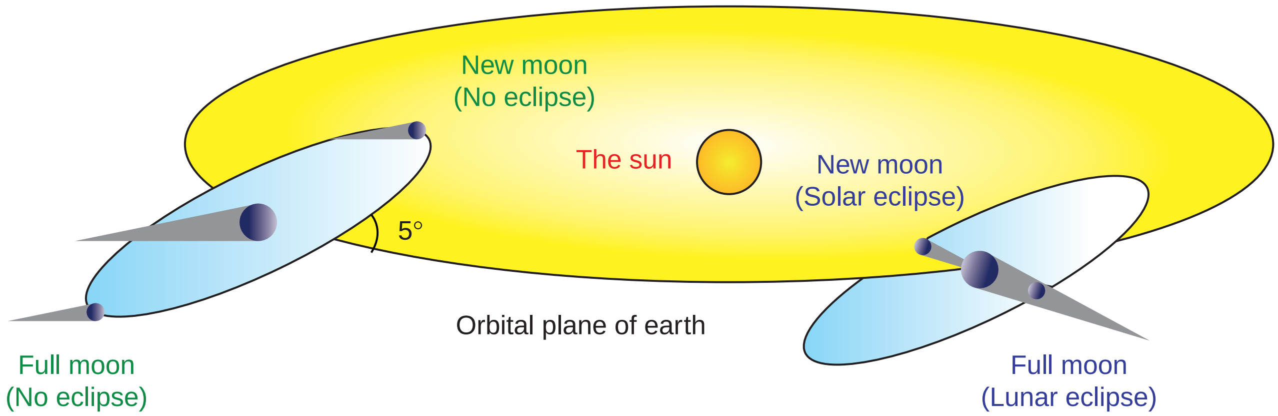


Figure 6.35 Orbital tilt of the Moon

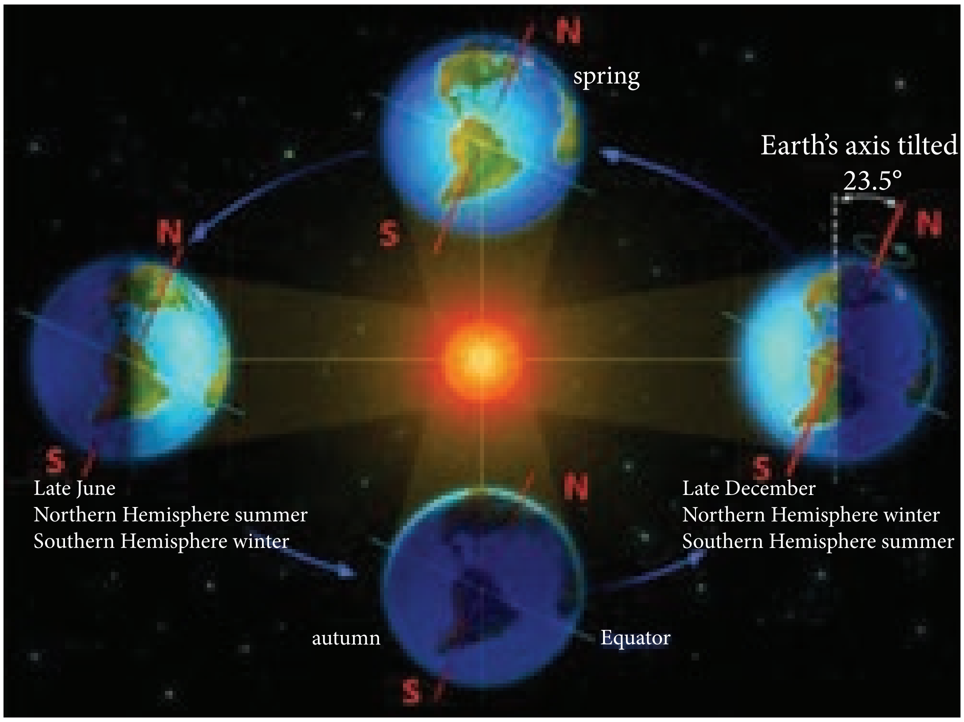


Figure 6.36 Seasons on Earth

Note

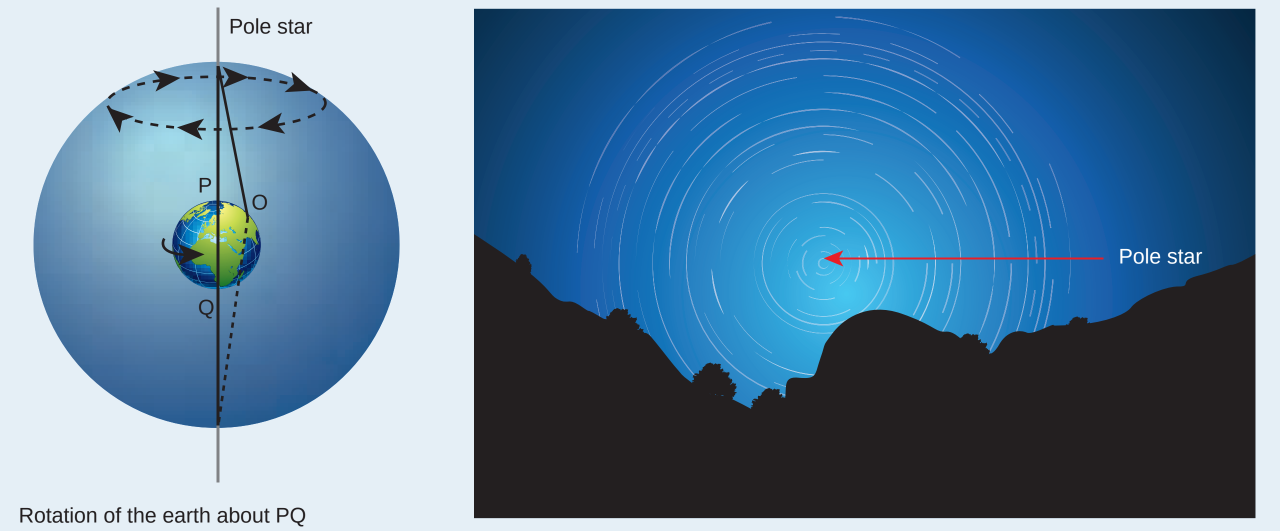
Pole star is a star located exactly above the Earth's axis of rotation, hence it appears to be stationary. The Star Polaris is our pole star.

4. Star's apparent motion and spinning of the Earth

The Earth's spinning motion can be proved by observing star's position over a night. Due to Earth's spinning motion, the stars in sky appear to move in circular motion about the pole star as shown in Figure 6.37

Point to ponder

Using Sun rays and shadows, How will you prove that the Earth's tilt is 23.5° ?



Rotation of the earth about PQ explains the apparent rotation of stars (in opposite direction) about PQ.

If we observe the stars, it slowly changes its position and appears to move in a circle over the night.

Figure 6.37 Star's apparent circular motion due to Earth's rotation.

6.5.5 Recent developments of astronomy and gravitation

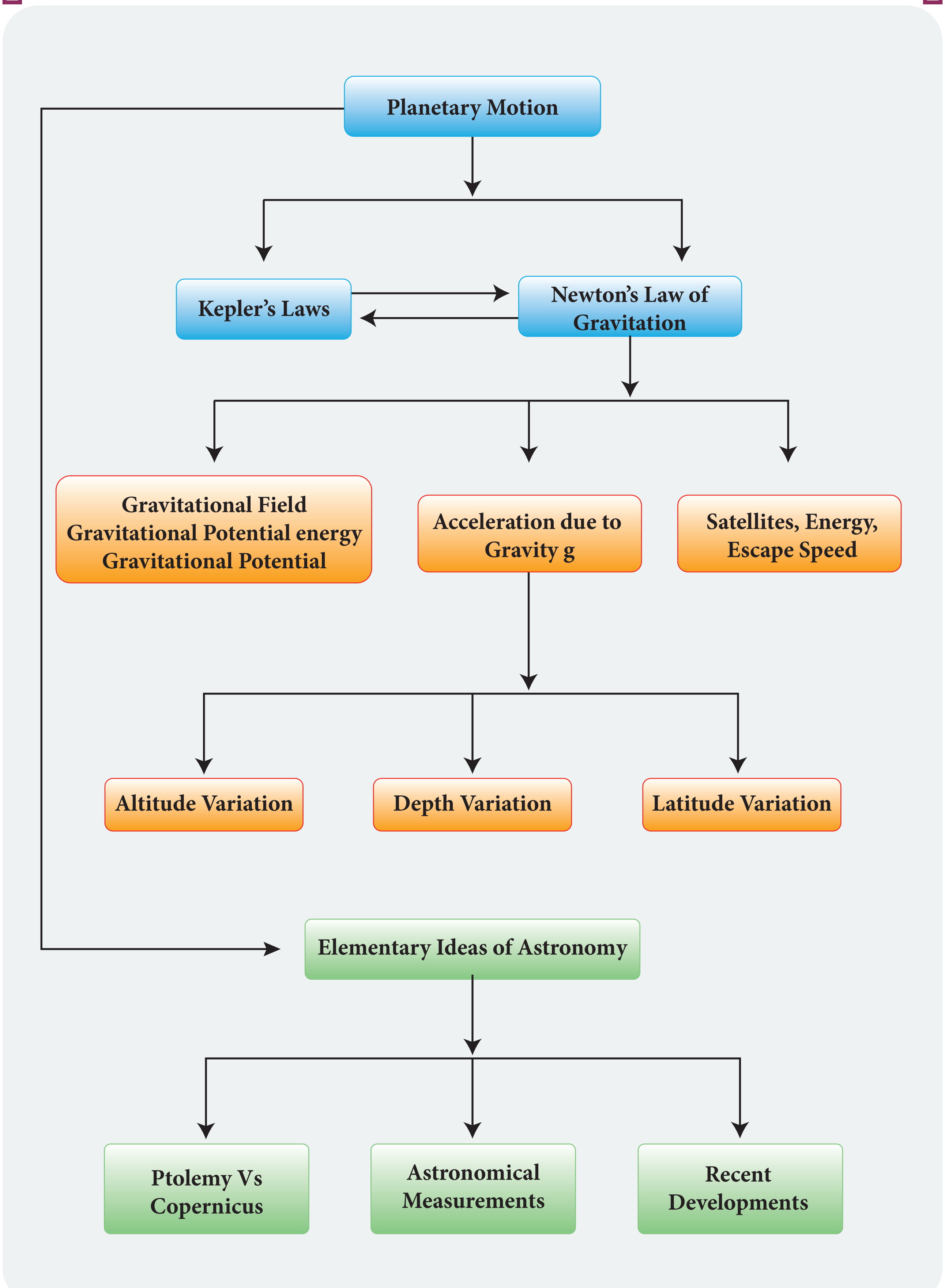
Till the 19th century astronomy mainly depended upon either observation with the naked eye or telescopic observation. After the discovery of the electromagnetic spectrum at the end of the 19th century, our understanding of the universe increased enormously. Because of this development in the late 19th century it was found that Newton's law of gravitation could not explain certain phenomena and showed some discrepancies. Albert Einstein formulated his 'General theory of relativity' which was one of the most successful theories of 20th century in the field of gravitation.

In the twentieth century both astronomy and gravitation merged together and have grown in manifold. The birth and death of stars were more clearly understood. Many Indian physicists made very important contributions to the field of astrophysics and gravitation. Subramanian Chandrasekar formulated the theory of black holes and explained the life of stars. These studies brought him the Nobel prize in the year 1983. Another very notable Indian astrophysicist Meghnad Saha discovered the ionization formula which was useful in classifying stars. This formula is now known as "Saha ionization formula". In the field of gravitation Amal Kumar Raychaudhuri solved an equation now known as "Raychaudhuri equation" which was a very important contribution. Another notable Indian Astrophysicist Jayant V Narlikar made pioneering contribution in the field of astrophysics and has written interesting books on astronomy and astrophysics. IUCAA (Inter University Center for Astronomy and Astrophysics) is one of the important Indian research institutes where active research in astrophysics and gravitation are conducted. The institute was founded by Prof. J.V. Narlikar. Students are encouraged to read more about the recent developments in these fields.

SUMMARY

- The motion of planets can be explained using Kepler's laws.
- **Kepler's first law**: All the planets in the solar system orbit the Sun in elliptical orbits with the Sun at one of the foci.
- **Kepler's second law:** The radial vector line joining the Sun to a planet sweeps equal areas in equal intervals of time.
- **Kepler's third law:** The ratio of the square of the time period of planet to the cubic power of semi major axis is constant for all the planets in the solar system.
- **Newton's law of gravitation** states that the gravitational force between two masses is directly proportional to product of masses and inversely proportional to square of the distance between the masses. In vector form it is given by $\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$
- Gravitational force is a central force.
- Kepler's laws can be derived from Newton's law of gravitation.
- The gravitational field due to a mass m at a point which is at a distance r from mass m is given by $\vec{E} = -\frac{Gm}{r^2}\hat{r}$. It is a vector quantity.
- The gravitational potential energy of two masses is given by $U = -\frac{Gm_1m_2}{r}$. It is a scalar quantity.
- The gravitational potential at a point which is at a distance r from mass m is given by $V = -\frac{Gm}{r}$. It is a scalar quantity.
- The acceleration due to Earth's gravity decreases as altitude increases and as depth increases.
- Due to rotation of the Earth, the acceleration due to gravity is maximum at poles and minimum at Earth's equator.
- The (escape) speed of any object required to escape from the Earth's gravitational field is $v_e = \sqrt{2gR_e}$. It is independent of mass of the object.
- The energy of the satellite is negative. It implies that the satellite is bound to Earth's gravitational force.
- Copernicus model explained that retrograde motion is due to relative motion between planets. This explanation is simpler than Ptolemy's epicycle explanation which is complicated
- Copernicus and Kepler measured the distance between a planet and the Sun using simple geometry and trigonometry.
- 2400 years ago, Eratosthenes measured the radius of the Earth using simple geometry and trigonometry.

CONCEPT MAP


Planetary Motion
Kepler's Laws
Newton's Law of Gravitation
Gravitational Field
Gravitational Potential energy
Gravitational Potential
Acceleration due to Gravity g
Satellites, Energy, Escape Speed
Altitude Variation
Depth Variation
Latitude Variation
Elementary Ideas of Astronomy
Ptolemy Vs Copernicus
Astronomical Measurements
Recent Developments

EVALUATION

I. Multiple Choice Questions

1. The linear momentum and position vector of the planet is perpendicular to each other at

(a) perihelion and aphelion

(b) at all points

(c) only at perihelion

(d) no point

2. If the masses of the Earth and Sun suddenly double, the gravitational force between them will

(a) remain the same

(b) increase 2 times

(c) increase 4 times

(d) decrease 2 times

3. A planet moving along an elliptical orbit is closest to the Sun at distance r_1 and farthest away at a distance of r_2. If v_1 and v_2 are linear speeds at these points respectively. Then the ratio $\frac{v_1}{v_2}$ is

(NEET 2016)

(a) $\frac{r_2}{r_1}$ (b) $\left(\frac{r_2}{r_1}\right)^2$

(c) $\frac{r_1}{r_2}$ (d) $\left(\frac{r_1}{r_2}\right)^2$

4. The time period of a satellite orbiting Earth in a cirular orbit is independent of .

(a) Radius of the orbit

(b) The mass of the satellite

(c) Both the mass and radius of the orbit

(d) Neither the mass nor the radius of its orbit

5. If the distance between the Earth and Sun were to be doubled from its present value, the number of days in a year would be

(a) 64.5 (b) 1032

(c) 182.5 (d) 730

6. According to Kepler's second law, the radial vector to a planet from the Sun sweeps out equal areas in equal intervals of time. This law is a consequence of

(a) conservation of linear momentum

(b) conservation of angular momentum

(c) conservation of energy

(d) conservation of kinetic energy

7. The gravitational potential energy of the Moon with respect to Earth is

(a) always positive

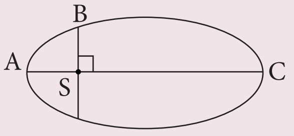
(b) always negative

(c) can be positive or negative

(d) always zero

8. The kinetic energies of a planet in an elliptical orbit about the Sun, at positions A, B and C are K_A, K_B and K_C respectively. AC is the major axis and SB is perpendicular to AC at the position of the Sun S as shown in the figure. Then

(NEET 2018)



(a) $K_A > K_B > K_C$

(b) $K_B < K_A < K_C$

(c) $K_A < K_B < K_C$

(d) $K_B > K_A > K_C$

9. The work done by the Sun's gravitational force on the Earth is
   (a) always zero
   (b) always positive
   (c) can be positive or negative
   (d) always negative
10. If the mass and radius of the Earth are both doubled, then the acceleration due to gravity g'
   (a) remains same (b) $\frac{g}{2}$
   (c) 2 g (d) 4 g
11. The magnitude of the Sun's gravitational field as experienced by Earth is
   (a) same over the year
   (b) decreases in the month of January and increases in the month of July
   (c) decreases in the month of July and increases in the month of January
   (d) increases during day time and decreases during night time.
12. If a person moves from Chennai to Trichy, his weight
   (a) increases
   (b) decreases
   (c) remains same
   (d) increases and then decreases
13. An object of mass 10 kg is hanging on a spring scale which is attached to the roof of a lift. If the lift is in free fall, the reading in the spring scale is
   (a) 98 N (b) zero
   (c) 49 N (d) 9.8 N
14. If the acceleration due to gravity becomes 4 times its original value, then escape speed
   (a) remains same
   (b) 2 times of original value
   (c) becomes halved
   (d) 4 times of original value
15. The kinetic energy of the satellite orbiting around the Earth is
   (a) equal to potential energy
   (b) less than potential energy
   (c) greater than kinetic energy
   (d) zero

Answers

1) a	**2)** c	**3)** a	**4)** b	**5)** b
6) b	**7)** b	**8)** a	**9)** c	**10)** b
11) c	**12)** b	**13)** b	**14)** b	**15)** b

II. Short Answer Questions

1. State Kepler's three laws.
2. State Newton's Universal law of gravitation.
3. Will the angular momentum of a planet be conserved? Justify your answer.
4. Define the gravitational field. Give its unit.
5. What is meant by superposition of gravitational field?
6. Define gravitational potential energy.
7. Is potential energy the property of a single object? Justify.
8. Define gravitational potential.
9. What is the difference between gravitational potential and gravitational potential energy?
10. What is meant by escape speed in the case of the Earth?
11. Why is the energy of a satellite (or any other planet) negative?
12. What are geostationary and polar satellites?
13. Define weight

14. Why is there no lunar eclipse and solar eclipse every month?

15. How will you prove that Earth itself is spinning?

III. Long Answer Questions

1. Discuss the important features of the law of gravitation.
2. Explain how Newton arrived at his law of gravitation from Kepler's third law.
3. Explain how Newton verified his law of gravitation.
4. Derive the expression for gravitational potential energy.
5. Prove that at points near the surface of the Earth, the gravitational potential energy of the object is $U = mgh$
6. Explain in detail the idea of weightlessness using lift as an example.
7. Derive an expression for escape speed.
8. Explain the variation of g with lattitude.
9. Explain the variation of g with altitude.
10. Explain the variation of g with depth from the Earth's surface.
11. Derive the time period of satellite orbiting the Earth.
12. Derive an expression for energy of satellite.
13. Explain in detail the geostationary and polar satellites.
14. Explain how geocentric theory is replaced by heliocentric theory using the idea of retrograde motion of planets.
15. Explain in detail the Eratosthenes method of finding the radius of Earth.
16. Describe the measurement of Earth's shadow (umbra) radius during total lunar eclipse

IV. Exercises

1. An unknown planet orbits the Sun with distance twice the semi major axis distance of the Earth's orbit. If the Earth's time period is T_1, what is the time period of this unknown planet?

   Ans: $T_2 = 2\sqrt{2}T_1$

2. Assume that you are in another solar system and provided with the set of data given below consisting of the planets' semi major axes and time periods. Can you infer the relation connecting semi major axis and time period?

Planet (imaginary)	Time period(T) (in year)	Semi major axis (a) (in AU)
Kurinji	2	8
Mullai	3	18
Marutham	4	32
Neithal	5	50
Paalai	6	72

   Ans: $a \propto 2T^2$

3. If the masses and mutual distance between the two objects are doubled, what is the change in the gravitational force between them?

   Ans: No change

4. Two bodies of masses m and 4m are placed at a distance r. Calculate the gravitational potential at a point on the line joining them where the gravitional field is zero.

   Ans: $V = -\dfrac{9Gm}{r}$

5. If the ratio of the orbital distance of two planets $\frac{d_1}{d_2}=2$, what is the ratio of gravitational field experienced by these two planets?

Ans: $E_2 = 4\,E_1$

6. The Moon Io orbits Jupiter once in 1.769 days. The orbital radius of the Moon Io is 421700 km. Calculate the mass of Jupiter?

Ans: 1.898×10^{27} kg

7. If the angular momentum of a planet is given by $\vec{L} = 5t^2\hat{i} - 6t\hat{j} + 3\hat{k}$ What is the torque experienced by the planet? Will the torque be in the same direction as that of the angular momentum?

Ans: $\vec{\tau} = 10t\hat{i} - 6\hat{j}$

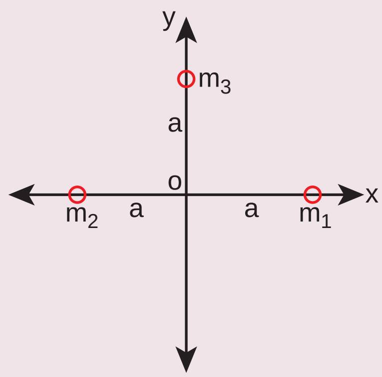
8. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction. Calculate the speed of each particle

Ans: $\frac{1}{2}\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$

9. Suppose unknowingly you wrote the universal gravitational constant value as $G = 6.67\times10^{11}$ instead of the correct value $G = 6.67\times10^{-11}$, what is the acceleration due to gravity g' for this incorrect G? According to this new acceleration due to gravity, what will be your weight W'?

Ans: $g' = 10^{22}\,g$, $W' = 10^{22}\,W$

10. Calculate the gravitational field at point O due to three masses m_1,m_2 and m_3 whose positions are given by the following figure. If the masses m_1 and m_2 are equal what is the change in gravitational field at the point O?



Ans: $\vec{E} = \frac{G}{a^2}\left[(m_1 - m_2)i + m_3 j\right]$

if $m_1 = m_2$, $\vec{E} = \frac{G}{a^2}\left[m_3\,\hat{j}\right]$

11. What is the gravitational potential energy of the Earth and Sun? The Earth to Sun distance is around 150 million km. The mass of the Earth is 5.9×10^{24} kg and mass of the Sun is 1.9×10^{30} kg.

Ans: U= -49.84×10^{32} Joule

12. Earth revolves around the Sun at 30 km s^{-1}. Calculate the kinetic energy of the Earth. In the previous example you calculated the potential energy of the Earth. What is the total energy of the Earth in that case? Is the total energy positive? Give reasons.

Ans: K.E = 26.5×10^{32} J

E = -23.29×10^{32} J

(–) ve implies that Earth is bounded with Sun

13. An object is thrown from Earth in such a way that it reaches a point at infinity with non-zero kinetic energy

$\left[K.E(r=\infty)=\frac{1}{2}Mv_{\infty}^{2}\right]$, with what velocity should the object be thrown from Earth?

Ans: $v_e = \sqrt{v_{\infty}^2 + 2gR_E}$

14. Suppose we go 200 km above and below the surface of the Earth, what are the g values at these two points? In which case, is the value of g small?

Ans: $g_{down} = 0.96\ g$

$g_{up} = 0.94\ g$

15. Calculate the change in g value in your district of Tamil nadu. (Hint: Get the latitude of your district of Tamil nadu from the Google). What is the difference in g values at Chennai and Kanyakumari?

Ans: $g_{chennai} = 9.7677\ m\ s^{-2}$

$g_{Kanyakumari} = 9.7667\ m\ s^{-2}$

$\Delta g = 0.001\ m\ s^{-2}$

V. Conceptual Questions

1. In the following, what are the quantities which that are conserved?

a) Linear momentum of planet

b) Angular momentum of planet

c) Total energy of planet

d) Potential energy of a planet

2. The work done by Sun on Earth in one year will be

a) Zero b) Non zero

c) positive d) negative

3. The work done by Sun on Earth at any finite interval of time is

a) positive, negative or zero

b) Strictly positive

c) Strictly negative

d) It is always zero

4. If a comet suddenly hits the Moon and imparts energy which is more than the total energy of the Moon, what will happen?

5. If the Earth's pull on the Moon suddenly disappears, what will happen to the Moon?

6. If the Earth has no tilt, what happens to the seasons of the Earth?

7. A student was asked a question 'why are there summer and winter for us? He replied as 'since Earth is orbiting in an elliptical orbit, when the Earth is very far away from the Sun(aphelion) there will be winter, when the Earth is nearer to the Sun(perihelion) there will be winter'. Is this answer correct? If not, what is the correct explanation for the occurrence of summer and winter?

8. The following photographs are taken from the recent lunar eclipse which occurred on January 31, 2018. Is it possible to prove that Earth is a sphere from these photographs?

BOOKS FOR REFERENCE

1. Mechanics by Charles Kittel, Walter Knight, Malvin Ruderman, Carl Helmholtz and Moyer
2. Newtonian Mechanics by A.P. French
3. Introduction to Mechanics by Daniel Kepler and Robert Kolenkow
4. Mechanics by Somnath Datta
5. Concepts of Physics volume 1 and Volume 2 by H.C. Verma
6. Physics for Scientist and Engineers with Modern physics by Serway and Jewett
7. Physics for Scientist and Engineers by Paul Tipler and Gene Mosca
8. Physics for the Inquiring Mind by Eric Rogers
9. Fundamental laws of Mechanics by Irodov.
10. Question and Problems in School Physics by Tarasov and Tarasova

ICT CORNER

Gravitation

Through this activity you will be able to learn about the gravitational force and orbital paths.

STEPS:

- Click the URL or scan the QR code to launch the activity page. Click on the box labelled "Model" to start the activity.
- In the activity window, a diagram of sun and earth is given. Click the play icon to see the motion of earth.
- We can change the objects by selecting objects from the table given in the right side window.
- The path of the gravity, velocity and the object in motion can be viewed. Check on the relevant boxes given in the table.

Step1

Step2

Step3

Step4

URL:

https://phet.colorado.edu/sims/html/gravity-and-orbits/latest/gravity-and-orbits_en.html

* Pictures are indicative only.

* If browser requires, allow **Flash Player** or **Java Script** to load the page.