2.2 Limits

2.2.1 Limit of a Function

Let $y = f(x)$ be a function of *x*. If at $x = a, f(x)$ takes indeterminate form, then we consider the values of the function which are very near to '*a*'. If these values tend to a definite unique number as *x* tends to '*a*', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

(1) **Meaning of '$x \to a$':** Let *x* be a variable and *a* be the constant. If *x* assumes values nearer and nearer to '*a*' then we say '*x* tends to *a*' and we write '$x \to a$'. It should be noted that as $x \to a$, we have $x \neq a$. By '*x* tends to *a*' we mean that

(i) $x \neq a$

(ii) *x* assumes values nearer and nearer to '*a*' and

(iii) We are not specifying any manner in which *x* should approach to '*a*'. *x* may approach to *a* from left or right as shown in figure.

(2) **Left hand and right hand limit :** Consider the values of the functions at the points which are very near to *a* on the left of *a*. If these values tend to a definite unique number as *x* tends to *a*, then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

Similarly we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as $f(a+0) = \lim_{x \to a^+} f(x)$ $= \lim_{h \to 0} f(a+h)$.

(3) **Method for finding L.H.L. and R.H.L.**

(i) For finding right hand limit (R.H.L.) of the function, we write $x + h$ in place of *x*, while for left hand limit (L.H.L.) we write $x - h$ in place of *x*.

(ii) Then we replace *x* by '*a*' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

(4) **Existence of limit :** $\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Note : ❒ If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty, \infty \times \infty, 0 \times \infty, 1^\infty, 0^0, \infty^0$

❒ In short, we write L.H.L. for left hand limit and R.H.L. for right hand limit.

❒ It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

(5) **Sandwich theorem :** If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x) \le g(x) \le h(x)\ \forall x \in$ neighborhood of $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$ (say), then $\lim_{x\to a} g(x) = l$. This theorem is normally applied when the $\lim_{x\to a} g(x)$ can't be obtained by using conventional methods as function $f(x)$ and $h(x)$ can be easily found.

Example: 1 If $f(x) = \begin{cases} x, & \text{when } x > 1 \\ x^2, & \text{when } x < 1 \end{cases}$, then $\lim_{x\to 1} f(x) =$ **[MP PET 1987]**

(a) x^2 (b) x (c) -1 (d) 1

Solution: (d) To find L.H.L. at $x = 1$. *i.e.,*

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h)^2 = \lim_{h\to 0}(1+h^2-2h) = 1$ i.e., $\lim_{x\to 1^-} f(x) = 1$ ....(i)

Now find R.H.L. at x = 1 i.e., $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = 1$ i.e., $\lim_{x\to 1^+} f(x) = 1$ .....(ii)

From (i) and (ii), L.H.L. = R.H.L. $\Rightarrow \lim_{x\to 1} f(x) = 1$.

Example: 2 $\lim_{x\to 2} \frac{|x-2|}{x-2} =$

(a) 1 (b) −1 (c) Does not exist (d) None of these

Solution: (c) L.H.L.= $\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{h\to 0} \frac{|2-h-2|}{2-h-2} = \lim_{h\to 0} \frac{h}{-h} = -1$ .....(i)

and, R.H.L.= $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{h\to 0} \frac{|2+h-2|}{2+h-2} = \lim_{h\to 0} \frac{h}{h} = 1$ .....(ii)

From (i) and (ii) L.H.L. ≠ R.H.L. *i.e.* $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

Example: 3 If $f(x) = \begin{cases} \frac{2}{5-x}, & \text{when } x < 3 \\ 5-x, & \text{when } x > 3 \end{cases}$, then

(a) $\lim_{x\to 3^+} f(x) = 0$ (b) $\lim_{x\to 3^-} f(x) = 0$ (c) $\lim_{x\to 3^+} f(x) \ne \lim_{x\to 3^-} f(x)$ (d) None of these

Solution: (c) $\lim_{x\to 3+} f(x) = 5-3 = 2$ and $\lim_{x\to 3-} f(x) = \frac{2}{5-3} = 1$

Example: 4 Let the function *f* be defined by the equation $f(x) = \begin{cases} 3x, & \text{if } 0 \le x \le 1 \\ 5-3x, & \text{if } 1 < x \le 2 \end{cases}$, then **[SCRA 1996]**

(a) $\lim_{x\to 1} f(x) = f(1)$ (b) $\lim_{x\to 1} f(x) = 3$ (c) $\lim_{x\to 1} f(x) = 2$ (d) $\lim_{x\to 1} f(x)$ does not exist

Solution: (d) L.H.L. = $\lim_{x\to 1-0} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 3(1-h) = \lim_{h\to 0}(3-3h) = 3-3.0 = 3$

R.H.L. = $\lim_{x\to 1+0} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[5-3(1+h)] = \lim_{h\to 0}(2-3h) = 2-3.0 = 2$

Hence $\lim_{x\to 1} f(x)$ does not exists.

Example: 5 $\lim_{x\to 0} \frac{|x|}{x} =$ **[Roorkee 1982; UPSEAT 2001]**

(a) 1 (b) −1 (c) 0 (d) Does not exist

Solution: (d) $\because \lim_{x \to 0-} \frac{|x|}{x} = -1$ and $\lim_{x \to 0+} \frac{|x|}{x} = 1$, hence limit does not exists.

2.2.2 Fundamental Theorems on Limits

The following theorems are very useful for evaluation of limits if $\lim_{x \to 0} f(x) = l$ and $\lim_{x \to 0} g(x) = m$ (l and m are real numbers) then

(1) $\lim_{x \to a}(f(x) + g(x)) = l + m$ (Sum rule)

(2) $\lim_{x \to a}(f(x) - g(x)) = l - m$ (Difference rule)

(3) $\lim_{x \to a}(f(x).g(x)) = l.m$ (Product rule)

(4) $\lim_{x \to a} k\ f(x) = k.l$ (Constant multiple rule)

(5) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (Quotient rule)

(6) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$

(7) $\lim_{x \to a} \log\{f(x)\} = \log\{\lim_{x \to a} f(x)\}$

(8) If $f(x) \leq g(x)$ for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

(9) $\lim_{x \to a}[f(x)]^{g(x)} = \{\lim_{x \to a} f(x)\}^{\lim_{x \to a} g(x)}$

(10) If p and q are integers, then $\lim_{x \to a}(f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.

(11) If $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(m)$ provided 'f' is continuous at $g(x) = m$. *e.g.* $\lim_{x \to a} \ln[f(x)] = \ln(l)$, only if $l > 0$.

2.2.3 Some Important Expansions

In finding limits, use of expansions of following functions are useful :

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

(2) $a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + ...$

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...,|\ x| < 1$

(5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....,$ where $|\ x| < 1$

(6) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)} = e^{1 - \frac{x}{2} + \frac{x^2}{3}} .... = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ....\right)$

(7) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

(8) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(9) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(10) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

(11) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

(12) $\tanh x = x - \frac{x^3}{3} + 2x^5 - .....$

(13) $\sin^{-1} x = x + 1^2.\frac{x^3}{3!} + 3^2.1^2.\frac{x^5}{5!} + ...$ (14) $\cos^{-1} x = \left(\frac{\pi}{2}\right) - \sin^{-1} x$

(15) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

2.2.4 Methods of Evaluation of Limits

We shall divide the problems of evaluation of limits in five categories.

(1) **Algebraic limits :** Let $f(x)$ be an algebraic function and '*a*' be a real number. Then $\lim_{x \to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method :** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when $x \to \infty$:** In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

Step I : Write down the expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.

Step III : Use the result $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

Note : ❑ **An important result :** If *m, n* are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers,

then $$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + ... + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + ... + b_{n-1} x + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{if } m > n \end{cases}$$

Example: 6 $\lim_{x \to 1} (3x^2 + 4x + 5) =$

(a) 12 (b) −1 (c) Does not exist (d) None of these

Solution: (a) $\lim_{x \to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$.

Example: 7 The value of $\lim_{x \to 2} \frac{3^{x/2} - 3}{3^x - 9}$ is **[MP PET 2000]**

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\ln 3$

Solution: (c) $\lim_{x\to 2}\frac{3^{x/2}-3}{(3^{x/2})^2-(3)^2}=\lim_{x\to 2}\frac{(3^{x/2}-3)}{(3^{x/2}-3)(3^{x/2}+3)}=\frac{1}{6}$.

Example: 8 The value of $\lim_{x\to a}\frac{x^n-a^n}{x-a}$ is **[Rajasthan PET 1989, 92]**

(a) 0 (b) na^{n-1} (c) na^n (d) 1

Solution: (b) $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{(x-a)(x^{n-1}+x^{n-2}a+..+a^{n-1})}{(x-a)}=\lim_{x\to a}(x^{n-1}+x^{n-2}a+....+a^{n-1})=n.a^{n-1}$.

Example: 9 $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]$ equals **[Rajasthan PET 1987]**

(a) $\frac{1}{2x}$ (b) $-\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $-\frac{1}{x^2}$

Solution: (d) $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{(x+h)x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{(x+h)x}\right]=-\frac{1}{x^2}$.

Example: 10 The value of $\lim_{x\to 0}\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{x^2}$ is **[MP PET 1999]**

(a) 1 (b) –1 (c) – 2 (d) 0

Solution: (b) $\lim_{x\to 0}\frac{\left(\sqrt{1-x^2}-\sqrt{1+x^2}\right)}{x^2}\frac{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}=\lim_{x\to 0}\frac{(1-x^2)-(1+x^2)}{x^2\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}=\frac{-2}{2}=-1$.

Example: 11 $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ equals **[UPSEAT 1991]**

(a) 1 (b) $\frac{3}{2}$ (c) $\frac{1}{4}$ (d) None of these

Solution: (d) $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{\left(\sqrt{x-2}\right)^2-\left(\sqrt{4-x}\right)^2}$

$=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{(2x-6)}=\lim_{x\to 3}\frac{\sqrt{x-2}+\sqrt{4-x}}{2}=\frac{1+1}{2}=1$.

Example: 12 $\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=$

(a) $\frac{b}{e}$ (b) $\frac{c}{f}$ (c) $\frac{a}{d}$ (d) $\frac{d}{a}$

Solution: (c) Here the expression assumes the form $\frac{\infty}{\infty}$. We note that the highest power of x in both the numerator and denominator is 2. So we divide each terms in both the numerator and denominator by x^2.

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=\lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}}=\frac{a+0+0}{d+0+0}=\frac{a}{d}.$$

Example: 13 $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $\log 2$ (d) e^4

Solution: (b) $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right] = \lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1} = \frac{1}{2}$.

Example: 14 The values of constants a and b so that $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$ is

(a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=-1, b=1$ (d) $a=2, b=-1$

Solution: (b) We have $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0 \Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=0$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than that of denominator. As the denominator is a first degree polynomial. So, numerator must be a constant *i.e.*, a zero degree polynomial. $\therefore$ $1-a=0$ and $a+b=0 \Rightarrow a=1$ and $b=-1$. Hence, $a=1$ and $b=-1$.

Example: 15 $\lim_{x\to 1} x^x =$

(a) 1 (b) ∞ (c) Not defined (d) None of these

Solution: (a) $\lim_{x\to 1} x^x = \left(\lim_{x\to 1} x\right)^{\lim_{x\to 1} x} = 1^1 = 1$

Example: 16 $\lim_{x\to 1}(1+x)^{1/x} =$

(a) 2 (b) e (c) Not defined (d) None of these

Solution: (a) $\lim_{x\to 1}(1+x)^{1/x} = \left(\lim_{x\to 1}(1+x)\right)^{\lim_{x\to 1}\left(\frac{1}{x}\right)} = 2$

Example: 17 The value of the limit of $\frac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(a) 3 (b) 9 (c) 18 (d) 21

Solution: (d) Let $y = \lim_{x\to 3}\frac{x^3-x^2-18}{x-3} = \lim_{x\to 3}(x^2+2x+6) = 9+6+6 = 21$

Example: 18 The value of the limit of $\frac{x^3-8}{(x^2-4)}$ as x tends to 2 is

(a) 3 (b) $\frac{3}{2}$ (c) 1 (d) 0

Solution: (a) $\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x^2+2x+4)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{4+4+4}{2+2} = 3$.

Example: 19 $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}$ is equal to **[Rajasthan PET 1988]**

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\right) = \lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$= \lim_{x\to 0}\left(\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{1+x-1+x}\right) = \lim_{x\to 0}\left(\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2}\right) = \frac{2}{2} = 1$

Example: 20 $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ equals **[IIT 1978; Kurukshetra CEE 1998]**

(a) $\frac{2a}{3\sqrt{3}}$ (b) $\frac{2}{3\sqrt{3}}$ (c) 0 (d) None of these

Solution: (b) $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}=\lim_{x\to a}\left(\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right)\times\left(\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\right)\times\left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right)$

$$=\lim_{x\to a}\left\{\frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}\right\}=\frac{2}{3\sqrt{3}}.$$

Example: 21 $\lim_{n\to\infty}\frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}}=$ **[EAMCET 1994]**

(a) $\frac{99}{100}$ (b) $\frac{1}{100}$ (c) $\frac{1}{99}$ (d) $\frac{1}{101}$

Solution: (b) $\lim_{n\to\infty}\frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}}=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^{99}}{n^{100}}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{99}=\int_0^1 x^{99}\,dx=\left[\frac{x^{100}}{100}\right]_0^1=\frac{1}{100}.$

Example: 22 The values of constants 'a' and 'b' so that $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2$ is

(a) $a=0,b=0$ (b) $a=1,b=-1$ (c) $a=1,b=-3$ (d) $a=2,b=-1$

Solution: (c) $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2\Rightarrow\lim_{x\to\infty}x-1-ax-b=2\Rightarrow\lim_{x\to\infty}x(1-a)-(1+b)=2$.

Comparing the coefficient of both sides, $1-a=0$ and $1+b=-2\Rightarrow a=1,b=-3$

Example: 23 $\lim_{n\to\infty}\left[\frac{\Sigma n^2}{n^3}\right]=$ **[Rajasthan PET 1999, 2002]**

(a) $-\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{-1}{3}$

Solution: (c) $\lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3}\right]=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}=\frac{1}{3}$

Note : ❐ Students should remember that,

$\lim_{n\to\infty}\frac{\Sigma n}{n^2}=\frac{1}{2}$ and $\lim_{n\to\infty}\frac{\Sigma n^2}{n^3}=\frac{1}{3}$.

Example: 24 $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+......+\frac{n}{1-n^2}\right]$ is equal to **[IIT 1984; DCE 2000]**

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) None of these

Solution: (b) $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+......+\frac{n}{1-n^2}\right]=\lim_{n\to\infty}\frac{\Sigma n}{1-n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2+n}{1-n^2}=-\frac{1}{2}.$

Example: 25 If $f(x)=\frac{2}{x-3}, g(x)=\frac{x-3}{x+4}$ and $h(x)=-\frac{2(2x+1)}{x^2+x-12}$ then $\lim_{x\to 3}[f(x)+g(x)+h(x)]$ is

(a) – 2 (b) – 1 (c) $-\frac{2}{7}$ (d) 0

Solution: (c) We have $f(x)+g(x)+h(x)=\frac{x^2-4x+17-4x-2}{x^2+x-12}=\frac{x^2-8x+15}{x^2+x-12}=\frac{(x-3)(x-5)}{(x-3)(x+4)}$

$\therefore\lim_{x\to 3}[f(x)+g(x)+h(x)]=\lim_{x\to 3}\frac{(x-3)(x-5)}{(x-3)(x+4)}=-\frac{2}{7}.$

Example: 26 If $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$ equal **[Kurukshetra CEE 1998]**

(a) e (b) $\frac{1}{e}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$

Solution: (b) Let $P = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n} \Rightarrow P = \lim_{n\to\infty}\left(\frac{1}{n}.\frac{2}{n}.\frac{3}{n}.\frac{4}{n}..........\frac{n}{n}\right)^{1/n}$

$\therefore \log P = \frac{1}{n}\lim_{n\to\infty}\left(\log\frac{1}{n}+\log\frac{2}{n}+.........+\log\frac{n}{n}\right) \Rightarrow \log P = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\frac{r}{n}$

$\log P = \int_0^1 \log x\,dx = [x\log x - x]_0^1 = (-1) \Rightarrow P = \frac{1}{e}$.

Example: 27 If $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$, then **[Karnataka CET 2000]**

(a) $a=1$ and $b=1$ (b) $a=1$ and $b=-1$ (c) $a=1$ and $b=-2$ (d) $a=1$ and $b=2$

Solution: (c) $\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right) = 2 \Rightarrow \lim_{x\to\infty}\left(\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}\right) = 2 \Rightarrow \lim_{x\to\infty}[x^3(1-a)-bx^2-ax+(1-b)] = 2(x^2+1)$.

Comparing the coefficients of both sides, $1-a=0$ and $-b=2$ or $a=1, b=-2$.

Example: 28 $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+.....+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to **[AMU 2000]**

(a) 0 (b) 1 (c) 10 (d) 100

Solution: (d) $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+......+(x+100)^{10}}{x^{10}+10^{10}} = \lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+...+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]} = 100$.

Example: 29 Let $f(x)=4$ and $f'(x)=4$, then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ equals **[Rajasthan 2000; AIEEE 2002]**

(a) 2 (b) -2 (c) -4 (d) 3

Solution: (c) $y = \lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2} \Rightarrow y = \lim_{x\to 2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}$

$\Rightarrow y = \lim_{x\to 2}\frac{-2f(x)+2f(2)+xf(2)-2f(2)}{(x-2)} \Rightarrow y = \lim_{x\to 2}-2\frac{[f(x)-f(2)]}{x-2}+\lim_{x\to 2}\frac{f(2).(x-2)}{(x-2)}$

$\Rightarrow y = -2\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}+f(2) \Rightarrow y = -2\lim_{x\to 2}f'(x)+f(2) = -8+4 = -4$.

(2) **Trigonometric limits :** To evaluate trigonometric limits the following results are very important.

(i) $\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{x}{\sin x}$

(ii) $\lim_{x\to 0}\frac{\tan x}{x} = 1 = \lim_{x\to 0}\frac{x}{\tan x}$

(iii) $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1 = \lim_{x\to 0}\frac{x}{\sin^{-1}x}$

(iv) $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1 = \lim_{x\to 0}\frac{x}{\tan^{-1}x}$

(v) $\lim_{x\to 0}\frac{\sin x^0}{x} = \frac{\pi}{180}$

(vi) $\lim_{x\to 0}\cos x = 1$

(vii) $\lim_{x\to a}\frac{\sin(x-a)}{x-a} = 1$

(viii) $\lim_{x\to a}\frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x\to a}\sin^{-1}x = \sin^{-1}a, |a| \le 1$

(x) $\lim_{x\to a}\cos^{-1}x = \cos^{-1}a; |a| \le 1$

(xi) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a; -\infty < a < \infty$

(xii) $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x \to \infty} \frac{\sin(1/x)}{(1/x)} = 1$

Example: 30 $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right) =$ **[IIT 1978, 84; Rajasthan PET 1997, 2001; UPSEAT 2003]**

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{2}{\pi}$ (d) 0

Solution: (c) $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$, Put $1 - x = y \Rightarrow$ as $x \to 1, y \to 0$

Thus $\lim_{y \to 0} y \tan \frac{\pi(1-y)}{2} = \lim_{y \to 0} \frac{2}{\pi} . \frac{\left(\frac{\pi y}{2}\right)}{\tan\left(\frac{\pi y}{2}\right)} = \frac{2}{\pi} \times 1 = \frac{2}{\pi}$.

Example: 31 $\lim_{x \to 1} \frac{\sqrt{1 - \cos 2(x-1)}}{x-1}$ **[IIT 1998; UPSEAT 2001]**

(a) Exists and it equal $\sqrt{2}$

(b) Exists and it equals $-\sqrt{2}$

(c) Does not exist because $x - 1 \to 0$

(d) Does not exist because left hand limit is not equal to right hand limit

Solution: (d) $f(1+) = \lim_{h \to 0} f(1+h) = = \lim_{h \to 0} \frac{\sqrt{1 - \cos 2h}}{h} = \lim_{h \to 0} \sqrt{2} \frac{\sinh}{h} = \sqrt{2}$

$f(1-) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{\sqrt{1 - \cos(-2h)}}{-h} = \lim_{h \to 0} \sqrt{2} \frac{\sinh}{-h} = -\sqrt{2}$.

$\therefore$ limit does not exist because left hand limit is not equal to right hand limit.

Example: 32 $\lim_{x \to 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x} =$ **[MP PET 2000; UPSEAT 2000; Karmataka CET 2002]**

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{6}{5}$ (d) $\frac{5}{6}$

Solution: (a) $\lim_{x \to 0} \frac{2 \sin^2 x}{x^2} \frac{\sin 5x}{\sin 3x} \frac{3x}{3x} \frac{5x}{5x} = \lim_{x \to 0} \frac{2 \sin^2 x}{x^2} . \frac{3x}{\sin 3x} . \frac{\sin 5x}{5x} . \frac{5x}{3x} = 2 . \frac{5}{3} = \frac{10}{3}$.

Example: 33 $\lim_{x \to 0} \frac{x^3}{\sin x^2} =$

(a) 0 (b) $\frac{1}{3}$ (c) 3 (d) $\frac{1}{2}$

Solution: (a) $\lim_{x \to 0} \frac{x^3}{\sin x^2} = \lim_{x \to 0} \frac{x^2}{\sin x^2} . x = \left(\lim_{x \to 0} \frac{x^2}{\sin x^2}\right)\left(\lim_{x \to 0} x\right) = 1.0 = 0$.

Example: 34 $\lim_{x \to 0} \frac{\sin 3x + \sin x}{x} =$

(a) $\frac{1}{3}$ (b) 3 (c) 4 (d) $\frac{1}{4}$

Solution: (c) $\lim_{x \to 0} \frac{\sin 3x + \sin x}{x} = \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} .3 + \lim_{x \to 0} \frac{\sin x}{x} = 1.3 + 1 = 4$.

Example: 35 If $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, then $\lim_{x\to 0} f(x)=$ **[IIT 1988; UPSEAT 1988; SCRA 1996]**

(a) 1 (b) 0 (c) –1 (d) None of these

Solution: (b) $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=\left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\sin\frac{1}{x}\right)$ = 0 × (A number oscillating between – 1 and 1) = 0.

Example: 36 If $f(x)=\begin{cases} \frac{\sin[x]}{[x]}, & [x]\neq 0 \\ 0, & [x]=0 \end{cases}$, then $\lim_{x\to 0} f(x)$ equals **[IIT 1985; Rajasthan PET 1995]**

(a) 1 (b) 0 (c) –1 (d) Does not exist

Solution: (d) In closed interval of $x=0$ at right hand side $[x]$ =0 and at left hand side $[x]=-1$. Also [0] =0.

Therefore function is defined as $f(x)=\begin{cases} \frac{\sin[x]}{[x]}, & (-1\leq x<0) \\ 0, & (0\leq x<1) \end{cases}$

$\therefore$ Left hand limit $=\lim_{x\to 0-} f(x)=\lim_{x\to 0-}\frac{\sin[x]}{[x]}=\frac{\sin(-1)}{-1}=\sin 1^c$

Right hand limit = 0, Hence, limit doesn't exist.

Example: 37 $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ **[IIT 1974; Rajasthan PET 2000]**

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{2}{3}$ (d) None of these

Solution: (a) $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3\cos x}=\lim_{x\to 0}\frac{\sin x\left(2\sin^2\frac{x}{2}\right)}{x^3\cos x}=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{2}{\cos x}\cdot\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{1}{4}\right]=\frac{1}{2}$

Example: 38 If $f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, then $\lim_{x\to 2} f(x)$ is given by

(a) – 2 (b) –1 (c) 0 (d) 1

Solution: (d) $\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(t+1)}=\lim_{t\to 0}\frac{\sin(e^t-1)}{\log(t+1)}$. (Putting $x = 2 + t$)

$=\lim_{x\to\infty}\frac{\sin(e^t-1)}{e^t-1}\cdot\frac{e^t-1}{t}\cdot\frac{t}{\log(1+t)}=\lim_{t\to 0}\frac{\sin(e^t-1)}{e^t-1}\left(\frac{1}{1!}+\frac{t}{2!}+.....\right)\left[\frac{1}{\left(1-\frac{1}{2}t+\frac{1}{3}t^2-.....\right)}\right]$

= 1.1.1 = 1 $[\because$ As $t\to 0, e^t-1\to 0$, $\therefore\frac{\sin(e^t-1)}{(e^t-1)}=1]$

Example: 39 $\lim_{x\to\pi/2}\frac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}=$ **[Kerala (Engg.) 2001]**

(a) $\log a$ (b) log 2 (c) a (d) $\log x$

Solution: (a) $\lim_{x\to\pi/2}\left(\frac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}\right)=\lim_{x\to\pi/2} a^{\cos x}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)$

$=a^{\cos(\pi/2)}\lim_{x\to\pi/2}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)=1\log a=\log a$.

Example: 40 If $f(x)=\begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim_{x\to 0}\frac{f(x)}{x^2}$ is **[Karnataka CET 2002]**

(a) 3 (b) –1 (c) 0 (d) 1

Solution: (d) $f(x) = x(x-1)\sin x - (x^3 - 2x^2)\cos x - x^3 \tan x$

$= x^2 \sin x - x^3 \cos x - x^3 \tan x + 2x^2 \cos x - x \sin x$

Hence, $\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0}\left(\sin x - x\cos x - x\tan x + 2\cos x - \frac{\sin x}{x}\right) = 0-0-0+2-1 = 1$.

Example: 41 If $f(x) = \cot^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$ and $g(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$, $0 < a < \frac{1}{2}$ is **[Orissa JEE 2003]**

(a) $\frac{3}{2(1+a^2)}$ (b) $\frac{3}{2(1+x^2)}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

Solution: (d) $f(x) = \cot^{-1}\left\{\frac{3x-x^3}{1-3x^2}\right\}$ and $g(x) = \cos^{-1}\left\{\frac{1-x^2}{1+x^2}\right\}$

Put $x = \tan\theta$ in both equation

$f(\theta) = \cot^{-1}\left\{\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right\} = \cot^{-1}\{\tan 3\theta\}$

$f(\theta) = \cot^{-1}\cot\left(\frac{\pi}{2} - 3\theta\right) = \frac{\pi}{2} - 3\theta \Rightarrow f'(\theta) = -3$ ......(i)

and $g(\theta) = \cos^{-1}\left\{\frac{1-\tan^2\theta}{1+\tan^2\theta}\right\} = \cos^{-1}(\cos 2\theta) = 2\theta \Rightarrow g'(\theta) = 2$ ..... (ii)

Now $\lim_{x\to a}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\frac{1}{\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)} = f'(x).\frac{1}{g'(x)} = -3\times\frac{1}{2} = -\frac{3}{2}$.

Example: 42 $\lim_{x\to\frac{\pi}{2}}\frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi - 2x]^3}$ is **[AIEEE 2003]**

(a) $\frac{1}{8}$ (b) 0 (c) $\frac{1}{32}$ (d) ∞

Solution: (c) $\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$

Let $x = \frac{\pi}{2} + y$, then $y \to 0$ $\Rightarrow \lim_{y\to 0}\frac{\tan\left(\frac{-y}{2}\right)(1-\cos y)}{(-2y)^3} = \lim_{y\to 0}\frac{-\tan\frac{y}{2}.2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y\to 0}\frac{1}{32}\frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2 = \frac{1}{32}$.

Example: 43 If $\lim_{x\to 0}\frac{[(a-n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

Solution: (d) $\lim_{x\to 0} n\frac{\sin nx}{nx}.\lim_{x\to 0}\left((a-n)n - \frac{\tan x}{x}\right) = 0 \Rightarrow n[(a-n)n - 1] = 0 \Rightarrow (a-n)n = 1 \Rightarrow a = n + \frac{1}{n}$.

(3) **Logarithmic limits :** To evaluate the logarithmic limits we use following formulae

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....\ ..\text{to } \infty$ where $-1 < x \le 1$ and expansion is true only if base is *e*.

(ii) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(iii) $\lim_{x\to e} \log_e x = 1$

(iv) $\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$

(v) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \neq 1$

Example: 44 $\lim_{h\to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$ **[IIT Screening 1997]**

(a) –1 (b) 1 (c) 2 (d) –2

Solution: (a) $\lim_{h\to 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2} = \lim_{x\to a} \frac{\left((2h) - \frac{(2h)^2}{2} + \frac{(2h)^3}{3} - .....\infty\right) - 2\left(h - \frac{h^2}{2} + \frac{h^3}{3} - ......\right)}{h^2}$

$= \lim_{h\to 0} \frac{-h^2 + 2h^3 - ....}{h^2} = \lim_{h\to 0} \frac{h^2\{-1 + 2h - ....\}}{h^2} = \lim_{h\to 0} \{-1 + 2h + ...\} = -1$.

Example: 45 $\lim_{x\to a} \frac{\log\{1+(x-a)\}}{(x-a)} =$

(a) –1 (b) 2 (c) 1 (d) –2

Solution: (c) Let $x - a = y$, when $x \to a$, $y \to 0$,

$\therefore$ The given limit = $\lim_{y\to 0} \frac{\log\{1+y\}}{y} = 1$.

Example: 46 $\lim_{h\to 0} \frac{\log_{10}(1+h)}{h} =$

(a) 1 (b) $\log_{10} e$ (c) $\log_e 10$ (d) None of these

Solution: (b) $\lim_{h\to 0} \frac{\log_e(1+h)}{h} \cdot \frac{1}{\log_e 10} = \log_{10} e$.

Example: 47 If $\lim_{x\to 0} \frac{\log(3+x) - \log(3-x)}{x} = k$, then the value of *k* is **[AIEEE 2003]**

(a) 0 (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Solution: (c) $\lim_{x\to 0} \frac{\log(3+x) - \log(3-x)}{x} = \lim_{x\to 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x} = \lim_{x\to 0} \frac{\log\left(\frac{1+(x/3)}{1-(x/3)}\right)}{x}$

$= \lim_{x\to 0} \frac{\log(1+(x/3))}{x} - \lim_{x\to 0} \frac{\log(1-(x/3))}{x} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$.

(4) **Exponential limits :**

(i) **Based on series expansion :** We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\ ...\infty$

To evaluate the exponential limits we use the following results –

(a) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (b) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ (c) $\lim_{x\to 0} \frac{e^{\lambda x} - 1}{x} = \lambda \quad (\lambda \neq 0)$

(ii) **Based on the form 1^∞ :** To evaluate the exponential form 1^∞ we use the following results.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$, or when $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$. Then $\lim_{x\to a}\{f(x)\}^{g(x)} = \lim_{x\to a}[1 + f(x) - 1]^{g(x)} = e^{\lim_{x\to a}(f(x)-1)g(x)}$

(b) $\lim_{x\to 0}(1+x)^{1/x} = e$ (c) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (d) $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^{\lambda}$ (e) $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^x = e^{\lambda}$

Note : ❑ $\lim_{x\to\infty} a^x = \begin{cases}\infty, \text{if } a > 1\\ 0, \text{if } a < 1\end{cases}$ i.e., $a^{\infty} = \infty$, if $a > 1$ and $a^{\infty} = 0$ if $a < 1$.

Example: 48 $\lim_{x\to 0}\frac{e^{\alpha x} - e^{\beta x}}{x} =$ **[MP PET 1994]**

(a) $\alpha + \beta$ (b) $\frac{1}{\alpha} + \beta$ (c) $\alpha^2 - \beta^2$ (d) $\alpha - \beta$

Solution: (d) $\lim_{x\to 0}\frac{e^{\alpha x} - e^{\beta x}}{x} = \lim_{x\to 0}\frac{(e^{\alpha x}-1)-(e^{\beta x}-1)}{x} = \lim_{x\to 0}\frac{e^{\alpha x}-1}{x} - \lim_{x\to 0}\frac{e^{\beta x}-1}{x} = \alpha - \beta$.

Example: 49 The value of $\lim_{x\to 0}\frac{e^x - (1+x)}{x^2}$ is **[Karnataka CET 1995]**

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{4}$

Solution: (b) $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2} = \lim_{x\to 0}\frac{(1+x+\frac{x^2}{2!}+.....)-(1+x)}{x^2} = \lim_{x\to 0}\frac{x^2\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+.....\right)}{x^2} = \frac{1}{2!} = \frac{1}{2}$.

Example: 50 $\lim_{x\to 0}\frac{a^x - 1}{\sqrt{1+x}-1}$ is equal to

(a) $2\log_e a$ (b) $\frac{1}{2}\log_e a$ (c) $a\log_e 2$ (d) None of these

Solution: (a) $\lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \lim_{x\to 0}\frac{(a^x-1)(\sqrt{1+x}+1)}{1+x-1} = \lim_{x\to 0}\left(\frac{a^x-1}{x}\right).(\sqrt{1+x}+1)$

$= \left(\lim_{x\to 0}\frac{a^x-1}{x}\right).\left(\lim_{x\to 0}(\sqrt{1+x}+1)\right) = (\log_e a).(2) = 2\log_e a$.

Example: 51 The value of $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}$ is **[UPSEAT 2003]**

(a) e^4 (b) 0 (c) 1 (d) e^2

Solution: (d) $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}.(x+2).\frac{2}{(x+1)}} = \lim_{x\to\infty}\left(\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right)^{2.\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)} = e^{2\lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)/\left(1+\frac{1}{x}\right)\right]} = e^2$.

Alternative method : $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{x+2} = e^{\lim_{x\to\infty}\frac{2}{x+1}(x+2)} = e^{\lim_{x\to\infty}2\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)} = e^2$

Example: 52 If a, b, c, d are positive, then $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$ **[EAMCET 1992]**

(a) $e^{d/b}$ (b) $e^{c/a}$ (c) $e^{(c+d)/(a+b)}$ (d) e

Solution: (a) $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx} = \lim_{x\to\infty}\left\{\left(1+\frac{1}{a+bx}\right)^{a+bx}\right\}^{\frac{c+dx}{a+bx}} = e^{d/b}$ $\left\{\because \lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{a+bx} = e \text{ and } \lim_{x\to\infty}\frac{c+dx}{a+bx}=\frac{d}{b}\right\}$

Alternative method : $e^{\lim_{x\to\infty}\left(\frac{1}{a+bx}\right)\left(\frac{c+dx}{1}\right)} = e^{d/b}$.

Example: 53 $\lim_{x\to 0} x^x =$ **[Roorkee 1987]**

(a) 0 (b) 1 (c) e (d) None of these

Solution: (b) Let $y = x^x \Rightarrow \log y = x\log x$; $\therefore \lim_{y\to 0}\log y = \lim_{x\to 0} x\log x = 0 = \log 1 \Rightarrow \lim_{x\to 0} x^x = 1$

Example: 54 The value of $\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$ is **[DCE 2001]**

(a) $\frac{11e}{24}$ (b) $\frac{-11e}{24}$ (c) $\frac{e}{24}$ (d) None of these

Solution: (a) $(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} = e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-....\right)} = e^{1-\frac{x}{2}+\frac{x^2}{3}-.......} = e\, e^{-\frac{x}{2}+\frac{x^2}{3}-......}$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)^2+.....\right] = e\left[1-\frac{x}{2}+\frac{11}{24}x^2-..........\,..\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2} = \frac{11e}{24}$$

Example: 55 $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ equals **[UPSEAT 2001]**

(a) $\pi/2$ (b) 0 (c) $2/e$ (d) $-e/2$

Solution: (d) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}[\log(1+x)]} = e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....\right)} = e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+....\right)} = e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+....\right)}$

$$= e\left[1+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+.....\right)}{1!}+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+...\right)^2}{2!}+...\right] = \left[e-\frac{ex}{2}+\frac{11e}{24}x^2-..........\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}\left[\frac{e-\frac{ex}{2}+\frac{11e}{24}x^2..........-e}{x}\right] \Rightarrow \lim_{x\to 0}\left(-\frac{e}{2}-\frac{11e}{24}x+...\right) = -\frac{e}{2}.$$

Example: 56 $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m =$ **[AMU 2001]**

(a) 0 (b) e (c) $1/e$ (d) 1

Solution: (d) $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m = \lim_{m\to\infty}\left[1+\left(\cos\frac{x}{m}-1\right)\right]^m = \lim_{m\to\infty}\left[1-\left(-\cos\frac{x}{m}+1\right)\right]^m$

$$= \lim_{m\to\infty}\left[1-2\sin^2\frac{x}{2m}\right]^m = e^{\lim_{m\to\infty}-\left(2\sin^2\frac{x}{2m}\right)m} = e^{\lim_{m\to\infty}-2\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^2\left(\frac{x^2}{4m^2}\right)m} = e^{-2\lim_{m\to\infty}\frac{x^2}{4m}} = e^0 = 1.$$

Example: 57 $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)} =$ **[AMU 2002]**

(a) e (b) e^2 (c) e^{-1} (d) 1

Solution: (b) $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)} = \lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\frac{1}{n(n-1)}\right)^{n(n-1)}} = \frac{e}{e^{-1}} = e^2$.

Alternative Method: $\lim_{n\to\infty}\left(1+\frac{2}{n^2-n-1}\right)^{n(n-1)} = e^{\lim_{n\to\infty}\frac{2n(n-1)}{n^2-n-1}} = e^2$.

(5) **L' Hospital's rule :** If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

(ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$

Note : ❐ The above rule is also applicable if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

❐ If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get, $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

Example: 58 $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} =$ **[Kerala (Engg.) 2002]**

(a) m/n (b) n/m (c) $\frac{m^2}{n^2}$ (d) $\frac{n^2}{m^2}$

Solution: (c) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}}\right\} = \lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}\cdot\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\cdot\frac{4}{n^2x^2}\right] = \frac{m^2}{n^2}\times 1 = \frac{m^2}{n^2}$

Trick : Apply L-Hospital rule ,

$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\frac{m\sin mx}{n\sin nx} = \lim_{x\to 0}\frac{m^2\cos mx}{n^2\cos nx} = \frac{m^2}{n^2}$.

Example: 59 The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is **[IIT Screening 2002]**

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) n cannot be negative integer for then the limit $=0$

$$\text{Limit} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2^2(x/2)^2}\frac{e^x-\cos x}{x^{n-2}} = \frac{1}{2}\lim_{x\to 0}\frac{e^x-\cos x}{x^{n-2}} \quad (n\neq 1 \text{ for then the limit = 0})$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{e^x+\sin x}{(n-2)x^{n-3}}.$$ So, if $n=3$, the limit is $\frac{1}{2(n-2)}$ which is finite. If $n=4$, the limit is infinite.

Example: 60 Let $f:R\to R$ be such that $f(1)=3$ and $f'(1)=6$. Then $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{\frac{1}{x}}$ equals **[IIT Screening 2002]**

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

Solution: (c) $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}[\log f(1+x)-\log f(1)]} = e^{\lim_{x\to 0}\frac{f'(1+x)/f(1+x)}{1}} = e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$.

Example: 61 $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4} =$ **[IIT Screening 1997; AMU 1997]**

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 1 (d) None of these

Solution: (a) $\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$ $(\frac{0}{0}$ form$)$ $= \lim_{\alpha\to\pi/4}\frac{\cos\alpha+\sin\alpha}{1}$ (By 'L' Hospital rule)

$$= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Example: 62 $\lim_{x\to a}\frac{x^3-a^3}{x^2-a^2} =$

(a) 0 (b) Not defined (c) $2a$ (d) $\frac{3a}{2}$

Solution: (d) $\lim_{x\to a}\frac{x^3-a^3}{x^2-a^2}$ $\left(\frac{0}{0}\text{ form}\right)$ $= \lim_{x\to a}\frac{3x^2}{2x}$ (By 'L' Hospital rule) $= \frac{3a^2}{2a} = \frac{3a}{2}$.

Example: 63 $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} =$ **[Roorkee 1983]**

(a) $1/2\sqrt{x}$ (b) $1/2\sqrt{h}$ (c) Zero (d) None of these

Solution: (a) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$

Trick : Applying 'L' Hospital's rule, [Differentiating N^r and D^r with respect to h]

We get, $\lim_{h\to 0}\frac{\frac{1}{2\sqrt{x+h}}-0}{1} = \frac{1}{2\sqrt{x}}$.

Example: 64 $\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}$ **[MP PET 2001]**

(a) 0 (b) 1 (c) $\frac{\sin\beta}{\beta}$ (d) $\frac{\sin 2\beta}{2\beta}$

Solution: (d) $\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2} = \lim_{\alpha-\beta\to 0}\frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{(\alpha+\beta)\,(\alpha-\beta)} = \lim_{\alpha-\beta\to 0}\frac{\sin(\alpha-\beta)}{(\alpha-\beta)}\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha+\beta)}{(\alpha+\beta)} = \lim_{\alpha\to\beta}\frac{\sin(\alpha+\beta)}{(\alpha+\beta)} = \frac{\sin 2\beta}{2\beta}$.

Trick : By L' Hospital's rule, $\lim_{\alpha\to\beta}\frac{2\sin\alpha\cos\alpha}{2\alpha} = \frac{\sin 2\beta}{2\beta}$.

Example: 65 $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$ equals **[IIT 1971]**

(a) 2/3 (b) 1/3 (c) 1/2 (d) 0

Solution: (c) $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \to 0} \left\{ \frac{\frac{2\tan 2x}{2x} - 1}{3 - \frac{\sin x}{x}} \right\} = \frac{1}{2}$.

Example: 66 If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1}$ equals **[IIT 1983]**

(a) 1/24 (b) 1/5 (c) $-\sqrt{24}$ (d) None of these

Solution: (d) $\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1} = \lim_{x \to 1} \frac{-\sqrt{25 - x^2} + \sqrt{24}}{x - 1}$ [Multiply both numerator and denominator by ($\sqrt{24} + \sqrt{25 - x^2}$)]

$= \lim_{x \to 1} \frac{x + 1}{\sqrt{24} + \sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Alternative method: By L'-Hospital rule, $\lim_{x \to 1} \frac{G'(x)}{1} = \lim_{x \to 1} \frac{-1(-2x)}{2\sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Example: 67 If $f(a) = 2, f'(a) = 1, g(a) = 1, g'(a) = 2,$ then $\lim_{x \to a} \frac{g(x) f(a) - g(a) f(x)}{x - a}$ equals

[IIT 1983; Rajasthan PET 1990; MP PET 1995; DCE 1999; Karnataka CET 1999, 2003]

(a) –3 (b) $\frac{1}{3}$ (c) 3 (d) $-\frac{1}{3}$

Solution: (c) Applying L – Hospital's rule, we get, $\lim_{x \to a} \frac{g(x) f(a) - g(a) f(x)}{x - a} = \lim_{x \to a} \frac{g'(x) f(a) - g(a) f'(x)}{1}$

$= g'(a) f(a) - g(a) f'(a) = 2 \times 2 - 1 \times (1) = 3.$

Example: 68 $\lim_{x \to 0} \frac{(1 + x)^n - 1}{x} =$ **[Kurukshetra CEE 2002]**

(a) n (b) 1 (c) –1 (d) None of these

Solution: (a) $\lim_{x \to 0} \frac{(1 + nx + {}^nC_2 x^2 + \text{......higher powers of } x \text{ to } x^n) - 1}{x} = n$

Trick : Apply L- Hospital rule.

Example: 69 $\lim_{x \to 0} \frac{\sin x + \log(1 - x)}{x^2}$ is equal to **[Roorkee 1995]**

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) None of these

Solution: (c) Apply L- Hospital rule, we get, $\lim_{x \to 0} \frac{\cos x - \frac{1}{1 - x}}{2x} = \lim_{x \to 0} \frac{-\sin x - \frac{1}{(1 - x)^2}}{2} = -\frac{1}{2}$

Alternative method : $\lim_{x \to 0} \frac{\sin x + \log(1 - x)}{x^2} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{......} \right)}{x^2} + \lim_{x \to 0} \frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{......} \right)}{x^2}$

$\left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{......} \text{ and } \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \text{........} \right)$

Hence, $\lim_{x\to 0}\dfrac{\dfrac{-x^2}{2}-x^3\left(\dfrac{1}{3!}+\dfrac{1}{3}\right)-\dfrac{x^4}{4}....}{x^2}=-\dfrac{1}{2}$.

Example: 70 $\lim_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$ equals **[Rajasthan PET 1996]**

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

Solution: (d) Let $y=\lim_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Applying L–Hospital's rule, $y=\lim_{x\to 0}\dfrac{e^x+xe^x-\dfrac{1}{1+x}}{2x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$y=\lim_{x\to 0}\dfrac{1}{2}\left[e^x+e^x+xe^x+\dfrac{1}{(1+x)^2}\right]=\lim_{x\to 0}\dfrac{1}{2}[1+1+0+1]=\dfrac{3}{2}$

Example: 71 $\lim_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to **[Rajasthan PET 2000]**

(a) 0 (b) 1 (c) – 1 (d) $\frac{1}{2}$

Solution: (d) $\lim_{x\to 0}\dfrac{\sin^{-1}x-\tan^{-1}x}{x^3}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule,

$=\lim_{x\to 0}\dfrac{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{1+x^2}}{3x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$=\lim_{x\to 0}\dfrac{\dfrac{-1}{2}\times\dfrac{-2x}{(1-x^2)^{3/2}}+\dfrac{2x}{(1+x^2)^2}}{6x}=\lim_{x\to 0}\dfrac{1}{6}\left[\dfrac{1}{(1-x^2)^{3/2}}+\dfrac{2}{(1+x^2)^2}\right]=\dfrac{1}{2}$.

Example: 72 $\lim_{x\to 1}\dfrac{1+\log x-x}{1-2x+x^2}=$ **[Karnataka CET 2000]**

(a) 1 (b) – 1 (c) 0 (d) $-\frac{1}{2}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x\to 1}\dfrac{1+\log x-x}{1-2x+x^2}=\lim_{x\to 1}\dfrac{\dfrac{1}{x}-1}{-2+2x}=\lim_{x\to 1}\dfrac{1-x}{2x(x-1)}$

Again applying L-Hospital's rule, we get $\lim_{x\to 1}\dfrac{-1}{4x-2}=-\dfrac{1}{2}$

Example: 73 $\lim_{x\to 0}\dfrac{4^x-9^x}{x(4^x+9^x)}=$ **[EAMCET 2002]**

(a) $\log\left(\frac{2}{3}\right)$ (b) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (c) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (d) $\log\left(\frac{3}{2}\right)$

Solution: (a) $y=\lim_{x\to 0}\dfrac{4^x-9^x}{x(4^x+9^x)}$ $\left(\dfrac{0}{0}\text{ form}\right)$

Using L-Hospital's rule, $y=\lim_{x\to 0}\dfrac{4^x\log 4-9^x\log 9}{(4^x+9^x)+x(4^x\log 4+9^x\log 9)}\Rightarrow y=\dfrac{\log 4-\log 9}{2}\Rightarrow y=\dfrac{\log\left(\dfrac{2}{3}\right)^2}{2}=\log\dfrac{2}{3}$.

Example: 74 If $f(a)=2$, $f'(a)=1$, $g(a)=-3$, $g'(a)=-1$, then $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}=$ **[Karnataka CET 2003]**

(a) 1 (b) 6 (c) – 5 (d) – 1

Solution: (a) $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}$ $\left(\frac{0}{0}\text{ form}\right)$

Using L-Hospital's rule, $\lim_{x\to a}\frac{f(a)g'(x)-f'(x)g(a)}{1-0}=f(a)\times g'(a)-f'(a)\times g(a)=2\times(-1)-1\times(-3)=1$.

Example: 75 The value of $\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}$ is **[MP PET 2003]**

(a) $\frac{2}{9}$ (b) $-\frac{2}{49}$ (c) $\frac{1}{56}$ (d) $-\frac{1}{56}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x\to 7}\frac{0-\frac{1}{2\sqrt{x-3}}}{2x}=\lim_{x\to 7}\frac{-1}{4x\sqrt{x-3}}=\frac{-1}{4.7\sqrt{7-3}}=\frac{-1}{56}$.

Example: 76 Let $f(a)=g(a)=k$ and their n^{th} derivatives $f^n(a), g^n(a)$ exist and are not equal for some n. If $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$, then the value of k is **[AIEEE 2003]**

(a) 4 (b) 2 (c) 1 (d) 0

Solution: (a) $\lim_{x\to a}\frac{k\,g(x)-k\,f(x)}{g(x)-f(x)}=4$

By L-Hospital' rule, $\lim_{x\to a}k\left[\frac{g'(x)-f'(x)}{g'(x)-f'(x)}\right]=4$, $\therefore\ k=4$.

Example: 77 The value of $\lim_{x\to 0}\left(\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right)$ is **[AIEEE 2003]**

(a) 3 (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^2}\sec^2 t\,dt}{\frac{d}{dx}(x\sin x)}=\lim_{x\to 0}\frac{\sec^2 x^2.2x}{\sin x+x\cos x}$ (By L' –Hospital's rule)

$=\lim_{x\to 0}\frac{2\sec^2 x^2}{\left(\frac{\sin x}{x}+\cos x\right)}=\frac{2\times 1}{1+1}=1$.

Example: 78 $\lim_{x\to\pi/6}\left[\frac{3\sin x-\sqrt{3}\cos x}{6x-\pi}\right]$ **[EAMCET 2003]**

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Solution: (b) Using L–Hospital's rule, $\lim_{x\to\pi/6}\frac{3\cos x+\sqrt{3}\sin x}{6}=\frac{3.\frac{\sqrt{3}}{2}+\sqrt{3}.\frac{1}{2}}{6}=\frac{1}{\sqrt{3}}$.

Example: 79 Given that $f'(2) = 6$ and $f'(1) = 4$, then $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} =$ **[IIT Screening 2003]**

(a) Does not exist (b) $-\frac{3}{2}$ (c) $\frac{3}{2}$ (d) 3

Solution: (d) $\lim_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} = \lim_{h \to 0} \frac{f'(2h + 2 + h^2)(2 + 2h)}{f'(h - h^2 + 1)(1 - 2h)} = \frac{6 \times 2}{4 \times 1} = 3$.