Chapter - 3
Electric Potential

In previuos chapter we have studied electric field due to a point charge or group of charges and described it in terms of a vector $\vec{E}$ called intensity of electric field. We have also studied continous charge distribution,force between charges and to calculate $\vec{E}$ using Coulomb's law and Gauss's law. In this chapter we will see that the electric field can also be decribed in terms of a scalar quantity called "electrostatic potenial" V. This is an important concept. It is so because electric field vector $\vec{E}$ and electrostatic potential V can be related to each other. Since V is a scalar quantity and addition of scalars is far easier than addition of vectors, in many problems it is much easier to find V first and then use it to find $\vec{E}$ rather than obtaining $\vec{E}$ from direct calculations. The concept of potential is also important from the point of view that potential is related to potential energy. Thus, by using the law of conservation of energy we can solve many problems in electrostatics without goining into the details of forces involved like we did in study of mechanics.

In this chapter first of all we will define electric potential. After that we will learn to calculate electric potential due to a point charge and system of charges. Subsequently we will study the relation between electric field and potential. After studying about electric potential due to some specific charge configurations, we will study about electrocstatic potential energies of such systems. In the end of this chapter we will learn about the work done in rotating an electric dipole in some external electric field and calculate its potential energy.

3.1 Electrostatic Potential and Potential Difference

From our understanding of conservative forces we know that a potential energy is associated with a conservative force. From experiments it is known that the electric field (force) is conserative and thus has an associated potential energy U called as the electrostatic potential energy. (Unless stated otherwise, in this chapter and chapters to follow the term electric field refers to electrostatic field. In the study related with phenomenon of electro magnetism we shall see that a changing magnetic field also produced an electric field which is not conservative and potential energy can not be associated with such a field).

Consider a positive test charge q_o being brought from some point A to some point B in an electric field. Here we are assuming that in the process the test charge does not distrub source charge(s) which produce electric field i.e. all other charges present in the surrounding remain at their respective places. If in this process the potential energy changes by $U_B - U_A$ then the potential difference between points A and B is defined as

$$V_B - V_A = \frac{U_B - U_A}{q_0} = \frac{\Delta U}{q_0} \qquad \ldots (3.1)$$

The above equation defines the electrostatic potential difference between two points for a given electric field. To define absolute potential (which from now on we call simply electric potential) at a point we can select a reference point at which we consider both potential energy and potential as zero. Generally we take this reference position to be at infinity. Thus if we consider the point A to be at infinity so $V_A = V_\infty = 0$ and $U_A = U_\infty = 0$ and then from equation (3.1)

$$V_B = U_B / q_0$$

Since point B is arbitrary so in general the above equation can be written as

$$V = U / q_0 \qquad \ldots (3.2)$$

Thus electric potential at a point is defined as electric potential energy per unit charge. Clearly electric potential is a scalar quantity and is independent of test charge. It is a characteristic of electric field only.

We know that energy and work are related with each other so we can define potential difference and potential in terms of work. If work done by conservative electric field in moving the system from initial position to final position is denoted by W_e then

$$\Delta U = -W_e$$

therefore the potential difference between points A and B is

$$\Delta V = V_B - V_A = -\frac{W_e}{q_0} \qquad \ldots(3.3)$$

So, the potential difference between two points in an electric field is equal to the negative of the work done by the electric field in bringing a unit positive charge from initial position (A) to final position (B). Depending upon the signs and magnitudes of W_{ba} and q_0 potential difference between two points can be positive, negative or zero. If we assume the potential at infinity (reference point) to be zero then from equation 3.3 we can define potential at a point by

$$V = -\frac{W_{e\infty}}{q_0} \qquad \ldots(3.4)$$

Where $W_{e\infty}$ is the work done by electric field on test charge q_0 in bringing it from infinity (reference point) to the point under consideration. Thus, the electric potential at a point is equal to the negative of the work done by electric field in bringing a unit positive charge from infinity (reference position) to given point.

Suppose we move a particle of charge q_0 from point A to point B in an electric field with the help of some external agent (force). If the motion of the particle does not involve any charge in its kinetic energy i.e $\Delta K = 0$ then by the work-kinetic energy theorm

$$W_{ext} = -W_e \qquad \ldots(3.5)$$

Where W_{ext} refers to the work done by external force during the move, then from equations (3.3) and (3.5)

$$W_{ext} = -W_e \qquad \ldots(3.6)$$

and as assumed earlier if we take point A to be at infinity (reference position) then potential at a point is

$$\Delta V = V_B - V_A = \frac{W_{ext}}{q} \qquad \ldots(3.6)$$

Where $W_{\infty A}$ refers to the work done by external force in bringing the charge q_0 from infinity to point under consideration. Accordingly the electric potential at a point is equal to the work done by the external agent (without changing Kinetic energy) on a unit positive charge in bringing it from infinity (reference position) to the desired point.

From above discussion it is obvious that there are many equivalent definitions for electric potential. (In forth coming subsection we will define electric potential as a line integral of electric field). However , from each definition it is apparent that electric potential is a scalar quantity.

The SI unit of electric potential is volt with

$$1 \text{ volt (V)} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}}$$

The electric potential at a point is 1 volt; if in moving a unit charge from infinity to that point the work done is 1 joule. Dimensions of electric potential are

$$[V] = \left[\frac{ML^2T^{-2}}{TA}\right] = \left[ML^2T^{-3}A^{-1}\right]$$

It is obvious that unit of potential difference is also volt. This unit of electric potential allows us to adopt a more conventional unit V / m for the electric field E (which till now we have described in unit of N/C) i.e

$$1\,N/C = 1V/m$$

We leave the verification of this expression as an excerise for readers. After defining 'volt' we can now define an energy unit called electron volt (eV) which is convenient for measuring atomic and nuclear energies. One electron volt (eV) is the energy equal to the work needed to move a single elementary charge e (electron or proton) through a potential difference of exactly one volt. From equation

$$W = q(\Delta V)$$

$$1\,eV = e(1V) = \left(1.6\times10^{-19}\,C\right)(J/C)$$

$$= 1.6\times10^{-19}\,J$$

3.1.1 Electric potential Derived From Electric Field

Consider an arbitrary electric field for which electric field lines are as shown in Fig 3.1. Let a positive test charge q_0 move in this field along the curved path shown from point A to B. At any point on this path an electric force $\vec{F}_e = q_0\vec{E}$ acts on it for a differential displacement $d\vec{\ell}$, here $\vec{E}$ is the electric field intensity at the location of differential element. So the work done by electric force during this displacement is given by

$$dW = \vec{F}\cdot d\vec{\ell} = q_0\vec{E}\cdot d\vec{\ell} \qquad \ldots(3.8)$$

Therefore the total work W done by the electric force as the particle moves from point A to point B is

$$W_e = \int_A^B q_0\vec{E}\cdot d\vec{\ell} = q_0\int_A^B \vec{E}\cdot d\vec{\ell} \qquad \ldots(3.9)$$

From equations (3.3) and (3.9)then

$$V_B - V_A = -\frac{q_0}{q_0}\int_A^B \vec{E}\cdot d\vec{\ell}$$

$$= -\int_A^B \vec{E}\cdot d\vec{\ell} \qquad \ldots(3.10)$$

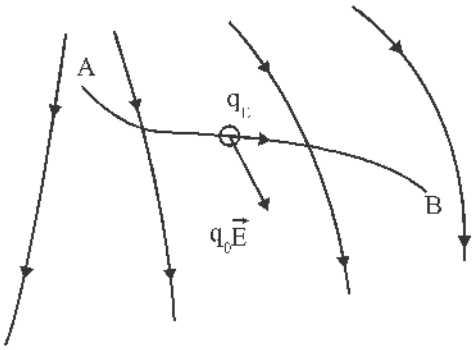


Fig 3.1 : Motion of a test charge from point A to point B in a non uniform field

The integral present in right hand side of equation 3.10 is called line integral (meaning the integral along a particular path) of $\vec{E}$ from A to B. However, since the electric field is consvertive all paths (between A and B) give the same result. Thus the potential difference between any two points A and B in an electric field is equal to the negative of the line intgral of $\vec{E}$ from A to B.

If electric field is along $d\vec{\ell}$ then integral in equation 3.10 is positive and potential difference negative i.e $V_B < V_A$. Electric field tends to move a positive charge from high potential to low potential and tend to move a negative charge from low potential to high potential.

In equation 3.10 if we consider A to be at infinity (reference position) and set $V_A = 0$ then

$$V = -\int_\infty^B \vec{E}\cdot d\vec{\ell} \qquad \ldots(3.11)$$

Equation 3.11 gives us the potential at any point realtive to zero potential at infinity (reference point).

3.2 Potential Due to a Point Charge

For a point charge Q electric field at a distance r is given by

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad \ldots(3.12)$$

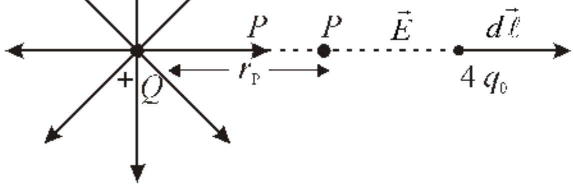


Fig 3.2 : Determinations potential due to a point charge here a test charge q_0 is being moved from point p to infinity

If Q is positive $\vec{E}$ is directed radially outward from the charge. Now we will make use of equations 3.10 for obtaining expression of potential at some point in this field. To do so let us imagine that a test charge q is being moved from a point P to infinity on a radial line along the direction of $\vec{E}$ as shown in fig 3.2. For such a path differntial displacement $d\vec{\ell}$ can be written as $d\vec{\ell} = d\vec{r}$ and as $\vec{E}$ and are in same direction $\vec{E}\cdot d\vec{\ell} = Ed\ell = Edr$. Using equation 3.10 (along with above mentioned charges) between the limits r_P to ∞ we obtain

$$V_\infty - V_P = -\int_{r_P}^{\infty} \vec{E}\cdot d\vec{r} = -\int_{r_P}^{\infty} Edr$$

or $-V_P = -\int_{r_P}^{\infty} \frac{Q}{4\pi \in_0 r^2} dr \quad [\because V_\infty = 0]$

or $V_P = \frac{Q}{4\pi \in_0}\left[\frac{-1}{r}\right]_{r_P}^{\infty} \qquad \left[\because \int \frac{1}{x^2}dx = -\frac{1}{x}\right]$

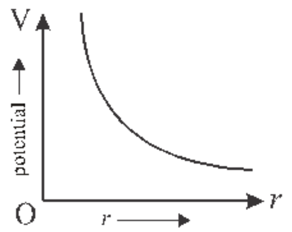
$$= \frac{Q}{4\pi \in_0}\left[\frac{1}{r}\right]_{\infty}^{r_P}$$

$$V_P = \frac{Q}{4\pi \in_0 r_P}$$

as P can be any arbitrary point so in general

$$V = \frac{Q}{4\pi \in_0 r} \qquad \ldots (3.13)$$

Therefore the potential due to a point charge is inversely proptional to the distance between point charge and the observation point. It does not depend on the direction of observation point relative to the point charge. This variation is shown graphicallyin fig 3.3.



r = distance of observation (r) from point charge

Fig 3.3 : Graphical variation of potential due to a point charge with distance

If the source charge is negative

$$V = \frac{1}{4\pi \in_0}\frac{(-Q)}{r} \quad \ldots (3.14)$$

For an isolated positive charge (Q > O) electric potential is positive while for an isolated negative charge (Q < O) it is negative. For a given charge and a given distance potential in some medium is less then the potential in free space and is given by

$$V_m = \frac{V}{\in_0 \in_r}$$

3.3 Potential due to a Group of Point Charges

Electric potential is a scalar quantity so we can find the net potential at a point due to a group of point charges by algebraic sum of potentials due to individual charges at that point. Suppose we wish to determine the potential at a point P due to a group of point charges. Let distance of point P from charges, $q_1, q_2, q_3 \ldots q_n$ be $r_1, r_2, r_3,$ respectively (Fig 3.4. Then as stated above the net potential at P is given by

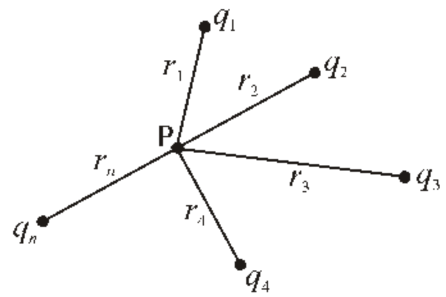


Fig 3.4 : The potential at point P due to a system of point charges

$$V = V_1 + V_2 + V_3 + \ldots + V_n$$

$$V = \frac{1}{4\pi \in_0}\frac{q_1}{r_1} + \frac{1}{4\pi \in_0}\frac{q_2}{r_2} + \ldots + \frac{1}{4\pi \in_0}\frac{q_n}{r_n}$$

$$V = \frac{1}{4\pi \in_0}\left[\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots + \frac{q_n}{r_n}\right]$$

$$V = \frac{1}{4\pi \in_0}\sum_{i=1}^{n}\frac{q_i}{r_i} \qquad \ldots (3.15)$$

Example 3.1 : The electric potential at some point is -15 V and at some other point is V(volt). If 150 J of work is needed to move a 6 coulomb charge from the first point to the second then find the value of V.

Solution : Here ; $V_A = -15$ Volt, $W_{exe} = 150$ J

$$V_B = V \ , \ q_0 = 6\,\text{C}$$

$$\because \quad V_B - V_A = \frac{W_{ext}}{q_0}$$

$$V - (-15) = \frac{150}{6} = 25 \text{ volt}$$

$V = 25 - 15 = 10$ volt

Example 3.2 The work done in displacing a 20 C charge through 0.2m for moving it from a point A to a point B is 2 J. Find the potential difference between the two points.

Solution : Here $q_0 = 20C$

$W_{ext} = 2\text{ J}$

therefore, $V_B - V_A = \dfrac{W_{ext}}{q_0} = \dfrac{2}{20} = 0.1\text{ V}$

Example 3.3 Calculate the electric potential at a distance 10 cm from a charge $1.1\times10^{-9}\,C$ in air.

Solution : Here $Q = 1.1\times10^{-9}\,C$

$r = 10\ cm = 0.1\ m$

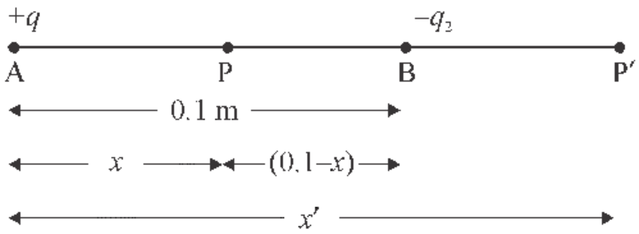
So electric potential

$$V = \frac{1}{4\pi\epsilon_0}\frac{(Q)}{r}$$

$$V = \frac{9\times10^9\times1.1\times10^{-9}}{0.1} = 99\text{ V}$$

Example 3.4 The distance between two charges $4\times10^{-9}\,C$ and $-3\times10^{-9}\,C$ is 0.1 m. Where on the line joining the two charges potential is zero? Assume the potential to be zero at infinity.

Solution : Refer figure shown, here q_1 and q_2 are of opposite nature but $|q_1 > q_2|$. Therefore there will not be any point in the region left of q_1 where the potentials due to q_1 and q_2 are canceling each other. Such points can be in between q_1 and q_2 or to the right of q_2 on the line joining. Such points are shown in Fig by P and P'



If distance of P from q_o is x then for potential at P to be zero

$$V_A + V_B = O$$

$$\frac{1}{4\pi\epsilon_0}\cdot\frac{q_1}{x} = -\frac{1}{4\pi\epsilon_0}\frac{q_2}{(0.1-x)}$$

$$\frac{4\times10^{-9}}{x} = -\frac{-(-3\times10^{-9})}{(0.1-x)}$$

$$3x = 0.4 - 4x$$

$$x = \frac{0.4}{7} = 0.057m$$

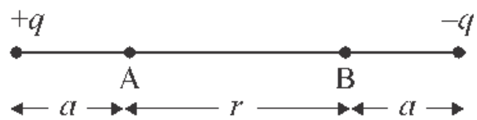
If P'is at distance x' from q_1 then for potential at P' to be zero

$$\frac{4\times10^{-9}}{x'} = \frac{3\times10^{-9}}{(x'-0.1)}$$

$\Rightarrow \quad 3x' = 4x - 0.4$

$\Rightarrow \quad x' = 0.4m$

Example 3.5 Two charges + q and - q are arranged as shown, the potentials at points A and B are V_A and V_B then Calculate $V_A - V_B$



Solution : Potential at A due to +q

$$= \frac{1}{4\pi\epsilon_0}\times\frac{q}{a}$$

potential at A due to - q

$$= \frac{1}{4\pi \epsilon_0} \frac{(-q)}{(a+r)}$$

total potential at A

$$V_A = \frac{1}{4\pi \epsilon_0}\left(\frac{q}{a} - \frac{q}{a+r}\right)...$$

Potential at B due to + q

$$= \frac{1}{4\pi \epsilon_0} \frac{q}{a+r}$$

Potential at B due to - q

$$= \frac{1}{4\pi \epsilon_0}\left(\frac{-q}{a}\right)$$

total potential at B

$$V_B = \frac{1}{4\pi \epsilon_0}\left(\frac{q}{a+r} - \frac{q}{a}\right)...$$

So

$$V_A - V_B = \frac{1}{4\pi \epsilon_0}\left[\left(\frac{q}{a} - \frac{q}{a+r}\right) - \left(\frac{q}{a+r} - \frac{q}{a}\right)\right]$$

$$= \frac{1}{4\pi \epsilon_0} 2q\left(\frac{1}{a} - \frac{1}{a+r}\right)$$

$$= \frac{1}{4\pi \epsilon_0} 2q\left(\frac{(a+r)-a}{a(a+r)}\right)$$

$$V_A - V_B = \frac{1}{4\pi \epsilon_0} \frac{2qr}{a(a+r)}$$

3.4 Electric Potential Due to Electric Dipole

Fig 3.5 depicts an electric dipole AB. Charges at A and B are -q and + q and separation between them is 2a. We wish to calculate potential at point P at a distnace r from the centre O of the dipole. Line OP makes an angle θ with the axis of dipole. Potential at P due to charge - q at A

$$V_1 = \frac{1}{4\pi \epsilon_0} \frac{(-q)}{r_1} \quad ...(3.16)$$

and potential at P due to charge + q at B

$$V_2 = \frac{1}{4\pi \epsilon_0} . \frac{q}{r_2} \quad ...(3.17)$$

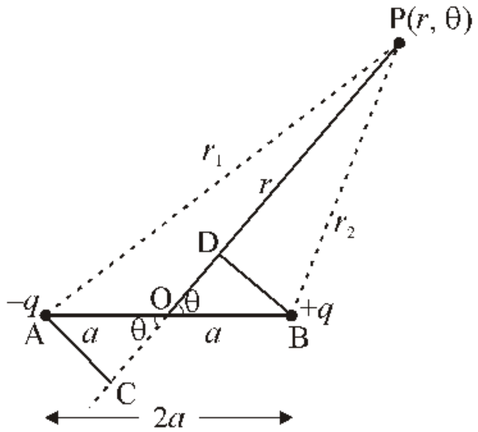


Fig 3.5 : Potential due to dipole at point **P** (r, θ)

So the net electric potential at P

$$V = \frac{q}{4\pi \epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{q}{4\pi \epsilon_0}\left[\frac{r_1 - r_2}{r_1 r_2}\right] \quad ...(3.18)$$

To determine $r_1 - r_2$ and $r_1 r_2$ we draw perpendiculars AC and BD respectively on line OP from points A and B. If r > > a $AP \approx PC$ and $PB \approx PD$. From Fig 3.5

$$OP = OD + DP$$

$$r = a\cos\theta + r_2$$

$$\because DP \approx BP = r_2$$

$$r_2 = r - a\cos\theta \quad \left(\because \cos\theta = \frac{OD}{a}\right)$$

$$AP \approx CP = OP + OC \qquad (OD = a\cos\theta)$$

$$r_1 = r + a\cos\theta$$

$$\cos\theta = \frac{OC}{a} \text{ or } OC = a\cos\theta$$

So $\quad r_1 - r_2 = 2a\cos\theta$

and $\quad r_1 r_2 = r^2 - a^2\cos^2\theta = r^2 \quad$ as $r^2 >> a^2$ substituting these in Egn. (3.18)

$$V = \frac{q}{4\pi\in_0}\frac{2a\cos\theta}{(r^2)}$$

As $\quad 2aq = p$

So $$V = \frac{1}{4\pi\in_0}\frac{p\cos\theta}{r^2} \qquad \text{. . . (3.19)}$$

Thus the electric dipole potential falls off, at large distance, as $r^{\ 2}$, not as $r^{\ 1}$, characteristic of the potential due to a point charge.

Special Cases

(i) For axial points where $\theta = 0^\circ \cos\theta = 1$, from equation (3.19)

$$V = \frac{1}{4\pi\in_0}\frac{p}{r^2}$$

(ii) For equatorial points where $\theta = 90^\circ \cos\theta = 0$, from equation (3.19) $V = 0$

Dipole potential can also be expressed as

$$V = \frac{1}{4\pi\in_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\in_0}\cdot\frac{\vec{p}\cdot\hat{r}}{r^2}$$

Thus it is clear that

1. For equatorial points the electric potential due to a dipole is zero however field is not zero.
2. Under indential conditions the electric potential due to a dipole is $\frac{2a}{r}\cos\theta$ times of that due to a single charge.
3. For a small dipole (or at large disrances) electic potential is inversely proportional to the square of distance.
4. The potential due to a dipole depends not just on r but also on the angle between the position vector $\vec{r}$ and the dipole moment vector p.

Example 3.6 Two point charges $8\times10^{-19}\,C$ and $-8\times10^{-19}\,C$ are separated by a distance of $2\times10^{-10}\,m$. For such a dipole find electric potential at a point at a distance of $4\times10^{\ 6}\,m$ when the point is (a) on the dipole axis (b) on equatorial line and (c) at an orientation of 60° from dipolemoment.

Solution : Here $q = 8\times10^{\ 19}C$

$2a = 2\times10^{\ 10}\,m$

$r = 4\times10^{\ 6}\,m$

So dipole moment

$p = q.2a = 8\times10^{\ 19}\times2\times10^{\ 10}$

$P = 16\times10^{\ 29}\,C\cdot m$

(a) For axial position

$$V = \frac{1}{4\pi\in_0}\frac{p}{r^2} = \frac{9\times10^9\times16\times10^{\ 29}}{(4\times10^{\ 6})^2}$$

$V = 9\times10^{\ 8}$ Volt

(b) For equatorial positions V = O

(c) When $\theta = 60^\circ$

$$V = \frac{1}{4\pi\in_0}\frac{p\cos\theta}{r^2}$$

$$V = \frac{9\times10^9\times16\times10^{\ 29}\cos60^\circ}{(4\times10^{\ 6})^2}$$

$V = 4.5\times10^{-8}$ Volt

3.5 Equipotential Surface

In some electric field a surface having same potential at all points is called an equipotential surface. As the potential difference between any two points on an equipotential surface is zero, no work is done in moving a charge from one point to other on an equipotential surface. As the work done is zero when electic force (electric field) is perpendicular to displacement, so electric field must be normal to an equipotential surface.

For illustration of equipotential surfaces following examples can be considered.

1. For a uniform electric field $\vec{E}$ equipotential surfaces are flat and perpendicular to field lines. According to fig 3.6, surfaces labelled as I, II, III are equipotentials.

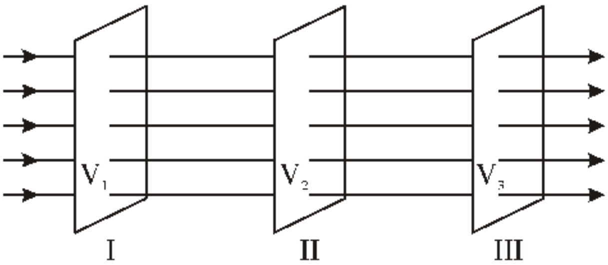


Fig 3.6 : Equipotential surfaces for a uniform electric field

2. For an isolated point charge : For an isolated point charge + q potential at a distance r

$$V = \frac{1}{4\pi \in_0} \frac{q}{r}$$

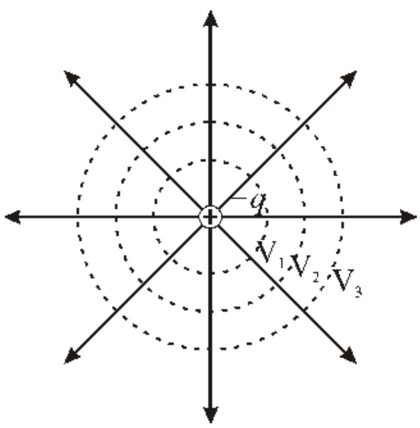


Fig 3.7 : Equipotential surfaces for a point charge

Imagine a spherical surface of radius r the position of point charge +q, is at its center it is obvious that the electric potential at all points on this surface is same. Thus for a point charge equipotential surfaces are spherical around point charge. For a positive point charge with the increases in radius of spherical surface the potential of surface decreases.

3.5.1 Properties of Equipotential Surfaces

1. No net work is done on a charge by an electric field as the charge moves between two points on the same equipotential surface.
2. Electric field is always directed normal to equipotential surface.
3. Two equipotential surfaces can never intersect each other because otherwise the point of interesection will have two potentials which is unacceptable.
4. The surface of a charged conductor is always equipotential. In fact the entire volume of a conductor is equipotential.

3.6 Relation Between Electric Field and Electric Potential

We have already discussed that if we know electric field in some region of space then potential difference between two points can be found using equation (3.14) which is a relation between electric field and electric potential. In this section our aim is to determine electric field for a known potential function V.

In some arbitrary electric field $\vec{E}$ for a differential displacement $d\vec{\ell}$ equation (3.10) can be written in differential form as follows

$$dV = -\vec{E} \cdot d\vec{\ell} = -E d\ell \cos\theta \qquad \ldots (3.20)$$

Where θ is the angle between $\vec{E}$ and $d\vec{\ell}$

$$-\frac{dV}{d\ell} = E\cos\theta \qquad \ldots (3.21)$$

The quantity $-\frac{dV}{d\ell}$ gives rate of loss (fall) of potential with distance. From above equation it is clear that if angle between $\vec{E}$ and $d\vec{\ell}$, $\theta = 0°$ then the space rate of loss of potential will be maximum. Thus in general $-\frac{dV}{d\ell}$ is a scalar quantity but its maximum value $-\left(\frac{dV}{d\ell}\right)_{max}$ occur for a specific direction $(\theta = 0_i)$ i.e. in direction of $\vec{E}$. Thus the maximum rate of loss of potential with distance can be treated as a vector in direction of $\vec{E}$. In language of mathematics $\left(\frac{dV}{d\ell}\right)_{max}$ is called gradient of V and written as Grad V

$$\left(\frac{dV}{d\ell}\right)_{max} = Grad\,V \qquad \ldots (3.22)$$

Accordingly $\vec{E} = -Grad\,V \qquad \ldots (3.23)$

For an equipotential surface direction of Grad V is along the normal to the surface. This can be explained using Fig 3.8. Here two equipotential surface S_1 and S_2 are shown with potentials V and V – dV respectively. In moving from a point A on surface S_1 to either point B or C on surface S_2 for both the paths AB and AC change in potential is same. However the rate of change of potential with distance i.e. $\frac{dV}{AB}$ and $\frac{dV}{AC}$ are different. As AB < AC so $\frac{dV}{AB} > \frac{dV}{AC}$ and because AB is normal to the surface the rate of loss of potential is maximum along normal.

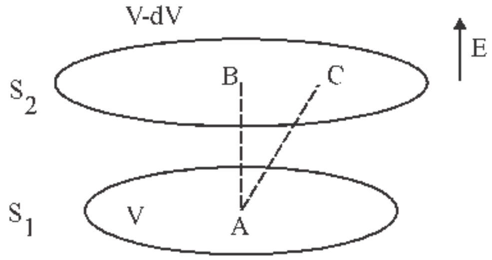


Fig 3.8 : Gradient of potential

Equation (3.21) can be rewritten as

$$-\frac{\delta V}{\delta \ell} = E\cos\theta = E_\ell$$

Here $E_\ell = E\cos\theta$, is the compoment of $\vec{E}$ in direction of $d\vec{\ell}$. Note that here we have used partial derivative which shows that above equation involves only the variation of V along a specified axis (here called l axis) and only the compoment of $\vec{E}$ along that axis. If we take the l axis to be in turn, the x, y and z axes, we find that the x, y and z compoments of $\vec{E}$ at any point are

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z} \ldots (3.24)$$

In cartesian coordinates

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$

so $\vec{E} = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$

or $\vec{E} = -\nabla V \qquad \ldots (3.25)$

where $\nabla = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \ldots (3.26)$

is called 'del operator: With the help of equation (3.25) $\vec{E}$ can be determined if potential function V (x, y, z) is known.

If the potential function is spherically symmetric i.e. a function of radial distance r then electric field is given by

$$E_r = -\frac{dV}{dr} \qquad \ldots (3.27)$$

Example 3.7 For some electric field the electric potential is given by the following expression

$$V = \frac{343}{r}\, volt$$

Determine electric field at position given by position vector $\vec{r} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ m

Solution : As $\vec{E} = -\frac{dV}{dr}\hat{r}$

Here $\frac{dV}{dr} = \frac{d}{dr}\left[\frac{343}{r}\right] = -\frac{343}{r^2}$

$$r = |\vec{r}| = \sqrt{(3)^2 + (2)^2 + (-6)^2}$$

$$= \sqrt{49} = 7\,\text{m}$$

$$\text{so } \vec{E} = -\left(\frac{-343}{r^2}\right)\hat{r}$$

$$\text{but} \qquad \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

$$\therefore \qquad \vec{E} = \frac{343}{r^3}\vec{r}$$

$$\vec{E} = \frac{343}{(7)^3}\left(3\hat{i} + 2\hat{j} - 6\hat{k}\right)$$

$$\vec{E} = \left(3\hat{i} + 2\hat{j} - 6\hat{k}\right)\text{V/m}$$

Example 3.8 For some electric field represented by potential function

$$V(x, y, z) = 6x - 8xy - 8y + 6yz$$

where V is in volt and x, y, z are in meters. Find magnitude of electric field at point (1, 1, 1) m.

Solution :

$$\vec{E} = -\vec{\nabla}V = \hat{i}\left(-\frac{\partial V}{\partial x}\right) + \hat{j}\left(-\frac{\partial V}{\partial x}\right) + \hat{k}\left(-\frac{\partial V}{\partial z}\right)$$

$$\text{or } \vec{E} = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial Z}\right]$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial x}(6x - 8xy - 8y + 6yz) = (6 - 8y)$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial y}(6x - 8xy - 8y + 6yz) = (-8x - 8 + 6z)$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(6x - 8xy - 8y + 6yz) = 6y$$

$$\vec{E} = -\left[(6 - 8y)\hat{i} + (-8x - 8 + 6z)\hat{j} + 6y\hat{k}\right]$$

at point (1, 1, 1)

$$\text{or } \vec{E} = -\left[(6 - 8)\hat{i} + (-8 - 8 + 6)\hat{j} + 6\hat{k}\right]$$

$$\text{or } \vec{E} = \left(2\hat{i} + 10\hat{j} - 6\hat{k}\right)\text{V/m}$$

$$|\vec{E}| = \sqrt{(2)^2 + (10)^2 + (-6)^2} = \sqrt{140} = 2\sqrt{35}\ \text{V/m}$$

3.7 Calculation of Electric Potential

3.7.1 Electric Potential due to a charged spherical shell

Consider a spherical shell of radius R and charge q. We wish to calculate electric potential due to such a shell at an internal point, a point on its surface and a point outside the shell. Let the distance of the point of observation from the centre of the shell be r.

(a) For points outside the charged sphere (r > R)

From the definition of potential

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$$

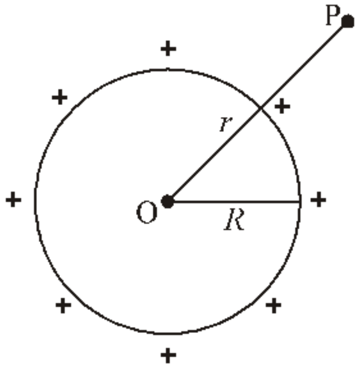


Fig 3.9 : Potential at outer point of shell ($r > R$)

However, for points external to the shell

$$\vec{E} = \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}$$

$$\therefore \quad V = -\int_{\infty}^{r} \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}.d\vec{r}$$

but $\hat{r} \cdot d\vec{r} = dr$ as $\hat{r}$ and $d\vec{r}$ are in some direction

$$\therefore \quad V = -\frac{q}{4\pi \in_0} \left[-\frac{1}{r} \right]_{\infty}^{r}$$

$$V = +\frac{q}{4\pi \in_0} \left[\frac{1}{r} - \frac{1}{\infty} \right]$$

$$V = \frac{1}{4\pi \in_0} \frac{q}{r} \quad \ldots (3.28)$$

We see that the potential due to uniformly charged shell is same as that due to a point charge q at the centre, for points external to shell. It is inversely proptional to of distance (r) and tends to zero as r tends to infinity.

(b) Potential at a point on the surface (r = R)

For this case $V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r}$

can be obtained by substituting r = R in equation (3.28)

$$V_s = \frac{1}{4\pi \in_0} \frac{q}{R} \quad \ldots (3.29)$$

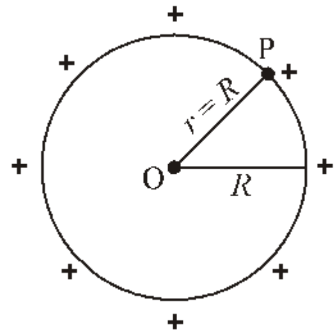


Fig 3.10 : Potential at the surface of shell (r = R)

(c) Potential at an internal point of the shell (r < R)

For determining potential at an internal point of a charged shell we must note that in moving from infinity to an internal point, the dependence of $\vec{E}$ is different for parts of the path outside and inside the charged shell.

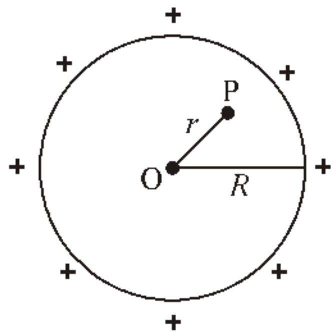


Fig: 3.11 : Determination of potential of an internal point ($r < R$)

Therefore for evaluation of

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$$

the path of intergration is to be considered to be made of two parts :-

(i) From infinity to distance R from centre (i.e. up to surface) and

(ii) From distance R (surface) to internal point r.

Thus we can write

$$V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{r} - \int_{R}^{r} \vec{E} \cdot d\vec{r}$$

First of these two integral has already been solved and its value can be obtained from equation (3.29) and as for internal points (r < R) electric field is zero for a charged shell the second integral reduces to zero i.e.

$$V_{in} = \frac{1}{4\pi \epsilon_0} \frac{q}{R} + \int_R^r -\vec{0} \cdot d\vec{r}$$

$$V_{in} = \frac{1}{4\pi \epsilon_0} \frac{q}{R} \quad \ldots (3.30)$$

From equation (3.30) it is clear that the potential has a fixed value for all points with in the shell equal to the value of potential at the surface. In fact it is the maximum value of potential due to a uniformly charged spherical shell.

The variation of potential with distance from centre is shown in fig 3.12

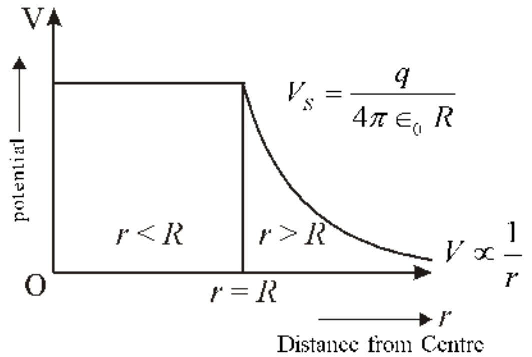


Fig 3.12 : The variation of potential for a charged spherical shell

3.7.2 Electric Potential due to a Charged Conducting Sphere

On charging a conductor as the charge resides on its outer surface, the behaviour of the field intensity due to a charged spherical conductor is same as that of a charged spherical shell. Therefore electric potential due to a charged conducting sphere is same as that due to a charged spherical shell. Thus results derived in section 3.7.1 for spherical shell are applicable in this case.

3.7.3 Electric Potential due to a Uniformly Charged Non Conducting Sphere

Consider a uniformly charged non conducting sphere of radius R having a charge q. For such a sphere expressions of the electric field at external point, point at surface and internal points are as follows.

External points $\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \hat{r} \qquad (r > R)$

At surface $\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{q}{R^2} \hat{r} \qquad (r = R)$

internal points and $\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{q}{R^3} \vec{r} \qquad (r < R)$

And the general realtion for calculating V from E is

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$$

Now we will calculate the electric potential for various positions of observation point.

(A) For points outside the charged sphere (r > R)

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$$

as for such points $\vec{E} = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2} \hat{r}$

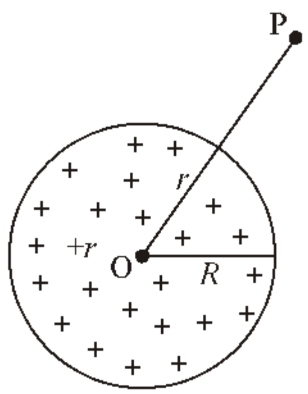


Fig 3.13 : A uniformly charged non conducting sphere point P is external to sphere

So, $V = -\int_{\infty}^{r} \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r} \cdot d\vec{r}$

or $V = -\frac{1}{4\pi \in_0} \int_{\infty}^{r} \frac{1}{r^2} dr \qquad (\hat{r} \cdot d\vec{r} = dr)$

$$V = -\frac{1}{4\pi \in_0} \left[-\frac{1}{r} \right]_{\infty}^{r}$$

$$V = \frac{q}{4\pi \in_0} \left[\frac{1}{r} - \frac{1}{\infty} \right] \qquad \because \frac{1}{\infty} = o$$

$$V = \frac{q}{4\pi \in_0 r} \qquad \text{. . . (3.31)}$$

Thus for points external to charged sphere $V \propto \frac{1}{r}$

(B) At the surface of Charged Sphere (r = R)

On substituting r = R in equation 3.31, we obtain

$$V_S = \frac{1}{4\pi \in_0} \frac{q}{R} \qquad \text{. . . (3.32)}$$

(C) At a point inside the charged non conducting sphere

As in moving from infinity to a point inside the sphere the variation of electric field with distance is non uniform i.e it varies from infinity to surface in accordance with E 1/2while from surface upto an internal point accodring to E r. Therefore we have to evalute the integral by breaking it into two parts (i) from infinity to R and (ii) from R to r

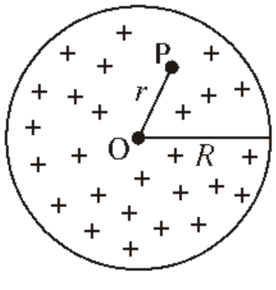


Fig 3.14 : Inside non-conducting sphare r < R

Thus $V = \left(-\int_{\infty}^{R} \vec{E} \cdot d\vec{r} \right) + \left(-\int_{R}^{r} \vec{E} \cdot d\vec{r} \right)$

or $V = \frac{1}{4\pi \in_0} \frac{q}{R} - \int_{R}^{r} \frac{1}{4\pi \in_0} \frac{q}{r^3} r \, \hat{r} \cdot d\vec{r}$

$$\left(\because \vec{E} = \frac{1}{4\pi \in_0} \frac{q}{R^3} r \, \hat{r} \right)$$

or $V = \frac{1}{4\pi \in_0} \frac{q}{R} - \frac{1}{4\pi \in_0} \frac{q}{R^3} \left(\frac{r^2}{2} \right)_R^r$

or $V = \frac{1}{4\pi \in_0} \frac{q}{R} - \frac{1}{4\pi \in_0} \frac{q}{R^3} \left(\frac{r^2}{2} - \frac{R^2}{2} \right)$

or $V = \frac{1}{4\pi \in_0} q \left(\frac{1}{R} - \frac{r^2}{2R^3} + \frac{1}{2R} \right)$

$$V = \frac{1}{4\pi \in_0} \frac{q}{2R} \left(3 - \frac{r^2}{R^2} \right) \; V = \frac{q}{4\pi \in_0} \left(\frac{3R^2 - r^2}{2R^3} \right)$$

. . . (3.33)

to obtain potential at the centre on putting r = 0 in equation (3.33) we obtain

or

$$V_{centre} = \frac{3}{2} \left(\frac{q}{4\pi \in_0 R} \right) \qquad \text{. . . (3.34)}$$

or $V_{centre} = \frac{3}{2} V_S = 1.5 V_S$

Thus, potential at the centre of uniformly charged solid non conducting sphere is 1.5 times the value of potential at surface .

From above discussion we conclude that inside such a charged sphere potential decreases from. Centre to surface according to r^2 specific dependence and outside surface it decreases with r^{-1} dependence to become zero a infinity. This variation is shown graphically in Fig 3.15

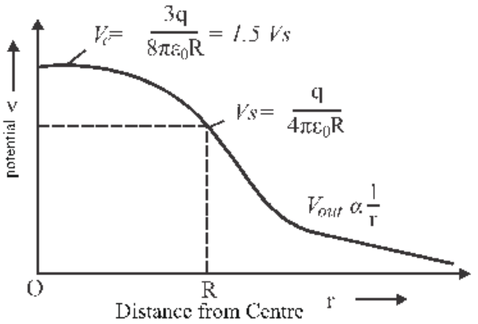


Fig 3.15 : Graph between electric potential versus the distance from centre of a uniformly charged insulating sphere

Example 3.9 A solid insulating sphere of radius 10 cm is given a charge of $3.2 \times 10^{\ 19}$ C. Determine electric potential at following points from the centre (i) at 14 cm (ii) at 10 cm (iii) at 4 cm.

Solution : Here $R = 10cm = 0.10m$

$q = 3.2 \times 10^{-19}\ C$

(i) $r = 14cm = 0.14m$

i.e. the observation point is outside the shpere

$$V = \frac{1}{4\pi \in_0} \frac{q}{r}$$

or $V = 9 \times 10^{9} \times \frac{3.2 \times 10^{\ 19}}{0.14} = 2.057 \times 10^{-8}\ \text{V}$

(ii) $r = 10\,\text{cm} = 0.10\,\text{m}$ $r = R$ i.e point is on the surface of sphere

$$V = \frac{1}{4\pi \in_0} \frac{q}{R} = \frac{9 \times 10^{9} \times 3.2 \times 10^{\ 19}}{0.10}$$

or $V = 2.88 \times 10^{\ 8}$ Volt

(iii) $r = 4\,\text{cm} = 0.04\,\text{m}$ i.e the point is inside the sphere

$P_3(r_3)$

$$V = \frac{1}{4\pi \in_0} \frac{q}{2R}\left[3 - \frac{r^2}{R^2}\right]$$

$$V = \frac{9 \times 10^{9} \times 3.2 \times 10^{-19}}{2 \times 0.10}\left[3 - \frac{(0.04)^2}{(0.10)^2}\right]$$

$$V = 1.44 \times 10^{-8}\left[3 - \frac{16}{100}\right]$$

$$V = 1.44 \times 10^{\ 8}[2.84]$$

$$V = 4.09 \times 10^{\ 8}\ \text{Volt}$$

3.8 Potential Energy of a System of Charges

In the beginning of this chapter we have seen that a potential energy is associated with a (conservative) electrocstatic field. Now we will discuss about the potential energy for a system of charges. Charges in such system exerts electrostatic force on each other. If positions of one or more charges is changed i.e the configuration of the system is changed then work is done by electrocstatic forces. If the system changes its configuration from an initial state i to a different final state f electrostatic force does work W_e on the particles, then by definition the change in potential energy of the system is

$$\Delta U = U_f - U_i = -W_e$$

i.e the change in potential energy as system changes its configuration is equal to the negative of the work done by the electrostatic force. We can also define U in terms of, W_{ext} the work done by external force. If we assume that the kinetic energy of system in both intitial and final states are zero, then

$$\Delta U = U_f - U_i = W_{ext}$$

For convenience we normally take the reference zero potential energy configuration of a system of charged particles to be that in which the particle are infinitely separated from one another. If this is so then $U_i = U_\infty = 0$ then final potential energy of the system is regarded as its potential energy, i.e.

$$U = -W_{\infty}$$

Where W_{∞} is the work done by electric forces on the particles during the move from infinity. If we wish to define U in terms of work done by external forces a change in kinetic energy, then

$$U = -W_{ex} = +W_{ext}$$

Where W_{ext} is now work done by external force in bringing the charges from infinity to the final configuration.

Based on above definitions let us first determine the potential energy of a two point charge system. Fig 3.16 depicts two point charges q_1 and q_2 of same nature separated by a distance r. First we assume that initially both the charges are infinity (for away) and at rest. When we bring q_1 from infinitely to its present (final) position no work is done either by electrostatic force or external force as no electric field was present. However when we bring q_2 from infinity then work has to be done by external agent because electrostatic force acts on the charge q_2 by q_1 during the move.

The electire potential due to charge q_1 at the location of charge q_2 is

$$V = \frac{1}{4\pi \in_0} \frac{q_1}{r}$$

and from the definition of potential

$$W_{\infty} = -W_e = qV$$

$$U = W_{\infty} = qV$$

On substituting $\quad q = q_2$

$$U = \frac{1}{4\pi \in_0} \frac{q_1 q_2}{r} \qquad \ldots (3.35)$$

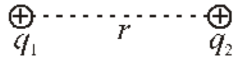


Fig 3.16 : System of two point charges

If the charges have the same sign external agent has to do positive work to push them together against their mutual repulsion. This work gets stored in the form of potential energy of system which is positive in this case. If the system is released then as the charges move apart the potential energy of the system now changes into the kinetic energies of the charges.

If the charges q_1 and q_2 have opposite signs the potential energy of system is negative. Note that depending upon the nature of charges electrostatic potential energy can either be positive or negative on the contrary the gravitational potential energy for a pair of particles is always negative.

3.8.1 Electrostatic Potential Energy of a System of more than Two point Charges

The total potential energy of a system of charged particles can be obtained by calculating potential energy for every pair of charges and summing the terms algebrically (with signs).

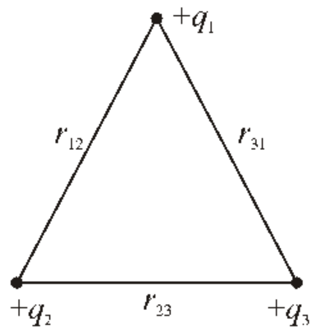


Fig 3.17 : A system of three point charges

Consider three point carges q_1, q_2 and q_3 fixed at points P_1, P_2 and P_3 as shown in fig 3.17, for such a system the potential energy can be determined as follows -

In bringing the charge q_1 from infinity to its position $P_1(\vec{r}_1)$ (while other charges are still at infinity) no work is done as no other charge is present in the region i.e

$$W_1 = 0$$

Then we bring charge q_2 in from infinity to its position P$_2$ (at a distance r_{12} from P$_1$) then the work done

$$W_2 = (\text{potential due to q}) ´\ q_2$$

or $$W_2 = \frac{1}{4\pi \in_0} \frac{q_1 q_2}{r_{12}}$$

Like wise the work done in moving charge q$_3$ from infinity and place it at P$_3$

$$W_3 = (\textit{Potential due to } q_1 \textit{ and } q_2) \times q_3$$

$$W_3 = \left(\frac{1}{4\pi \in_0} \frac{q_1}{r_{13}} = \frac{1}{4\pi \in_0} \frac{q_2}{r_{23}} \right) \times q_3$$

$$W_3 = \frac{1}{4\pi \in_0} \left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

So the potential energy of this system of charges is

$$U = W_1 + W_2 + W_3$$

$$U = O + \frac{1}{4\pi \in_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi \in_0} \left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

$$U = \frac{1}{4\pi \in_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

$$U = \frac{1}{4\pi \in_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \quad ...(3.36)$$

The above equation represents the potential energy of a system of three point charges, note that this expression contains three terms. The process can be extended to a four charge system and expression for potential energy can be determined by

$$U = W_1 + W_2 + W_3 + W_4$$

$$U = \frac{1}{4\pi \in_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)$$

. . . (3.37)

Which contains six term corresponding to six possible pairs of charges. If there are N charges in a system, the expression for potential energy will contain terms and we can write

$$U = \frac{1}{2} \sum_{j=1}^{j=N} \sum_{k=1}^{k=N} \frac{1}{4\pi \in_0} \frac{q_j q_k}{r_{jk}} \quad ...(3.38)$$

In above equation the factor of 1/2 before the summation sign ensures that although the pairs of charges are appearing twice but their contribution to the sum is effectively considered only once.

Example 3.10 Two protons are separated from each other by a distance of $6 \times 10^{\ 15}$ m . Find the electrostatic potential energy of the system in electron volt units.

Solution : Here $r = 6 \times 10^{-15}\, m$

$$q_1 = q_2 = 1.6 \times 10^{-19} C$$

As $$U = \frac{1}{4\pi \in_0} \frac{q_1 q_2}{r}$$

$$U = \frac{9 \times 10^9 \times 1.6 \times 10^{-19} \times 1.6 \times 10^{-19}}{6 \times 10^{\ 15}}$$

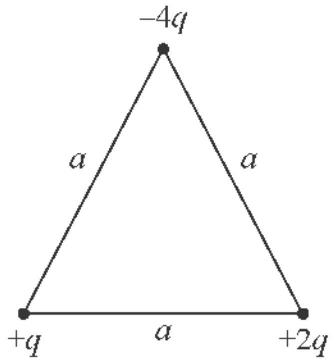
$$U = 3.84 \times 10^{-14} J$$

$$U = \frac{3.84 \times 10^{\ 14}}{1.6 \times 10^{-19}} \text{eV}$$

$$U = 0.24 \times 10^6 \text{eV} = 0.24 \text{MeV}$$

Example 3.11 Three charges are arranged as shown in Fig. Calculate the electrostatic potential energy of the system. Consider

$q = 1.0\times10^{-7}$ C and $a = 0.10$ m .



Solution : Total potential energy

$$U = U_{12} + U_{23} + U_{31}$$

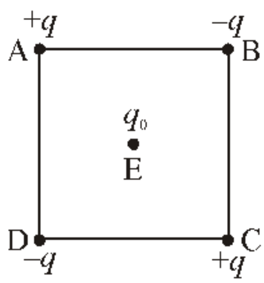
$$= \left(9.0\times10^{9}\right)\left[\frac{(+q)(-4q)}{a} + \frac{(+q)(+2q)}{a} + \frac{(-4q)(+2q)}{a}\right]$$

$$= 9.0\times10^{9}\times\left(-10q^{2}\right)/a$$

$$= -\frac{9.0\times10^{9}\times10\times\left(1\times10^{-7}\right)^{2}}{0.10} = -9.0\times10^{-3}\text{ J}$$

The negative potential energy means that an external agent would have to do 9×10^{-3} J of work to disassemble this configuration completely ending with three charges infinitely far apart.

Example 3.12 As shown in Fig four charges are placed at the vertices of a square of edge d. (a) calculate the work done in assembling this system (b) If some other charge q_0 is taken from infinity to the centre E of the square and all the four remain fixed at their location, how much additional work is to be done in the process.



Solution : (a) The work done in assembling the system is equal to the potential energy of the system.

Here there will be $\frac{4\times(4-1)}{2} = 6$ pairs of charges for such a four charge system.

$$W = U = k\left[\frac{q(-q)}{AB} + \frac{q(-q)}{AD} + \frac{qq}{AC} + \frac{(-q)q}{BC} + \frac{(-q)(-q)}{BD} + \frac{q(-q)}{CD}\right]$$

$$AB = BC = CD = AD = d$$

$$AC = BD = d\sqrt{2}$$

$$\therefore \quad W = -\frac{4kq^{2}}{d} + \frac{2kq^{2}}{d\sqrt{2}}$$

$$= -\frac{kq^{2}}{d}\left[4-\sqrt{2}\right] \text{ where } k = \frac{1}{4\pi\epsilon_0}$$

(b) Potential at centre E of the square due to charges at four corners

$$V = \frac{+Kq}{(AE)} + \frac{-Kq}{(BE)} + \frac{+Kq}{(CE)} + \frac{-Kq}{(DE)}$$

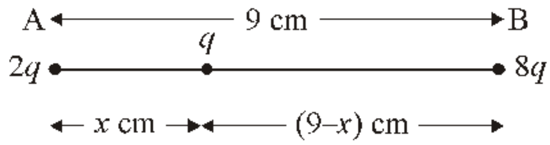
$$V = \frac{Kq}{d/\sqrt{2}} - \frac{Kq}{d/\sqrt{2}} + \frac{Kq}{d/\sqrt{2}} - \frac{Kq}{d/\sqrt{2}} = 0$$

Thus potential at point E is zero so no additional work is needed to be done in moving charge q_0 from infinity to E.

Example 3.13 Three point charges q, $2q$ and $8q$ are to be placed on a 9 cm long straight line. Find the position where the charge should be placed such that the potential energy of this system is minimum?

Solution : To have minimum potential energy, charges of greater value should be kept farthest thus charges $2q$ and $8q$ should be farthest separated by 9 cm.

Let the charge q is placed at a distance of x cm from $2q$ [Fig] then the potential energy of system



$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{2q\times q}{x\times 10^{-2}} + \frac{8q\times q}{(9-x)\times 10^{-2}} + \frac{2q\times 8q}{9\times 10^{-2}}\right]$$

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2}{10^{-2}}\right]\left[\frac{2}{x} + \frac{8}{9-x} + \frac{16}{9}\right]$$

For U to be minimum $\frac{dU}{dx} = 0$

$$\frac{dU}{dx} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{10^{-2}}\left[-\frac{2}{x^2} + \frac{8}{(9-x)^2} + 0\right] = 0$$

$$\frac{2}{x^2} = \frac{8}{(9-x)^2}$$

$$\frac{1}{x^2} = \frac{4}{(9-x)^2}$$

$$(9-x)^2 = 4x^2$$

$$(9-x) = \pm 2x$$

$$x = 3cm$$

or $x = -9cm$

here x = -9 cm is not possible so charge q should be placed in between $2q$ and $8q$ at a distance of 3 cm from $2q$.

3.9 Work Done in Rotating an Electric Dipole in Electric Field

When a electric dipole is placed in an electric field a torque acts on it. This torque has a tendency to align the dipole along the field. So work has to be done to rotate a dipole if it is in equilibrium under the field.

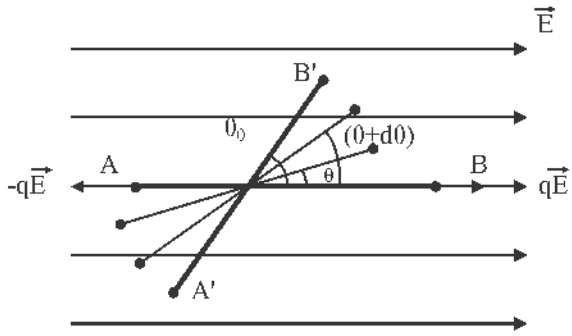


Fig 3.18 : Rotating a dipole in a uniform electric field

Consider a dipole in a uniform electric field $\vec{E}$ as shown in fig 3.13. Initially the dipole is in equilibrium in position AB. Now the dipole is rotated to bring it into position 'AB' making angle θ_0 with direction of $\vec{E}$.

At an angular position θ the torque acting on the dipole is $\tau = pE\sin\theta$

work done is rotating the dipole through a small angle $d\theta$ is then

dW = torque × angular displacement

$$dW = \tau d\theta$$

$$dW = pE\sin\theta d\theta$$

So the work done in rotating the dipole from angular position $\theta = 0^\circ$ to $\theta = \theta_0$ is

$$W = \int_{\theta=0^\circ}^{\theta=\theta_0} pE\sin\theta\, d\theta$$

$$W = pE\left[-\cos\theta\right]_0^{\theta_0}$$

$$W = pE\left(\cos 0^\circ - \cos\theta_0\right)$$

$$W = pE\left(1-\cos\theta_0\right)$$

If $\theta = 0^\circ$ then

$$W = pE\left(1-\cos\theta\right) \quad \ldots (3.39)$$

Also note that

(i) Work done in rotating a dipole from angular position θ_1 to θ_2 with respect to field

$$W = pE(\cos\theta_1 - \cos\theta_2) \quad \ldots (3.40a)$$

(ii) If $\theta_1 = 0^\circ$ and $\theta_2 = 90^\circ$ then $W = pE$

$$\ldots (3.40b)$$

(iii) If $\theta_1 = 0^\circ$ and $\theta_2 = 180^\circ$ then $W = 2pE$

$$\ldots (3.40c)$$

3.10 Potential Energy of an Electric Dipole in Electric Field

The potential energy of a dipole in an electric field is equal to the work done in bringing the dipole from infinity into the field.

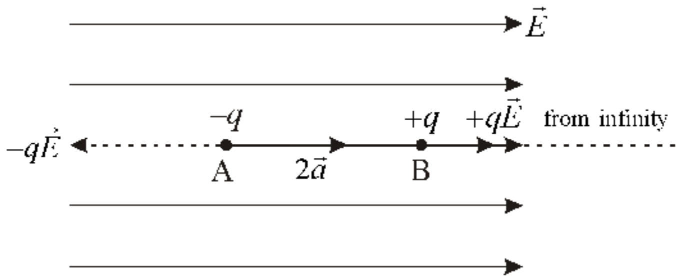


Fig 3.19 : Dipole in external electric field

In fig 3.19 an electric dipole is brought from infinity to its final position in a uniform electric field such that during the move dipole moment $\vec{P}$ always points in direction of $\vec{E}$. Due to electric field force on charge q_1 $\vec{F} = q\vec{E}$ is in direction of $\vec{E}$ and that on -q is $\vec{F} = -q\vec{E}$ is opposite to $\vec{E}$. Thus in bringing the dipole in field $\vec{E}$ external work is to be done on the charge + q while electric field does work on -q. In the move from infinity to their respective final position in field, charge -q covers a distance 2a more than charge +q. Thus work done by electric field is more and negative. This work done is given by

$W =$ Force on $(-q) \times$ additional distance covered by -q

$$W = -qE \times 2a = -2qaE$$

$$W = -pE \qquad \because p = 2qa$$

So, the potential energy of dipole, aligned with with $\vec{E}$ is

$$U_1 = -pE \quad \ldots (3.41)$$

Now, the additional work done in rotating dipole from this position to angular position θ is

$$U_2 = pE(1 - \cos\theta) \quad \ldots (3.42)$$

Thus the potential energy of dipole placed at angle θ with respect to the field is

$$U = U_1 + U_2$$

$$U = -pE + pE(1 - \cos\theta)$$

$$U = -pE\cos\theta$$

which can be rewritten as

$$U = -\vec{p}\cdot\vec{E} \quad \ldots (3.43)$$

Equation (3.4 1) is the expression for the potential energy of electric dipole placed in uniform electric field.

Special Cases

(a) If the dipole is aligned with electric field

$$\theta = 0^\circ$$

$$U = -pE\cos\theta$$

$$U = -pE\cos 0^\circ$$

or $U = -pE \quad \ldots (3.44)$

In this position the electric dipole is in stable equilibrium, as in this case potential energy is minimum.

(b) If the dipole moment is perpendicular to electric field

$$U = -pE\cos 90^\circ$$

$$U = 0 \quad \ldots (3.45)$$

(c) If the dipole moment makes angle $\theta = 180^\circ$ with direction of $\vec{E}$.

$U = -pE\cos 180^\circ$

$U = pE$ . . . (3.46)

This position is called position of unstable equilibrium as in this situation potential energy of dipole is maximum.

Example 3.14 An electric dipole consits of two point charges $+1.0\times10^{-6}$ and -1.0×10^{-6} at a separation of 2 cm. This dipole is placed in a uniform electric field 1.0×10^{5} V/m. Find

(a) maximum torque on it due to electric field

(b) potential energy of dipole in position of stable equilibrium

(c) potential energy of dipole in angular position of 180° with respect to the position of stable equilibrium.

(d) energy needed to rotate the dipole through 90° with respect to the stable equilibrium position.

Solution : Here $q = 1\times10^{-6}$ C

$2a = 2cm = 2\times10^{-2}m, E = 1\times10^{5}$ V/m

dipole moment $p = q2a = 1\times10^{-6}\times2\times10^{-2}$

$= 2\times10^{-8}\,C - m$

(a) maximum torque $\tau = pE = 2\times10^{-8}\times1\times10^{5}$

$= 2\times10^{-3}\,N - m$

(b) potential energy in stable equilibrium position

$U = -pE$

$U = -2\times10^{-8}\times1\times10^{5} = -2\times10^{-3}J$

(c) potential energy in rotated position (relative to stable equilibrium)

$U = +pE = +2\times10^{-3}J$

(d) Work done in rotating dipole through angle 90° relative to stable equilibrium position

$W = pE(1-\cos\theta)$

$= 2\times10^{-8}\times1\times10^{5}(1-\cos 90^\circ)$

$W = 2\times10^{-3}J$

Important Points

1. **Electric potential :** The electric potential at a point in an electric field is equal to the work done against the electric field in bringing a unit positive charge from infinity to the point without changing the kinetic energy of the charge. Its S.I. unit is volt

$$V = -\int_{\infty}^{R}\vec{E}\cdot d\vec{r}$$

2. **Potential difference :** The potential difference between two points in an electric field is equal to the negative of the work done by electric field or work done by external agent (without change in kinetic energy) in moving a unit positive charge from initial to final point. Its S.I. unit is volt

$$V_A - V_B = \frac{W_{AB}}{q_o} = \int_{B}^{A} -\vec{E}\cdot d\vec{r}$$

3. Potential due to a point charge

$$V = \frac{Q}{4\pi \in_0 r}$$

4. Potential due to a system of point charges

$$V = V_1 + V_2 + V_3 ... + V_n$$

5. Electric potential due to an electric dipole at position (r, θ) [r > > a]

$$V = \frac{1}{4\pi \in_0} \frac{p\cos\theta}{r^2} \quad \text{if } a^2 << r^2$$

6. Electric potential at equatorial points of a dipole is zero.

7. Equipotential surface : An equipotential surface in an electic field is a surface at all points of which electric potential is same.

8. The electric field $\vec{E}$ is always directed perpendicular to corresponding equipotential surfaces.

9. $\vec{E} = -grad\ v = -\nabla V$

10. Electric potential due to a uniformly charged spherical shell or charged spherical conductor

(i) external points $(r > R)$ $V = \frac{1}{4\pi \in_0} \frac{q}{r}$

(ii) surface $(r = R)$ $V = \frac{1}{4\pi \in_0} \frac{q}{R}$

(iii) internal point $(r < R)$ $V = \frac{1}{4\pi \in_0} \frac{q}{R}$

11. Electric potential due to a uniformly charged insulating solid sphere

(i) external point $(r > R)$ $V = \frac{1}{4\pi \in_0} \frac{q}{r}$

(ii) surface $(r = R)$ $V = \frac{1}{4\pi \in_0} \frac{q}{R}$

(iii) internal point $(r < R)$ $V = \frac{1}{4\pi \in_0} \left[\frac{3R^2 - r^2}{2R^3} \right]$

12. Electric potential energy of a system of two point charges

$$U=\frac{1}{4\pi\in_0}\frac{q_1q_2}{r}$$

13. Potential energy of a system of N point charges

$$U=\frac{1}{2}\sum_{k=1}^{j=N}\sum_{\substack{k=1\\ j\neq k}}^{k=N}\frac{1}{4\pi\in_0}\frac{q_jq_k}{r_{jk}}$$

14. Work done in rotating an electric dipole in external field

(i) from $\theta=0$ to θ

$$W=pE(1-\cos\theta)$$

(ii) from $\theta=\theta_1$ to $\theta=\theta_2$

$$W=pE(\cos\theta_1-\cos\theta_2)$$

15. Potential energy of dipole in external electic field $U=-pE\cos\theta$

Questions For Practice

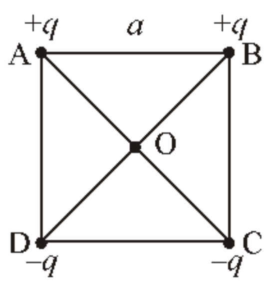
Multiple Choice Questions

1. At certain distance from a point charge electric field and potential are 50 V / m and 300 V respectively, this distance is

(a) 9 m (b) 15 m

(c) 6 m (d) 3 m

2. Four charges are placed on corners of a square as shown in fig. Let electric field and potential at its centre are $\vec{E}$ and V. If the charges at A and B are interchanged with charges placed at C and D. Then



(a) $\vec{E}$ remains the same but V is changed

(b) both $\vec{E}$ and V are changed

(c) both $\vec{E}$ and V are unchanged

(d) $\vec{E}$ is changed but V remains the same

3. The electric potential at some point in an electric field is 200 V. The work done in moving an electron from infinity to that point is -

(a) -3.2×10^{-17} J (b) 200 J

(c) - 200 J (d) 100 J

4. Two charged conducting spheres of radii r_1 and r_2 are at some potential. The ratio of their surface charge densities is

(a) $\frac{r_2}{r_1}$ (b) $\frac{r_1}{r_2}$

(c) $\frac{r_2^2}{r_1^2}$ (d) $\frac{r_1^2}{r_2^2}$

5. A charge of 10 μC is located at the origin of X - Y coordinate system. The potential difference between points (a, 0) and $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ is

(a) 9×10^4 (b) Zero

(c) $\frac{9\times10^4}{a}$ (d) $\frac{9\times10^4}{\sqrt{2}}$

6. The electric potential at the surface of a charged spherical hollow conductor of radius 2 m is 500 V. The potential at a distance 1.5 m from centre is

(a) 375 V (b) 250 V

(c) Zero (d) 500 V

7. An α particle is moved from rest from a point where potential is 70 V to another point having potential 50 V. The kinetic energy of α particle at the second point is

(a) 20 eV (b) 40 eV

(c) 20 MeV (d) 40 MeV

8. In some region where electric field intensity E is zero, the electric potential varies with distance according to

(a) $V \propto \frac{1}{r}$ (b) $V \propto \frac{1}{r^2}$

(c) V = Zero (d) V = Constant

9. Two conducting spheres of radii R_1 and R_2 respectively have the same surface charge density. If the electric potentials at their surface are V_1 and V_2 respectively then V_1 / V_2 is equal to

(a) $\frac{r_1}{r_2}$ (b) $\frac{r_2}{r_1}$

(c) $\frac{r_1^2}{r_2^2}$ (d) $\frac{r_2^2}{r_1^2}$

10. The electric potential function for some electric field is defined by $V = -5x + 3y + \sqrt{15}z$. The intensity of electric field (in S.I. units) at point (x, y, z) is

(a) $3\sqrt{2}$ (b) $4\sqrt{2}$

(c) $5\sqrt{2}$ (d) 7

11. A unit charge is moved in a circular path of radius r having a point charge q at the centre. The work done in one complete rotation is

(a) Zero (b) $\frac{1}{4\pi\in_0}\frac{q}{r^2}$

(c) $2\pi r.J$ (d) $2\pi rq.J$

12. For a system of two electrons on bringing one elctron nearer to other, the electrostatic potential energy of system

(a) increases (b) decreases

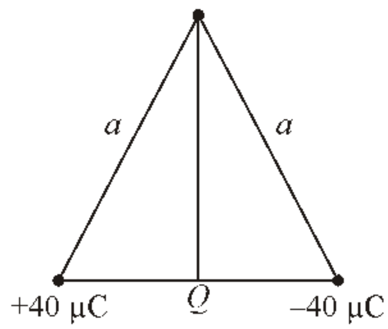
(c) remains the same (d) become zero

13. 1000 tiny water droplets each of radius r and charge q combine to form a big drop. The electric potential of the bigger drop as compared to a tiny droplet is increased by a factor of

(a) 1000 (b) 100

(c) 10 (d) 1

14. For the arrangement of charges as shown in adjoining diagram, the work done in moving a 1 C charge from P to Q (in joule) is



(a) 10 (b) 5

(c) infinite (d) zero

15. 64 identical mercury drops (each having a potential 10 V) are combined to form a bigger drop. The potential at the surface of this bigger drop will be

(a) 80 V (b) 160 V

(c) 640 V (d) 320 V

Very Short Answer Question

1. Mention whether electric potential is a scalar or vector quantity.
2. Give definition of electric potential.
3. Can two equipotential surfaces intersect each other at some point?
4. What will be the electric potential due to a charge, at infinity.
5. Can electric potential at some point in free space be zero though electric field may not be zero at that point? Give exapmle.
6. Can electric field at some point be zero though electric potential is not zero? Give example.
7. What will be the work done in moving a 200 μC charge in moving it form one point to another point at a distance of 10 cm on the same equipotential surface?
8. What is the shape of equipotential surface for the following

   (a) due to a point charge

   (b) for a uniform electric field
9. What is the potential energy of an electric dipole placed parallel to an electric field?
10. The electric potential at the surface of a charged spherical conductor of radius 10 cm is 15 V. What is the electric potential at its centre.
11. The electric potential at the surface of a uniformly charged non conducting sphere of radius 5 cm is 10 V. What is the electric field at its centre.
12. The electric potential at a point (x, y, z) (all in meters) in free space is given by $V = 2x^2$ volt. Determine electric field intensity at point (1m, 2m, 3m).
13. Write the expression for potential energy of a two point charge system.
14. Write the expression for potential energy for a system of three point charges.
15. Write the S.I. unit for potential gradient.
16. How much work is to be done in moving an electron between two points having a potential difference of 20 V.
17. The electric potential due to a point charge at some point in free space is 10V. If the entire system is now placed in a dielectric medium of dielectric constant 2. What will be the potential at the same point?
18. Write the expression for work done in rotating an electric dipole from 0^0 to 180^0 in a uniform field.
19. Write the value of electric potential of earth as has been assumed?
20. If potential function is V = 4x + 3y volt then calculate magnitude of electric field intensity at point (2,1) meter.

Short Answer Question

1. What is meant by electric potential? Write its formula and S.I. unit.
2. Show that the potential inside the potential inside a charged spherical shell is same as that on its surface.
3. What is meant by equipotential surface. Draw equipotential surfaces for a point charge.
4. Determine the expression for potential energy for a system of three point charges.
5. The electric potential in complete volume of a charged conductor is same as that on its surface, why?
6. Derive relation between electric potential and field.
7. Derive expression for work done in rotating an

electric dipole in a uniform electric field.

8. Show that no work is done in moving a test charge from one point to other on an equipotential surface.

9. What is meant by electrostatic potential energy? Derive expression for potential energy of a system of point charges.

10. Determine expression for potential energy of an electric dipole in external electric field.

11. Write expression for electrostatic potential energy for a system of two point charges q_1 and q_2 placed at positions given by position vectors $\vec{r}_1$ and $\vec{r}_2$ in a uniform electric field.

12. Write two properties of equipotential surface.

13. Show that the electric potential due to a point charge when surrounded by some dielectric medium is $1/\in_r$ times of the electric potential when the charge is in free space.

14. Show that the electric potential at the centre of a uniformly charged insulating solid sphere is 1.5 times the value of potential at its surfaces.

15. Two charges 10 μC and 5 μC are 1 m apart. To decreases the separation to 0.5 m how much work has to be done.

16. Define electric potential difference. Distinguish between potential difference and potential.

Essay type Questions

1. Derive an expression for electric potential at a point due to a point charge.

2. Derive an expression for electric potential at a point due to an electric dipole. Show that potential is maximum for axial point while is zero for equatorial line.

3. Derive expression for electric potential due to a charged spherical shell at points (i) outside, (ii) at the surface and (iii) inside the shell. Draw the graph showing variation of potential with distance.

4. Derive expressions for electric potential due to a uniformly charged spherical non conducting solid sphere at points (i) outside (ii) at the surface and (iii) inside the sphere. Draw the graph showing variation of potential with distance.

5. Define electrostatic potential energy. Dervie expression for an electric dipole in a uniform electric field. For what positions; states of stable and unstable equilibrium are obtained?

Answers (Multiple Choice Questions)

1. (c) 2. (d) 3. (a) 4. (a) 5. (b) 6. (d)
7. (b) 8. (d) 9. (a) 10. (d) 11. (a) 12. (a)
13. (b) 14. (d) 15. (b)

Very Short Answer Questions

1. Scalar quantity

2. It is equal to the work done by external agent in brining a unit positive charge from infinity to the point under consideration without changing the kinetic energy of the charge.

3. No, otherwise there would be two values of electric potential which is absured.

4. Zero

5. Yes, electric potential at points on equatorial line of on electric dipole is zero but electric field is not zero.

6. Yes, (a) the electric field at the mid point on the line joining two identical charges is zero but electric potential is not.

   (b) electric field inside a charged spherical shell is zero but electric potential is not.

7. As the potential difference between any two points on a given equipotential surface $\Delta V = 0$

   Work $W = q\ \Delta V = 0$

8. (a) Spherical surfaces centred at the point charge

   (b) parallel planes oriented normal to the electric field

9. $U = -pE$

10. 15 V

11. 15 V

12. $E_x = -\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\left(2x^2\right) = -4x$

$E_y = E_z = 0$

$\vec{E} = -4x\hat{i}$

$\vec{E}_{(\text{at }1,2,3)} = -4\times 1\hat{i} = -4\hat{i} \;\; v/m$

13. $U = \frac{1}{4\pi\in_0}\frac{q_1q_2}{r}$

14. $U = \frac{1}{4\pi\in_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_3q_1}{r_{31}}\right]$

15. V/m

16. $W = qV = 1.6\times10^{\;19}\times 20 = 32\times 10^{\;19} J$

17. $V_m = \frac{V}{E_r} = \frac{10}{2} = 5v$

18. $W = pE\left(\cos\theta_1 - \cos\theta_2\right) = pE\left(\cos 0^\circ - \cos 180^\circ\right)$

$W = 2pE$ joule

19. Zero

20. $\vec{E} = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y}\right]$

$\vec{E} = -4i - 3j$

$\left|\vec{E}\right| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5V/m$

Numberical Problems

1. 6 J of work is needed to move a 3 C point charge between two points. Find the potential difference between the two points. [Ans : 2 V]

2. If the electric potentials at two points A and B are 2 V and 4 V respectively then find the work need to move a 8 μC charge from A to B.

[Ans : $\left(1.6\times10^{-5} J\right)$]

3. Four charges, 100 μC , -50 μC , 20 μC and -60 μC respectively are placed on four corners of a square of edge $\sqrt{2}$ m. Find the electric potential at the centre of square.

[Ans : $\left(9\times10^4 V\right)$]

4. Two point charges, $3\times10^{-8} C$ and $-2\times10^{-8} C$ are 15 cm apart. At what point(s) on the line joining the charges the electric potential is zero? Assume the electric potential to be zero at infinity.

[Ans : 9 cm away from positive charge, and 45 cm away from it towards negative charge]

5. Four charges, –2 μc, +3 μc, –4 μc and +5 μc respectively are placed on corners of a square of edge 0.9 m. Find the electric potential at the centre of the square. Ans : $\left(2.8\times10^4 V\right)$

6. A charge of 5 μC is placed on each of the vertices of a regular hexagon of side 10 cm. Find the electric potential at the centre of hexagon.

Ans : $\left(2.7\times10^6 V\right)$

7. Four charges each 2 μC is placed on four corners of a square of side $2\sqrt{2}$ m. Find the potential at the centre of square. Ans : $\left(36\times10^5 V\right)$

8. Three charges 1 μC, 2 μC and 3 μC respectively are placed on the vertices of an equilateral triangle of 100m cm side. Calculate the electric potential at the centre of the triangle. [Ans : 93.6 V]

9. Two charges –1 μC and +1 μC at a separation of $4\times10^{\;14} m$ forms an electic dipole. Calculate electric potential at an axial point located at a distance $2\times10^{\;6} m$ from centre.

Ans : $\left(9\times10^{\;2} V\right)$

10. (a) Calculate eletric potential due to a $4\times10^{-7} C$ point charge at a point at a distance 9 cm from it.

(b) now, Find the work done in bringing another $2\times10^{-9}\,C$ charge from infinity to this point.

11. A 30 μC charge is located at the origin of X - Y coordinate system. Find the potential difference between points $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and (a, 0).

12. Three charges -q, + q and + q are located in X - Y plane at points (0, -a) (0,0) and (0, a). Show that potential at point at a distance r on a line inclined at angle θ to the axis is given by

$$V = \frac{1}{4\pi \in_0}\left(\frac{q}{r} + \frac{2qa\cos}{r^2}\right), r >> a$$

13. How much work has to be done in putting charges $+q$, $2q$ and $+4q$ respectively at the corners of an equilateral triangle of side 'a'.

Ans : $\left(\frac{1}{4\pi \in_0}\frac{14q^2}{a}\right)$

14. No external electric field is applied on a system of two charges 7 μC and –2 μC located at (–9 cm, 0,0) and (+9 cm, 0, 0). Determine the electrostatic potential energy of this system.

(b) how much work is needed to make the charges separate by infinite distance.

[Ans : = 0.7 J, 0.7 J]

15. For some electric field, potential at a point (x,y) is given as $V = 6xy + y^2 - x^2$. Determine the electric field at this point.

Ans : $\vec{E} = (2x - 6y)\hat{i} - (6x + 2y)\hat{j}$

16. A hollow metallic sphere of radius 0.2 m is given a charge of + 15μc . Find (i) electric potential at its surface (ii) potential at its centre (iii) electric potential at (iv) potential at 0.3 m from centre.

Ans : (i) 8.75×10^5 V ; (ii) 8.75×10^5 V

(iii) 8.75×10^5 V ; (iv) 4.5×10^5 V

17. Three charges, $+q$, $+2q$ and xq respectively are placed at the vertices of an equilateral triangle of side r. Find the value of x for which the potential energy of system becomes zero.

Ans : $(x = -2/3)$