Integrals

Case Study Based Questions

Case Study 1

Following paragraph given to student by the teacher.
The given integral $\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.
Consider $I = \int f(x)\,dx$

Put $x = g(t)$ so that $\frac{dx}{dt} = g'(t)$

We write $dx = g'(t)\,dt$
Thus, $I = \int f(x)\,dx = \int f[g(t)]\,g'(t)\,dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution.

Based on the above information, solve the following questions:

Q 1. $\int 2x \sin(x^2 + 1)\,dx$ **is equal to:**

a. $-\sin(x^2+1) + C$ b. $-\cos(x^2+1) + C$
c. $\sin(x^2+1) + C$ d. None of these

Q 2. $\int \frac{x}{\sqrt{32 - x^2}}\,dx$ **is equal to:**

a. $-\sqrt{32 - x^2} + C$ b. $\sqrt{32 + x^2} + C$
c. $\sqrt{64 - x^2} + C$ d. $\sqrt{32 - x^2} + C$

Q 3. $\int \frac{\sin(2\tan^{-1} x)}{1 + x^2}\,dx$ **is equal to:**

a. $-\frac{\sin 2(\tan^{-1} x)}{2} + C$ b. $-\frac{\cos 2(\tan^{-1} x)}{2} + C$

c. $\frac{\cos 2(\tan^{-1} x)}{2} + C$ d. $\frac{\sin 2(\tan^{-1} x)}{2} + C$

Q 4. $\int \frac{\sqrt{1+x^2}}{x^4} dx$ **is equal to:**

a. $-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$

b. $\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C$

c. $\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{1/2}+C$

d. None of these

Q 5. $\int \frac{\cos^{-1} x}{\sqrt{1-x^2}} dx$ **is equal to:**

a. $\frac{(\sin^{-1} x^2)^2}{2}+C$

b. $\frac{(-\sin^{-1} x)^3}{2}+C$

c. $\frac{-(\cos^{-1} x)^2}{2}+C$

d. $\frac{(\tan^{-1} x)^2}{x}+C$

Solutions

1. Let $I = \int 2x \sin(x^2+1)\, dx$

Put $x^2+1 = t \Rightarrow 2x\, dx = dt$

[Differentiating both sides w.r.t. x]

Now, $I = \int \sin t\, dt$

$= -\cos t + C = -\cos(x^2+1) + C$

So, option (b) is correct.

2. Let $I = \int \frac{x}{\sqrt{32-x^2}} dx$

Put $32 - x^2 = t$

$\Rightarrow$ $-2x\, dx = dt$

$\Rightarrow$ $x\, dx = \frac{-1}{2} dt$

Now, $I = -\frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= \frac{-1}{2}\cdot 2\sqrt{t} + C$

$= -\sqrt{t} + C$

$= -\sqrt{32-x^2} + C$

So, option (a) is correct.

3. Let $$I = \int \frac{\sin(2\tan^{-1} x)}{1+x^2}\,dx$$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}\,dx = dt$

$\therefore$ $$I = \int \sin 2t\,dt = -\frac{\cos 2t}{2} + C$$

$$= -\frac{\cos 2(\tan^{-1} x)}{2} + C$$

So, option (b) is correct.

4. Let $$I = \int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \frac{\sqrt{1+x^2}}{x}\cdot\frac{1}{x^3}\,dx$$

$$= \int \sqrt{\frac{1+x^2}{x^2}}\cdot\frac{1}{x^3}\,dx = \int \sqrt{\frac{1}{x^2}+1}\cdot\frac{1}{x^3}\,dx$$

Now, putting $1+\frac{1}{x^2} = t^2$

$\Rightarrow$ $$\frac{-2}{x^3}\,dx = 2t\,dt \Rightarrow -\frac{1}{x^3}\,dx = t\,dt$$

$\therefore$ $$I = \int -t^2 dt = -\frac{t^3}{3} + C$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2} + C$$

So, option (a) is correct.

5. Let $$I = \int \frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx$$

Put $\cos^{-1} x = t$

$\Rightarrow$ $$\frac{-1}{\sqrt{1-x^2}}\,dx = dt \Rightarrow \frac{1}{\sqrt{1-x^2}}\,dx = -\,dt$$

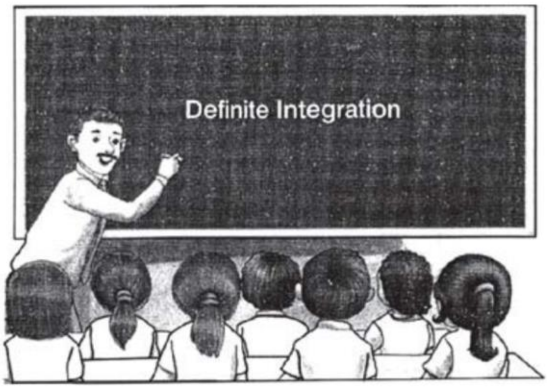
$\therefore$ $$I = \int t\,(-\,dt) = -\int t\,dt$$

$$= \frac{-\,t^2}{2} + C = \frac{-\,(\cos^{-1} x)^2}{2} + C$$

So, option (c) is correct.

Case Study 2

In Presidency Public School, class teacher of XIIth class teaches the topic of definite integration.



If $f(x)$ is the continuous function, integral of $f(x)$ over the interval $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and $\int_a^b f(x)\,dx = [F(x)]_a^b = [F(b) - F(a)]$.

Based on the above information, solve the following questions:

Q 1. $\int_4^7 x^2 dx$ is equal to:

a. $\frac{7}{3}$ b. $\frac{278}{3}$ c. 93 d. $\frac{407}{3}$

Q 2. $\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$ is equal to:

a. $\frac{\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{\pi}{12}$ d. $\frac{\pi}{6}$

Q 3. $\int_{-1}^{1} (x+3)\,dx$ is equal to:

a. 2 b. 6 c. –1 d. –6

Q 4. $\int_4^6 e^x dx$ is equal to:

a. 1 b. $e^6 - e^4$ c. $e^5 - 1$ d. $e^5 - e^3$

Q 5. $\int_2^3 \frac{1}{x} dx$ is equal to:

a. $\log\frac{2}{3}$ b. $\log 3$ c. $\log\frac{3}{2}$ d. $\log 2$

Solutions

1. $\int_4^7 x^2\,dx = \left[\frac{x^3}{3}\right]_4^7 = \left[\frac{(7)^3}{3} - \frac{(4)^3}{3}\right]$

$$= \left[\frac{343}{3} - \frac{64}{3}\right] = \frac{279}{3} = 93$$

So, option (c) is correct.

2. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1} x]_1^{\sqrt{3}}$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

So, option (c) is correct.

3. $\int_{-1}^{1} (x+3)\,dx$

$$= \left[\frac{x^2}{2} + 3x\right]_{-1}^{1} = \left[\frac{1}{2} + 3\right] - \left[\frac{1}{2} - 3\right] = 6$$

So, option (b) is correct.

4. $\int_4^6 e^x\,dx = [e^x]_4^6 = e^6 - e^4$

So, option (b) is correct.

5. $\int_2^3 \frac{1}{x}\,dx = [\log|x|]_2^3 = [\log 3 - \log 2]$

$$= \log\frac{3}{2} \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

So, option (c) is correct.

Case Study 3

For any function $f(x)$, we have

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx$$

where, $a < c_1 < c_2 < c_3 \ldots < c_{n-1} < c_n < b$.

Based on the above information, solve the following questions:

Q 1. $\int_0^{3/2} |4x-5|\, dx =$

a. $\frac{13}{10}$ b. $\frac{13}{4}$ c. $\frac{11}{10}$ d. $\frac{11}{4}$

Q 2. $\int_0^{2\pi} |\cos x|\, dx =$

a. 1 b. 2 c. 3 d. 4

Q 3. $\int_{-1}^{2} e^{|x|}\, dx =$

a. $e-2$ b. $2e-2$ c. e^2+e-2 d. $2e^2-1$

Q 4. $\int_0^5 [x]\, dx =$

a. 10 b. 14 c. 17 d. 20

Q 5. $\int_{-2}^{1} f(x)\, dx =$

where, $f(x)=\begin{cases} 3-5x, & x<0 \\ 4+3x, & x \geq 0 \end{cases}$

a. $\frac{23}{2}$ b. $\frac{35}{2}$ c. $\frac{43}{2}$ d. $\frac{47}{2}$

Solutions

1. Integrand $f(x)=|4x-5|$ can be defined as:

$$|4x-5| = \begin{cases} -(4x-5), & x \leq \frac{5}{4} \\ 4x-5, & x > \frac{5}{4} \end{cases}$$

Let $I = \int_0^{3/2} |4x-5|\, dx$

$$= \int_0^{5/4} -(4x-5)\, dx + \int_{5/4}^{3/2} (4x-5)\, dx$$

$$= \int_0^{5/4} (5-4x)\, dx + \int_{5/4}^{3/2} (4x-5)\, dx$$

$$= [5x-2x^2]_0^{5/4} + [2x^2-5x]_{5/4}^{3/2}$$

$$= 5\left(\frac{5}{4}-0\right)-2\left\{\left(\frac{5}{4}\right)^2-0\right\}+2\left\{\left(\frac{3}{2}\right)^2-\left(\frac{5}{4}\right)^2\right\}$$

$$-5\left(\frac{3}{2}-\frac{5}{4}\right)$$

$$= \frac{25}{4}-2\times\frac{25}{16}+2\left(\frac{3}{2}+\frac{5}{4}\right)\left(\frac{3}{2}-\frac{5}{4}\right)-5\times\frac{1}{4}$$

$$= \frac{25}{4}-\frac{25}{8}+2\times\frac{11}{4}\times\frac{1}{4}-\frac{5}{4}$$

$$= \frac{(100-50+22-20)}{16}=\frac{52}{16}=\frac{13}{4}$$

So, option (b) is correct.

2. Integrand $f(x) = |\cos x|$ can be defined as:

$$|\cos x| = \begin{cases} \cos x, & 0 \le x \le \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ \cos x, & \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

Let $\quad I = \int_0^{2\pi} |\cos x|\, dx$

$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\, dx + \int_{3\pi/2}^{2\pi} \cos x\, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$

$$= \left(\sin\frac{\pi}{2}-\sin 0\right)-\left(\sin\frac{3\pi}{2}-\sin\frac{\pi}{2}\right)+\left(\sin 2\pi-\sin\frac{3\pi}{2}\right)$$

$$= (1-0)-(-1-1)+[0-(-1)] = 1+2+1 = 4$$

So, option (d) is correct.

3. Integrand $f(x) = e^{|x|}$ can be defined as:

$$e^{|x|} = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

Let $\quad I = \int_{-1}^{2} e^{|x|}\, dx = \int_{-1}^{0} e^{-x} dx + \int_0^2 e^x\, dx$

$$= [-e^{-x}]_{-1}^{0} + [e^{x}]_{0}^{2}$$
$$= -(e^{0} - e^{1}) + (e^{2} - e^{0})$$
$$= -(1 - e) + (e^{2} - 1)$$
$$= -1 + e + e^{2} - 1 = e^{2} + e - 2$$

So, option (c) is correct.

4. Integrand $f(x) = [x]$ can be defined as:

$$[x] = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \\ 4, & 4 \le x < 5 \end{cases}$$

Let $I = \int_0^5 [x]\, dx$

$$= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \int_3^4 3\, dx + \int_4^5 4\, dx$$
$$= 0 + [x]_1^2 + 2[x]_2^3 + 3[x]_3^4 + 4[x]_4^5$$
$$= (2-1) + 2(3-2) + 3(4-3) + 4(5-4)$$
$$= 1 + 2 \times 1 + 3 \times 1 + 4 \times 1$$
$$= 1 + 2 + 3 + 4 = 10$$

So, option (a) is correct.

5. Let $I = \int_{-2}^{1} f(x)\, dx = \int_{-2}^{0} f(x)\, dx + \int_0^1 f(x)\, dx$

$$= \int_{-2}^{0} (3 - 5x)\, dx + \int_0^1 (4 + 3x)\, dx$$
$$= \left[3x - \frac{5}{2}x^2\right]_{-2}^{0} + \left[4x + \frac{3}{2}x^2\right]_0^1$$
$$= 3(0+2) - \frac{5}{2}(0-4) + 4(1-0) + \frac{3}{2}(1-0)$$

$$= 6 + 10 + 4 + \frac{3}{2} = 20 + \frac{3}{2} = \frac{43}{2}.$$

So, option (c) is correct.

Case Study 4

If $f(x)$ is a continuous function defined on $[-a, a]$, then

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

A function $f(x)$ is even, when $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

Based on the above information, solve the following questions:

Q 1. If $f(x)$ is an odd function, then the value of $\int_{-1}^{1} \{f(x) + f(-x)\}\,dx$ is:

a. 0　　b. 1　　c. $\frac{1}{2}$　　d. –1

Q 2. If $f(x)$ is even function, then the value of $\int_{-c}^{c} \{f(x) - f(-x)\}\,dx$ is:

a. –1　　b. 0　　c. $\frac{1}{2}$　　d. 1

Q 3. $\int_{-2}^{2} \log\left(\frac{4-x}{4+x}\right) dx =$

a. 0　　b. –1　　c. $\frac{1}{2}$　　d. 2

Q 4. $\int_{-\pi/2}^{\pi/2} x \sin x \, dx =$

a. π　　b. $\frac{\pi}{2}$　　c. 2　　d. $\frac{1}{2}$

Q 5. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \cos 2x}$ **is equal to:**

a. 1　　b. 2　　c. 3　　d. 4

Solutions

1. Given that, $f(x)$ is odd when

$$f(-x) = -f(x)$$

$$\therefore \quad \int_{-1}^{1}\{f(x) + f(-x)\}\,dx = \int_{-1}^{1}\{f(x) - f(x)\}\,dx$$

$$= \int_{-1}^{1} 0\,dx = 0$$

So, option (a) is correct.

2. Given that, $f(x)$ is even when

$$f(-x) = f(x)$$

$$\therefore \quad \int_{-c}^{c}\{f(x) - f(-x)\}\,dx = \int_{-c}^{c}\{f(x) - f(x)\}\,dx$$

$$= \int_{-c}^{c} 0\,dx = 0$$

So, option (b) is correct.

3. Let the integrand $f(x) = \log\left(\frac{4-x}{4+x}\right)$

$$\therefore \quad f(-x) = \log\left(\frac{4+x}{4-x}\right) = \log\left(\frac{4-x}{4+x}\right)^{-1}$$

$$= -\log\left(\frac{4-x}{4+x}\right)$$

$$= -f(x)$$

$\Rightarrow f(x)$ is an odd function.

$$\therefore \quad \int_{-2}^{2}\log\left(\frac{4-x}{4+x}\right)dx = 0$$

So, option (a) is correct.

4. Let the integrand $f(x) = x\sin x$

$$\therefore \quad f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x\sin x = f(x)$$

$\Rightarrow \quad f(x)$ is an even function.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} x\sin x\,dx = 2\int_{0}^{\pi/2} x\sin x\,dx$$

$$= 2\left[x\int \sin x\, dx\right]_0^{\pi/2} - 2\left[\int\left\{\frac{d}{dx}(x)\int \sin x\, dx\right\}dx\right]_0^{\pi/2}$$

$$= 2\left[-x\cos x\right]_0^{\pi} - 2\left[\int -\cos x\, dx\right]_0^{\pi}$$

$$= -2\left[x\cos x\right]_0^{\pi/2} + 2\left[\sin x\right]_0^{\pi/2}$$

$$= -2\left[\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - 0\cdot\cos 0\right] + 2\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= -2\left[\frac{\pi}{2}\times 0 - 0\times 1\right] + 2\left[1 - 0\right]$$

$$= -2\times 0 + 2\times 1 = 0 + 2 = 2$$

So, option (c) is correct.

5. Let the integrand $f(x) = \dfrac{1}{1+\cos 2x}$.

$$\Rightarrow \quad f(x) = \frac{1}{2\cos^2 x}$$

$$\because \quad f(-x) = \frac{1}{2}\cdot\frac{1}{\cos^2(-x)} = \frac{1}{2\cos^2 x} = f(x)$$

$\Rightarrow$ $f(x)$ is an even function.

$$\because \quad \int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x} = 2\int_0^{\pi/4}\frac{dx}{1+\cos 2x}$$

$$= 2\int_0^{\pi/4}\frac{dx}{2\cos^2 x}$$

$$= \int_0^{\pi/4}\sec^2 x\, dx = \left[\tan x\right]_0^{\pi/4}$$

$$= \tan\frac{\pi}{4} - \tan 0$$

$$= 1 - 0 = 1$$

So, option (a) is correct.

Case Study 5

The mathematics teacher teaches the following type of integration.

In this type of integral, integrand is the product of two functions. One is in exponential form and second function is the sum of two functions in which one is derivative of other function. Then, to evaluate such integrals, we directly use the following formula

$$\int e^x [f(x)+f'(x)]\,dx = e^x f(x)+C$$

Based on the above information, solve the following questions:

Q 1. Evaluate $\int e^x (\sin x + \cos x)\,dx$.

Q 2. Evaluate $\int [\sin(\log x) + \cos(\log x)]\,dx$. *(NCERT)*

Q 3. Evaluate $\int \frac{x-3}{(x-1)^3} e^x\,dx$.

Solutions

1. Let $I = \int e^x (\sin x + \cos x)\,dx$

and $f(x) = \sin x$, then $f'(x) = \cos x$

So, the given integral is of the form

$$I = \int e^x [f(x) + f'(x)]\,dx$$

We know that,

$$\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + C$$

$\therefore \quad I = e^x \sin x + C$

2. Let $I = \int [\sin(\log x) + \cos(\log x)]\,dx$

Put $\log x = t \Rightarrow x = e^t$

$\Rightarrow \quad dx = e^t dt$

$\therefore \; I = \int e^t (\sin t + \cos t)\,dt$

$[\because f(x) = \sin t \text{ and } f'(x) = \cos t]$

So, the given integral is of the form

$$I = \int e^x [f(x) + f'(x)]\,dx$$

$\therefore \quad I = e^t \sin t + C = x \sin(\log x) + C$

3. Let $I = \int \frac{x-3}{(x-1)^3} e^x\,dx = \int \frac{x-1-2}{(x-1)^3} e^x\,dx$

$$= \int e^x \left(\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right) dx$$

$$= \int e^x \left(\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right) dx \qquad \ldots(1)$$

Now, let $\quad f(x) = \frac{1}{(x-1)^2}$

$\Rightarrow \quad f'(x) = \frac{-2}{(x-1)^3}$

Then, eq. (1) becomes of the form

$$I = \int e^x \, [f(x) + f'(x)] \, dx$$

Also, we know that,

$$\int e^x \, [f(x) + f'(x)] \, dx = e^x \, f(x) + C$$

Hence, $\quad I = \frac{e^x}{(x-1)^2} + C.$

Case Study 6

Mr. Rohan Gupta of Nalanda Public School is teaching the integration by parts to his student in the classroom.

Let $f(x)$ and $g(x)$ be the differentiable function, then

$$\int f(x) \, g(x) \, dx = f(x) \int g(x) \, dx - \int \left[\frac{d}{dx} f(x) \cdot \int g(x) \, dx \right] dx$$

If $f(x) = u$ and $g(x) = v$, then

$$\int uv \, dx = u \int v \, dx - \int \left[\frac{du}{dx} \cdot \int v \, dx \right] dx$$

Based on the above information, solve the following questions: (CBSE 2020)

Q 1. Evaluate $\int \left(1 + x - \frac{1}{x} \right) e^{x + \frac{1}{x}} \, dx.$

Q 2. Evaluate $\int \frac{e^x}{x+1} [1 + (x+1) \log (x+1)] \, dx.$

Solutions

1. Let $I = \int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$

$$= \int e^{x+\frac{1}{x}}\, dx + \int \underset{\text{I}}{x} \left[\left(1 - \underset{\text{II}}{\frac{1}{x^2}}\right) e^{x+\frac{1}{x}}\right] dx$$

$$= \int e^{x+\frac{1}{x}}\, dx + x e^{x+\frac{1}{x}} - \int 1 \times e^{x+\frac{1}{x}}\, dx$$

$$\left[\begin{array}{l} \because \int u \cdot v\, dx = u \int v\, dx - \int \left\{\frac{d}{dx}(u) \int v\, dx\right\} dx \\ I_1 = \int \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}}\, dx \\ \text{put } x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt \\ \therefore \int e^t\, dt = e^t = e^{x+\frac{1}{x}} \end{array}\right]$$

$$= x\, e^{x+\frac{1}{x}} + C$$

2. Let $I = \int \frac{e^x}{x+1} [1 + (x+1) \log (x+1)]\, dx$

$$= \int e^x \left[\frac{1}{x+1} + \log (x+1)\right] dx$$

$$= \int e^x \left(\frac{1}{x+1}\right) dx + \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log (x+1)}\, dx$$

$$= \int e^x \left(\frac{1}{x+1}\right) dx + [\log (x+1) \int e^x\, dx$$

$$- \int \left[\left(\frac{d}{dx} \{(\log (x+1)\} \int e^x\, dx\right) dx\right]$$

[Using integration by parts]

$$= \int \frac{e^x}{x+1}\, dx + e^x \log (x+1) - \int \frac{e^x}{x+1}\, dx$$

$$= e^x \log (x+1) + C$$

Case Study 7

If $f(x)$ is a continuous function defined on $[0, a]$, then $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Based on the above information, solve the following questions:

Q 1. Evaluate $\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$.

Q 2. If $f(x)=\frac{\sin x-\cos x}{1+\sin x\cos x}$, then find the value of $\int_0^{\pi/2} f(x)\,dx$.

Q 3. If $g(x)=\log(1+\tan x)$, then find the value of $\int_0^{\pi/4} g(x)\,dx$.

Solutions

1. Let $$I=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx \quad \text{...(1)}$$

$$\Rightarrow \quad I=\int_0^a \frac{f(a-x)}{f(a-x)+f(a-(a-x))}dx$$

$$\Rightarrow \quad I=\int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx \quad \text{...(2)}$$

On adding eqs. (1) and (2), we get

$$2I=\int_0^a 1\,dx=[x]_0^a=a \Rightarrow I=\frac{1}{2}a$$

2. Let $$I=\int_0^{\pi/2} f(x)\,dx=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx \quad \text{...(1)}$$

$$\therefore \quad I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x \sin x}\, dx$$

$$\Rightarrow \quad I = -\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}\, dx = -I \quad \text{[From eq. (1)]}$$

$$\Rightarrow \quad 2I = 0$$

$$\therefore \quad I = \int_0^{\pi/2} f(x)\, dx = 0$$

3. Let $\quad I = \int_0^{\pi/4} g(x)\, dx = \int_0^{\pi/4} \log(1+\tan x)\, dx \quad ...(1)$

$$\therefore \quad I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log 2 - \log(1+\tan x)\, dx$$

$$= \int_0^{\pi/4} \log 2\, dx - \int_0^{\pi/4} g(x)\, dx$$

$$\Rightarrow \quad I = \log 2\left(\frac{\pi}{4}-0\right) - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \int_0^{\pi/4} g(x)\, dx = \frac{\pi}{8}\log 2$$

Case Study 8

If $f(x)$ is a continuous function defined on $[a, b]$, then $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$.

Based on the above information, solve the following questions:

Q 1. Evaluate $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx.$

Q 2. Evaluate $\int_{\pi/6}^{\pi/3} \log \tan x\, dx.$

Q 3. If $g(x)=\frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}$**, then find the value of** $\int_a^b g(x)\,dx.$

Solutions

1. Let $$I=\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx \quad ...(1)$$

$$\Rightarrow \quad I=\int_a^b \frac{f(a+b-x)}{f(a+b-x)+f[a+b-(a+b-x)]}dx$$

$$\Rightarrow \quad I=\int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx \quad ...(2)$$

On adding eqs. (1) and (2), we get

$$2I=\int_a^b 1\,dx=[x]_a^b=b-a$$

$$\Rightarrow \quad I=\frac{1}{2}(b-a)$$

2. Let $$I=\int_{\pi/6}^{\pi/3} \log \tan x\, dx$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3} \log \tan\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)dx$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3} \log \cot x\, dx=-\int_{\pi/6}^{\pi/3} \log \tan x\, dx$$

$$\Rightarrow \quad I=-I$$

$$\Rightarrow \quad 2I=0 \Rightarrow I=0$$

3. Let $$I=\int_0^b g(x)\,dx=\int_a^b \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}\,dx \quad \ldots(1)$$

$\therefore$ $$I=\int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+[a+b-(a+b-x)]^{1/n}}\,dx$$

$$=\int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+x^{1/n}}\,dx \quad \ldots(2)$$

On adding eqs. (1) and (2), we get

$$2I=\int_a^b 1\,dx=[x]_a^b=b-a$$

$\Rightarrow$ $$I=\frac{1}{2}(b-a)$$