

SUPPLEMENTARY MATERIAL

CHAPTER 7

7.6.3 $\int (px + q)\sqrt{ax^2 + bx + c} \, dx.$

We choose constants A and B such that

$$\begin{aligned} px + q &= A \left[\frac{d}{dx}(ax^2 + bx + c) \right] + B \\ &= A(2ax + b) + B \end{aligned}$$

Comparing the coefficients of x and the constant terms on both sides, we get

$$2aA = p \text{ and } Ab + B = q$$

Solving these equations, the values of A and B are obtained. Thus, the integral reduces to

$$\begin{aligned} &A \int (2ax + b)\sqrt{ax^2 + bx + c} \, dx + B \int \sqrt{ax^2 + bx + c} \, dx \\ &= AI_1 + BI_2 \end{aligned}$$

$$\text{where } I_1 = \int (2ax + b)\sqrt{ax^2 + bx + c} \, dx$$

Put $ax^2 + bx + c = t$, then $(2ax + b)dx = dt$

$$\text{So } I_1 = \frac{2}{3}(ax^2 + bx + c)^{\frac{3}{2}} + C_1$$

$$\text{Similarly, } I_2 = \int \sqrt{ax^2 + bx + c} \, dx$$

is found, using the integral formulae discussed in [7.6.2, Page 328 of the textbook].

Thus $\int (px + q)\sqrt{ax^2 + bx + c} dx$ is finally worked out.

Example 25 Find $\int x\sqrt{1+x-x^2} dx$

Solution Following the procedure as indicated above, we write

$$\begin{aligned} x &= A \left[\frac{d}{dx}(1+x-x^2) \right] + B \\ &= A(1-2x) + B \end{aligned}$$

Equating the coefficients of x and constant terms on both sides,

We get $-2A = 1$ and $A + B = 0$

Solving these equations, we get $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Thus the integral reduces to

$$\begin{aligned} \int x\sqrt{1+x-x^2} dx &= -\frac{1}{2} \int (1-2x)\sqrt{1+x-x^2} dx + \frac{1}{2} \int \sqrt{1+x-x^2} dx \\ &= -\frac{1}{2} I_1 + \frac{1}{2} I_2 \end{aligned} \quad (1)$$

Consider $I_1 = \int (1-2x)\sqrt{1+x-x^2} dx$

Put $1+x-x^2 = t$, then $(1-2x)dx = dt$

Thus $I_1 = \int (1-2x)\sqrt{1+x-x^2} dx = \int t^{\frac{1}{2}} dt = \frac{2}{3} t^{\frac{3}{2}} + C_1$

$$= \frac{2}{3} (1+x-x^2)^{\frac{3}{2}} + C_1, \text{ where } C_1 \text{ is some constant.}$$

Further, consider
$$I_2 = \int \sqrt{1+x-x^2} \, dx = \int \sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2} \, dx$$

Put $x - \frac{1}{2} = t$. Then $dx = dt$

Therefore,
$$\begin{aligned} I_2 &= \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} \, dt \\ &= \frac{1}{2}t\sqrt{\frac{5}{4} - t^2} + \frac{1}{2} \cdot \frac{5}{4} \sin^{-1} \frac{2t}{\sqrt{5}} + C_2 \\ &= \frac{1}{2} \frac{(2x-1)}{2} \sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + C_2 \\ &= \frac{1}{4} (2x-1) \sqrt{1+x-x^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + C_2, \end{aligned}$$

where C_2 is some constant.

Putting values of I_1 and I_2 in (1), we get

$$\begin{aligned} \int x \sqrt{1+x-x^2} \, dx &= -\frac{1}{3} (1+x-x^2)^{\frac{3}{2}} + \frac{1}{8} (2x-1) \sqrt{1+x-x^2} \\ &\quad + \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + C, \end{aligned}$$

where $C = -\frac{C_1+C_2}{2}$ is another arbitrary constant.

Insert the following exercises at the end of EXERCISE 7.7 as follows:

$$12. x\sqrt{x+x^2} \quad 13. (x+1)\sqrt{2x^2+3} \quad 14. (x+3)\sqrt{3-4x-x^2}$$

Answers

$$12. \frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{(2x+1)\sqrt{x^2+x}}{8} + \frac{1}{16} \log \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + C$$

$$13. \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3\sqrt{2}}{4} \log \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + C$$

$$14. -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}} + \frac{7}{2} \sin^{-1} \left(\frac{x+2}{\sqrt{7}} \right) + \frac{(x+2)\sqrt{3-4x-x^2}}{2} + C$$

CHAPTER 10

10.7 Scalar Triple Product

Let \vec{a} , \vec{b} and \vec{c} be any three vectors. The scalar product of \vec{a} and $(\vec{b} \times \vec{c})$, i.e., $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product of \vec{a} , \vec{b} and \vec{c} in this order and is denoted by $[\vec{a}, \vec{b}, \vec{c}]$ (or $\vec{a}\vec{b}\vec{c}$). We thus have

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

Observations

1. Since $(\vec{b} \times \vec{c})$ is a vector, $\vec{a} \cdot (\vec{b} \times \vec{c})$ is a scalar quantity, i.e. $[\vec{a}, \vec{b}, \vec{c}]$ is a scalar quantity.
2. Geometrically, the magnitude of the scalar triple product is the volume of a parallelepiped formed by adjacent sides given by the three

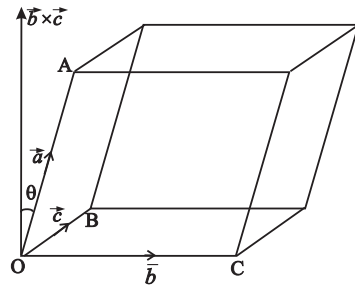


Fig. 10.28

vectors \vec{a} , \vec{b} and \vec{c} (Fig. 10.28). Indeed, the area of the parallelogram forming the base of the parallelepiped is $|\vec{b} \times \vec{c}|$. The height is the projection of \vec{a} along the normal to the plane containing \vec{b} and \vec{c} which is the magnitude of the component of \vec{a} in the direction of $\vec{b} \times \vec{c}$ i.e., $\frac{|\vec{a} \cdot (\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|}$. So the required

volume of the parallelepiped is $\frac{|\vec{a} \cdot (\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|} |\vec{b} \times \vec{c}| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$,

3. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

and so

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

4. If \vec{a} , \vec{b} and \vec{c} be any three vectors, then

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

(cyclic permutation of three vectors does not change the value of the scalar triple product).

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then, just by observation above, we have

$$\begin{aligned}
 [\vec{a}, \vec{b}, \vec{c}] &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
 &= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1) \\
 &= b_1 (a_3 c_2 - a_2 c_3) + b_2 (a_1 c_3 - a_3 c_1) + b_3 (a_2 c_1 - a_1 c_2) \\
 &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
 &= [\vec{b}, \vec{c}, \vec{a}]
 \end{aligned}$$

Similarly, the reader may verify that

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{a}, \vec{b}, \vec{c}]$$

$$\text{Hence } [\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

5. In scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$, the dot and cross can be interchanged. Indeed,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}] = \vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

6. $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}]$. Indeed

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= \vec{a} \cdot (-\vec{c} \times \vec{b})$$

$$= -(\vec{a} \cdot (\vec{c} \times \vec{b}))$$

$$= -[\vec{a}, \vec{c}, \vec{b}]$$

7. $[\vec{a}, \vec{a}, \vec{b}] = 0$ Indeed

$$\begin{aligned} [\vec{a}, \vec{a}, \vec{b}] &= [\vec{a}, \vec{b}, \vec{a}] \\ &= [\vec{b}, \vec{a}, \vec{a}] \\ &= \vec{b} \cdot (\vec{a} \times \vec{a}) \\ &= \vec{b} \cdot \vec{0} = 0. \end{aligned} \quad (\text{as } \vec{a} \times \vec{a} = \vec{0})$$

Note: The result in 7 above is true irrespective of the position of two equal vectors.

10.7.1 Coplanarity of Three Vectors

Theorem 1 Three vectors \vec{a} , \vec{b} and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Proof Suppose first that the vectors \vec{a} , \vec{b} and \vec{c} are coplanar.

If \vec{b} and \vec{c} are parallel vectors, then, $\vec{b} \times \vec{c} = \vec{0}$ and so $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

If \vec{b} and \vec{c} are not parallel then, since \vec{a} , \vec{b} and \vec{c} are coplanar, $\vec{b} \times \vec{c}$ is perpendicular to \vec{a} .

$$\text{So } \vec{a} \cdot (\vec{b} \times \vec{c}) = 0.$$

Conversely, suppose that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. If \vec{a} and $\vec{b} \times \vec{c}$ are both non-zero, then we conclude that \vec{a} and $\vec{b} \times \vec{c}$ are perpendicular vectors. But $\vec{b} \times \vec{c}$ is perpendicular to both \vec{b} and \vec{c} . Therefore, \vec{a} and \vec{b} and \vec{c} must lie in the plane, i.e. they are coplanar.

If $\vec{a} = \vec{0}$, then \vec{a} is coplanar with any two vectors, in particular with \vec{b} and \vec{c} . If $(\vec{b} \times \vec{c}) = \vec{0}$, then \vec{b} and \vec{c} are parallel vectors and so, \vec{a} , \vec{b} and \vec{c} are coplanar since any two vectors always lie in a plane determined by them and a vector which is parallel to any one of it also lies in that plane.

Note: Coplanarity of four points can be discussed using coplanarity of three vectors.

Indeed, the four points A, B, C and D are coplanar if the vectors \overrightarrow{AB} , \overrightarrow{AC} and \overrightarrow{AD} are coplanar.

Example 26 Find $\vec{a} \cdot (\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

Solution We have $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -10$.

Example 27 Show that the vectors

$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

Solution We have $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$.

Hence, in view of Theorem 1, \vec{a}, \vec{b} and \vec{c} are coplanar vectors.

Example 28 Find λ if the vectors

$\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} + 7\hat{j} + 3\hat{k}$ are coplanar.

Solution Since \vec{a}, \vec{b} and \vec{c} are coplanar vectors, we have $[\vec{a}, \vec{b}, \vec{c}] = 0$, i.e.,

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ \lambda & 7 & 3 \end{vmatrix} = 0.$$

$$\Rightarrow 1(-3 + 7) - 3(6 + \lambda) + 1(14 + \lambda) = 0$$

$$\Rightarrow \lambda = 0.$$

Example 29 Show that the four points A, B, C and D with position vectors

$4\hat{i} + 5\hat{j} + \hat{k}$, $-(\hat{j} + \hat{k})$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$, respectively are coplanar.

Solution We know that the four points A, B, C and D are coplanar if the three vectors \overline{AB} , \overline{AC} and \overline{AD} are coplanar, i.e., if

$$[\overline{AB}, \overline{AC} \text{ and } \overline{AD}] = 0$$

$$\text{Now } \overrightarrow{AB} = -(\hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -4\hat{i} - 6\hat{j} - 2\hat{k}$$

$$\overrightarrow{AC} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\text{and } \overrightarrow{AD} = 4(-\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -8\hat{i} - \hat{j} + 3\hat{k}$$

$$\text{Thus } [\overrightarrow{AB}, \overrightarrow{AC} \text{ and } \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0.$$

Hence A, B, C and D are coplanar.

Example 30 Prove that $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2 [\vec{a}, \vec{b}, \vec{c}]$.

Solution We have

$$\begin{aligned} [\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] &= (\vec{a} + \vec{b}) \cdot ((\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})) \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}) \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) \quad (\text{as } \vec{c} \times \vec{c} = \vec{0}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a}) \\ &= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{a}] + [\vec{a}, \vec{c}, \vec{a}] + [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{a}] \\ &= 2[\vec{a}, \vec{b}, \vec{c}] \quad (\text{Why ?}) \end{aligned}$$

Example 31 Prove that $[\vec{a}, \vec{b}, \vec{c} + \vec{d}] = [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]$

Solution We have

$$\begin{aligned} [\vec{a}, \vec{b}, \vec{c} + \vec{d}] &= \vec{a} \cdot (\vec{b} \times (\vec{c} + \vec{d})) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{d}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{d}) \\ &= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]. \end{aligned}$$

Exercise 10.5

- Find $[\vec{a} \vec{b} \vec{c}]$ if $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$
(Ans. 24)
- Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.
- Find λ if the vectors $\hat{i} - \hat{j} + \hat{k}$, $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + \lambda\hat{j} - 3\hat{k}$ are coplanar.
(Ans. $\lambda = 15$)
- Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then
 - If $c_1 = 1$ and $c_2 = 2$, find c_3 which makes \vec{a} , \vec{b} and \vec{c} coplanar (Ans. $c_3 = 2$)
 - If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make \vec{a} , \vec{b} and \vec{c} coplanar.
- Show that the four points with position vectors $4\hat{i} + 8\hat{j} + 12\hat{k}$, $2\hat{i} + 4\hat{j} + 6\hat{k}$, $3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.
- Find x such that the four points A (3, 2, 1) B (4, x , 5), C (4, 2, -2) and D (6, 5, -1) are coplanar. (Ans. $x = 5$)
- Show that the vectors \vec{a} , \vec{b} and \vec{c} coplanar if $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.