3 Boolean Algebra

3.1 BOOLEAN ALGEBRA

In 1854, George Boole introduced the following formalism which eventually became Boolean Algebra.

Definition. *An algebraic system consisting of a set B of elements a, b, c ... and two binary operations called the* ***sum*** *and* ***product****, denoted respectively by + and $\bullet$, is called a* ***boolean algebra*** *iff for all a, b, c $\in$ B, the following axioms are satisfied :*

(1) $a + b,\ a \bullet b \in B$ *(closure property)*

(2) $a + b = b + a$ *and* $a \bullet b = b \bullet a$ *(commutative property)*

(3) $(a + b) + c = a + (b + c)$ *and* $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ *(associative property)*

(4) $a \bullet (b + c) = a \bullet b + a \bullet c$ *and* $a + (b \bullet c) = (a + b) \bullet (a + c)$ *(distributive laws)*

(5) *An additive identity 0 and a multiplicative identity 1 (both belonging to B) exist such that* $\forall\, a \in B,\ a + 0 = a$ *and* $a \bullet 1 = a$ *(identity)*

(6) *for every* $a \in B$*, there exists an element* $a' \in B$ *such that* $a + a' = 1$ *and* $a \bullet a' = 0$ *(complement or inverse)*

Remarks

1. Some authors include another axiom—that of *cardinality*—that there are atleast two elements a and b in B such that $a \neq b$.
2. Some axioms, especially (6) are quite different from usual arithmetic or algebraic structures. First, the additive and multiplicative inverses of an element are usually different, *e.g.*, in real numbers, additive inverse of 2 is (– 2) whereas multiplicative inverse of 2 is $\frac{1}{2}$. In boolean algebra, both inverses are the same. Secondly, in a usual algebra, we have $a + a' = 0$ and $a \bullet a' = 1$. In boolean algebra, we have $a + a' = 1$ and $a \bullet a' = 0$. This is more like sets, where $A \cup A' = \xi$ (universal set) and $A \cap A' = \phi$ (null set). Observe that ξ acts as multiplicative identity since $A \cap \xi = A$ and ϕ acts as additive identity (*i.e.* zero element) since $A \cup \phi = A,\ \forall\, A \subseteq \xi$. Also note that in boolean algebra, inverse of a is generally denoted as a' and not a^{-1}.
3. The distributive property $a \bullet (b + c) = a \bullet b + a \bullet c$ is similar to that of real numbers, but the distribution of + over $\bullet$ *i.e.* $a + (b \bullet c) = (a + b) \bullet (a + c)$ does not hold for conventional algebra.
4. In conventional algebra, $x + x + x + \ldots$ *n times* $= n\,x$, but in boolean algebra, $x + x + x + \ldots$ *n times* $= x$. This is called idempotent law.
5. Since family of sets is a classical example of a boolean algebra, many texts use $\cup$ and $\cap$ instead of + and $\bullet$.

6. *Modern computers and telecommunications use boolean algebra a lot* — binary digits (or bits) 0 and 1 correspond to electrical switch off or on, current absent or flowing, bulbs off or on, capacitor discharged or charged etc. This will become clearer as we proceed.
7. Sometimes $a \bullet b$ is written simply as ab.
8. When parentheses are not used, the operation • has precedence over +. Thus, in $a + b \bullet c$, we first evaluate $b \bullet c$.

Example. *Let B = {0, 1} and let two operations + and • be defined on B as follows :*

+	1	0
1	1	1
0	1	0

•	1	0
1	1	0
0	0	0

Then B, or more precisely the triplet (B, +, •) is a boolean algebra. (Check ! Are all axioms (1) to (6) satisfied?) Here 1 is the multiplicative identity and 0 is the additive identity. This is the smallest possible Boolean Algebra called Switching Algebra.

More examples of Boolean Algebras

We have already given one example of a Boolean Algebra B = {0, 1}.

Now as second example, let A be a family of sets which is closed under the operations of union, intersection and complement. Then (A, ∪, ∩) is a Boolean Algebra. Note that universal set ξ is the unit element and the null set ϕ is the zero element.

ILLUSTRATIVE EXAMPLES

Example 1. *If V = {1, 2, 3}, A = {1, 2}, then A′ = {3}. Show that the set T = {V, A, A′, φ} along with operations ∪ and ∩ forms Boolean algebra.*

Solution. We have to show that this system satisfies basic axioms of Boolean Algebra.

The composition tables for ∪ and ∩ are as follows :

∪	V	A	A′	φ
V	V	V	V	V
A	V	A	V	A
A′	V	V	A′	A′
φ	V	A	A′	φ

∩	V	A	A′	φ
V	V	A	A′	φ
A	A	A	φ	φ
A′	A′	φ	A′	φ
φ	φ	φ	φ	φ

(1) Closure : all unions and intersections of sets V, A, A′, φ belong to T.

(2) Operations ∪ and ∩ are commutative in sets.

(3) Operations ∪ and ∩ are associative in sets.

(4) Operations ∪ and ∩ are distributive in sets
(*i.e.* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ etc.)

(5) Identity element for ∪, that is, zero element (or additive identity) is φ, since $V \cup \phi = V$; $A \cup \phi = A$; $A' \cup \phi = A'$; $\phi \cup \phi = \phi$. Identity element for operation ∩ (that is, unit element) is V since $V \cap V = V$; $A \cap V = A$; $A' \cap V = A'$; $\phi \cap V = \phi$.

(6) Inverse : Inverse of A is A′ and inverse of V is φ (check).

Since all basic axioms are satisfied, the system (T, ∪, ∩) is a boolean algebra.

Example 2. *Let D_n denote the set of divisors of n, where n is a natural number, define operations +, $\bullet$ and ′ on D_n as*

$$a + b = \text{lcm } \{a, b\} \text{ i.e. least common multiple of } a \text{ and } b,$$

$$a \bullet b = \text{gcd } \{a, b\} \text{ i.e. greatest common divisor of } a \text{ and } b, \text{ and } a' = \frac{n}{a}.$$

Prove that D_4 is not a boolean algebra, while D_6 is a boolean algebra.

Solution. (*i*) For D_4 = {1, 2, 4}, the tables for given operations are

+	1	2	4
1	1	2	4
2	2	2	4
4	4	4	4

•	1	2	4
1	1	1	1
2	1	2	2
4	1	2	4

′	1	2	4
	4	2	1

From these we observe that operations + and • are closed, commutative and associative.

For distribution of + over •, we see that

$1 + (2 \bullet 4) = 1 + 2 = 2$;

$(1 + 2) \bullet (1 + 4) = 2 \bullet 4 = 2$ etc.

Similarly, we can verify for other elements, and also the distribution of • over +.

Now zero element (additive identity) is such that

a + zero element = $a \ \forall \ a \in D_4$.

From table of '+', we see that $a + 1 = a \ \forall a$, so that 1 acts as zero element (additive identity), while from table of '•' we see that $a \bullet 4 = a \ \forall a$, so that 4 acts as unit element (multiplicative identity).

Thus, we should have $a + a'$ = unit element, $a \bullet a'$ = zero element $\forall \ a$.

From table of ′, we see that $2' = 2$.

However from table of +, we see that $2 + 2 = 2 \neq 4$ (unit element).

Also from table of •, we see that $2 \bullet 2 = 2 \neq 1$ (zero element).

Hence, D_4 is not a boolean algebra.

(*ii*) For D_6 = {1, 2, 3, 6}, we construct tables for operations as

+	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

•	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

′	1	2	3	6
	6	3	2	1

As above, we can verify closure, commutative, associative and distributive properties.

From + table, we see that $a + 1 = a \ \forall \ a$, so that 1 acts as zero element (additive identity), while from • table, we see that $a \bullet 6 = a \ \forall \ a$, so that 6 acts as unit element (multiplicative identity).

Now we verify $a + a'$ = unit element, $a \bullet a'$ = zero element $\forall \ a$:

$1 + 6 = 6, \ 1 \bullet 6 = 1$

$2 + 3 = 6, \ 2 \bullet 3 = 1$

$3 + 2 = 6, \ 3 \bullet 2 = 1$

$6 + 1 = 6, \ 6 \bullet 1 = 1$

Thus, all properties of boolean algebra are satisfied.

Note. In general, if n is a product of distinct prime numbers, then D_n = {set of positive divisors of n} is a boolean algebra, with least common multiple acting as +, greatest common divisor acting as •, $\frac{n}{a}$ acting as complement of a, integer 1 acting as zero element (additive identity) and integer n acting as unit element (multiplicative identity). However, if in the prime factorisation of n, any number is repeated, then D_n is not a boolean algebra.

EXERCISE 3.1

1. If B = {ϕ, ξ, S, S′} where S is any non-empty subset of ξ, then show that B along with the operations $\cup$ and $\cap$ forms a boolean algebra.
2. Let B = {{1}, {2}, {1, 2}, ϕ}. Show that (B, $\cap$, $\cup$, ′, ϕ, {1, 2}) is a boolean algebra.
3. Let B = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}. Show that (B, $\cup$, $\cap$, ′, ϕ, {1, 2, 3}) is a boolean algebra.
4. Let D_n = set of divisors of n, where n is a natural number.
   Define operations +, $\bullet$ and ′ as
   $$a + b = \text{lcm}\ \{a, b\},\ a \bullet b = \text{gcd}\ \{a, b\},\ a' = \frac{n}{a}.$$
   Prove that D_8 is not a boolean algebra while D_{15} is a boolean algebra.

3.2 DUALITY IN A BOOLEAN ALGEBRA

By definition, the **dual** of any statement in a boolean algebra is the statement derived by interchanging + and $\bullet$ and also interchanging the identities 1 and 0 in the original statement.

For example, the dual of statement $a + b = b + a$ is $a \bullet b = b \bullet a$, and dual of statement $a + a = 1$ is $a \bullet a = 0$.

The **Principle of Duality** says that the *dual of any theorem in a Boolean algebra is also a theorem.*

A theorem is derived from the basic axioms; dual statement of theorem can be proved by using the dual of each step of the proof of original theorem. For example, if we can prove that $a + a = a$, it will follow that $a \bullet a = a$; if we can prove that $a + 1 = 1$, it will follow that $a \bullet 0 = 0$. Thus, when we prove a theorem in boolean algebra, we get its dual for free.

3.3 ELEMENTARY PROPERTIES OF BOOLEAN ALGEBRA

Starting from basic axioms (1) to (6), a number of properties of Boolean Algebra can be proved. Important of these are :

(*i*) **(Idempotent Law) : $a + a = a$ and $a \cdot a = a$.**
(*ii*) **(Boundedness Laws) $a + 1 = 1$ and $a \cdot 0 = 0$.**
(*iii*) **(Involution Law) : $(a')' = a$.**
(*iv*) **$1' = 0$ and $0' = 1$.**
(*v*) **(De Morgan's Laws) : $(a + b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$.**
(*vi*) **(Law of absorption) : $a + (a \cdot b) = a$ and $a \cdot (a + b) = a$.**
(*vii*) **(Uniqueness of inverse) : for every $a \in$ B, a' is unique.**

Proofs of properties:

(*i*) To prove that $a + a = a$

$a = a + 0$	(by Identity)
$= a + (a \bullet a')$	(by Inverse)
$= (a + a) \bullet (a + a')$	(by Distributive Law)
$= (a + a) \bullet 1$	(by Inverse)
$= a + a$	(by Identity)

To prove that $a \bullet a = a$

$a = a \bullet 1$	(by Identity)
$= a \bullet (a + a')$	(by Inverse)
$= a \bullet a + a \bullet a'$	(by Distributive Law)
$= a \bullet a + 0$	(by Inverse)
$= a \bullet a$	(by Identity)

Aliter, the property $a \bullet a = a$ follows by duality from $a + a = a$.

(ii) To prove that $a + 1 = 1$

1	$= a + a'$	(by Inverse)
	$= a + a' \bullet 1$	(by Identity)
	$= (a + a') \bullet (a + 1)$	(by Distributive Law)
	$= 1 \bullet (a + 1)$	(by Inverse)
	$= a + 1$	(by Identity)

The property $a \bullet 0 = 0$ follows by duality.

Exercise : Prove $a \bullet 0 = 0$ from basic axioms!

(iii) To prove Involution Law : $(a')' = a$

As a' is inverse (complement) of a, we have

$a + a' = 1$ and $a \bullet a' = 0$.

It can be written as $a' + a = 1$ and $a' \bullet a = 0$.

Therefore, by axiom (6), $(a')' = a$.

(iv) To prove that $1' = 0$ and $0' = 1$

$1'$	$= 1' \bullet 1$	(by Identity)
	$= 1 \bullet 1'$	(by Commutative Law)
	$= 0$	(by Inverse)

The property $0' = 1$ follows by duality.

Exercise : Prove $0' = 1$ from basic axioms!

Remark. To prove that $a' = b$ or $b' = a$ *i.e.* to prove that a and b are inverse of each other, we should prove $a + b = 1$ and $a \bullet b = 0$.

(v) De Morgan's Laws : To prove that $(a + b)' = a' \bullet b'$ (*I.S.C. 2000*)

$(a + b) \bullet (a' \bullet b')$	$= (a' \bullet b') \bullet (a + b)$	(by Commutative Law)
	$= ((a' \bullet b') \bullet a) + ((a' \bullet b') \bullet b)$	(by Distributive Law)
	$= ((b' \bullet a') \bullet a) + ((a' \bullet b') \bullet b)$	(why?)
	$= (b' \bullet (a' \bullet a)) + (a' \bullet (b' \bullet b))$	(why?)
	$= (b' \bullet 0) + (a' \bullet 0) = 0 + 0 = 0.$	

Also $(a + b) + (a' \bullet b') = (a + b + a') \bullet (a + b + b')$

$= (a + a' + b) \bullet (a + 1) = (1 + b) \bullet 1 = 1 \bullet 1 = 1.$

Hence by definition of inverse, $(a + b)' = a' \bullet b'$.

To prove that $(a \bullet b)' = a' + b'$

$(a \bullet b) + (a' + b')$	$= a' + b' + a \bullet b$	(by Commutative Law)
	$= (a' + b' + a) \bullet (a' + b' + b)$	(why ?)
	$= (a + a' + b') \bullet (a' + b + b')$	(why ?)
	$= (1 + b') \bullet (a' + 1)$	(why ?)
	$= 1 \bullet 1 = 1.$	

Also $(a \bullet b) \bullet (a' + b') = (a \bullet b) \bullet a' + (a \bullet b) \bullet b'$

$= (a \bullet a') \bullet b + a \bullet (b \bullet b')$

$= 0 \bullet b + a \bullet 0 = 0 + 0 = 0.$

Hence by definition of inverse, $(a \bullet b)' = a' + b'$.

Aliter, the property $(a \bullet b)' = a' + b'$ follows by duality from $(a + b)' = a' \bullet b'$.

(vi) Law of Absorption : To prove that $a + a \bullet b = a$

$a + (a \bullet b)$	$= a \bullet 1 + a \bullet b$	(why?)
	$= a \bullet (1 + b)$	(by Distributive Law)
	$= a \bullet 1$	(by property (ii) above)
	$= a$	(why?)

To prove that $a \bullet (a + b) = a$

$a \bullet (a + b) = (a + 0) \bullet (a + b)$ (why?)

$= a + 0 \bullet b$ (by Distributive Law)

$= a + 0$ (why?)

$= a$ (why?)

Aliter, the property $a \bullet (a + b) = a$ follows by duality from $a + a \bullet b = a$.

(*vii*) Uniqueness of complement (inverse) : (*I.S.C. 2003*)

Let us assume that a_1' and a_2' are two complements of a. Then by definition, $a + a_1' = 1$, $a \bullet a_1' = 0$, $a + a_2' = 1$, $a \bullet a_2' = 0$.

$$a_1' = a_1' \bullet 1 = a_1' \bullet (a + a_2')$$
$$= (a_1' \bullet a) + (a_1' \bullet a_2') = 0 + (a_1' \bullet a_2')$$
$$= (a \bullet a_2') + (a_1' \bullet a_2') = (a + a_1') \bullet a_2'$$
$$= 1 \bullet a_2' = a_2'.$$

Now try to prove it by starting from $a_1' = a_1' + 0$.

ILLUSTRATIVE EXAMPLES

Example 1. *Write the dual of each of the following statements :*

(*i*) $x + (y \bullet x) = x$

(*ii*) $x \bullet y' + y = x + y$

(*iii*) $(x + y) \bullet (x + 1) = x + x \bullet y + y$

(*iv*) $(x' + y) \bullet (x + y') = x' \bullet y' + x \bullet y$

(*v*) $[(x' + y) \bullet (y' + z)] \bullet (x' + z)' = 0.$

Solution. Since the dual statement is derived by interchanging + and • and also interchanging the identities 1 and 0 in the original statement, therefore, the dual statements of the given statements are :

(*i*) $x \bullet (y + x) = x$

(*ii*) $(x + y') \bullet y = x \bullet y$

(*iii*) $(x \bullet y) + (x \bullet 0) = x \bullet (x + y) \bullet y$

(*iv*) $(x' \bullet y) + (x \bullet y') = (x' + y') \bullet (x + y)$

(*v*) $[(x' \bullet y) + (y' \bullet z)] + (x' \bullet z)' = 1.$

Example 2. *Prove that* $a + (a' \bullet b) = a + b$ *and* $a \bullet (a' + b) = a \bullet b$.

Solution. $a + (a' \bullet b) = (a + a') \bullet (a + b)$ (by Distributive Law)

$= 1 \bullet (a + b)$ (by Complement)

$= a + b$ (by Identity)

Similarly, $a \bullet (a' + b) = (a \bullet a') + (a \bullet b)$ (by Distributive Law)

$= 0 + (a \bullet b)$ (by Complement)

$= a \bullet b$ (by Identity)

Example 3. *Prove that* $a + (a' \bullet c + b) = a + b + c$.

Solution. $a + (a' \bullet c + b) = (a + a' \bullet c) + b$ (by Associative Law)

$= ((a + a') \bullet (a + c)) + b$ (by Distributive Law)

$= (1 \bullet (a + c)) + b$ (by Complement)

$= (a + c) + b$ (by Identity)

$= a + (c + b)$ (by Associativity)

$= a + (b + c)$ (by Commutative Law)

$= a + b + c.$

Example 4. *Prove that* $ab + c(a' + b') = ab + c$. (*I.S.C. 2004*)

Solution. $ab + c(a' + b') = ab + c(ab)'$ (by De Morgan's Law)

$= (ab + c) \cdot (ab + (ab)')$ (by Distributive Law)

$= (ab + c) \cdot 1$ (by Complement)

$= ab + c$ (by Identity)

Example 5. *Prove that* $(x + y) \cdot (x + 1) = x + x \cdot y + y$.

Solution. $(x + y) \cdot (x + 1) = (x + y) \cdot 1$ (by Boundedness)

$= x + y$ (by Identity)

$= (x + x \cdot y) + y$ (by Absorption Law)

$= x + x \cdot y + y$ (by Associative Law)

Example 6. *Show that in a boolean algebra, the zero element 0 and the unit element 1 are unique.*

Solution. Let 0, $\bar{0}$ be two zero elements in B.

As 0 is an additive identity, by definition,

$a + 0 = a$ for all $a \in$ B.

$\therefore \quad \bar{0} + 0 = \bar{0}$ ...(*i*) (taking $a = \bar{0}$)

Also as $\bar{0}$ is an additive identity, by definition,

$a + \bar{0} = a$ for all $a \in$ B.

$\therefore \quad 0 + \bar{0} = 0$ ...(*ii*) (taking $a = 0$)

However, by commutativity, we have $\bar{0} + 0 = 0 + \bar{0}$

$\Rightarrow \quad \bar{0} = 0.$ (using (*i*) and (*ii*))

Hence, zero element in a boolean algebra is unique.

Let 1, $\bar{1}$ be two unit elements in B. Then, by definition

$a \cdot 1 = a$ and $a \cdot \bar{1} = a$ for all $a \in$ B.

$\therefore \quad \bar{1} \cdot 1 = \bar{1}$ and $1 \cdot \bar{1} = 1$.

However, by commutativity, we have $\bar{1} \cdot 1 = 1 \cdot \bar{1}$.

Hence $\bar{1} = 1$.

Example 7. *Prove the following :*

(*i*) *If* $x + y = 0$, *then* $x = 0 = y$

(*ii*) $x \cdot y' = 0$ *if and only if* $x \cdot y = x$

(*iii*) $x = 0$ *if and only if* $y = x \cdot y' + x' \cdot y$ *for all* y.

Solution. (*i*) Given $x + y = 0$

Now $x = x \cdot (x + y)$ (by Absorption Law)

$= x \cdot 0$ (given, $x + y = 0$)

$= 0.$

Similarly, $y = y \cdot (y + x) = y \cdot (x + y) = y \cdot 0 = 0.$

(*ii*) First assume $x \cdot y' = 0$

$\Rightarrow \quad (x \cdot y')' = 0' \Rightarrow x' + y = 1$

Now $x = x \cdot 1 = x \cdot (x' + y)$ (using $x' + y = 1$)

$= x \cdot x' + x \cdot y = 0 + x \cdot y$

$= x \cdot y.$

Now to prove the other part, assume $x \cdot y = x$. We wish to prove that

$x \cdot y' = 0$

Now $x \cdot y' = (x \cdot y) \cdot y'$ (using $x = x \cdot y$)

$= x \cdot (y \cdot y') = x \cdot 0 = 0.$

(*iii*) First assume that $x = 0$

Then $x \bullet y' + x' \bullet y = 0 \bullet y' + 0' \bullet y = 0 + 1 \bullet y = 0 + y = y$ for all y.

To prove the other part, assume that $y = x \bullet y' + x' \bullet y$ for all y.

In particular, it is true for $y = 0$.

$\therefore \quad 0 = x \bullet 0' + x' \bullet 0 = x \bullet 1 + x' \bullet 0 = x + 0 = x.$

Thus, $x = 0$.

Example 8. *Prove that if $x + y = x + z$ and $x' + y = x' + z$, then $y = z$.*

Solution. Given $x + y = x + z$ ...(*i*)

and $x' + y = x' + z$ ...(*ii*)

Now
$$\begin{aligned} y &= y + 0 = y + x \bullet x' && (\because x \bullet x' = 0) \\ &= (y + x) \bullet (y + x') && \text{(by Distributive Law)} \\ &= (x + y) \bullet (x' + y) && \text{(by Commutative Law)} \\ &= (x + z) \bullet (x' + z) && \text{(by using (i) and (ii))} \\ &= x \bullet x' + z && \text{(by Distributive Law)} \\ &= 0 + z = z. \end{aligned}$$

Example 9. *A boolean algebra cannot have exactly three elements.*

Solution. Let a boolean algebra B have exactly three elements and let B = {0, 1, x}, where $x \neq 0, x \neq 1$ be a boolean algebra under the operation + and $\bullet$.

Since $x \in$ B, there exists $x' \in$ B such that

$x + x' = 1$ ...(*i*) and $x \bullet x' = 0$ ...(*ii*)

As $x' \in$ B and B has exactly three elements, namely 0, 1 and x.

Three cases arise :

Case I. If $x' = 0$

then $(x')' = 0' \Rightarrow x = 1$, which is wrong.

Case II. If $x' = 1$

then $(x')' = 1' \Rightarrow x = 0$, which is wrong.

Case III. If $x' = x$, then using (*i*), we get

$x' + x = 1 \Rightarrow x + x = 1$

$\Rightarrow \quad x = 1$ $\quad (\because x + x = x$, Idempotent law)

which is wrong.

It follows that a boolean algebra cannot have exactly three elements.

3.4 BOOLEAN EXPRESSIONS AND FUNCTIONS

In a boolean algebra (B, +, $\bullet$, ′, 0, 1), the specific elements like 0 and 1, or other elements of B, are called **constants. Variables** like x and y may denote any element of B. Thus in boolean algebra {0, 1}, there are only two constants 0, 1, while we may define any number of variables like x, y, z etc. on B, which may take values 0 or 1.

A **boolean expression** is any expression built from variables and constants by applying the operations +, $\bullet$, ′ a finite number of times. Examples of boolean expressions are

$x + 1,\ x + y,\ x' + y \bullet 0,\ (x + y)(y' + z),\ [(x_1 \bullet x_2) + x_3]'$ etc.

A **boolean function** is determined by a boolean expression.

For example,

$f(x) = x'$,

$f(x, y) = x + y$

$g(x, y) = x \bullet y' + 1$

Two different boolean expressions may represent the same boolean function. For example, $x \bullet (y + z)$ and $x \bullet y + x \bullet z$ represent the same boolean function in 3 variables.

A boolean function of n variables $x_1, x_2, \ldots, x_n$ is a mapping from B^n to B. We write it as $f(x_1, x_2, \ldots, x_n)$ or $g(x_1, x_2, \ldots, x_n)$ etc.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the value of the boolean expression* $(x_1 \bullet x_2)' + x_3$ *if*

(i) $x_1 = 1, x_2 = 0, x_3 = 1$

(ii) $x_1 = 1, x_2 = 1, x_3 = 0$.

Solution. (*i*) $(x_1 \bullet x_2)' + x_3 = (1 \bullet 0)' + 1 = 0' + 1$

$= 1 + 1 = 1.$

(*ii*) $(x_1 \bullet x_2)' + x_3 = (1 \bullet 1)' + 0 = 1' + 0$

$= 0 + 0 = 0.$

Example 2. *Construct input/output table for the boolean function f on boolean algebra {0, 1} defined by :*

(i) $f(x_1, x_2) = x_1 \bullet x'_2$

(ii) $f(x_1, x_2, x_3) = (x_1 \bullet x_2)' + x_3$.

Solution. (*i*) As the given boolean function has two variables and is defined on the boolean algebra {0, 1}, its input/output table is :

Input			*Output*
x_1	x_2	x_2'	$x_1 \bullet x_2'$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

(*ii*) As the given boolean function has three variables and is defined on the boolean algebra {0, 1}, its input/output table is :

Input					*Output*
x_1	x_2	x_3	$x_1 \bullet x_2$	$(x_1 \bullet x_2)'$	$(x_1 \bullet x_2)' + x_3$
1	1	1	1	0	1
1	1	0	1	0	0
1	0	1	0	1	1
1	0	0	0	1	1
0	1	1	0	1	1
0	1	0	0	1	1
0	0	1	0	1	1
0	0	0	0	1	1

Example 3. *Define a boolean function f on boolean algebra {0, 1} corresponding to the expression* $x + y'$. *Give its domain and range. Construct input/output table (or truth table).*

Solution. We define $f : B^2 \to B$ as

$f(x, y) = x + y'$, where $x, y \in \{0, 1\}$

Domain is B^2 *i.e.* $(\{0, 1\})^2 = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

From $f(x, y) = x + y'$,

putting $x = 0, y = 0$, we get $f(0, 0) = 0 + 0' = 0 + 1 = 1$.

Similarly, $f(0, 1) = 0 + 1' = 0 + 0 = 0,$

$f(1, 0) = 1 + 0' = 1 + 1 = 1,$

$f(1, 1) = 1 + 1' = 1 + 0 = 1.$

$\therefore$ Range is B *i.e.* {0, 1}.

The input/output table or truth table is :

x	y	y'	$f(x, y) = x + y'$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

Example 4. *(i) If $f(x, y, z) = xy + yz' + x'yz$ in boolean algebra $B = \{0, 1\}$, evaluate $f(0, 0, 0)$ and $f(0, 1, 1)$. How many rows would the truth table have?*

(ii) Write a boolean expression in terms of x, y or their complements which has value 1 when $x = 0$, $y = 1$ and assumes value 0 otherwise.

Solution. (*i*) $f(x, y, z) = xy + yz' + x'yz$

$\therefore \quad f(0, 0, 0) = 0 \cdot 0 + 0 \cdot 0' + 0' \cdot 0 \cdot 0 = 0 + 0 \cdot 1 + 1 \cdot 0 \cdot 0$

$= 0 + 0 + 0 = 0$

$f(0, 1, 1) = 0 \cdot 1 + 1 \cdot 1' + 0' \cdot 1 \cdot 1 = 0 + 1 \cdot 0 + 1 \cdot 1 \cdot 1 = 0 + 0 + 1 = 1.$

As there are 3 variables, each of which can have two values, the truth table will have $2 \times 2 \times 2 = 8$ rows.

(*ii*) Here we use the fact that $x \cdot y$ has value 1 only when $x = 1$, $y = 1$; otherwise $x \cdot y$ has value 0. Hence, the expression which has value 1 when $x = 0$, $y = 1$ is $x'y$.

Note. This idea can be better understood by constructing the following truth table :

x	y	x'	y'	xy	xy'	$x'y$	$x'y'$
1	1	0	0	1	0	0	0
1	0	0	1	0	1	0	0
0	1	1	0	0	0	1	0
0	0	1	1	0	0	0	1

It is very clear from this table that expression $x'y$ has value 1 when $x = 0$, $y = 1$ and assumes value 0 otherwise.

EXERCISE 3.2

1. Write the dual statement of each of the following :

(*i*) $(1 + x) \cdot (0 + y) = y$ (*ii*) $(x \cdot y)' = x' + y'$

(*iii*) $a + 1 = 1$ (*iv*) $x + x'y = x + y$

(*v*) $x + [x \cdot (y + 1)] = x$ (*vi*) $x + [(y' + x) \cdot y]' = 1$

(*vii*) $(x + y)' = x' \cdot y'$ (*viii*) $(x' + y')' = x \cdot y$

(*ix*) $(1 \cdot x) + 0 = x$ (*x*) $(1 \cdot x)' = 0 + x'$

(*xi*) If $x + y = 0$ then $x = 0 = y$ (*xii*) $x \cdot y' = 0$ if and only if $x \cdot y = x$

(*xiii*) $x = 0$ if and only if $y = x \cdot y' + x' \cdot y$ for all y.

2. Using elementary properties of boolean algebra, prove that :

(*i*) $x + x \cdot (y + 1) = x$ (*ii*) $(a + b)\, a'\, b' = 0$

(*iii*) $(x + y) + (x' \cdot y') = 1$.

3. Prove the following statements :

(*i*) If $x \cdot y = 1$, then $x = 1 = y$ (*ii*) $x + y' = 1$ if and only if $x + y = x$.