# Chapter 5

# Continuity and Differentiability

## Exercise 5.7

Q. 1 Find the second order derivatives of the function

$x^2 + 3x + 2$

Answer:

Let us take y= $x^2 + 3x + 2$

Now,

$$\frac{dy}{dx} = \frac{d\left(x^2\right)}{dx} + \frac{d(3x)}{dx} + \frac{d(2)}{dx}$$

= 2x + 3

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d(2x+3)}{dx} = \frac{d(2x)}{dx} + \frac{d(3)}{dx}$$

= 2 + 0

= 2

Q. 2 Find the second order derivatives of the function

$x^{20}$

Answer:

Let us take y = $x^{20}$

Now,

$$\frac{dy}{dx} = \frac{d(x^{20})}{dx}$$

$= 20x^{19}$

Therefore,

$\frac{d^2y}{dx^2} = \frac{d(20x^{19})}{dx} = 20\frac{d(x^{19})}{dx}$

$= 20 \times 19 \times x^{18}$

$= 380\, x^{18}$

Q. 3 Find the second order derivatives of the function

x. cos x

Answer:

: Let us take y= x. cos x

Now,

$\frac{dy}{dx} = \frac{d(x\cos x)}{dx}$

$= \cos x \frac{d(x)}{dx} + x\frac{d(\cos x)}{dx}$

= cosx.1 + x (-sin x)

= cos x –x sin x

Therefore,

$\frac{d^2y}{dx^2} = \frac{d(\cos x - x\sin x)}{dx}$

$= \frac{d(\cos x)}{dx} - \frac{d(x\sin x)}{dx}$

$= -\sin x - \left[\sin x.\frac{d(x)}{dx} + x.\frac{d(\sin x)}{dx}\right]$

= -sin x – (sin x + x cos x)

= - (x cos x + 2sinx)

Q. 4 Find the second order derivatives of the function log x

Answer:

Let us take y = logx

Now,

$$\frac{dy}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{x}\right)}{dx} = \left(-\frac{1}{x^2}\right)$$

Q. 5 Find the second order derivatives of the function

$x^3$ log x

Answer:

Let us take y = $x^3$ log x

Now,

$$\frac{dy}{dx} = \frac{d(x^3 \log x)}{dx}$$

$= \log x. \frac{d(x^3)}{dx} + x^3.\frac{d(\log x)}{dx}$

= logx.3$x^2$ + $x^3$.1/x

= logx .3$x^2$ + $x^2$

= $x^2$(1 + 3logx)

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d[x^2(1+3\log x)]}{dx}$$

$= (1 + 3 \log x). \frac{d(x^2)}{dx} + x^2 \frac{d(1+3\log x)}{dx}$

$= (1 + 3 \log x). 2x + x^2. \frac{3}{x}$

$= 2x + 6x\log x + 3x$

$= 5x + 6x\log x$

$= x (5 + 6\log x)$

Q. 6 Find the second order derivatives of the function

$e^x \sin 5x$

Answer:

Let us take $y = e^x \sin 5x$

Now,

$\frac{dy}{dx} = \frac{d(e^x \sin 5x)}{dx}$

$= \sin 5x. \frac{d(e^x)}{dx} + e^x. \frac{d(\sin 5x)}{dx}$

$= \sin 5x. e^x + e^x. \cos 5x. \frac{d(5x)}{dx}$

$= e^x \sin 5x + e^x \cos 5x.5$

$= e^x (\sin 5x + 5\cos 5x)$

$\frac{d^2y}{dx^2} = \frac{d[e^x(\sin 5x + 5\cos 5x)]}{dx}$

$= (\sin 5x + 5 \cos 5x). \frac{d(e^x)}{dx} + e^x. \frac{d(\sin 5x + 5\cos 5x)}{dx}$

$= (\sin 5x + 5 \cos 5x)e^x + e^x \left[\cos 5x. \frac{d(5x)}{dx} + 5(-\sin 5x). \frac{d(5x)}{dx}\right]$

$= e^x (\sin 5x + 5\cos 5x) + e^x (5\cos 5x – 25\sin 5x)$

$= e^x (10\cos 5x – 24\sin 5x)$

$= 2e^x (5\cos 5x – 12\sin 5x)$

Q. 7 Find the second order derivatives of the function

$e^{6x}$ cos 3x

Answer:

Let us take y = $e^{6x}$ cos 3x

Now,

$$\frac{dy}{dx} = \frac{d(e^{6x}\cos 3x)}{dx}$$

$$= \cos 3x.\frac{d(e^{6x})}{dx} + e^{6x}\frac{d(\cos 3x)}{dx}$$

$$=\cos 3x.\ e^{6x}.\frac{d(6x)}{dx} + e^{6x}.(-\sin 3x).\frac{d(3x)}{dx}$$

= 6e6xcos3x - 3e6xsin3x

$$\frac{d^2y}{dx^2} = \frac{d[6e^{6x}\cos 3x - 3e^{6x}\sin 3x]}{dx}$$

$$= 6.\frac{d(e^{6x}\cos 3x)}{dx} - 3.\frac{d(e^{6x}\sin 3x)}{dx}$$

$$= 6.[6e^{6x}\cos 3x - 3e^{6x}\sin 3x] - 3\left[\sin 3x.\frac{d(e^{6x})}{dx} + e^{6x}\frac{d(\sin 3x)}{dx}\right]$$

= $36e^{6x}\cos3x - 18e^{6x}\sin3x - 3[\sin3x.e^{6x}.6 + e^{6x}.\cos3x.3]$

= $36e^{6x}\cos3x - 18e^{6x}\sin3x - 18e^{6x}\sin3x - 9e^{6x}\cos3x$

= $27e^{6x}\cos3x - 36e^{6x}\sin3x$

= $9e^{6x}(3\cos3x - 4\sin3x)$

Q. 8 Find the second order derivatives of the function

$\tan^{-1} x$

Answer:

Let us take y = $\tan^{-1}$ x Now,

$$\frac{dy}{dx} = \frac{d(tan^{-1})}{dx} = \frac{1}{1+x^2}$$

$\frac{d^2y}{dx^2} = \frac{d\left[\frac{1}{1+x^2}\right]}{dx}$

$= \frac{d(+x^2)^{-1}}{dx} = (-1).(1 + x^2).\frac{d(1+x^2)}{dx}$

$= \frac{1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2}$

Q. 9 Find the second order derivatives of the function

log (log x)

Answer:

Let us take y = log (log x)

Now,

$\frac{dy}{dx} = \frac{d[\log(\log x)]}{dx}$

$= \frac{1}{\log x}.\frac{d(\log x)}{dx} = \frac{1}{x \log x}$

= $(x \log x)^{-1}$

$\frac{d^2y}{dx^2} = \frac{d(x \log x)^{-1}}{dx}$

= (-1). $(x \log x)^{-2}.\frac{d(x \log x)}{dx}$

$= \frac{-1}{(x log\, x)^2}.\left[\log x.\frac{d(x)}{dx} + x.\frac{d(\log x)}{dx}\right]$

$= \frac{-1}{(x \log x)^2}.\left[\log x. 1 + x.\frac{1}{x}\right]$

$= \frac{-(1+\log x)}{(x \log x)^2}$

Q. 10 Find the second order derivatives of the function

sin (log x)

Answer:

Let us take y = sin (log x)

Now,

$$\frac{dy}{dx} = \frac{d[\sin(\log x)]}{dx}$$

$= \cos(\log x).\frac{d(\log x)}{dx}$

$= \frac{\cos(\log x)}{x}$

Then

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{\cos(\log x)}{x}\right)}{dx}$$

$= \frac{x.\frac{d[\cos(logx)]}{dx} - \cos(\log x).\frac{d(x)}{dx}}{x^2}$

$= \frac{x.\left[-\sin(\log x).\frac{d(\log x)}{dx}\right] - \cos(\log x).1}{x^2}$

$= \frac{-x\sin(\log x).\frac{1}{x}.\cos(\log x)}{x^2}$

$= \frac{-\sin(\log x) + \cos(\log x)}{x^2}$

Q. 11 If y = 5 cos x – 3 sin x, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer:

It is given that y = 5 cos x – 3 sin x

Now, on differentiating we get,

$$\frac{dy}{dx} = \frac{d[5\cos x - 3\sin x]}{dx}$$

$= \frac{d(5\cos x)}{dx} - \frac{d(3\sin x)}{dx}$

$= \frac{5d(\cos 5x)}{dx} - \frac{3d(\sin x)}{dx}$

= 5(-sin x) – 3(cos x)

= -(5sinx + cos x)

Then,

$\frac{d^2y}{dx^2} = \frac{d(-(5\sin x + \cos x))}{dx}$

$= -\left[5.\frac{d(\sin x)}{dx} + 3.\frac{d(\cos x)}{dx}\right]$

= - [5cosx +3(-sin x)]

= -[5cosx-3sinx]

= -y

Therefore,

$\frac{d^2y}{dx^2} + y = 0$

Hence Proved.

Q. 12 If y = $\cos^{-1}$ x, Find $d^2y/dx^2$ in terms of y alone.

Answer:

It is given that y = $\cos^{-1}$ x

Now,

$\frac{dy}{dx} = \frac{d(cos^{-1})}{dx} = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}}$

Therefore,

$\frac{d^2y}{dx^2} = \frac{d(-(1-x^2)^{-\frac{1}{2}})}{dx}$

$= -\left(-\frac{1}{2}\right).(1-x^2)^{-\frac{3}{2}}.\frac{d(1-x^2)}{dx}$

$= \frac{1}{2\sqrt{1-x^2}^3} \times (-2x)$

$\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)}^3}$ ……… (1)

Now it is given that $y = \cos^{-1} x$

$\Rightarrow x = \cos y$

Now putting the value of x in equation (1), we get

$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{1-cos^2y}^3}$

$= \frac{-\cos y}{\sqrt{sin^2y}^3}$

$= \frac{-\cos y}{(\sin y)^3} = \frac{-\cos y}{\sin y}.\frac{1}{sin^2y}$

$= \frac{d^2y}{dx^2} = -\cot y.cosec^2y$

Q. 13 If y = 3 cos (log x) + 4 sin (log x), show that $x^2 y_2 + xy_1 + y = 0$

Answer:

It is given that y = 3 cos (log x) + 4 sin (log x)

Now, on differentiating we get,

$\frac{dy}{dx} = \frac{d(3\cos(\log x))+4\sin(\log x))}{dx}$

$= 3.\frac{d(\cos(\log x))}{dx} + 4.\frac{d(\sin(\log x))}{dx}$

$= 3.\left[-\sin(\log x).\frac{d(\log x)}{dx}\right] + 4.\left[\cos(\log x).\frac{d(\log x)}{dx}\right]$

$= \frac{dy}{dx} = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$

Again differentiating we get,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)}{dx}$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Therefore,

$x^2 y_2 + xy_1 + y$

$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) +$
$3\cos(\log x) + 4\sin(\log x)$

= -sin(logx) – 7cos(logx) + 4cos(logx) – 3sin(logx) + 3cos(logx) + 4sin(logx)

= 0

So, $x^2 y_2 + xy_1 + y = 0$

Hence Proved

Q. 14

If y = Ae$^{mx}$ + Be$^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer:

According to given equation, we have,

$y = Ae^{mx} + Be^{nx}$

Then, $\frac{dy}{dx} = \frac{d(Ae^{mx}+Be^{nx})}{dx}$

$= A.\frac{d(e^{mx})}{dx} + B.\frac{d(e^{nx})}{dx}$

$= A.e^{mx}\frac{d(mx)}{dx} + B.e^{nx}\frac{d(nx)}{dx}$

$= Ame^{mx} + Bne^{nx}$

Now, on again differentiating we get,

$\frac{d^2y}{dx^2} = \frac{d(Ame^{mx}+Bne^{nx})}{dx}$

$= Am.\frac{d(e^{mx})}{dx} + Bn.\frac{d(e^{nx})}{dx}$

$= Am.e^{mx}\frac{d(mx)}{dx} + Bn.e^{nx}\frac{d(nx)}{dx}$

$= Am^2e^{mx} + Bn^2e^{nx}$

$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= Am^2e^{mx} + Bn^2e^{nx} - (m + n)(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$

$= 0$

$= \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Hence Proved

Q. 15 If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Answer:

According to given equation, we have,

$y = 500e^{7x} + 600e^{-7x}$

$$\frac{dy}{dx} = \frac{d\left(500e^{7x}+600e^{-7x}\right)}{dx}$$

$$= 500.\frac{d\left(e^{7x}\right)}{dx} + 600.\frac{d(-7x)}{dx}$$

$$= 500.e^{7x}\frac{d(7x)}{dx} + 600.e^{-7x}\frac{d(-7x)}{dx}$$

$= 3500e^{7x} - 4200e^{-7x}$

Now, on again differentiating we get,

$$\frac{d^2y}{dx^2} = \frac{d(3500e^{7x}-4200e^{-7x})}{dx}$$

$$= 3500.\frac{d(e^{7x})}{dx} - 4200\frac{d(e^{-7x})}{dx}$$

$$= 3500\ e^{7x}\frac{d(7x)}{dx} - 42500e^{-7x}\frac{d}{dx(-7x)}$$

$= 7\times3500.e^{7x} + 7\times4200.e^{-7x}$

$= 49\times500e^{7x} + 49\times600e\text{-}^{7x}$

$= 49(500e^{7x} + 600e^{-7x})$

$= 49y$

$$\therefore \frac{d^2y}{dx^2} = 49\,y$$

Hence Proved

Q. 16 If $e^y (x + 1) = 1$, show that=

Answer:

It is given that

$e^y (x + 1) = 1$

$= e^y = \frac{1}{x+1}$

Now, taking logarithm on both the sides we get,

y = log $\frac{1}{x+1}$

On differentiating both sides, we get,

$\frac{dy}{dx} = (x+1)\frac{d\left(\frac{1}{x+1}\right)}{dx}$

= (x + 1). $\frac{-1}{(x+1)^2} = \frac{-1}{x+1}$

Again, on differentiating we get,

$\therefore \frac{d^2y}{dx^2} = -\frac{d\left(\frac{1}{x+1}\right)}{dx}$

$= -\left(\frac{d^2y}{dx^2}\right) = \frac{1}{(x+1)^2}$

$= \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

$= \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Hence Proved

Q. 17 If y = $(\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

Answer:

: It is given that

y = $(\tan^{-1} x)^2$

On differentiating we get,

$\frac{dy}{dx} = \frac{d[(tan^{-1}x)^2]}{dx}$

= 2 $\tan^{-1}$ x $\frac{d[tan^{-1}x]}{dx}$

$= 2 \tan^{-1}x \frac{1}{1+x^2}$

$= (1 + x^{2)} \frac{dy}{dx} = 2 \tan^{-1} x$

Again differentiating, we get,

$(1 + x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 2\left(\frac{1}{1+x^2}\right)$

$= (1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(1 + x^2) \frac{dy}{dx} = 2$

So, $(1+x^2)2y^2 + 2x(1+x^2)\, y_1 = 2$

where, $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

Hence Proved