Exercise 7.1

Answer 1E.

Consider the following integral:

$$\int x^2 \ln x dx$$

Evaluate the given integral using integration by parts with the indicated choices of u and dv.

Here $u = \ln x,$ and $dv = x^2 dx.$

The formula for integration by parts is given by

$$\int u dv = uv - \int v du$$

Let $u = \ln x$ and $dv = x^2 dx.$ Then

$$\frac{du}{dx} = \frac{1}{x} \quad \text{and} \quad \int dv = \int x^2 dx$$

$$du = \frac{1}{x} dx \quad \text{and} \quad v = \frac{x^3}{3}$$

Plug the values of u, v, du and dv into the integration by parts formula.

$$\int u dv = uv - \int v du$$

$$\int x^2 \ln x dx = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \left(\frac{x^3}{3} + C \right) \quad \text{Use the formula } \int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C$$

Therefore, the value of $\int x^2 \ln x dx$ is,

$$\boxed{\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C}.$$

Answer 2E.

Consider the integral $\int \theta \cos\theta d\theta$

To solve the integral use the integration by parts formula: $\int udv = uv - \int vdu$

Let, $u = \theta$, $dv = \cos\theta d\theta$

Then, $du = d\theta$, $v = \sin\theta$

$$\int \theta\cos\theta d\theta = \theta(\sin\theta) - \int \sin\theta d\theta$$

$= \theta(\sin\theta) + \cos\theta + C$ Since $\int \sin\theta d\theta = -\cos\theta + C$

Therefore, $\int \theta\cos\theta d\theta = \boxed{\theta(\sin\theta) + \cos\theta + C}$

Answer 3E.

Consider the integral $\int x\cos 5x\,dx.$

Evaluate the given integral.

Use integration by parts, $\int u\,dv = uv - \int v\,du$ to evaluate the integral.

Let $u = x,\quad dv = \cos 5x\,dx$

Find du.

$du = dx$

Find v.

$$\begin{aligned} v &= \int dv \\ &= \int \cos 5x dx \\ &= \frac{\sin 5x}{5} \end{aligned}$$

Substitute the known values in the formula $\int u\,dv = uv - \int v\,du.$

Then the integral becomes the following:

$$\int x\cos 5x\,dx = \overset{u}{x}\left(\overbrace{\frac{\sin 5x}{5}}^{v}\right) - \int \overbrace{\frac{\sin 5x}{5}}^{v} dx$$

$$= x\left(\frac{\sin 5x}{5}\right) - \frac{1}{5}\int \sin 5x dx$$

$$= \frac{x\sin 5x}{5} - \frac{1}{5}\left[\frac{(-\cos 5x)}{5} + C_1\right]$$

$$= \frac{x\sin 5x}{5} + \frac{\cos 5x}{25} - \frac{1}{5}C_1$$

$$= \frac{x\sin 5x}{5} + \frac{\cos 5x}{25} + C \qquad \text{Where } C = -\frac{1}{5}C_1$$

Hence, the required value of the given integral is $\boxed{\frac{x\sin 5x}{5} + \frac{\cos 5x}{25} + C}$.

Answer 4E.

Consider the integration

$$\int y e^{0.2y} dy$$

We have to evaluate the integral using integration by parts

Let

$$u = y$$

Then taking derivative on both sides

$$du = dy$$

And

$$dv = e^{0.2y} dy$$

Then take integration on both sides

$$\int dv = \int e^{0.2y} dy$$

$$v = \frac{e^{0.2y}}{0.2} \quad \left(\text{Since } \int e^{nx} dx = \frac{e^{nx}}{n}\right)$$

The formula for integration by parts is

$$\int u\, dv = uv - \int v\, du$$

Here

$$\int ye^{0.2y} dy = y\left[\frac{e^{0.2y}}{0.2}\right] - \int\left(\frac{e^{0.2y}}{0.2}\right)dy$$

$$= \frac{ye^{0.2y}}{0.2} - \frac{1}{0.2}\left(\frac{e^{0.2y}}{0.2}\right) + C$$

$$= \frac{ye^{0.2y}}{0.2} - \left(\frac{e^{0.2y}}{(0.2)^2}\right) + C$$

$$= \frac{e^{0.2y}}{0.2}\left[y - \frac{1}{0.2}\right] + C$$

Hence $\boxed{\int ye^{0.2y} dy = \frac{e^{0.2y}}{0.2}\left[y - \frac{1}{0.2}\right] + C}$

Answer 5E.

Given $\int t e^{-3t} dt$

We have to evaluate the given integral using integration by parts

Let $u = t$, $dv = e^{-3t} dt$

$du = dt$, $v = -\frac{e^{-3t}}{3}$

We know that the formula for integration by parts is

$$\int u\, dv = uv - \int v\, du$$

$$\therefore \int te^{-3t} dt = t\left[\frac{e^{-3t}}{-3}\right] - \int\left(\frac{e^{-3t}}{-3}\right)dt$$

$$= \frac{te^{-3t}}{-3} + \frac{1}{3}\left(\frac{e^{-3t}}{-3}\right) + C$$

$$= -\frac{te^{-3t}}{3} - \left(\frac{e^{-3t}}{9}\right) + C$$

$$= -\frac{e^{-3t}}{3}\left[t + \frac{1}{3}\right] + C$$

Hence $\boxed{\int te^{-3t} dt = -\frac{e^{-3t}}{3}\left[t + \frac{1}{3}\right] + C}$

Answer 6E.

Given $\int (x-1)\sin \pi x\, dx$

We have to evaluate the given integral using integration by parts

Let $u = x-1$, $dv = \sin \pi x\, dx$

$du = dx$, $v = -\dfrac{\cos \pi x}{\pi}$

We know that the formula for integration by parts is

$\int u\, dv = uv - \int v\, du$

$$\therefore \int (x-1)\sin \pi x\, dx = (x-1)\left[-\frac{\cos \pi x}{\pi}\right] - \int \left(-\frac{\cos \pi x}{\pi}\right) dx$$

$$= -\frac{(x-1)\cos \pi x}{\pi} + \frac{1}{\pi}\int (\cos \pi x)\, dx$$

$$= -\frac{(x-1)\cos \pi x}{\pi} + \frac{1}{\pi}\left(\frac{\sin \pi x}{\pi}\right) + C$$

$$= -\frac{(x-1)\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} + C$$

Hence $\boxed{\int (x-1)\sin \pi x\, dx = -\frac{(x-1)\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} + C}$

Answer 7E.

Given $\int (x^2+2x)\cos x\, dx$

We have to evaluate the given integral using integration by parts

Let $u = x^2+2x$, $dv = \cos x\, dx$

$du = (2x+2)dx$, $v = \sin x$

We know that the formula for integration by parts is

$\int u\, dv = uv - \int v\, du$

$$\therefore \int (x^2+2x)\cos x\, dx = (x^2+2x)[\sin x] - \int (2x+2)(\sin x)\, dx \quad \ldots\ldots (1)$$

Now for $\int (2x+2)(\sin x)\, dx$, let $u_1 = 2x+2$, $dv_1 = \sin x dx$

$du_1 = 2\, dx$, $v_1 = -\cos x$

$$\therefore \int (2x+2)(\sin x)\, dx = (2x+2)(-\cos x) - \int (-\cos x)\, 2dx$$

$$= -(2x+2)\cos x + 2\sin x + C$$

Substituting this in (1), we get

$$\int (x^2+2x)\cos x\, dx = (x^2+2x)[\sin x] - [-(2x+2)\cos x + 2\sin x + C]$$

$$= (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C_1$$

Hence $\boxed{\int (x^2+2x)\cos x\, dx = (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C_1}$

Answer 8E.

Given $\int t^2 \sin \beta t\, dt$

We have to evaluate the given integral using integration by parts

Let $u = t^2$, $dv = \sin \beta t\, dt$

$du = 2t\, dt$, $v = -\dfrac{\cos \beta t}{\beta}$

We know that the formula for integration by parts is

$\int u\, dv = uv - \int v\, du$

$$\therefore \int t^2 \sin \beta t\, dt = t^2\left[-\frac{\cos \beta t}{\beta}\right] - \int \left(-\frac{\cos \beta t}{\beta}\right)(2t)\, dt$$

$$= -\frac{t^2 \cos \beta t}{\beta} + \frac{2}{\beta}\int t\cos \beta t\, dt \quad \ldots\ldots (1)$$

Now for $\int t\cos\beta t\,dt$, let $u_1 = t$, $dv_1 = \cos\beta t\,dt$

$du_1 = dt$, $v_1 = \dfrac{\sin\beta t}{\beta}$

$$\therefore \int t\cos\beta t\,dt = t\left(\frac{\sin\beta t}{\beta}\right) - \int\left(\frac{\sin\beta t}{\beta}\right)dt$$

$$= \frac{t\sin\beta t}{\beta} - \frac{1}{\beta}\left(-\frac{\cos\beta t}{\beta}\right) + C$$

$$= \frac{t\sin\beta t}{\beta} + \frac{\cos\beta t}{\beta^2} + C$$

Substituting this in (1), we get

$$\therefore \int t^2\sin\beta t\,dt = -\frac{t^2\cos\beta t}{\beta} + \frac{2}{\beta}\left[\frac{t\sin\beta t}{\beta} + \frac{\cos\beta t}{\beta^2} + C\right]$$

$$= -\frac{t^2\cos\beta t}{\beta} + \frac{2t\sin\beta t}{\beta^2} + \frac{2\cos\beta t}{\beta^3} + C_1$$

Hence $\boxed{\int t^2\sin\beta t\,dt = -\dfrac{t^2\cos\beta t}{\beta} + \dfrac{2t\sin\beta t}{\beta^2} + \dfrac{2\cos\beta t}{\beta^3} + C_1}$

Answer 9E.

Consider the following integral:

$$\int \ln\left(\sqrt[3]{x}\right)dx$$

To evaluate the integration, use integration by parts.

Formula for the integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \quad \text{......(1)}$$

Rewrite the integration as follows:

$$\int \ln\left(\sqrt[3]{x}\right)dx = \int 1\cdot\ln\left(\sqrt[3]{x}\right)dx$$

Suppose $f(x) = \ln\left(\sqrt[3]{x}\right)$ and $g'(x) = 1$.

Then, $g(x) = x$ and

$$f'(x) = \frac{1}{\sqrt[3]{x}}\frac{d}{dx}\left(x^{\frac{1}{3}}\right)$$

$$= \frac{1}{x^{\frac{1}{3}}}\cdot\frac{1}{3}\left(x^{-\frac{2}{3}}\right)$$

$$= \frac{1}{3}\frac{1}{x^{\frac{1}{3}}\cdot x^{\frac{2}{3}}}$$

$$= \frac{1}{3}\frac{1}{x^{\frac{3}{3}}}$$

$$= \frac{1}{3x}$$

Therefore, $f'(x) = \dfrac{1}{3x}$.

Substitute $f(x)=\ln\left(\sqrt[3]{x}\right) f'(x)=\frac{1}{3x}, g(x)=x,$ and $g'(x)=1$ into the equation (1) as follows:

$$\int f(x)g'(x)dx = f(x)g(x)-\int g(x)f'(x)dx$$

$$\begin{aligned}\int \ln\left(\sqrt[3]{x}\right)\cdot 1dx &= \ln\left(\sqrt[3]{x}\right)\cdot x-\int x\cdot\frac{1}{3x}dx\\ &= x\ln\left(\sqrt[3]{x}\right)-\frac{1}{3}\int dx\\ &= x\ln\left(\sqrt[3]{x}\right)-\frac{1}{3}x+C \qquad \text{Where } C \text{ is integrand constant}\end{aligned}$$

Thus, $\int \ln\left(\sqrt[3]{x}\right)dx = \boxed{x\ln\left(\sqrt[3]{x}\right)-\frac{1}{3}x+C}$.

Answer 10E.

We have to evaluate $\int \sin^{-1} x dx$

Let $u=\sin^{-1}x,\ dv=dx$

Then $du=\frac{1}{\sqrt{1-x^2}}dx,\ \ v=x$

Using the equation $\int u\,dv = uv-\int v du$

$$\begin{aligned}\int \sin^{-1} x dx &= \left(\sin^{-1}x\right).x-\int x.\frac{1}{\sqrt{1-x^2}}dx\\ &= x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx\end{aligned}$$

Now evaluate the $\int \frac{x}{\sqrt{1-x^2}}dx$ by substituting

$1-x^2=t$
$-2xdx=dt$
Or $xdx=-\frac{1}{2}dt$

Thus
$$\begin{aligned}\int\frac{x}{\sqrt{1-x^2}}dx &= \int\frac{1}{\sqrt{t}}\left(-\frac{1}{2}dt\right)\\ &= -\frac{1}{2}\int t^{-1/2}dt\\ &= -\frac{1}{2}\frac{t^{-1/2+1}}{(-1/2+1)}+C_1\\ &= -\frac{1}{2}\frac{t^{1/2}}{1/2}+C_1\\ &= -t^{1/2}+C_1\\ &= -\left(1-x^2\right)^{1/2}+C_1\end{aligned}$$

Therefore
$$\begin{aligned}\int \sin^{-1} x\,dx &= x\sin^{-1}x-\left(-\left(1-x^2\right)^{1/2}+C_1\right)\\ &= \boxed{x\sin^{-1}x+\sqrt{\left(1-x^2\right)}+C} \qquad \text{Where } C= -C_1\end{aligned}$$

Answer 11E.

$$\int \arctan 4t\,dt = \int \tan^{-1}(4t)\,dt$$

Let $u=\tan^{-1}(4t),\quad dv=dt$

Then $du=\frac{1}{1+(4t)^2}.4dt=\frac{4}{1+16t^2}dt,\quad v=t$

Using the equation $\int u\,dv = uv - \int v\,du$

$$\int \arctan 4t\,dt = \tan^{-1}(4t)t - \int \frac{4}{1+16t^2} t\,dt$$

$$= t\tan^{-1}(4t) - 4\int \frac{t}{1+16t^2}\,dt \qquad \text{--- (1)}$$

Substitute $1+16t^2 = \theta \Rightarrow 32t\,dt$

$$= d\theta \Rightarrow t\,dt = \frac{d\theta}{32}$$

Then $4\int \frac{t\,dt}{1+16t^2} = 4\int \frac{1}{\theta}\frac{d\theta}{32}$

$$= \frac{1}{8}\ln|\theta| + C_1$$

$$= \frac{1}{8}\ln\left|1+16t^2\right| + C_1$$

Therefore equation (1) becomes

$$\int \arctan(4t)\,dt = t.\tan^{-1}(4t) - \frac{1}{8}\ln\left|1+16t^2\right| - C_1$$

$$= \boxed{t\arctan(4t) - \frac{1}{8}\ln\left(1+16t^2\right) + C}$$

Since $\left(1+16t^2\right) > 0$ for all t

Where $C = -C_1$

Answer 12E.

We have to evaluate $\int p^5 \ln p\,dp$

Let $u = \ln p, \quad dv = p^5 dp$

Then $du = \frac{1}{p}dp, \quad v = \frac{p^6}{6}$

Using the equation $\int u\,dv = uv - \int v\,du$

$$\int p^5 \ln p\,dp = \frac{p^6}{6}.\ln p - \int \frac{p^6}{6}.\frac{1}{p}\,dp$$

$$= \frac{p^6}{6}\ln p - \frac{1}{6}\int p^5 dp$$

$$= \frac{p^6}{6}\ln p - \frac{1}{6}\left(\frac{p^6}{6} + C_1\right)$$

$$= \boxed{\frac{p^6}{6}\ln p - \frac{p^6}{36} + C}, \qquad \text{Where } C = -\frac{C_1}{6}$$

Answer 13E.

Consider the following integral:

$$\int t\sec^2(2t)\,dt.$$

To evaluate the integration, use integration by parts.

Formula for the integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \quad \text{......(1)}$$

Suppose $f(x)=t$ and $g'(x)=\sec^2(2t)$.

Then, $g(x)=\frac{1}{2}\tan 2x$ and $f'(x)=1$

Substitute $f(x)=t, f'(x)=1, g(x)=\frac{1}{2}\tan 2x, g'(x)=\sec^2(2t)$, and $dx=dt$ into equation (1) as follows:

$$\begin{aligned}\int t\sec^2(2t)\,dt &= t\cdot\frac{1}{2}\tan(2t)-\int\frac{1}{2}\tan(2t)\cdot 1dt\\ &= \frac{t}{2}\tan(2t)-\frac{1}{2}\int\tan(2t)\,dt\\ &= \frac{t}{2}\tan(2t)-\frac{1}{2}\left(\frac{1}{2}\ln(\sec(2t))\right)+C \qquad \text{Where } C \text{ is integrand constant}\\ &= \frac{t}{2}\tan(2t)-\frac{1}{4}\ln(\sec(2t))+C\end{aligned}$$

Thus, $\int t\sec^2(2t)\,dt = \boxed{\frac{t}{2}\tan(2t)-\frac{1}{4}\ln(\sec(2t))+C}$.

Answer 14E.

Consider the following integral:

$$\int s2^s ds$$

By using integration by parts, $\int udv = uv-\int vdu$.

Let, $u=s$, $dv=2^s ds$.

Then, $du=ds$, $v=\frac{2^s}{\ln 2}$.

$$\begin{aligned}\int s2^s ds &= s\left(\frac{2^s}{\ln 2}\right)-\int\frac{2^s}{\ln 2}ds\\ &= s\left(\frac{2^s}{\ln 2}\right)-\frac{1}{\ln 2}\int 2^s ds\end{aligned}$$

$= s\left(\frac{2^s}{\ln 2}\right)-\frac{1}{\ln 2}\left(\frac{2^s}{\ln 2}\right)+C$ Since $\int 2^s ds = \left(\frac{2^s}{\ln 2}\right)+C$

$$=\boxed{s\left(\frac{2^s}{\ln 2}\right)-\frac{2^s}{(\ln 2)^2}+C}$$

Answer 15E.

We have to evaluate $\int(\ln x)^2 dx$

Let $u=(\ln x)^2$, $dv=dx$

Then $du=2(\ln x).\frac{1}{x}dx$, $v=x$

Using the equation $\int u\,dv = uv-\int vdu$

$$\begin{aligned}\int(\ln x)^2 dx &= (\ln x)^2 x-\int x2(\ln x).\frac{1}{x}dx\\ &= x(\ln x)^2-2\int\ln x dx\end{aligned}$$

Now integrating $\int\ln x dx$ by parts

$$\begin{aligned}\int(\ln x)^2 dx &= x(\ln x)^2-2\left[(\ln x)x-\int\frac{1}{x}xdx\right]\\ &= x(\ln x)^2-2(\ln x)x+2\int dx\\ &= \boxed{x(\ln x)^2-2x\ln x+2x+C}\end{aligned}$$

Answer 16E.

Consider the following integral:

$$\int t \sinh mt \; dt$$

Evaluate the above integral.

Use the integration by parts formula to evaluate the integral

Let $u = t,\ dv = \sinh mt\, dt$; then $du = dt,\ v = \dfrac{\cosh mt}{m}$.

The formula for integration by parts is as follows:

$$\int u\, dv = uv - \int v\, du$$

Substitute the values in the above formula.

$$\begin{aligned}\int t \sinh mt \; dt &= t \cdot \frac{\cosh mt}{m} - \int \frac{\cosh mt}{m} dt \\ &= \frac{t \cosh mt}{m} - \frac{1}{m}\int \cosh mt\, dt \\ &= \frac{t \cosh mt}{m} - \frac{1}{m}\frac{\sinh mt}{m} + C \\ &= \frac{t \cosh mt}{m} - \frac{\sinh mt}{m^2} + C\end{aligned}$$

Therefore, $\int t \sinh mt \; dt = \boxed{\dfrac{t \cosh mt}{m} - \dfrac{\sinh mt}{m^2} + C}$.

Answer 17E.

Consider the following indefinite integral:

$$\int e^{2\theta} \sin(3\theta) d\theta.$$

The object is to evaluate the above integral.

Use integration by parts: $\int u\, dv = uv - \int v\, du$. ……(1)

Let $I = \int e^{2\theta} \sin(3\theta)\, d\theta$ ……(2)

Assume $u = e^{2\theta}$ and $dv = \sin 3\theta$.

Then,

$$du = 2e^{2\theta} d\theta \text{ and } v = \int \sin 3\theta\, d\theta$$

It follows that,

$$\begin{aligned} v &= \int \sin 3\theta\, d\theta \\ &= \frac{-\cos 3\theta}{3} \qquad \text{Use } \int \sin a\theta\, d\theta = \frac{-\cos a\theta}{a}\end{aligned}$$

Substitute all the values in equation (1), then the integral becomes,

$$\begin{aligned} I &= \int e^{2\theta} \sin(3\theta)\, d\theta \\ &= e^{2\theta}\left(\frac{-\cos 3\theta}{3}\right) - \int \left(\frac{-\cos 3\theta}{3}\right) 2e^{2\theta} d\theta \\ &= -\frac{1}{3} e^{2\theta} \cos 3\theta + \frac{2}{3}\int e^{2\theta} \cos 3\theta\, d\theta \quad \text{……(2)}\end{aligned}$$

Again consider the integral $\int e^{2\theta}\cos 3\theta\, d\theta$.

Again use the integration by parts.

Assume $u_1 = e^{2\theta}$ and $dv_1 = \cos 3\theta$.

Then,

$$du_1 = 2e^{2\theta}d\theta \text{ and } v_1 = \int \cos 3\theta d\theta$$

It follows that,

$$v_1 = \int \cos 3\theta\, d\theta$$

$= \frac{\sin 3\theta}{3}$ Use $\int \cos a\theta\, d\theta = \frac{\sin a\theta}{a}$

Substitute all the values in equation (1), then the integral becomes,

$$\begin{aligned}\int e^{2\theta}\cos 3\theta\, d\theta &= \frac{1}{3}e^{2\theta}\sin 3\theta - \int\left(\frac{\sin 3\theta}{3}\right)2e^{2\theta}d\theta \\ &= \frac{1}{3}e^{2\theta}\sin 3\theta - \frac{2}{3}\int e^{2\theta}\sin 3\theta d\theta \\ &= \frac{1}{3}e^{2\theta}\sin 3\theta - \frac{2}{3}I \qquad \cdots\cdots(3)\end{aligned}$$

Use equation (3) in equation (2) to obtain that,

$$\begin{aligned}I &= -\frac{1}{3}e^{2\theta}\cos 3\theta + \frac{2}{3}\left[\frac{1}{3}e^{2\theta}\sin 3\theta - \frac{2}{3}I\right] \\ &= -\frac{1}{3}e^{2\theta}\cos 3\theta + \frac{2}{9}e^{2\theta}\sin 3\theta - \frac{4}{9}I\end{aligned}$$

Add $\frac{4}{9}I$ on both sides of the above equation.

$$\begin{aligned}I + \frac{4}{9}I &= -\frac{1}{3}e^{2\theta}\cos 3\theta + \frac{2}{9}e^{2\theta}\sin 3\theta \\ \frac{13}{9}I &= \frac{1}{9}\left(-3e^{2\theta}\cos 3\theta + 2e^{2\theta}\sin 3\theta\right) \\ I &= \frac{1}{13}\left[-3e^{2\theta}\cos 3\theta + 2e^{2\theta}\sin 3\theta\right] \\ &= \frac{1}{13}e^{2\theta}\left(2\sin 3\theta - 3\cos 3\theta\right) + C\end{aligned}$$

Thus, the value of the integral is $\int e^{2\theta}\sin 3\theta d\theta = \boxed{\frac{1}{13}e^{2\theta}\left(2\sin 3\theta - 3\cos 3\theta\right) + C}$.

Answer 18E.

Consider the integral:

$$I = \int e^{-\theta}\cos 2\theta\, d\theta$$

Make the assumption as shown below:

$$\begin{aligned}u &= e^{-\theta} \\ du &= -e^{-\theta}d\theta \\ dv &= \cos 2\theta\, d\theta \arcsin\theta \\ v &= \frac{\sin 2\theta}{2}\end{aligned}$$

Use the equation shown below to integrate by parts:

$$\int u\,dv = uv - \int v\,du$$

$$\int e^{-\theta}\cos 2\theta\,d\theta = e^{-\theta}\frac{\sin 2\theta}{2} - \int \frac{\sin 2\theta}{2}\left(-e^{-\theta}\right)d\theta$$

$$I = \frac{e^{-\theta}\sin 2\theta}{2} + \frac{1}{2}\int e^{-\theta}\sin 2\theta d\theta \quad \text{...... (1)}$$

Now consider $\int e^{-\theta}\sin 2\theta d\theta$

Integrate by parts

$$u_1 = e^{-\theta}$$
$$du_1 = -e^{-\theta}d\theta$$
$$dv_1 = \sin 2\theta d\theta$$
$$v_1 = -\frac{\cos 2\theta}{2}$$

$$\int u_1 dv_1 = u_1 v_1 - \int v_1 du_1$$

$$\int e^{-\theta}\sin 2\theta d\theta = e^{-\theta}\frac{\left(-\cos 2\theta\right)}{2} - \int\left(-e^{-\theta}\right)\frac{\left(-\cos 2\theta\right)}{2}d\theta$$

$$\int e^{-\theta}\sin 2\theta d\theta = -\frac{e^{-\theta}\cos 2\theta}{2} - \frac{1}{2}\int e^{-\theta}\cos 2\theta d\theta \quad \text{...... (2)}$$

Substitute equation (2) in (1).

$$I = \frac{e^{-\theta}\sin 2\theta}{2} + \frac{1}{2}\left[-\frac{e^{-\theta}\cos 2\theta}{2} - \frac{1}{2}\int e^{-\theta}\cos 2\theta d\theta\right]$$

$$I = \frac{e^{-\theta}\sin 2\theta}{2} - \frac{e^{-\theta}\cos 2\theta}{4} - \frac{1}{4}\int e^{-\theta}\cos 2\theta\,d\theta$$

$$I = \frac{e^{-\theta}\sin 2\theta}{2} - \frac{e^{-\theta}\cos 2\theta}{4} - \frac{1}{4}I$$

Simplify;

$$I + \frac{1}{4}I = \frac{1}{2}\left(e^{-\theta}\sin 2\theta\right) - \frac{1}{4}\left(e^{-\theta}\cos 2\theta\right) + C_1$$

$$\frac{5}{4}I = \frac{1}{2}\left(e^{-\theta}\sin 2\theta\right) - \frac{1}{4}\left(e^{-\theta}\cos 2\theta\right) + C_1$$

$$\frac{5}{4}I = \frac{1}{2}\left(e^{-\theta}\sin 2\theta\right) - \frac{1}{4}\left(e^{-\theta}\cos 2\theta\right) + C_1$$

$$I = \frac{1}{2}.\left(\frac{4}{5}\right)\left(e^{-\theta}\sin 2\theta\right) - \frac{1}{4}.\left(\frac{4}{5}\right)\left(e^{-\theta}\cos 2\theta\right) + \frac{4}{5}C_1$$

(C_1 is an integrating constant)

$$I = \boxed{\frac{2}{5}\left(e^{-\theta}\sin 2\theta\right) - \frac{1}{5}\left(e^{-\theta}\cos 2\theta\right) + C}$$

Where $C = \frac{4}{5}C_1$

Hence, the final value of the integral is $\boxed{\frac{2}{5}\left(e^{-\theta}\sin 2\theta\right) - \frac{1}{5}\left(e^{-\theta}\cos 2\theta\right) + C}$.

Answer 19E.

Given $\int z^3 e^z dz$

We have to evaluate the given integral using integration by parts

Let $u = z^3$, $dv = e^z dz$

$du = 3z^2 dz$, $v = e^z$

We know that the formula for integration by parts is

$\int u\,dv = uv - \int v\,du$

$\therefore \int z^3 e^z dz = z^3 e^z - \int 3z^2 e^z dz$

$= z^3 e^z - 3\int z^2 e^z dz$ ...... (1)

Now for $\int z^2 e^z dz$, we use integration by parts

Let $u_1 = z^2$, $dv_1 = e^z dz$

$du_1 = 2z\,dz$, $v_1 = e^z$

$\therefore \int z^2 e^z dz = z^2 e^z - \int 2z e^z dz$

$= z^2 e^z - 2\int z e^z dz$...... (2)

Again for $\int z e^z dz$, we use integration by parts

Let $u_2 = z$, $dv_2 = e^z dz$

$du_2 = dz$, $v_2 = e^z$

$\therefore \int z e^z dz = z e^z - \int e^z dz$

$= z e^z - e^z + C$ ...... (3)

Substituting (3) in (2), we get

$\int z^2 e^z dz = z^2 e^z - 2\left[z e^z - e^z + C\right]$

$= z^2 e^z - 2z e^z + 2e^z + C_1$

$\Rightarrow \int z^2 e^z dz = z^2 e^z - 2z e^z + 2e^z + C_1$ ...... (4)

Substituting (4) in (1), we get

$\int z^3 e^z dz = z^3 e^z - 3\left[z^2 e^z - 2z e^z + 2e^z + C_1\right]$

$= z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z + C_2$

Hence $\boxed{\int z^3 e^z dz = z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z + C_2}$

Answer 20E.

Evaluate: $\int x \tan^2 x\,dx$

Use integration by parts $\int u\,dv = uv - \int v\,du$

Let $u = x$, $dv = \tan^2 x\,dx$

$dv = \tan^2 x\,dx$

$\Rightarrow dv = \left(\sec^2 x - 1\right) dx$

$\Rightarrow v = \tan x - x$

$du = dx$, $v = \tan x - x$

Consider,

$$\int x\tan^2 x\,dx = x(\tan x - x) - \int(\tan x - x)\,dx$$

$$= x(\tan x - x) - \int \tan x\,dx + \int x\,dx \quad \left\{\begin{array}{l}\text{Use the formulae:}\\ \int \tan x dx = -\ln|\cos x| \text{ and } \int x^n\,dx = \dfrac{x^{n+1}}{n+1}\end{array}\right.$$

$$= x\tan x - x^2 + \ln|\cos x| + \frac{x^2}{2} + c \quad \text{where } c \text{ is a constant of integration}$$

$$= x\tan x + \ln|\cos x| + \frac{x^2}{2} + c$$

Therefore $\int x\tan^2 x\,dx = \boxed{x\tan x + \ln|\cos x| + \frac{x^2}{2} + c}$.

Answer 21E.

Consider the integral $\int \frac{xe^{2x}}{(1+2x)^2}dx$.

Evaluate the indefinite integral.

Let $u = 2x$.

Differentiate on both sides

$du = 2dx$

Substitute the values and solve the integral.

$$\int \frac{xe^{2x}}{(1+2x)^2}dx = \int \frac{\frac{u}{2}\cdot e^u}{(1+u)^2}\frac{du}{2}$$

$$= \frac{1}{4}\int \frac{ue^u}{(1+u)^2}du$$

$$= \frac{1}{4}\int \frac{(1+u-1)e^u}{(1+u)^2}du \quad (\text{Add 1 and Subtract 1 in the numerator})$$

$$= \frac{1}{4}\int \frac{(1+u)e^u - e^u}{(1+u)^2}du$$

Separate two parts of the following integral:

$$\frac{1}{4}\int \frac{(1+u)e^u - e^u}{(1+u)^2}du$$

$$= \frac{1}{4}\int\left[\frac{e^u}{(1+u)} - \frac{e^u}{(1+u)^2}\right]du \quad \left(\text{Since } \int \frac{f(u)-g(u)}{h(u)}du = \int \frac{f(u)}{h(u)}du - \int \frac{g(u)}{h(u)}du\right)$$

$$= \frac{1}{4}\int e^u\left[\frac{1}{(1+u)} - \frac{1}{(1+u)^2}\right]du$$

$$= \frac{1}{4}e^u\cdot\frac{1}{(1+u)} + C \quad \left(\text{Since } \int e^u\left(f(u) + f'(u)\right)du = e^u f(u) + C\right)$$

$$= \frac{1}{4}e^{2x}\cdot\frac{1}{(1+2x)} + C \quad (\text{Replace } u \text{ by } 2x)$$

Therefore, the integral of $\int \frac{xe^{2x}}{(1+2x)^2}dx$ is $\boxed{\frac{e^{2x}}{4(1+2x)} + C}$.

Answer 22E.

Consider the integral

$$\int (\arcsin x)^2\, dx$$

We have to evaluate the indefinite integral

The integration by parts formula is $\int u\, dv = uv - \int v\, du$

Write $\int (\arcsin x)^2\, dx = \int (\sin^{-1} x)^2\, dx$

Put

$$x = \sin\theta$$
$$dx = \cos\theta d\theta$$

And $\left(\sin^{-1}(\sin\theta)\right)^2 = \theta^2 \qquad \left(\text{Since } \sin^{-1}(\sin\alpha) = \alpha\right)$

Therefore,

$$\int (\sin^{-1} x)^2\, dx = \int \theta^2 \cos\theta\, d\theta$$

Now, solve the integral $\int \theta^2 \cos\theta\, d\theta$ by using Integration by parts

Let

$$u = \theta^2, \qquad dv = \cos\theta\, d\theta$$
$$du = 2\theta\, d\theta, \quad v = \sin\theta \qquad (\text{Take integration on both sides})$$
$$\left(\text{Since } \int \cos\theta d\theta = \sin\theta + C\right)$$

Therefore,

$$\int (\sin^{-1} x)^2\, dx = \int \theta^2 \cos\theta\, d\theta$$
$$= \theta^2 \sin\theta - \int 2\theta\, \sin\theta\, d\theta$$
$$\int (\sin^{-1} x)^2\, dx = \theta^2 \sin\theta - 2\int \theta\, \sin\theta\, d\theta \qquad \text{......(1)}$$

Now, take $\int \theta \sin\theta d\theta$ in equation (1)

Again for $\int \theta\, \sin\theta\, d\theta$,

Let $u = \theta$, $dv = \sin\theta\, d\theta$

$$du = d\theta, \quad v = -\cos\theta \qquad \left(\text{Since } \int \sin\theta d\theta = -\cos\theta + C\right)$$

Therefore,

$$\int (\sin^{-1} x)^2\, dx = \theta^2 \sin\theta - 2\int \theta\, \sin\theta\, d\theta$$
$$= \left[\theta^2 \sin\theta - 2\left[\theta(-\cos\theta) - \int -\cos\theta d\theta\right]\right] + c$$
$$\left(\text{Since } \int u\, dv = uv - \int v\, du\right)$$
$$= \left[\theta^2 \sin\theta - 2\left[-\theta\cos\theta + \int \cos\theta d\theta\right]\right] + c$$
$$= \left[\theta^2 \sin\theta - 2\left[-\theta\cos\theta + \sin\theta\right]\right] + c$$
$$= \left[\theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta\right] + c$$
$$= \left(\theta^2 - 2\right)\sin\theta + 2\theta\cos\theta + c$$
$$= \left[\left(\sin^{-1} x\right)^2 - 2\right]x + 2\left(\sin^{-1} x\right)\cos\left(\sin^{-1} x\right) + c$$

Hence $\boxed{\int (\arcsin x)^2\, dx = x\left[\left(\sin^{-1} x\right)^2 - 2\right] + 2\sin^{-1} x \cos\left(\sin^{-1} x\right) + c}$

Answer 23E.

Given $\int_0^{1/2} x\cos\pi x\,dx$

We have to evaluate the given integral

We know that $\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$

Let $f(x) = x$, $g'(x) = \cos\pi x$

$f'(x) = dx$, $g(x) = \dfrac{\sin\pi x}{\pi}$

Now $\int_0^{1/2} x\cos\pi x\,dx = \left\{x\left[\dfrac{\sin\pi x}{\pi}\right]\right\}_0^{1/2} - \int_0^{1/2}\dfrac{\sin\pi x}{\pi}dx$

$= \dfrac{1}{2\pi}\sin\dfrac{\pi}{2} - \left[\dfrac{-\cos\pi x}{\pi^2}\right]_0^{1/2}$

$= \dfrac{1}{2\pi} + \dfrac{1}{\pi^2}\left(\cos\dfrac{\pi}{2} - \cos 0\right)$ $\qquad \left(\because \sin\dfrac{\pi}{2} = 1\right)$

$= \dfrac{1}{2\pi} - \dfrac{1}{\pi^2}$ $\qquad \left(\begin{array}{l}\because \cos\dfrac{\pi}{2} = 0 \\ \cos 0 = 1\end{array}\right)$

$= \dfrac{\pi - 2}{2\pi^2}$

Hence $\boxed{\int_0^{1/2} x\cos\pi x\,dx = \dfrac{\pi-2}{2\pi^2}}$

Answer 24E.

Evaluate the integral, $\int_0^1 (x^2+1)e^{-x}dx$.

Use differentiation to reduce the factor x^2+1 in the integrand to a constant.

Use integration by parts:

$\int u\,dv = uv - \int v\,du$

Assume $u = (x^2+1)$ and $dv = e^{-x}dx$.

Then $du = 2x\,dx$ and $v = \dfrac{e^{-x}}{-1}$.

Consider the integral,

$$\int_0^1 (x^2+1)e^{-x}dx = \left[(x^2+1)\left(\frac{e^{-x}}{-1}\right)\right]_0^1 - \int_0^1 \frac{e^{-x}}{-1}2xdx \quad \text{Use integration by parts}$$

$$= \left[-(1^2+1)e^{-1}+(0^2+1)e^0\right] + 2\int_0^1 e^{-x}xdx \quad \text{Substitute lower and upper limits}$$

$$= \left(-\frac{2}{e}+1\right) + 2\left[\left(x\cdot\frac{e^{-x}}{-1}\right)_0^1 - \int_0^1 \left(\frac{e^{-x}}{-1}\right)dx\right] \quad \left\{\begin{array}{l}\text{Use integration by parts}\\ \text{Assume } u=x \text{ and } dv=e^{-x}dx\\ du=dx, \quad v=\frac{e^{-x}}{-1}\end{array}\right\}$$

$$= \left(-\frac{2}{e}+1\right) + 2\left[\left(-1\cdot e^{-1}+0\cdot e^{-0}\right) - \left(e^{-x}\right)_0^1\right] \quad \text{Since } \int_0^1 \left(\frac{e^{-x}}{-1}\right)dx = e^{-x}$$

$$= -\frac{2}{e}+1-\frac{2}{e}-2\left(e^{-x}\right)_0^1$$

$$= -\frac{2}{e}+1-\frac{2}{e}-2\left(e^{-1}-e^0\right)$$

$$= -\frac{2}{e}+1-\frac{2}{e}-\frac{2}{e}+2$$

$$= 3-\frac{6}{e}$$

Therefore, $\boxed{\int_0^1 (x^2+1)e^{-x}dx = 3-\frac{6}{e}}$

Answer 25E.

Consider the integration, $\int_0^1 t\cosh t dt$.

The formula for integration by parts is given by the following:

$$\int u dv = uv - \int v du.$$

Let $u=t$, $dv=\cosh t dt$.

Differentiate $u=t$ with respect x.

$$du = dt$$

Integrate, $dv=\cosh t dt$.

$$\int dv = \int \cosh t dt$$

$$v = \sinh t$$

Plug the values of u, v, du and dv in the integration by parts formula.

$$\int_0^1 t\cosh t dt = [t\sinh t]_0^1 - \int_0^1 \sinh t dt$$

$$= [t\sinh t]_0^1 - [\cosh t]_0^1$$

$$= [1\cdot\sinh(1) - 0\cdot\sinh(0)] - [\cosh(1)-\cosh(0)]$$

$$= \sinh(1)-\cosh(1)+\cosh(0)$$

$$= \frac{e^1-e^{-1}}{2} - \left(\frac{e^1+e^{-1}}{2}\right) + \frac{e^0+e^{-0}}{2}$$

$$= -2\frac{e^{-1}}{2}+1$$

$$= 1-e^{-1}$$

$$= 1-\frac{1}{e}$$

Therefore, $\int_0^1 t\cosh t dt = \boxed{1-\frac{1}{e}}$.

Answer 26E.

We have to evaluate the following integral

$$\int_4^9 \frac{\ln y}{\sqrt{y}}dy$$

Integration by parts is given by:

$$\int f(y)g'(y)dy = f(y)g(y) - \int f'(y)g(y)dy$$

Taking

$$\int_4^9 \frac{\ln y}{\sqrt{y}}dy$$

We can fit it into this formula to reduce it into a solvable integral.

Let's take:

$$g(y) = 2\sqrt{y}$$

$$g'(y) = \frac{1}{\sqrt{y}}$$

and

$$f(y) = \ln y$$

$$f'(y) = \frac{1}{y}$$

Plugging in:

$$\int_4^9 \frac{\ln y}{\sqrt{y}}dy = (2\sqrt{y})\ln y\Big|_4^9 - \int_4^9 \frac{2\sqrt{y}}{y}dy$$

$$= 6\ln(9) - 4\ln(4) - \int_4^9 \frac{2}{\sqrt{y}}dy$$

$$= 6\ln(9) - 4\ln(4) - 4\sqrt{y}\Big|_4^9$$

$$= 6\ln(9) - 4\ln(4) - 12 + 8$$

$$= \boxed{6\ln(9) - 4\ln(4) - 4}$$

Answer 27E.

Given $\int_1^3 r^3 \ln r \, dr$

We have to evaluate the given integral

We know that $\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$

Let $f(r) = \ln r$, $g'(r) = r^3$

$f'(r) = \frac{1}{r}\, dr$, $g(r) = \frac{r^4}{4}$

$$\text{Now } \int_1^3 r^3 \ln r\, dr = \left[\ln r \cdot \frac{r^4}{4}\right]_1^3 - \int_1^3 \left(\frac{1}{r}\cdot\frac{r^4}{4}\right)dr$$

$$= \left(\ln r \cdot \frac{r^4}{4}\right)_1^3 - \int_1^3 \frac{r^3}{4}dr$$

$$= \frac{81}{4}\ln 3 - \frac{1}{4}\left[\frac{r^4}{4}\right]_1^3$$

$$= \frac{81}{4}\ln 3 - \left(\frac{81}{16} - \frac{1}{16}\right)$$

$$= \frac{81}{4}\ln 3 - 5$$

Hence $\boxed{\int_1^3 r^3 \ln r\, dr = \frac{81}{4}\ln 3 - 5}$

Answer 28E.

Given $\int_0^{2\pi} t^2 \sin 2t\, dt$

We have to evaluate the given integral

We know that $\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$

Let $f(t) = t^2$, $g'(t) = \sin 2t\, dt$

$f'(t) = 2t\, dt$, $g(t) = -\frac{\cos 2t}{2}$

$$\therefore \int_0^{2\pi} t^2 \sin 2t\, dt = t^2\left[-\frac{\cos 2t}{2}\right]\Bigg|_0^{2\pi} - \int_0^{2\pi}\left(-\frac{\cos 2t}{2}\right)(2t)dt$$

$$= t^2\left[-\frac{\cos 2t}{2}\right]\Bigg|_0^{2\pi} + \int_0^{2\pi} t\cos 2t\, dt \qquad \text{...... (1)}$$

Now for $\int_0^{2\pi} t\cos 2t\, dt$, let $f_1(t) = t$, $g_1'(t) = \cos 2t\, dt$

$f_1'(t) = dt$, $g_1(t) = \frac{\sin 2t}{2}$

$$\therefore \int_0^{2\pi} t\cos 2t\, dt = t\left(\frac{\sin 2t}{2}\right)\Bigg|_0^{2\pi} - \int_0^{2\pi}\left(\frac{\sin 2t}{2}\right)dt$$

$$= t\left(\frac{\sin 2t}{2}\right)\Bigg|_0^{2\pi} - \frac{1}{2}\left(-\frac{\cos 2t}{2}\right)_0^{2\pi}$$

$$= t\left(\frac{\sin 2t}{2}\right)\Bigg|_0^{2\pi} + \frac{1}{2}\left(\frac{\cos 2t}{2}\right)_0^{2\pi}$$

$$= \frac{(2\pi)\sin 2(2\pi)}{2} - 0 + \frac{\cos 2(2\pi)}{4} - \frac{\cos 0}{4}$$

$$= \frac{1}{4} - \frac{1}{4} \qquad \begin{pmatrix} \because \cos(4\pi) = 1 \\ \cos 0 = 1 \\ \sin(4\pi) = 0 \end{pmatrix}$$

$$= 0$$

Substituting this in (1), we get

$$\therefore \int_0^{2\pi} t^2 \sin 2t\, dt = t^2\left[-\frac{\cos 2t}{2}\right]\Bigg|_0^{2\pi} + \int_0^{2\pi} t\cos 2t\, dt$$

$$= -\frac{(2\pi)^2\cos 2(2\pi)}{2} + \frac{0\cos 2(0)}{2} + 0$$

$$= -\frac{4\pi^2}{2} + 0 + 0 \qquad (\because \cos(4\pi) = 1)$$

$$= -\frac{4\pi^2}{2}$$

$$= -2\pi^2$$

Hence $\boxed{\int_0^{2\pi} t^2 \sin 2t\, dt = -2\pi^2}$

Answer 29E.

We have to evaluate $\int_0^1 \frac{y}{e^{2y}}dy = \int_0^1 ye^{-2y}dy$

Let $u = y$, $dv = e^{-2y}dy$

Then $du = dy$, $v = \frac{e^{-2y}}{-2}$

Using the equation $\int u\,dv = uv - \int v\,du$

$$\int_0^1 ye^{-2y}dy = y\frac{e^{-2y}}{-2}\Big]_0^1 - \int_0^1 \frac{e^{-2y}}{-2}dy$$

$$= -\frac{ye^{-2y}}{2}\Big]_0^1 + \frac{1}{2}\int_0^1 e^{-2y}dy$$

$$= -\frac{ye^{-2y}}{2}\Big]_0^1 + \frac{1}{2}\left(\frac{e^{-2y}}{-2}\right)\Big]_0^1$$

$$= -\frac{ye^{-2y}}{2}\Big]_0^1 - \frac{1}{4}e^{-2y}\Big]_0^1$$

Therefore $\int_0^1 ye^{-2y}dy = -\frac{1}{2}\left(1.e^{-2}-0\right) - \frac{1}{4}\left(e^{-2}-e^{0}\right)$

$$= -\frac{1}{2}e^{-2} - \frac{1}{4}\left(e^{-2}-1\right)$$

$$= \left(-\frac{1}{2}-\frac{1}{4}\right)e^{-2} + \frac{1}{4}$$

$$= \boxed{\frac{1}{4} - \frac{3}{4}e^{-2}}$$

Answer 30E.

Evaluate the following integral:

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx$$

Recall the definite integrals by parts:

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b g(x)f'(x)dx$$

Let $u = \tan^{-1}\left(\frac{1}{x}\right)$, $dv = dx$

Then, $du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)dx$, $\quad v = x$

$$du = \frac{-1}{x^2+1}dx, \quad v = x$$

The integration is as follows:

$$\int_1^{\sqrt{3}} \tan^{-1}\left(\frac{1}{x}\right)dx = \left[x\tan^{-1}\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{-1}{x^2+1}.x\,dx$$

$$= \left[x\tan^{-1}\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}dx$$

$$= \left[x\tan^{-1}\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \frac{1}{2}\int_1^{\sqrt{3}} \frac{2x}{x^2+1}dx \quad \text{...... (1)}$$

Now, evaluate $\int \frac{2x}{x^2+1}dx$.

Let $x^2+1=t$

$2x\,dx = dt$

$$\int \frac{2x}{x^2+1}dx = \int \frac{dt}{t}$$
$$= \ln|t|$$
$$= \ln(x^2+1)$$

The equation (1) becomes as follows:

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = \left[x\tan^{-1}\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$

$$= \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}(1) + \frac{1}{2}(\ln 4 - \ln 2)$$

$$= \sqrt{3}.\left(\frac{\pi}{6}\right) - \frac{\pi}{4} + \frac{1}{2}(\ln 2) \text{ Since } \ln a - \ln b = \ln\frac{a}{b}$$

$$= \boxed{\frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2}.$$

Answer 31E.

We have to evaluate $\int_0^{1/2} \cos^{-1} x\,dx$

Let $u = \cos^{-1} dx$, $dv = dx$

Then $du = -\frac{1}{\sqrt{1-x^2}}dx, \quad v = x$

Using the equation $\int u\,dv = uv - \int v\,du$

$$\int_0^{1/2} \cos^{-1} x\,dx = \left[x\cos^{-1} x\right]_0^{1/2} - \int_0^{1/2} x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$
$$= \left[x\cos^{-1} x\right]_0^{1/2} - \int_0^{1/2}\left(-\frac{x}{\sqrt{1-x^2}}\right)dx$$

Now evaluate $\int \frac{-x}{\sqrt{1-x^2}}dx$

Put $1-x^2 = t$

$-2x\,dx = dt \Rightarrow -x\,dx = \frac{1}{2}dt$

So $$\int \frac{-x}{\sqrt{1-x^2}}dx = \int\left(\frac{1}{\sqrt{t}}\right)\frac{1}{2}dt$$
$$= \frac{1}{2}\int t^{-1/2}dt$$
$$= \frac{1}{2}\frac{t^{-1/2+1}}{(-1/2+1)} + C \qquad \text{[C is an integration constant]}$$
$$= t^{1/2} + C = (1-x^2)^{1/2} + C$$

Therefore

$$\int_0^{1/2} \cos^{-1} x\,dx = \left[x\cos^{-1} x\right]_0^{1/2} - \left[\left(1-x^2\right)^{1/2} + C\right]_0^{1/2}$$

$$= \left(\frac{1}{2}\cos^{-1}\frac{1}{2} - 0\right) - \left[\left(1-\frac{1}{4}\right)^{1/2} - (1-0)^{1/2}\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{3}\right) - \left(\frac{3}{4}\right)^{1/2} + 1$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$$

$$= \boxed{\frac{1}{6}\left(\pi - 3\sqrt{3} + 6\right)}$$

Answer 32E.

Consider the following integral:

$$\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$$

Rewrite the given integral as $\int_1^2 (\ln x)^2 \cdot \frac{1}{x^3}\,dx.$

Use Integration by parts to evaluate the given integral.

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx \quad \text{...... (1)}$$

Here, $f(x) = (\ln x)^2$, $g'(x)\,dx = \frac{1}{x^3}\,dx.$

Then, evaluate as follows:

$$f'(x) = \frac{d}{dx}\left[(\ln x)^2\right]$$

$$= 2\ln x \cdot \frac{1}{x}$$

$$g(x) = \int g'(x)\,dx$$

$$= \int \frac{1}{x^3}\,dx$$

$$= -\frac{1}{2x^2}$$

Substitute the known values in the equation (1).

The value of the $\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$ integral is calculated as follows:

$$\int_1^2 (\ln x)^2 \cdot \frac{1}{x^3}\,dx = (\ln x)^2 \cdot \left(-\frac{1}{2x^2}\right)\Big]_1^2 - \int_1^2 \left(-\frac{1}{2x^2}\right)\cdot\left(2\ln x \cdot \frac{1}{x}\right)dx$$

$$= -\frac{(\ln x)^2}{2x^2}\Big]_1^2 + \int_1^2 \left(\frac{\ln x}{x^3}\right)dx$$

$$= -\left(\frac{(\ln 2)^2}{8}\right) + \int_1^2 \left(\frac{\ln x}{x^3}\right)dx \quad \text{...... (2)}$$

Evaluate the second integral using Integration by parts.

Here, $f(x)=\ln x$, $g'(x)dx=\frac{1}{x^3}dx.$

Then, evaluate as follows:

$$f'(x)=\frac{d}{dx}\left[(\ln x)\right]$$
$$=\frac{1}{x}$$

$$g(x)=\int g'(x)dx$$
$$=\int\frac{1}{x^3}dx$$
$$=-\frac{1}{2x^2}$$

Substitute the known values in the equation (1).

The value of the $\int_1^2 \ln x\cdot\frac{1}{x^3}dx$ integral is calculated as follows:

$$\int_1^2 \ln x\cdot\frac{1}{x^3}dx=\ln x\cdot\left(-\frac{1}{2x^2}\right)\Bigg]_1^2-\int_1^2\left(-\frac{1}{2x^2}\right)\cdot\left(\frac{1}{x}\right)dx$$
$$=\left[-\frac{\ln x}{2x^2}\right]_1^2+\int_1^2\left(\frac{1}{2x^3}\right)dx$$
$$=\left[-\frac{\ln x}{2x^2}\right]_1^2+\frac{1}{2}\int_1^2\left(x^{-3}\right)dx$$
$$=\left[-\frac{\ln x}{2x^2}\right]_1^2+\frac{1}{2}\left[\frac{x^{-3+1}}{-3+1}\right]_1^2$$
$$=\left[-\frac{\ln x}{2x^2}\right]_1^2+\frac{1}{2}\left[\frac{x^{-2}}{-2}\right]_1^2$$
$$=\left[-\frac{\ln x}{2x^2}\right]_1^2-\left[\frac{1}{4x^2}\right]_1^2$$
$$=\left[-\frac{1}{8}\ln(2)\right]-\left[\frac{1}{16}-\frac{1}{4}\right]$$
$$=\left[-\frac{1}{8}\ln(2)\right]-\left[\frac{1}{16}-\frac{1}{4}\right]$$
$$=\left[-\frac{1}{8}\ln(2)\right]-\left[\frac{1-4}{16}\right]$$
$$=\left[-\frac{1}{8}\ln(2)\right]-\left[\frac{-3}{16}\right]$$
$$=-\frac{1}{8}\ln(2)+\frac{3}{16}$$

Substitute, $\int_1^2 \ln x\cdot\frac{1}{x^3}dx=-\frac{1}{8}\ln(2)+\frac{3}{16}$ in (2).

Then the value of the $\int_1^2\frac{(\ln x)^2}{x^3}dx$ integral is calculated as follows:

$$\int_1^2\frac{(\ln x)^2}{x^3}dx=-\left(\frac{(\ln 2)^2}{8}\right)+\int_1^2\left(\frac{\ln x}{x^3}\right)dx$$
$$=-\left(\frac{(\ln 2)^2}{8}\right)-\frac{1}{8}\ln(2)+\frac{3}{16}$$

Therefore, $\int_1^2\frac{(\ln x)^2}{x^3}dx=\boxed{-\left(\frac{(\ln 2)^2}{8}\right)-\frac{1}{8}\ln(2)+\frac{3}{16}}.$

Answer 33E.

Consider the integral $\int \cos x \ln(\sin x)\,dx$. ...... (1)

Let $t = \sin x$

$dt = \cos x dx$

Substitute the values $\sin x$, $\cos x dx$ in equation (1).

$\int \cos x \ln(\sin x)\,dx = \int \ln(t)\,dt$ ...... (2)

Consider $\int \ln(t)\,dt$

$u = \ln(t)$ $\quad dv = dt$

$du = \frac{1}{t}dt$ $\quad v = t$

Use the integration by parts.

$$\int \ln(t)\,dt = \ln(t)\cdot t - \int t\cdot\frac{1}{t}dt$$
$$= t\ln(t) - t + c$$
$$= t(\ln(t) - 1) + c$$

From the equation (2).

$$\int \cos x \ln(\sin x)\,dx = t(\ln(t) - 1) + c$$
$$= \sin x(\ln(\sin x) - 1) + c$$

Therefore $\int \cos x \ln(\sin x)\,dx = \boxed{\sin x(\ln(\sin x) - 1) + c}$

Answer 34E.

We have to evaluate $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}dr = \int_0^1 \frac{r^2 r}{\sqrt{4+r^2}}dr$

Let $4 + r^2 = t$
Then $2rdr = dt$
When $r = 0, t = 4$
And when $r = 1, t = 5$

Then $$\int_0^1 \frac{r^2 r}{\sqrt{4+r^2}}dr = \int_4^5 \frac{(t-4)}{\sqrt{t}}\frac{dt}{2}$$
$$= \frac{1}{2}\int_4^5\left(\frac{t}{\sqrt{t}} - \frac{4}{\sqrt{t}}\right)dt$$
$$= \frac{1}{2}\int_4^5\left(t^{1/2} - 4t^{-1/2}\right)dt$$
$$= \frac{1}{2}\left[\frac{t^{3/2}}{3/2} - 4.\frac{t^{1/2}}{1/2}\right]_4^5$$

Therefore $$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}dr = \frac{1}{2}\left[\frac{2}{3}\left(5^{3/2} - 4^{3/2}\right) - 8\left(5^{1/2} - 4^{1/2}\right)\right]$$
$$= \frac{1}{3}\left(5^{3/2} - 8\right) - 4\left(5^{1/2} - 2\right)$$
$$= \frac{1}{3}5^{3/2} - \frac{8}{3} - 4.\left(5^{1/2}\right) + 8$$
$$= \left(\frac{5}{3} - 4\right)5^{1/2} + \frac{16}{3}$$
$$= \boxed{\frac{16}{3} - \frac{7}{3}\sqrt{5}}$$

Answer 35E.

We have to evaluate $\int_1^2 x^4 (\ln x)^2\, dx$

Let $u = (\ln x)^2$, $dv = x^4 dx$

Then $du = 2\ln x.\frac{1}{x}dx$, $v = \frac{x^5}{5}$

Using the equation $\int u\, dv = uv - \int v du$

$$\int_1^2 x^4 (\ln x)^2\, dx = \left[(\ln x)^2 . \frac{x^5}{5}\right]_1^2 - \int_1^2 2\ln x . \frac{1}{x} . \frac{x^5}{5} dx$$

$$= \left[(\ln x)^2 . \frac{x^5}{5}\right]_1^2 - \frac{2}{5}\int_1^2 (\ln x) . x^4 dx$$

$$= \left[(\ln x)^2 . \frac{x^5}{5}\right]_1^2 - \frac{2}{5}\left[\left\{\ln x . \frac{x^5}{5}\right\}_1^2 - \int_1^2 \frac{1}{x} . \frac{x^5}{5} dx\right] \quad \text{[Integrating by parts]}$$

$$= \left[(\ln x)^2 . \frac{x^5}{5}\right]_1^2 - \frac{2}{25}\left[(\ln x) . x^5\right]_1^2 + \frac{2}{25}\int_1^2 x^4 dx$$

$$= \left[(\ln x)^2 . \frac{x^5}{5}\right]_1^2 - \frac{2}{25}\left[x^5 (\ln x)\right]_1^2 + \frac{2}{25} . \left[\frac{x^5}{5}\right]_1^2$$

$$= \frac{1}{5}\left[2^5 (\ln 2)^2 - 1^5 (\ln 1)^2\right] - \frac{2}{25}\left[2^5 \ln 2 - 1^5 \ln 1\right] + \frac{2}{125}\left[2^5 - 1^5\right]$$

$$= \boxed{\frac{32}{5}(\ln 2)^2 - \frac{64}{25}(\ln 2) + \frac{62}{125}}$$

Answer 36E.

The formula for definite integral by parts is given by:

$$\int_a^b u dv = uv\Big]_a^b - \int_a^b v du$$

Consider the expression:

$$\int_0^t e^s \sin(t-s) ds$$

Make the assumption

$$I = \int_0^t e^s \sin(t-s) ds \quad \text{......} (1)$$

Substitute $u = \sin(t-s)$ in (1)

$$u = \sin(t-s)$$
$$du = \cos(t-s)(-1)$$
$$dv = e^s ds$$
$$v = e^s$$

Now the integral becomes,

$$\int_0^t e^s \sin(t-s) ds = \left[\sin(t-s) e^s\right]_0^t - \int_0^t -\cos(t-s) e^s ds$$

$$I = \left\{e^t (\sin 0) - e^0 \sin t\right\} + \int_0^t \cos(t-s) e^s ds \quad \text{......} (2)$$

$$= -\sin t + \int_0^t \cos(t-s) e^s ds$$

Again apply integration by parts for $\int_0^t \cos(t-s)e^s ds$

$$u=\cos(t-s)$$
$$du=-\sin(t-s)(-1)$$
$$dv=e^s ds$$
$$v=e^s$$

Then, the integral becomes:

$$\begin{aligned}\int_0^t \cos(t-s)e^s ds &= \left[\cos(t-s)e^s\right]_0^t-\int_0^t \sin(t-s)e^s ds\\ &=\cos(0)e^t-\cos t\, e^0-\int_0^t \sin(t-s)e^s ds\\ &=e^t-\cos t-I\end{aligned}$$

Substitute this value in equation (2).

$$I=-\sin t-\left(e^t-\cos t+I\right)$$
$$I=-\sin t+e^t-\cos t-I$$
$$2I=-\cos t-\sin t+e^t$$
$$I=\frac{1}{2}\left(-\cos t-\sin t+e^t\right)$$

Therefore, $\int_0^t e^s \sin(t-s)ds=\boxed{\frac{1}{2}\left(e^t-\sin t-\cos t\right)}$.

Answer 37E.

Consider the integral $\int \cos\sqrt{x}dx$ ...... (1)

Take $w=\sqrt{x}$

$$dw=\frac{1}{2}x^{-1/2}dx$$
$$dw=\frac{1}{2\sqrt{x}}dx$$
$$2\sqrt{x}dw=dx$$
$$2wdw=dx \qquad \left(\text{Since } \sqrt{x}=w\right)$$

Substitute the values $\sqrt{x}$, dx in equation (1)

$$\int \cos\sqrt{x}dx=2\int w\cos w dw$$

Consider $2\int w\cos w dw$

$$u=w \qquad dv=\cos w dw$$
$$du=dw \qquad v=\sin w$$

Write the integration by parts formula.

$$\int udv=uv-\int vdu$$

Use the integration by parts formula and evaluate.

$$\begin{aligned}2\int w\cos w dw &= 2\left(w\sin w-\int \sin w dw\right)\\ &=2\left(w\sin w+\cos w+c_1\right)\\ &=2\left(\sqrt{x}\sin\left(\sqrt{x}\right)+\cos\left(\sqrt{x}\right)+c_1\right) \qquad \left(\text{Since } \sqrt{x}=w\right)\\ &=\sqrt{x}\sin\left(\sqrt{x}\right)+\cos\left(\sqrt{x}\right)+2c_1\\ &=\sqrt{x}\sin\left(\sqrt{x}\right)+\cos\left(\sqrt{x}\right)+C\end{aligned}$$

Therefore $\int \cos\sqrt{x}dx=\boxed{\sqrt{x}\sin\left(\sqrt{x}\right)+\cos\left(\sqrt{x}\right)+C}$

Answer 38E.

Consider the integral $\int t^3 e^{-t^2} dt$

Rewrite the above integral as $\int t^2 \cdot t \cdot e^{-t^2} dt$ ...... (1)

Take,

$$w = t^2$$
$$dw = 2tdt$$
$$\frac{dw}{2} = tdt$$

Substitute the values of t^2, tdt in equation (1) , so that,

$$\int t^2 \cdot t \cdot e^{-t^2} dt = \int we^{-w} \frac{dw}{2}$$
$$= \frac{1}{2}\int we^{-w} dw$$

Consider,

$$\frac{1}{2}\int we^{-w} dw$$

$$u = w \quad dv = e^{-w}dw$$
$$du = dw \quad v = -e^{-w}$$

Write the integration by parts formula.

$$\int udv = uv - \int vdu$$

Use the integration by parts formula and evaluate further.

$$\int t^3 e^{-t^2} dt = \frac{1}{2}\left(-we^{-w} - \int\left(-e^{-w}\right)dw\right)$$
$$= \frac{1}{2}\left(-we^{-w} + \int\left(e^{-w}\right)dw\right)$$
$$= \frac{1}{2}\left(-we^{-w} - e^{-w}\right) + c$$
$$= -\frac{1}{2}e^{-w}\left(w+1\right) + c$$
$$= -\frac{1}{2}e^{-t^2}\left(t^2+1\right) + c \qquad \left(\text{Since } w = t^2\right)$$

Therefore $\int t^3 e^{-t^2} dt = \boxed{-\frac{1}{2}e^{-t^2}\left(t^2+1\right)+c}$

Answer 39E.

Consider integral $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) d\theta$ ...... (1)

Substitute $\theta^2 = t$

$$2\theta d\theta = dt$$
$$\theta d\theta = \frac{dt}{2}$$

If $\theta = \sqrt{\frac{\pi}{2}}$, then $t = \theta^2$

$$t = \left(\sqrt{\frac{\pi}{2}}\right)^2$$
$$= \frac{\pi}{2}$$

If $\theta = \sqrt{\pi}$, then $t = \theta^2$

$$t = \left(\sqrt{\pi}\right)^2$$
$$= \pi$$

Now

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) d\theta = \int_{\frac{\pi}{2}}^{\pi} t.\cos t \frac{dt}{2}$$ Substitute the values $\theta, t, d\theta, dt$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} t.\cos t dt$$

Now integrate by parts.

Using the equation $\int u\,dv = uv - \int v\,du$

Let $u = t,\ \ dv = \cos t dt$

Then $du = dt,\ v = \sin t$

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) d\theta = \frac{1}{2}\left[\left[t.\sin t\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin t\, dt\right]$$

$$= \frac{1}{2}\left[t\sin t\right]_{\pi/2}^{\pi} - \frac{1}{2}\int_{\pi/2}^{\pi} \sin t\, dt$$

$$= \frac{1}{2}\left[t\sin t\right]_{\pi/2}^{\pi} + \frac{1}{2}\left[\cos t\right]_{\pi/2}^{\pi}$$ $\left(\text{Since } \int \sin(t) = -\cos t dt\right)$

$$= \frac{1}{2}\left[\pi\sin\pi - \frac{\pi}{2}\sin\frac{\pi}{2}\right] + \frac{1}{2}\left[\cos\pi - \cos\frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left[0 - \frac{\pi}{2}\right] + \frac{1}{2}\left[-1-0\right]$$

$$= -\frac{\pi}{4} - \frac{1}{2}$$

$$= -\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

Therefore $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right) d\theta = \boxed{-\left(\frac{\pi}{4} + \frac{1}{2}\right)}$

Answer 40E.

Consider the integral $\int_0^{\pi} e^{\cos t} \sin 2t dt$.

Let $\cos t = x$

$-\sin t dt = dx$.

When $t = 0 \Rightarrow x = \cos(0) \Rightarrow x = 1$

When $t = \pi \Rightarrow x = \cos(\pi) \Rightarrow x = -1$.

Then

$$\int_0^{\pi} e^{\cos t} \sin 2t dt = 2\int_0^{\pi} e^{\cos t} \sin t \cos t dt$$
$$= -2\int_1^{-1} x e^x dx$$

$$\int_0^{\pi} e^{\cos t} \sin 2t dt = -2\int_1^{-1} x e^x dx \quad \text{...... (1)}$$

Consider $\int_{1}^{-1} xe^x dx$.

put $u = x$ and $dv = e^x dx$.

$du = dx$ and $v = e^x$

Use integration by parts.

$$\int_a^b u dv = (uv)_a^b - \int_a^b v du$$

$$\int_1^{-1} xe^x dx = (xe^x)_1^{-1} - \int_1^{-1} e^x dx$$

$$= (-1)e^{-1} - e - (e^x)_1^{-1}$$

$$= -e^{-1} - e - e^{-1} + e^1$$

$$= -2e^{-1}$$

$$= -\frac{2}{e}$$

$$\int_1^{-1} xe^x dx = -\frac{2}{e} \text{ ...... (2)}$$

Plug in (2) in (1).

$$\int_0^{\pi} e^{\cos t} \sin 2t dt = -2\left(-\frac{2}{e}\right)$$

$$= \frac{4}{e}$$

Therefore $\int_0^{\pi} e^{\cos t} \sin 2t dt = \boxed{\frac{4}{e}}$

Answer 41E.

Consider the integral $\int x \ln(1+x) dx$. ...... (1)

Let $1 + x = t \Rightarrow x = t - 1$

$dx = dt$

Substitute above values in (1).

$$\int x \ln(1+x) dx = \int (t-1) \ln t dt$$

let $u = \ln t$ and $dv = (t-1) dt$.

$du = \frac{1}{t} dt$ and $v = \frac{t^2}{2} - t$.

Use integration by parts.

$$\int x \ln(1+x) dx = \int (t-1) \ln t dt$$

$$= \ln t\left(\frac{t^2}{2} - t\right) - \int\left(\frac{t^2}{2} - t\right)\frac{1}{t} dt \quad \left[\text{Use formula } \int u dv = uv - \int v du\right]$$

$$= \ln t\left(\frac{t^2}{2} - t\right) - \int\left(\frac{t}{2} - 1\right) dt$$

$$= \ln t\left(\frac{t^2}{2} - t\right) - \frac{t^2}{4} + t + C$$

$$\int x \ln(1+x) dx = \ln t\left(\frac{t^2}{2} - t\right) - \frac{t^2}{4} + t + C$$

Substitute $1+x=t$ in (2)

$$\int x\ln(1+x)dx = \ln(1+x)\left(\frac{(1+x)^2}{2}-(1+x)\right)-\frac{(1+x)^2}{4}+(1+x)+C$$

$$= \ln(1+x)\left(\frac{1+2x+x^2-2-2x}{2}\right)-\frac{1+2x+x^2}{4}+(1+x)+C$$

$$= \ln(1+x)\left(\frac{x^2-1}{2}\right)-\frac{1+2x+x^2-4-4x}{4}+C$$

$$= \frac{1}{2}\ln(1+x)(x^2-1)-\frac{x^2-2x-3}{4}+C$$

$$= \frac{1}{2}\ln(1+x)(x^2-1)-\frac{1}{4}x^2+\frac{1}{2}x+\frac{3}{4}+C$$

Therefore $\int x\ln(1+x)dx = \boxed{\frac{1}{2}\ln(1+x)(x^2-1)-\frac{1}{4}x^2+\frac{1}{2}x+\frac{3}{4}+C}$

Answer 42E.

Consider the integral $\int \sin(\ln x)dx$. ...... (1)

Let $\ln x = t \Rightarrow x = e^t$

$\frac{1}{x}dx = dt \quad dx = e^t dt$

Substitute above values in (1).

$$\int \sin(\ln x)dx = \int e^t \sin t dt$$

let $u = \sin t$ and $dv = e^t dt$.

$du = \cos t dt$ and $v = e^t$

Use integration by parts.

$\int e^t \sin t dt = e^t \sin t - \int e^t \cos t dt \quad \left[\text{Use formula} \int u dv = uv - \int v du\right]$ ...... (2)

Consider $\int e^t \cos t dt$

let $u = \cos t$ and $dv = e^t dt$.

$du = -\sin t dt$ and $v = e^t$

$$\int e^t \cos t dt = e^t \cos t - \int e^t(-\sin t)dt$$

$$= e^t \cos t + \int e^t \sin t dt$$

Plug in value of $\int e^t \cos t dt$ in (2)

$$\int e^t \sin t dt = e^t \sin t - \left(e^t \cos t + \int e^t \sin t dt\right)$$

$$\int e^t \sin t dt = e^t \sin t - e^t \cos t - \int e^t \sin t dt$$

$$\int e^t \sin t dt + \int e^t \sin t dt = e^t \sin t - e^t \cos t$$

$$2\int e^t \sin t dt = e^t \sin t - e^t \cos t$$

$$\int e^t \sin t dt = \frac{1}{2}e^t(\sin t - \cos t)+C$$

$$= \frac{1}{2}e^{\ln x}(\sin(\ln x)-\cos(\ln x))+C$$

$$= \frac{1}{2}x(\sin(\ln x)-\cos(\ln x))+C$$

Therefore, $\int \sin(\ln x)dx = \boxed{\frac{1}{2}x(\sin(\ln x)-\cos(\ln x))+C}$.

Answer 43E.

Given $\int x e^{-2x} dx$

We have o evaluate the given indefinite integral

Let $u = x$, $dv = e^{-2x} dx$

$du = dx$, $v = \frac{e^{-2x}}{-2}$

We know that the formula for integration by parts is

$\int u\, dv = uv - \int v\, du$

$$\therefore \int x e^{-2x} dx = x\left[\frac{e^{-2x}}{-2}\right] - \int\left[\frac{e^{-2x}}{-2}\right]dx$$

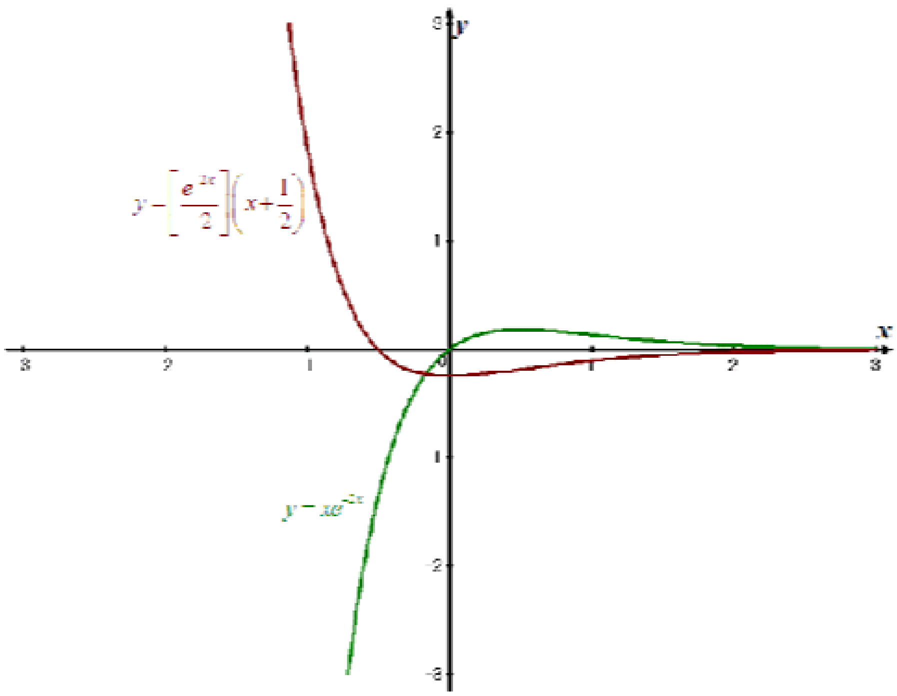
$$= x\left[\frac{e^{-2x}}{-2}\right] + \frac{1}{2}\int\left[e^{-2x}\right]dx$$

$$= x\left[\frac{e^{-2x}}{-2}\right] + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right] + C$$

$$= \left[\frac{e^{-2x}}{-2}\right]\left(x + \frac{1}{2}\right) + C$$

$$\therefore \boxed{\int x e^{-2x} dx = \left[\frac{e^{-2x}}{-2}\right]\left(x + \frac{1}{2}\right) + C}$$

By taking $C = 0$, the graph of the function and its anti derivative are shown in the below graph.



From the graph it is clear that $\boxed{\left[\frac{e^{-2x}}{-2}\right]\left(x + \frac{1}{2}\right)}$ is the anti derivative of the given integral

Answer 44E.

Consider the integral $\int x^{\frac{3}{2}} \ln(x)\,dx$

Let $u = \ln(x) \qquad dv = x^{\frac{3}{2}}dx$

$du = \frac{1}{x} \qquad v = \frac{2}{5}x^{\frac{5}{2}}$

Use integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned}\int x^{\frac{3}{2}} \ln(x)\,dx &= \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \int\left(\frac{2}{5}x^{\frac{5}{2}}\right)\frac{1}{x}dx \\ &= \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \int\left(\frac{2}{5}x^{\frac{5}{2}-1}\right)dx \\ &= \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \int\left(\frac{2}{5}x^{\frac{3}{2}}\right)dx\end{aligned}$$

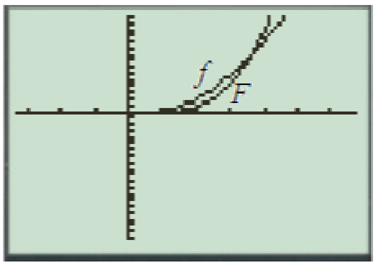
$$= \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \frac{2}{5}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)dx$$

$$= \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \frac{2}{5}\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)dx$$

$$= \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \frac{4}{25}\left(x^{\frac{5}{2}}\right)dx$$

Therefore $\int x^{\frac{3}{2}} \ln(x)\,dx = \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \frac{4}{25}\left(x^{\frac{5}{2}}\right)dx$ ...... (1)

Sketch the graphs of the $f(x) = x^{\frac{3}{2}}\ln(x)$ and $F(x) = \frac{2}{5}x^{\frac{5}{2}}\ln(x) - \frac{4}{25}\left(x^{\frac{5}{2}}\right)$



From the above graph, notice that $f(x) = 0$ when $F(x)$ has minimum.

Answer 45E.

Consider the indefinite integral $\int x^3\sqrt{1+x^2}\,dx$

Let $1+x^2 = t^2$ then $x\,dx = t\,dt$

Substitute the values in above integral,

$$\begin{aligned}\int x^3\sqrt{1+x^2}\,dx &= \int x^2\sqrt{1+x^2}\,(x\,dx) \\ &= \int\left(t^2-1\right)\sqrt{t^2}\,(t\,dt) \\ &= \int\left(t^4-t^2\right)dt \\ &= \frac{t^5}{5} - \frac{t^3}{3} + C\end{aligned}$$

Continuation of the above step,

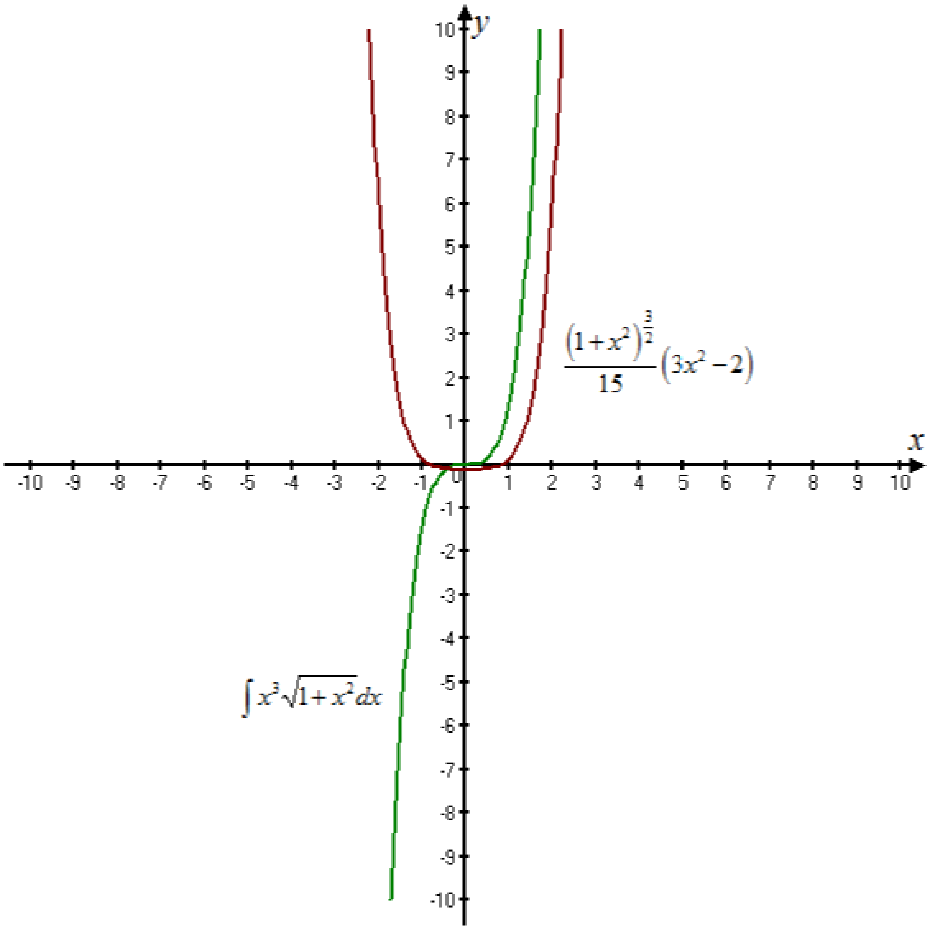
$$\int x^3\sqrt{1+x^2}\,dx = \frac{t^3}{15}\left(3t^2-5\right) \quad (\text{Take } C=0)$$

$$= \frac{\left(1+x^2\right)^{\frac{3}{2}}}{15}\left(3\left(1+x^2\right)-5\right) \qquad (\text{Back Substitution method})$$

$$= \frac{\left(1+x^2\right)^{\frac{3}{2}}}{15}\left(3x^2-2\right)$$

Hence, the value of the integral is $\boxed{\int x^3\sqrt{1+x^2}\,dx = \frac{\left(1+x^2\right)^{\frac{3}{2}}}{15}\left(3x^2-2\right)}$.

The graph of the function and its anti-derivative is as shown below.



Answer 46E.

Consider the integral is $\int x^2 \sin(2x)\,dx$

$$\int x^2 \sin(2x)\,dx = \int x^2 \sin(2x)\,dx$$

Let $u = x^2 \qquad dv = \sin(2x)$

$$du = 2x\,dx \quad v = -\frac{1}{2}\cos(2x)$$

Use integration by parts.

$$\int u\,dv = uv - \int v\,du$$

$$\int x^2 \sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x) - \int\left(-\frac{1}{2}\cos(2x)\right)2x\,dx$$

$$= -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,dx \quad \text{......} \ (1)$$

Consider $\int x\cos(2x)dx$

$t = x \qquad ds = \cos(2x)$

$dt = dx \qquad s = \frac{1}{2}\sin(2x)$

Use integration by parts.

$$\int tds = ts - \int sdt$$

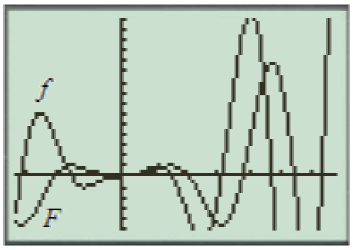
$$\int x\cos(2x)dx = \frac{x}{2}\sin(2x) - \int\frac{1}{2}\sin(2x)dx$$

$$= \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) \quad \text{...... (2)}$$

Substitute (2) in (1):

$$\int x^2\sin(2x)dx = -\frac{1}{2}x^2\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C$$

Sketch the graphs $f(x) = x^2\sin(2x)$ and $F(x) = -\frac{1}{2}x^2\cos(2x) + \frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x)$



From the above graph, notice that $f(x) = 0$ when $F(x)$ has maximum or minimum.

Answer 47E.

(A) The reduction formula is

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

Now evaluate $\int \sin^n x dx$ using the above formula

$$\int \sin^2 x dx = -\frac{1}{2}\cos x \sin^{2-1} x + \frac{2-1}{2}\int \sin^{2-2} x dx$$

$$= -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int \sin^0 x dx$$

$$= -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx$$

$$= -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

(B) By the reduction formula

$$\int \sin^4 x dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x dx$$

$$= -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{4}\sin 2x + C\right] \qquad \text{[From part (A)]}$$

$$= \boxed{-\frac{1}{4}\cos x \sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C_1}$$

Where $C_1 = \frac{3}{4}C$

Answer 48E.

(A) $\int \cos^n x dx = \int \cos^{n-1} x . \cos x dx$

Let $u = \cos^{n-1} x$, $dv = \cos x dx$

Then $du = (n-1)\cos^{n-2} x.(-\sin x)\,dx$, $v = \sin x$

Therefore

$$\int \cos^n x dx = \cos^{n-1} x \sin x - \int \sin x (n-1)\cos^{n-2} x(-\sin x)\,dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int\left(\cos^{n-2} x - \cos^n x\right)dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$
$$\int \cos^n x dx + (n-1)\int \cos^n x dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx$$
$$(1+n-1)\int \cos^n x dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx$$
$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

(B) We have to evaluate $\int \cos^2 x dx$

We evaluate the above integral by using the reduction formula

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Therefore

$$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{2-1}{2}\int \cos^0 x dx$$
$$= \frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx$$
$$= \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C_1$$
$$= \boxed{\frac{1}{4}\sin 2x + \frac{1}{2}x + C_1}$$

(C) We have to evaluate $\int \cos^4 x dx$

We evaluate the above integral by using the reduction formula

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

Therefore

$$\int \cos^4 x dx = \frac{1}{4}\cos^3 x \sin x + \frac{4-1}{4}\int \cos^2 x dx$$
$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left[\frac{1}{4}\sin 2x + \frac{1}{2}x + C_1\right] \qquad \text{[From part (B)]}$$
$$= \boxed{\frac{1}{4}\cos^3 x \sin x + \frac{3}{16}\sin 2x + \frac{3}{8}x + C}$$

Where $C = 3C_1/8$

Answer 49E.

(A) We have to evaluate $\int_0^{\pi/2} \sin^n x\,dx$

We know from reduction formula

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x\,dx$$

Therefore,

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= -\frac{1}{n}\left[\cos x \sin^{n-1} x\right]_0^{\pi/2} + \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= -\frac{1}{n}\left[\cos\frac{\pi}{2}\cdot\sin^{n-1}\left(\frac{\pi}{2}\right) - \cos 0\cdot\sin^{n-1}(0)\right] + \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= -\frac{1}{n}[0-0] + \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx \quad \text{Since } \cos\frac{\pi}{2} = 0 \text{ and } (\sin 0 = 0)$$

$$= \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx$$

Hence, $\boxed{\int_0^{\pi/2} \sin^n x\,dx = \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx}$

(B) From part (a) we have,

$$\int_0^{\pi/2} \sin^n x\,dx = \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx \qquad \text{---- (1)}$$

Putting n = 3 in (1) we get,

$$\int_0^{\pi/2} \sin^3 x\,dx = \left(\frac{3-1}{3}\right)\int_0^{\pi/2} \sin^{3-2} x\,dx$$

$$= \left(\frac{2}{3}\right)\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{2}{3}\cdot\left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{2}{3}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$= \frac{2}{3}[-0+1]$$

$$= \frac{2}{3}$$

Therefore, $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}$ ---- (2)

Also putting n = 5 in (1) we get,

$$\int_0^{\pi/2} \sin^5 x\,dx = \left(\frac{5-1}{5}\right)\int_0^{\pi/2} \sin^{5-2} x\,dx$$

$$= \frac{4}{5}\cdot\int_0^{\pi/2} \sin^3 x\,dx$$

$$= \left(\frac{4}{5}\right)\cdot\left(\frac{2}{3}\right) = \frac{8}{15} \qquad \text{using (2)}$$

Hence, $\boxed{\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}, \text{ and } \int_0^{\pi/2} \sin^5 x\,dx = \frac{8}{15}}$

(C) We have to show that for odd powers of sine

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\ldots\ldots(2n)}{3\cdot 5\cdot 7\cdot\ldots\ldots(2n+1)}$$

For n = 1 we have,

$$\int_0^{\pi/2} \sin^{2\times 1+1} x\,dx = \frac{2}{3}$$

$$\Rightarrow \int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}$$

$$\Rightarrow \frac{2}{3} = \frac{2}{3} \qquad \left[\text{since from part b } \int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\right]$$

$\Rightarrow$ The formula is true for n = 1.

Let the formula is true for n = k.
So, we have,

$$\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\ldots\ldots(2k)}{3\cdot 5\cdot 7\cdot\ldots\ldots(2k+1)} \qquad \text{---- (1)}$$

Now for n = k + 1. we have

$$\int_0^{\pi/2} \sin^{2(k+1)+1} x\,dx = \int_0^{\pi/2} \sin^{2k+3} x\,dx$$

$$= \frac{(2k+3-1)}{(2k+3)}\int_0^{\pi/2} \sin^{(2k+3)-2} x\,dx \qquad \text{using part (a)}$$

$$= \frac{(2k+2)}{(2k+3)}\int_0^{\pi/2} \sin^{2k+1} x\,dx$$

$$= \left(\frac{2k+2}{2k+3}\right)\frac{2\cdot 4\cdot 6\cdot\ldots(2k)}{3\cdot 5\cdot 7\cdot\ldots(2k+1)} \qquad \text{Using (1)}$$

$$= \frac{2\cdot 4\cdot 6\cdot\ldots(2k)\left[2(k+1)\right]}{3\cdot 5\cdot 7\cdot\ldots(2k+1)\left[2(k+1)+1\right]}$$

This implies that the formula is true for n = k + 1.
Hence, the given formula is true for all $n \geq 1$.

Answer 50E.

We have to prove for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{1\cdot 3\cdot 5\cdot\ldots\ldots(2n-1)}{2\cdot 4\cdot 6\cdot\ldots(2n)}\cdot\frac{\pi}{2}$$

We will prove the above formula by induction.

For n = 1, we have,

$$LHS = \int_0^{\pi/2} \sin^{2\times 1} x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}(1-\cos 2x)\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2}(1)\,dx - \frac{1}{2}\int_0^{\pi/2}(\cos 2x)\,dx$$

$$= \frac{1}{2}(x)_0^{\frac{\pi}{2}} - \frac{1}{2}\left(\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} \qquad \left[\because \int \cos 2x\,dx = \frac{\sin 2x}{2}\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}(\sin\pi - \sin 0)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) \qquad [\because \sin\pi = 0 = \sin 0]$$

$$= \text{RHS}$$

Let us assume that the given formula is true for n = k.
So we have,

$$\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot\ldots(2k-1)}{2\cdot 4\cdot 6\cdot\ldots(2k)}\left(\frac{\pi}{2}\right) \qquad \text{---- (1)}$$

Now for n = k + 1, we have,

$$\int_0^{\pi/2} \sin^{2(k+1)} x\,dx = \int_0^{\pi/2} \sin^{2k+2} x\,dx$$

$$= \left(\frac{2k+2-1}{2k+2}\right)\int_0^{\pi/2} \sin^{2k+2-2} x\,dx \qquad \left[\because \int_0^{\pi/2} \sin^n x\,dx = \left(\frac{n-1}{n}\right)\int_0^{\pi/2} \sin^{n-2} x\,dx\right]$$

$$= \left(\frac{2k+1}{2k+2}\right)\int_0^{\pi/2} \sin^{2k} x\,dx$$

$$=\frac{(2k+1)}{(2k+2)}\frac{1\cdot 3\cdot 5\cdot\ldots\ldots(2k-1)}{2\cdot 4\cdot 6\cdot\ldots\ldots(2k)}\left(\frac{\pi}{2}\right) \quad \text{[By Using (1)]}$$

$$=\frac{1\cdot 3\cdot 5\cdot\ldots\ldots(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\ldots\ldots(2k)(2k+2)}\left(\frac{\pi}{2}\right)$$

$$=\frac{1\cdot 3\cdot 5\cdot\ldots\ldots(2k-1)(2(k+1)-1)}{2\cdot 4\cdot 6\cdot\ldots\ldots(2k)(2(k+1))}\left(\frac{\pi}{2}\right)$$

This implies that the given formula is true for $n = k + 1$.

Therefore, the given formula is true for all $n \geq 1$

Answer 51E.

We have to evaluate $\int (\ln x)^n \, dx$

Let $u = (\ln x)^n \quad dv = dx$

Then $du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx \quad v = x$

Therefore

$$\int (\ln x)^n \, dx = (\ln x)^n \cdot x - \int n(\ln x)^{n-1} \cdot \frac{1}{x} \cdot x dx$$

$$= (\ln x)^n x - n\int (\ln x)^{n-1} \, dx$$

$$= \boxed{x(\ln x)^n - n\int (\ln x)^{n-1} \, dx}$$

Answer 52E.

We have to evaluate $\int x^n e^x dx$

Let $u = x^n \quad dv = e^x dx$

Then $du = nx^{n-1}dx \quad v = e^x$

Therefore

$$\int x^n e^x dx = x^n e^x - \int nx^{n-1} e^x dx \qquad \text{[Integration by part]}$$

$$= \boxed{x^n e^x - n\int x^{n-1} e^x dx}$$

Answer 53E.

Consider $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$ ...... (1)

Prove that the equation (1), by using integration by parts.

Rewrite the equation (1) as:

$$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$\int \tan^n x dx = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$ ...... (2)

Use integration by parts.

$\int u dv = uv - \int v du$ ...... (3)

Consider $\int \tan^{n-2} x \sec^2 x dx$

Let $u = \tan^{n-2} x \quad dv = \sec^2 x dx$

$du = (n-2)\tan^{n-3} x \sec^2 x dx \quad v = \tan x$

Substitute u, v, du, dv in (3).

$$\int \tan^{n-2} x \sec^2 x dx = \tan^{n-2} x(\tan x) - \int (n-2)\tan^{n-3} x \sec^2 x \tan x dx$$
$$= \tan^{n-1} x - \int (n-2)\tan^{n-2} x \sec^2 xdx$$
$$= \tan^{n-1} x - (n-2)\int \tan^{n-2} x \sec^2 xdx$$

$$\tan^{n-1} x = \int \tan^{n-2} x \sec^2 xdx + (n-2)\int \tan^{n-2} x \sec^2 xdx$$
$$\tan^{n-1} x = (n-2+1)\int \tan^{n-2} x \sec^2 xdx$$
$$\tan^{n-1} x = (n-1)\int \tan^{n-2} x \sec^2 xdx$$

$$\frac{\tan^{n-1} x}{n-1} = \int \tan^{n-2} x \sec^2 xdx \quad \text{...... (4)}$$

Substitute (4) in (2).

$$\int \tan^n xdx = \int \tan^{n-2} x \sec^2 xdx - \int \tan^{n-2} xdx$$
$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} xdx$$

Therefore $\int \tan^n xdx = \boxed{\frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} xdx}$

Answer 54E.

We have $\int \sec^n xdx = \int \sec^{n-2} x \sec^2 xdx$

Let $\quad u = \sec^{n-2} x \quad dv = \sec^2 xdx$

Then $\quad du = (n-2)\sec^{n-3} x \sec x \tan xdx \quad v = \tan x$

Or $\quad du = (n-2)\sec^{n-2} x \tan xdx\ , \ v = \tan x$

Therefore

$$\int \sec^n xdx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x.\tan^2 x.dx \qquad \text{[Integration by part]}$$
$$= \sec^{n-2} x.\tan x - (n-2)\int \sec^{n-2} x.(\sec^2 x - 1)dx$$
$$= \sec^{n-2} x.\tan x - (n-2)\int (\sec^n x - \sec^{n-2} x)dx$$
$$= \sec^{n-2} x.\tan x - (n-2)\int \sec^n xdx + (n-2)\int \sec^{n-2} xdx$$
$$(1+n-2)\int \sec^n xdx = \sec^{n-2} x.\tan x + (n-2)\int \sec^{n-2} xdx$$
$$\int \sec^n xdx = \frac{\sec^{n-2} x.\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} xdx \qquad \text{Where } n \neq 1$$

Answer 55E.

We have to evaluate $\int (\ln x)^3 dx$

Use the reduction formula

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

Thus

$$\int (\ln x)^3 dx = x(\ln x)^3 - 3\int (\ln x)^2 dx$$
$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int (\ln x)^1 dx\right] \quad \text{[Again by reduction formula]}$$
$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int (\ln x)dx\right]$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\int \ln xdx$$
$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x\ln x - \int (\ln x)^0 dx\right]$$

[Again by reduction formula]

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6\int dx$$
$$= \boxed{x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x + C}$$

Answer 56E.

We have to evaluate $\int x^4 e^x dx$

Use the reduction formula

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$$

Thus

$$\int x^4 e^x dx = x^4 e^x - 4\int x^3 e^x dx$$

$$= x^4 e^x - 4\left[x^3 e^x - 3\int x^2 e^x dx\right] \qquad \text{[Again by reduction formula]}$$

$$= x^4 e^x - 4x^3 e^x + 12\int x^2 e^x dx$$

Again by reduction formula

$$\int x^4 e^x dx = x^4 e^x - 4x^3 e^x + 12\left[x^2 e^x - 2\int x e^x dx\right]$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\int x e^x dx$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\left[x e^x - \int e^x dx\right]$$

Again by reduction formula

$$\int x^4 e^x dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24\int e^x dx$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C$$

$$= \boxed{e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C}$$

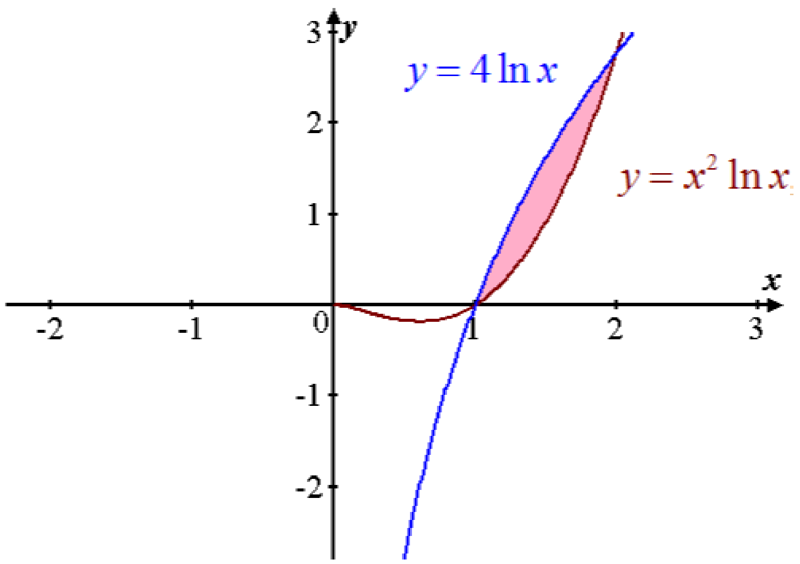
Answer 57E.

Consider the curves,

$y = x^2 \ln x,\ y = 4\ln x$

The objective to find the area of the region bounded by the curves $y = x^2 \ln x,\ y = 4\ln x$

The Graph of the region is shown below:



The formula for the area between the curves $f(x)$ and $g(x)$ for $a \le x \le b$ is

$$A = \int_a^b \left[f(x) - g(x)\right]dx$$

To find the intersection point, equate both curves as follows:

$$x^2 \ln x = 4\ln x$$

$$\Rightarrow \ln x\left(x^2 - 4\right) = 0$$

$$\Rightarrow \ln x = 0, x^2 - 4 = 0$$

$$\Rightarrow x = 1, x = 2$$

Thus, the region of the integration is $\{x \mid 1 \le x \le 2\}$.

Therefore area of the region bounded by the given curves is:

$$A=\int_a^b\left(f(x)-g(x)\right)dx$$

$$=\int_1^2\left[(4\ln x)-(x^2\ln x)\right]dx \qquad \text{......(1)}$$

$$\begin{cases}\text{Apply integration by parts}\\ \int u\,dv=uv-\int v\,du\end{cases}$$

First apply integration by parts for $\int_1^2 4\ln x\,dx$:

Let $u=\ln x, du=\frac{1}{x}dx,$

$dv=1, v=x$

Then, the integration becomes,

$$\int_1^2 4\ln x\,dx=4\left[\left[x\ln x\right]_1^2-\int_1^2\frac{1}{x}(x)\,dx\right]$$

$$=4\left(\left[x\ln x\right]_1^2-\left[x\right]_1^2\right)$$

$$=4\left[(2\ln 2)-\ln 1-(2-1)\right]$$

$$=8\ln 2-4$$

Next apply integration by parts for $\int_1^2 x^2\ln x\,dx$

Let $u=\ln x, du=\frac{1}{x}dx,$

$dv=x^2, v=\frac{x^3}{3}$

Then, the integration becomes,

$$\int_1^2 x^2\ln x\,dx=\left[\ln x\frac{x^3}{3}\right]_1^2-\int_1^2\frac{1}{x}\left(\frac{x^3}{3}\right)dx$$

$$=\left[\ln x\frac{x^3}{3}\right]_1^2-\left[\frac{x^3}{9}\right]_1^2$$

$$=\left[\frac{x^3}{3}\ln x-\frac{x^3}{9}\right]_1^2$$

$$=\left[\frac{8}{3}\ln 2-\frac{8}{9}+\frac{1}{9}\right]$$

$$=\frac{8}{3}\ln 2-\frac{7}{9}$$

Answer 58E.

Given curves are

$$y=x^2e^{-x}, y=xe^{-x}$$

We have to find the area of the region bounded by the given curves

To find the points of intersection, we have to solve

$$x^2e^{-x}=xe^{-x}$$

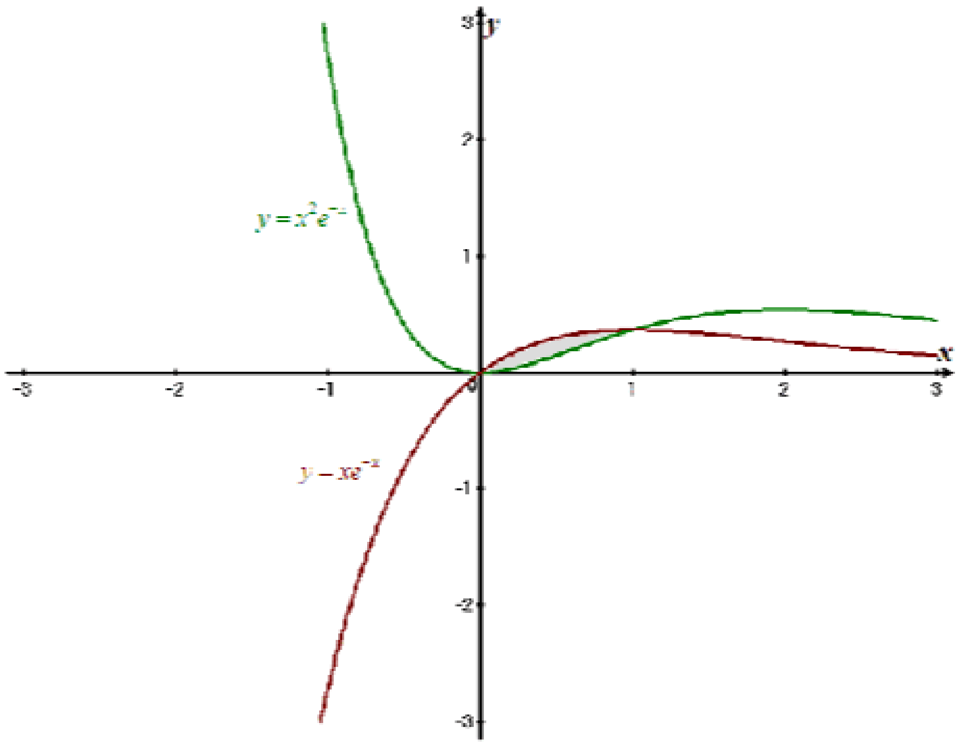
$$\Rightarrow \quad (x^2-x)e^{-x}=0$$

$$\Rightarrow \quad x^2-x=0$$

$$\Rightarrow \quad x=0,1$$

Therefore the two curves are intersect at $x=0,1$

The region bounded by the given curves is as shown below



We know that the area between the curves $f(x)$ and $g(x)$ for $a \le x \le b$ is

$$A=\int_a^b [f(x)-g(x)]dx$$

Therefore area of the region bounded by the given curves is

$$\int_0^1 xe^{-x}-x^2e^{-x}=\int_0^1 (x-x^2)e^{-x}$$

$$=\left[(x-x^2)(-e^{-x})-(1-2x)(e^{-x})+(-2)(-e^{-x})\right]_0^1$$

$$\left(\begin{array}{l}\text{By applying the formula}\\ \int_a^b f(x)g'(x)dx=f(x)g(x)\Big|_a^b-\int_a^b g(x)f'(x)dx\\ \text{Taking } f(x)=x-x^2\ ,\ g'(x)=e^{-x}\ dx\\ f'(x)=1-2x\ dx\ ,\ g(x)=-e^{-x}\end{array}\right)$$

$$=e^{-1}+2e^{-1}-[-1+2]$$

$$=3e^{-1}-1$$

Hence the area of the region bounded by the given curves is $\boxed{A=3e^{-1}-1}$

Answer 59E.

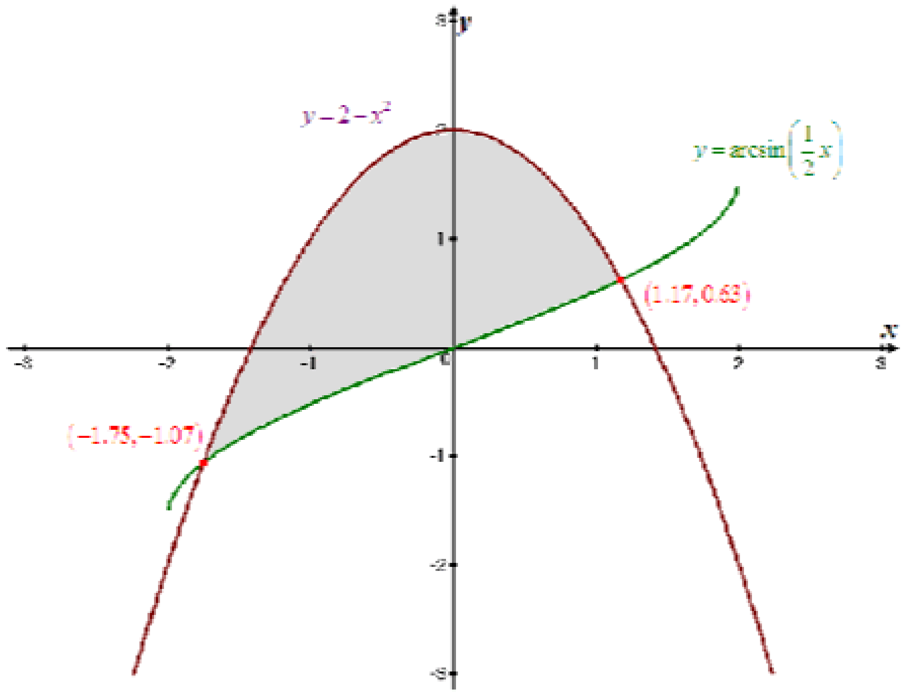
Given curves are

$$y=\arcsin\left(\frac{1}{2}x\right), y=2-x^2$$

We have to find the area of the region bounded by the given curves

Now we have to use the graph of the region to find the approximate x-coordinates of the points of intersection of the given curves

The graph of the region is as shown below



From the graph intersection points are $(-1.75,-1.07),(1.17,0.63)$

We know that the area between the curves $f(x)$ and $g(x)$ for $a \le x \le b$ is

$$A=\int_a^b[f(x)-g(x)]dx$$

The area of the region bounded the curves is

$$\int_{-1.75}^{1.17}\left(2-x^2-\sin^{-1}(0.5x)\right)dx$$

$$=\int_{-1.75}^{1.17}\left(2-x^2\right)dx-\int_{-1.75}^{1.17}\sin^{-1}(0.5x)dx$$

$$=\left[2x-\frac{x^3}{3}\right]_{-1.75}^{1.17}-\int_{-1.75}^{1.17}\sin^{-1}(0.5x)dx$$

$$=3.5196707+0.4795823$$

$$\approx 3.999253$$

Hence the area of the region bounded by the given curves is $\boxed{A\approx 3.999253}$

Answer 60E.

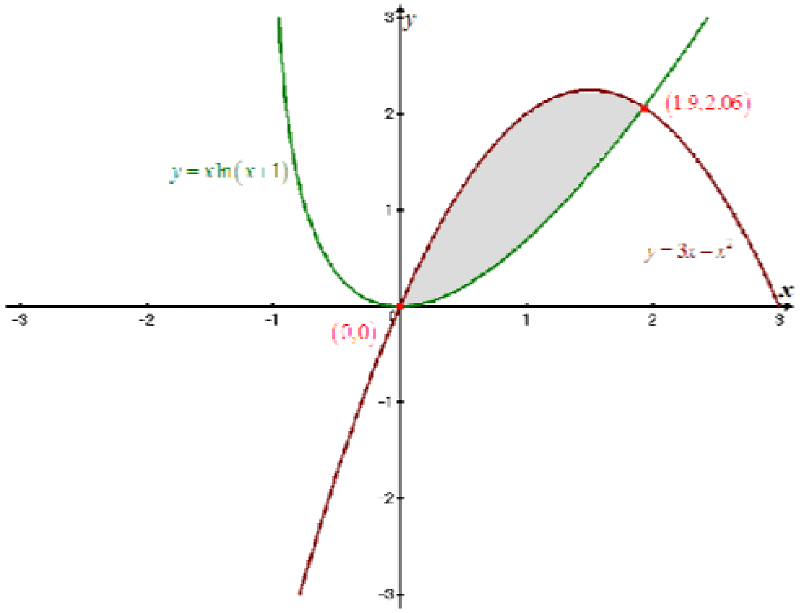
Given curves are

$$y=x\ln(x+1), y=3x-x^2$$

We have to find the area of the region bounded by the given curves

Now we have to use the graph of the region to find the approximate x-coordinates of the points of intersection of the given curves

The graph of the region is as shown below



From the graph, the points of intersection are $(0,0)$ and $(1.9, 2.06)$

We know that the area between the curves $f(x)$ and $g(x)$ for $a \le x \le b$ is

$$A = \int_a^b [f(x) - g(x)]dx$$

Therefore the area of the region bounded by the given curves is

$$A = \int_0^{1.9} \left[3x - x^2 - x\ln(x+1)\right]dx$$

$$= \left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1.9} - \int_0^{1.9} x\ln(x+1)\,dx$$

$$= \left(\frac{3(1.9)^2}{2} - \frac{(1.9)^3}{3} - 0\right) - \left[\left[\ln(x+1)\frac{x^2}{2}\right]_0^{1.9} - \int_0^{1.9} \frac{x^2}{(x+1)^2}dx\right]$$

$$\left(\begin{array}{l}\text{By applying the formula}\\ \int_a^b f(x)g'(x)\,dx = f(x)g(x)\big|_a^b - \int_a^b g(x)f'(x)\,dx\\ \text{Taking } f(x) = \ln(x+1)\ ,\ g'(x) = x\,dx\\ f'(x) = \frac{1}{x+1}\ dx\ ,\ g(x) = \frac{x^2}{2}\end{array}\right)$$

$$= 3.1287 - \left[\frac{(1.9)^2}{2}\log(2.9) - 0\right] + \int_0^{1.9} \frac{x^2}{2(x+1)}dx$$

$$= 2.1667 - 1.9218 + \int_0^{1.9} \frac{x^2}{2(x+1)}dx$$

$$= 1.2069 + \frac{1}{2}\left[\frac{1}{2}(x+1)^2 - 2(x+1) + \ln(x+1)\right]_0^{1.9}$$

$$= 1.2069 + \frac{1}{4}(2.9)^2 - (1.9) + \frac{1}{2}\ln(2.9)$$

$$= 1.2069 + 0.7348$$

$$= 1.9417$$

Hence the area of the region bounded by the given curves is $\boxed{A \approx 1.9417}$

Answer 61E.

Consider the curves about the specified axis:

$$y=\cos\left(\frac{\pi x}{2}\right), y=0, 0\le x\le 1$$

Use the Volume formula for cylindrical shells.

$$V=\int_a^b 2\pi x f(x)dx$$

Substitute the values.

$$V=\int_0^1 2\pi x\cos\left(\frac{\pi x}{2}\right)dx$$
$$=2\pi\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx$$

Apply Integral substitution.

Let,

$$u=\frac{\pi x}{2}$$
$$du=\frac{\pi}{2}dx$$

Therefore,

$$=2\pi\int_0^1 \frac{2\frac{2u}{\pi}\cos(u)}{\pi}du$$
$$=2\pi\times\frac{2}{\pi}\times\frac{2}{\pi}\int_0^1 u\cos(u)du$$

Apply integration by parts.

$$2\pi\times\frac{2}{\pi}\times\frac{2}{\pi}\int_0^1 u\cos(u)du=2\pi\times\frac{2}{\pi}\times\frac{2}{\pi}\left(u\sin(u)-\int_0^1\sin(u)du\right)$$
$$=2\pi\times\frac{2}{\pi}\times\frac{2}{\pi}(u\sin(u)+\cos(u))$$
$$=\frac{8}{\pi}(u\sin(u)+\cos(u))$$

Substitute $u=\frac{\pi x}{2}$.

$$\frac{8}{\pi}(u\sin(u)+\cos(u))=\frac{8}{\pi}\left(\frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)+\cos\left(\frac{\pi x}{2}\right)\right)+C$$

Now, apply the limit.

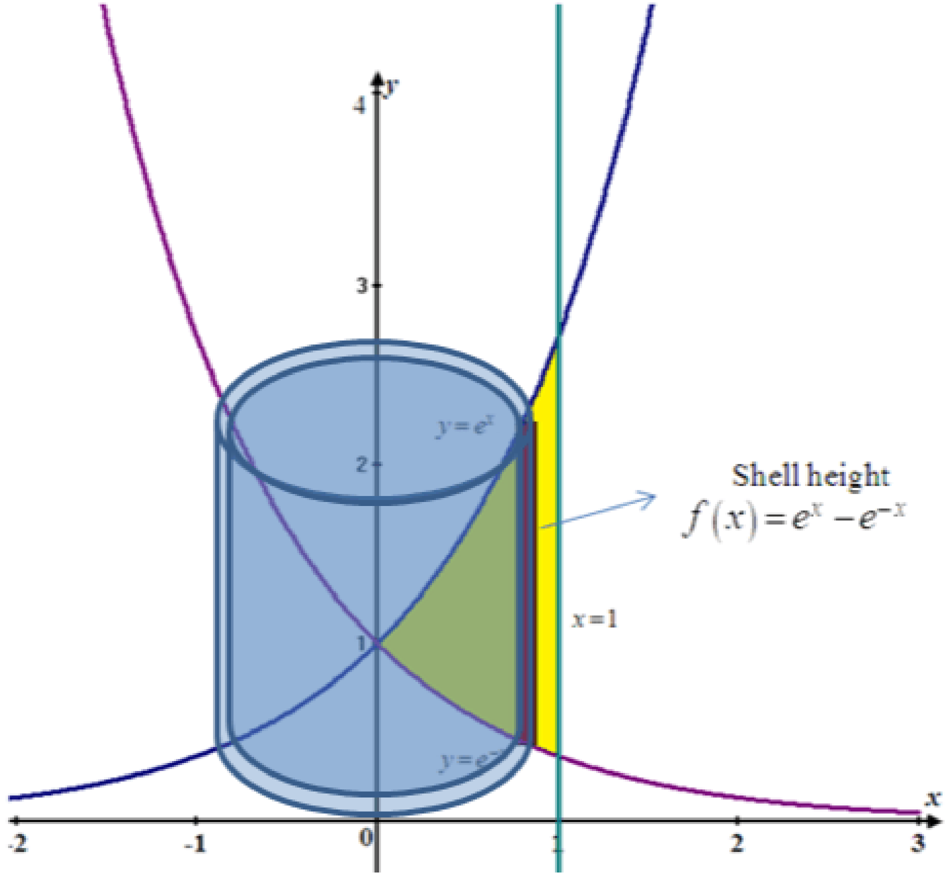
$$\left[\frac{8}{\pi}\left(\frac{\pi x}{2}\sin\left(\frac{\pi x}{2}\right)+\cos\left(\frac{\pi x}{2}\right)\right)\right]_0^1=\frac{8}{\pi}\left(\begin{array}{l}\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)\\ -\left(\frac{\pi\times 0}{2}\sin\left(\frac{\pi\times 0}{2}\right)+\cos\left(\frac{\pi\times 0}{2}\right)\right)\end{array}\right)$$
$$=\frac{8}{\pi}\left(\frac{\pi}{2}+0-(0+1)\right)$$
$$=\frac{8}{\pi}\left(\frac{\pi}{2}-1\right)$$
$$=\frac{8}{\pi}\times\frac{\pi}{2}-\frac{8}{\pi}$$
$$=4-\frac{8}{\pi}$$

Hence, the volume of the cylindrical shells is $\boxed{4-\frac{8}{\pi}}$.

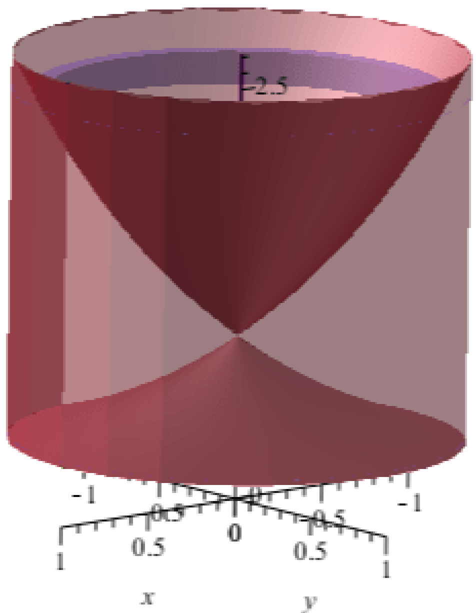
Answer 62E.

Consider the curves $y=e^{x}, y=e^{-x}$, and $x=1$

Sketch the region bounded by the above curves.



Rotate the above region about the $x-\text{axis}$.



Radius of shell has radius x, circumference $2\pi x$ and height $f(x)=e^{x}-e^{-x}$.

So, by the shell method, the volume generated by rotating the region bounded by the given curves is

$$V=\int_0^1 2\pi x\left(e^x-e^{-x}\right)dx$$

$$=2\pi\int_0^1\left(xe^x-xe^{-x}\right)dx$$

$$=2\pi\left[\int_0^1 xe^x dx-\int_0^1 xe^{-x}dx\right]$$

$$=2\pi\left[\left[xe^x\right]_0^1-\int_0^1 e^x dx-\left\{\left[-xe^{-x}\right]_0^1-\int_0^1 -e^{-x}dx\right\}\right]$$

$$=2\pi\left[e-\left[e^x\right]_0^1-\left\{-\frac{1}{e}-\left[e^{-x}\right]_0^1\right\}\right]$$

$$=2\pi\left[e-(e-1)-\left\{-\frac{1}{e}-\left(\frac{1}{e}-1\right)\right\}\right]$$

$$=2\pi\left[e-e+1+\frac{1}{e}+\left(\frac{1}{e}-1\right)\right]$$

$$=2\pi\left[1+\frac{1}{e}+\frac{1}{e}-1\right]$$

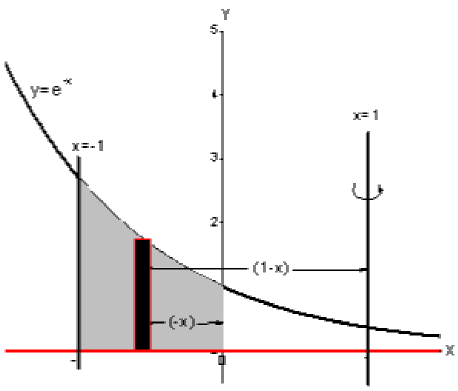
$$=2\pi\left[\frac{2}{e}\right]$$

$$=\frac{4\pi}{e}$$

Therefore volume of the region bonded by the curves $y=e^x, y=e^{-x}$, and $x=1$ is $\boxed{\frac{4\pi}{e}}$

Answer 63E.

First we sketch the region bounded by the curves $y=e^{-x}, y=0, x=-1, x=0$



We have to find the volume of the solid obtained by rotating the shaded region about the line x = -1.

If we consider a vertical strip at the distance (-x) from the origin, in this region then after a complete rotation about $x=1$, we get a cylindrical shell.

The radius of the shell = 1- x

And the height of the shell = e^{-x}

Then the volume of the solid is

$$V=2\pi\int_{-1}^{0}(1-x)e^{-x}dx$$

$$=2\pi\left[-(1-x)e^{-x}\right]_{-1}^{0}+2\pi\int_{-1}^{0}(-1)e^{-x}dx \quad \text{[Integration by parts]}$$

$$=2\pi\left[-(1-x)e^{-x}\right]_{-1}^{0}+2\pi\left[e^{-x}\right]_{-1}^{0}$$

$$=2\pi\left[-(1-0)e^0+(1+1)e^1\right]+2\pi\left[e^0-e^1\right]$$

$$=2\pi\left[-1+2e\right]+2\pi\left[1-e\right]$$

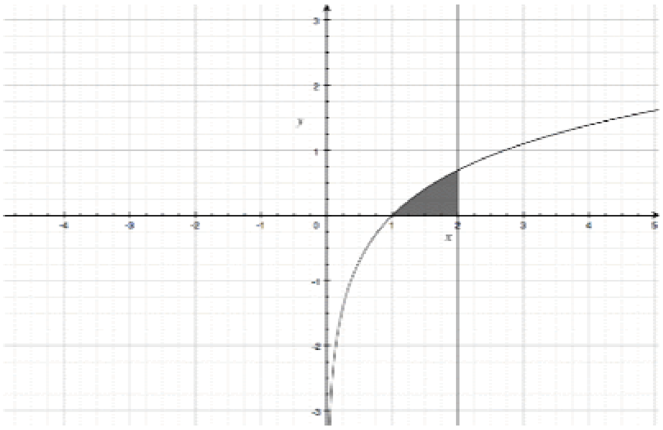
$$=-2\pi+4\pi e+2\pi-2\pi e$$

Or $\boxed{V=2\pi e}$

Answer 64E.

a)
We want to find the volume generated by rotating a region around the y-axis. This region is bounded by the curves $y = \ln x$, $y = 0$ and $x = 2$.

It often helps to first sketch the region we are dealing with.



Since we want to rotate around the y-axis, we can use shell method.

$$\text{Volume} = 2\pi\int x\, h(x)\,dx$$

The height of our shells ($h(x)$) will be given by $y = \ln x$ and their width by dx.

Thus, we can rewrite this integral as

$$\text{Volume} = 2\pi\int_1^2 x\ln(x)\,dx$$

To evaluate this integral however, we need integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Let us take:

$$f(x) = \ln(x)$$
$$f'(x) = 1/x$$

And

$$g(x) = \frac{1}{2}x^2$$
$$g'(x) = x$$

Consequently we can rewrite the integration by parts formula plugging in our expressions:

$$2\pi\int_1^2 x\ln(x)\,dx = 2\pi\left[\left((\log x)\left(\frac{1}{2}x^2\right)\right)\Big|_1^2 - \int_1^2 \frac{1}{2}x^2\left(\frac{1}{x}\right)dx\right]$$

We can then simplify:

$$= 2\pi\left[(\ln 2)\left(\frac{1}{2}(2^2)\right) - \ln(1)\left(\frac{1}{2}(1^2)\right) - \int_1^2 \frac{x}{2}\,dx\right]$$

$$= 2\pi\left[(2\ln 2) - \int_1^2 \frac{x}{2}\,dx\right]$$

And finally, evaluate the last integral and the expression:

$$= 2\pi\left[2\ln 2 - \left(\frac{x^2}{4}\right)\right]_1^2$$

$$= 2\pi\left[2\ln 2 - \left(1 - \frac{1}{4}\right)\right]$$

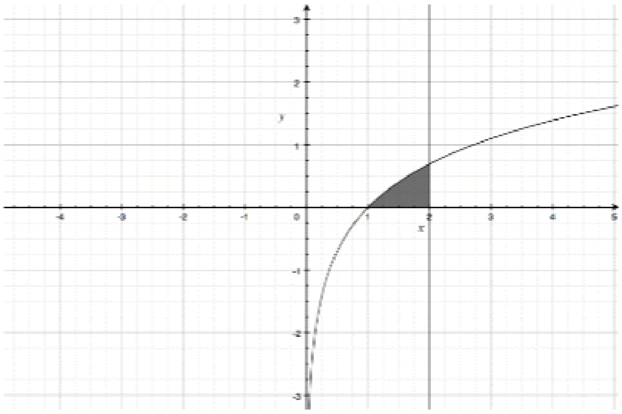
$$= 2\pi\left[2\ln 2 - \frac{3}{4}\right]$$

$$= \boxed{3.998}$$

b)

We want to find the volume generated by rotating a region around the x-axis. This region is bounded by: $y = \ln x$, $y = 0$ and $x = 2$.

It often helps to first sketch the region we are dealing with.



Since we want to rotate around the x-axis, we should use the disk method. Which, for x-axis rotation, is given by:

$\pi\int r(x)^2 dx$, where *r(x)* represents the radius of each infitesimal disk. For our purpose, $r(x) = \ln(x)$.

We can now set up our integral, with proper bounds 1 and 2:

$$\text{Volume} = \pi\int_1^2 (\ln x)^2 dx$$

To evaluate this integral however, we need integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Let us take:

$$f(x) = (\ln x)^2$$

$$f'(x) = 2\ln x\left(\frac{1}{x}\right)$$

And

$$g(x) = x$$

$$g'(x) = 1$$

Plugging into our integration by parts equation, we get:

$$\pi\int_1^2 (\ln x)^2 dx = \pi\left[(\ln x)^2 (x)_1^2 - \int_1^2 2x(\ln x)\left(\frac{1}{x}\right)dx\right]$$

$$= \pi\left[(\ln 2)^2 - \int_1^2 2(\ln x)dx\right]$$

Simplifying, we get:

At this stage, we will have to reapply integration by parts to solve $\int_1^2 2\ln(x)dx$.

Let us take:

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x}$$

And

$$g(x) = x$$

$$g'(x) = 1$$

Plugging back into the integration by parts equation, we get:

$$\int_1^2 2\ln(x)dx = 2(x\ln x)_1^2 - 2\int_1^2 x\left(\frac{1}{x}\right)dx$$

$$= 2\ln 2 - 2(x)_1^2$$

$$= 2\ln 2 - 2$$

Simplifying and plugging the integral back we get:
Volume

$= \pi\left[(\ln 2)^2 - 2\ln 2 + 2\right]$

$= 3.437401$

Answer 65E.

We have to calculate the average value of $f(x) = x\sec^2 x$ on the interval $\left[0, \frac{\pi}{4}\right]$

$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)dx$$

Thus, for our function:

$$\text{Average Value} = \frac{1}{\pi/4}\int_0^{\pi/4} x\sec^2 x\,dx$$

For this integral, we need to apply integration by parts, given by:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

We can set:

$g(x) = \tan x$

$g'(x) = \sec^2 x$

and

$f(x) = x$

$f'(x) = 1$

Plugging into our integration by parts formula:

$$\frac{1}{\pi/4}\int_0^{\pi/4} x\sec^2 x\,dx = \frac{4}{\pi}\left(x\tan(x)\Big|_0^{\pi/4} - \int_0^{\pi/4}\tan(x)dx\right)$$

$$= \frac{4}{\pi}\left(\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) - \ln\left|\sec\left(\frac{\pi}{4}\right)\right| + \ln\left|\sec(0)\right|\right)$$

$$= \frac{4}{\pi}\left(\frac{\pi}{4} - \frac{\ln(2)}{2}\right)$$

$$= 1 - \frac{2}{\pi}\ln(2)$$

Answer 66E.

Given that a rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket).

Also given that a model for the velocity of the rocket at time t is given by the equation $v(t) = -gt - v_e\ln\left(\frac{m-rt}{m}\right)$, where g is the acceleration due to gravity and t is not too large. If $g = 9.8\text{m/s}^2, m = 30{,}000\text{kg}, r = 160\text{kg/s}$, and $v_e = 3000\text{m/s}$, we have to find the height of the rocket on minute after liftoff.

The derivative of position is velocity, thus, given the velocity function we can integrate to find our position function.

$$y(t) = \int\left[-gt - v_e\ln\left(\frac{m-rt}{m}\right)\right]dt$$

$$= \int -gt\,dt - v_e\int\ln\left(\frac{m-rt}{m}\right)dt$$

$$= \int -gt\,dt + \frac{v_e}{r}\int\ln\left(\frac{u}{m}\right)du$$

(Take: $u = m - rt$ and $du = -r\,dt$)

To solve the second integral, we need to use integration by parts, given by:

$$\int f(u)g'(u)du = f(u)g(u) - \int f'(u)g(u)du$$

Taking:

$f(u)=\ln\left(\frac{u}{m}\right)$

$f'(u)=\frac{1}{u}$

and

$g(u)=u$

$g'(u)=1$

Plugging in:

$$\int \ln\left(\frac{u}{m}\right)du = u\ln\frac{u}{m}-\int dt$$
$$= u\ln\frac{u}{m}-u+C$$
$$= (m-rt)\ln\left(\frac{m-rt}{m}\right)-(m-rt)+C$$

Plugging back into our original formula, and evaluating the first integral:

$$y(t)=\int -gt\,dt+\frac{v_e}{r}\left[(m-rt)\ln\left(\frac{m-rt}{m}\right)-(m-rt)\right]+C$$
$$=-\frac{gt^2}{2}+\left(\frac{v_e m}{r}-v_e t\right)\ln\left(\frac{m-rt}{m}\right)-\left(\frac{v_e m}{r}-v_e t\right)+C$$

To figure out C, we can take the position at the beginning (height 0):

$$0=-\frac{g(0)^2}{2}+\left(\frac{v_e m}{r}-v_e(0)\right)\ln\left(\frac{m-r(0)}{m}\right)-\left(\frac{v_e m}{r}-v_e(0)\right)+C$$
$$=-\left(\frac{v_e m}{r}\right)+C$$
$$\frac{v_e m}{r}=C$$

Now we have a function for the height:

$$y(t)=-\frac{gt^2}{2}+\left(\frac{v_e m}{r}-v_e t\right)\ln\left(\frac{m-rt}{m}\right)-\left(\frac{v_e m}{r}-v_e t\right)+\frac{v_e m}{r}$$
$$=-\frac{gt^2}{2}+\left(\frac{v_e m}{r}-v_e t\right)\ln\left(\frac{m-rt}{m}\right)+(v_e t)$$

Plugging in our value, we can find $y(60\text{ s})$:

$$y(t)=-\frac{(9.8\text{ m/s}^2)(60\text{ s})^2}{2}+$$
$$\left(\frac{(3000\text{ m/s})(30000\text{ kg})}{(160\text{ kg/s})}-(3000\text{ m/s})(60\text{ s})\right)\ln\left(\frac{(30000\text{ kg})-(160\text{ kg/s})(60\text{ s})}{(30000\text{ kg})}\right)$$
$$+((3000\text{ m/s})(60\text{ s}))$$
$$=(-17640+(562500-180000)(-.386)+180000)\text{ meters}$$
$$=\boxed{14844.2\text{ m}\approx 10000\text{ m}}\text{, only one significant figure}$$

Answer 67E.

A particle that moves along a straight line has velocity $v(t)=t^2e^{-t}$ meters per second after t seconds.

Find the distance of the particle during the first t seconds.

The desired distance of the particle during the first t seconds is $s(t)=\int_0^t w^2e^{-w}\,dw$ since $v(t)>0$ for all t i.e. to avoid confusions, we have taken the velcoty in the form of $v(w)=w^2e^{-w}$.

The formula for integral by parts is $\int u\,dv=uv-\int v\,du$ where $u=f(x)$ and $v=g(x)$ then the differentials are $du=f'(x)dx$ and $dv=g'(x)dx$.

Compare the integral by parts with integral $s(t)=\int_0^t w^2e^{-w}\,dw$,

Let $u=w^2$ then $du=2w\,dw$ and $dv=e^{-w}dw$ then $v=-e^{-w}$

The desired distance of the particle during the first t seconds is,

$$s(t)=\int_0^t w^2e^{-w}\,dw$$

$$=-w^2e^{-w}\Big]_0^t-\int_0^t -2we^{-w}\,dw \qquad \left[\text{Since}\int_a^b u\,dv=uv\Big]_a^b-\int_a^b v\,du\right]$$

$$=-w^2e^{-w}\Big]_0^t+2\int_0^t we^{-w}\,dw \qquad ......(1)$$

Solve the integral $\int_0^t we^{-w}\,dw$ separately and then substitute it in equation (1).

Apply the integral by parts formula i.e. Let $u=w$ then $du=dw$ and $dv=e^{-w}dw$ then $v=-e^{-w}$.

Substitute these into the above integral,

$$\int_0^t we^{-w}\,dw=-we^{-w}\Big]_0^t-\int_0^t -e^{-w}\,dw \qquad \left[\text{Since}\int_a^b u\,dv=uv\Big]_a^b-\int_a^b v\,du\right]$$

$$=-we^{-w}\Big]_0^t+\int_0^t e^{-w}\,dw$$

$$=-we^{-w}\Big]_0^t-e^{-w}\Big]_0^t$$

Substitute the value of $\int_0^t we^{-w}\,dw$ in equation (1), the equation (1) becomes

$$s(t)=-w^2e^{-w}\Big]_0^t+2\int_0^t we^{-w}\,dw$$

$$=-w^2e^{-w}\Big]_0^t+2\left(-we^{-w}\Big]_0^t-e^{-w}\Big]_0^t\right)$$

$$=-\left[t^2e^{-t}-0^2\cdot e^{-0}\right]+2\left(-\left(te^{-t}-0\cdot e^{-0}\right)-\left(e^{-t}-e^{-0}\right)\right)$$

$$=-\left[t^2e^{-t}\right]+2\left(-te^{-t}-e^{-t}+1\right)$$

$$=-t^2e^{-t}-2te^{-t}-2e^{-t}+2$$

$$=2-\left(t^2+2t+2\right)e^{-t}\text{ meters}$$

Hence, the particle travels during the first t seconds is $\boxed{2-\left(t^2+2t+2\right)e^{-t}}$ meters.

Answer 68E.

We have to evaluate $\int_0^a f(x)g''(x)dx$

Let $\quad u=f(x) \qquad dv=g''(x)dx$

Then $\quad du=f'(x)dx \quad v=g'(x)$

Therefore

$$\int_0^a f(x)g''(x)dx=f(x)g'(x)\Big]_0^a-\int_0^a f'(x)g'(x)dx$$

$$=\left(f(a)g'(a)-f(0)g'(0)\right)-\int_0^a f'(x)g'(x)\,dx$$

Given that $f(0)=0, g(0)=0$

Then $\int_0^a f(x)g''(x)dx = f(a)g'(a) - \int_0^a f'(x)g'(x)\,dx$

Again integrating by pats, then

$$\int_0^a f(x)g''(x)dx = f(a)g'(a) - \left(f'(x)g(x)\Big]_0^a - \int_0^a f''(x)g(x)\,dx \right)$$

$$= f(a)g'(a) - \left[f'(x)g(x)\right]_0^a + \int_0^a f''(x)g(x)dx$$

$$= f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)dx$$

$(f(0)=0 \text{ and } g(0)=0, \text{ given})$

Answer 69E.

Given that

$f(1)=2, f(4)=7, f'(1)=5, f'(4)=3$

We need to find the value of $\int_1^4 x f''(x)\,dx$

Integrating parts by taking

$u = x,\ dv = f''(x)dx,$ Since f'' is continuous

$du = dx,\ v = f'(x)$

Then $\int_1^4 x f''(x)dx = x f'(x)\Big]_1^4 - \int_1^4 f'(x)\,dx$

$$= x f'(x)\Big]_1^4 - f(x)\Big]_1^4$$

$$= \left(4f'(4) - f'(1)\right) - \left(f(4) - f(1)\right)$$

$$= (4\times 3 - 5) - (7 - 2)$$

$$= 7 - 5$$

Thus $\boxed{\int_1^4 x f''(x)dx = 2}$

Answer 70E.

a)

Show the following equation by using integral by parts:

$$\int f(x)dx = xf(x) - \int xf'(x)dx$$

Formula for the integration by parts is, $\int u dv = uv - \int v du$ .... (1)

Let

$u = f(x) \Rightarrow du = f'(x)dx$

$dv = dx \Rightarrow v = x$

Substitute the above values in the equation (1).

$\int u dv = uv - \int v du$

$\int f(x)dx = xf(x) - \int xf'(x)dx$

Therefore, $\boxed{\int f(x)dx = xf(x) - \int xf'(x)dx}$.

b)

Prove the following equation by using part (a):

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

From part (a):

$$\int_a^b f(x)dx = [xf(x)]_a^b - \int_a^b xf'(x)dx$$

$$\int_a^b f(x)dx = [bf(b) - af(a)] - \int_a^b xf'(x)dx$$

Let $y = f(x) \Rightarrow x = f^{-1}(y)$

$dy = f'(x)dx$

If $a \le x \le b$ then $f(a) \le f(x) \le f(b)$

That is $f(a) \le y \le f(b)$ (Since $y = f(x)$)

$$\int_a^b f(x)dx = [bf(b) - af(a)] - \int_{f(a)}^{f(b)} f^{-1}(y)dy \quad \left(\begin{array}{l}\text{Since } x = f^{-1}(y), f(x)dx = dy \\ f(a) \le y \le f(b)\end{array}\right)$$

$$\int_a^b f(x)dx = [bf(b) - af(a)] - \int_{f(a)}^{f(b)} g(y)dy \quad \left(\begin{array}{l}\text{Since } f \text{ and } g \text{ are inverse functions} \\ f^{-1}(y) = g(y)\end{array}\right)$$

Hence, $\int_a^b f(x)dx = \boxed{[bf(b) - af(a)] - \int_{f(a)}^{f(b)} g(y)dy}$.

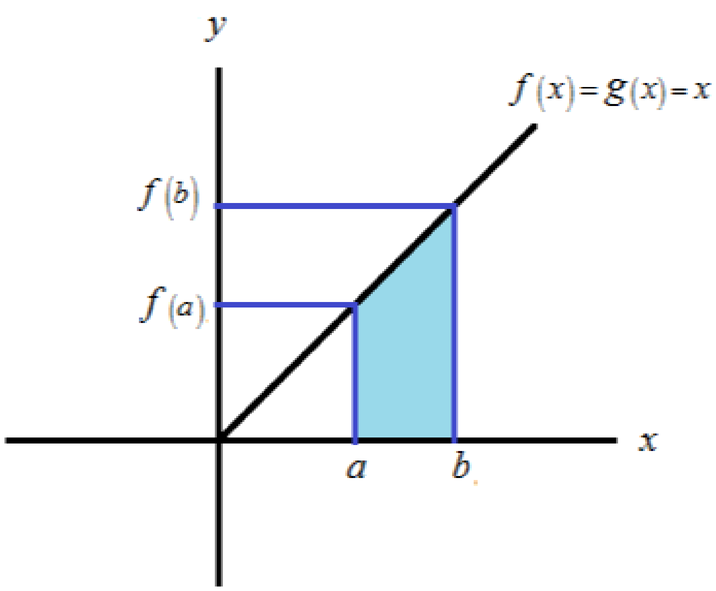
c)

Suppose f and g are positive functions and $0 < a < b$.

Sketch the diagram to give geometric interpretation of the following equation:

$$\int_a^b f(x)dx = [bf(b) - af(a)] - \int_{f(a)}^{f(b)} g(y)dy$$

Consider f and g are self-inverse functions, and the following diagram shows the geometric interpretation of the above equation.



d)

Consider the integral $\int_1^e \ln x dx$.

Let $f(x) = \ln x$

$f^{-1}(x) = g(x) = e^x$

Therefore, $g(y) = e^y$

Now, substitute the above values in $\int_a^b f(x)dx = [bf(b) - af(a)] - \int_{f(a)}^{f(b)} g(y)dy$.

$$\int_1^e \ln x dx = [ef(e) - 1f(1)] - \int_{f(1)}^{f(e)} e^y dy$$

$$\int_1^e \ln x dx = [e\ln e - \ln(1)] - [e^y]_{f(1)}^{f(e)}$$

$$\int_1^e \ln x dx = [e\ln e - 0] - [e^{f(e)} - e^{f(1)}]$$

$$\int_1^e \ln x dx = e\ln e - [e^{\ln e} - e^{\ln 1}]$$

$$\int_1^e \ln x dx = e - [e - 1]$$

$$\int_1^e \ln x dx = e - e + 1$$

$$\int_1^e \ln x dx = 1$$

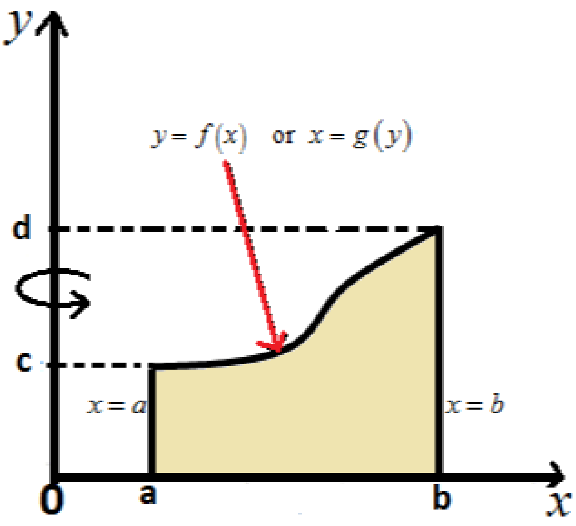
Therefore, the value of the integral $\int_1^e \ln x dx$ is $\boxed{1}$.

Answer 71E.

Consider the following region between $y = f(x)$ and $x = a, x = b$ is revolute with $y-\text{axis}$.

And it is given that the function f is one-to-one so f^{-1} is exists

Since $y = f(x)$ then inverse function can be written as $g(y) = x$



The region is perpendicular to revolution axis is $y-\text{axis}$, so to find volume use washer method

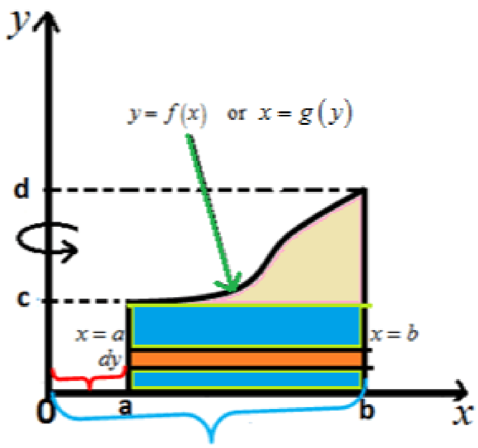
The region is split in two parts, below and above part of $y = c$

The entire volume $V = V_1 + V_2$

In case of volume V_1 , the inner radius of washer is a and outer radius is b

And thickness of washer is dy

The limits are $y = 0$ to c



Inner radius of washer is a
Outer radius of washer is b

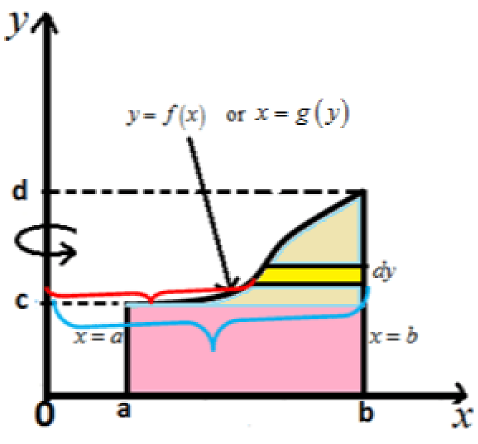
The lower portion of the region will give us the follow integral for the volume:

$$\begin{aligned} V_1 &= \pi\int_0^c \left(b^2 - a^2\right) dy \\ &= \pi\left(b^2 - a^2\right) y\Big|_0^c \\ &= \pi\left(b^2 - a^2\right)(c-0) \\ &= \pi b^2 c - \pi a^2 c \end{aligned}$$

In case of volume V_2 , the inner radius of washer is $x = g(y)$ and outer radius is b

And thickness of washer is dy

The limits are $y = c$ to d



Inner radius is $x = g(y)$
Outer radius is b

The upper portion of the region will give us the follow integral for the volume:

$$\begin{aligned} V_2 &= \pi\int_c^d \left(b^2 - \left[g(y)\right]^2\right) dy \\ &= \pi b^2 \, y\Big|_c^d - \pi\int_c^d \left[g(y)\right]^2 dy \\ &= \pi b^2 d - \pi b^2 c - \pi\int_c^d \left[g(y)\right]^2 dy \end{aligned}$$

The total volume V can be calculated as

$$V_1+V_2=\left(\pi b^2c-\pi a^2c\right)+\left(\pi b^2d-\pi b^2c-\pi\int_c^d\left[g(y)\right]^2dy\right)$$

$$V=\pi b^2d-\pi a^2c-\pi\int_c^d\left[g(y)\right]^2dy \qquad \text{......(1)}$$

Now substitute $y=f(x)$ in equation (1)

$$y=f(x)$$
$$x=g(y)$$
$$c=f(a)$$
$$a=g(c)$$
$$d=f(b)$$
$$b=g(d)$$
$$y=f(x)$$
$$dy=f'(x)dx$$

$$\begin{aligned}V&=\pi b^2d-\pi a^2c-\pi\int_c^d\left[g(y)\right]^2dy\\&=\pi b^2d-\pi a^2c-\pi\int_c^d\left[g(y)\right]^2dy\\&=\pi b^2d-\pi a^2c-\pi\int_a^b x^2f'(x)dx\end{aligned}$$

Consider the following integral to find using integration by parts

Recollect integration by parts

$$\int F(x)G(x)dx=F(x)\int G(x)dx-\int F'(x)\left(\int G(x)dx\right)dx$$

$$\begin{aligned}\pi\int_a^b x^2f'(x)dx&=\pi\left[x^2 f(x)\Big|_a^b-\int_a^b 2xf(x)dx\right]\\&\quad\left(F(x)=x^2,G(x)=f'(x)\right)\\&=\pi b^2f(b)-\pi a^2f(a)-\int_a^b 2\pi xf(x)dx\\&=\pi b^2d-\pi a^2c-\int_a^b 2\pi xf(x)dx\end{aligned}$$

Now, substituting back into the above formula for the volume:

$$\begin{aligned}\pi b^2d-\pi a^2c-\pi\int_a^b x^2f'(x)dx&=\pi b^2d-\pi a^2c-\left[\pi b^2d-\pi a^2c-\int_a^b 2\pi xf(x)dx\right]\\&=\int_a^b 2\pi xf(x)dx\end{aligned}$$

Thus, verifying the formula for the shell method, $V=\int_a^b 2\pi xf(x)dx$.