CHAPTER 6

# Binomial Theorem

- Introduction
- Analysis of Binomial Expansion
- Ratio of Consecutive Terms/Coefficients
- Applications of Binomial Expansion
- Use of Complex Numbers in Binomial Theorem
- Greatest Term in Binomial Expansion
- Sum of Series
- Miscellaneous Series
- Binomial Theorem for Any Index

## INTRODUCTION

Consider

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2$$
$$= x^2 + 2xy + y^2$$

Here, $xy$ can be written in two ways ($xy$ and $yx$). Hence the coefficient of $xy$ is equal to the number of ways $x$, $y$ can be arranged, which is 2! Consider

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Coefficient of $x^2y$ is 3 as for $x^2y$, we can arrange $x$, $x$, $y$ in $3!/2!$ ways.

With similar arguments, we have

$$(x+y)^4 = x^4 + \frac{4!}{3!}x^3y + \frac{4!}{2!2!}x^2y^2 + \frac{4!}{3!}xy^3 + y^4$$
$$= {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$

Now by this development, we can find the coefficient of any term in the binomial expansion of any positive integral power.

Thus in the expansion of $(x+y)^7$, the coefficient of $x^3y^4$ is equivalent to number of ways $x, x, x, y, y, y, y$ can be arranged which is $7!/(3!4!) = {}^7C_3$.

Hence, in general

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n, \text{ where } n \in N$$

**Note:**

- *This expansion has $(n+1)$ terms.*
- *Its general term is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$, where* $r = 0, 1, 2, 3, \ldots, n$.
- *In each term, the degree is n and the coefficient of $x^{n-r}y^r$ is equal to the number of ways $(n-r)$ x's and r y's can be arranged, which is given by*

$$\frac{n!}{(n-r)!r!} = {}^nC_r$$

- *$(p+1)^{th}$ term from the end is $(n-p+1)^{th}$ term from the beginning, i.e. $T_{n-p+1}$.*
- *$S = (x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$, where $n \in N$*

$$= \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

*Replacing r by $n-r$, we have*

$$S = \sum_{r=0}^{n} {}^nC_{n-r}x^{n-(n-r)}y^{n-r} \qquad (\because {}^nC_r = {}^nC_{n-r})$$

$$= \sum_{r=0}^{n} {}^nC_rx^ry^{n-r}$$

$$= {}^nC_0y^n + {}^nC_1y^{n-1}x + {}^nC_2y^{n-2}x^2 + \cdots + {}^nC_nx^n$$
$$= {}^nC_ny^n + {}^nC_{n-1}y^{n-1}x + {}^nC_{n-2}y^{n-2}x^2 + \cdots + {}^nC_0x^n$$

*Thus replacing r by $n-r$, we are in fact writing the binomial expansion in the reverse order.*

### Properties of Binomial Coefficient

- Sum of two consecutive binomial coefficients, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$
- Ratio of two consecutive binomial coefficients,

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

- If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x + y = n$. So,

$$\text{So, } {}^nC_r = {}^nC_{n-r} = \frac{n!}{r!(n-r)!}$$

### Pascal's Triangle

Coefficient of binomial expansion can also be easily determined by Pascal's triangle.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=0$ | | | | | | 1 | | | | | |
| $n=1$ | | | | | 1 | | 1 | | | | |
| $n=2$ | | | | 1 | | 2 | | 1 | | | |
| $n=3$ | | | 1 | | 3 | | 3 | | 1 | | |
| $n=4$ | | 1 | | 4 | | 6 | | 4 | | 1 | |
| $n=5$ | 1 | | 5 | | 10 | | 10 | | 5 | | 1 |

Construction of this triangle also justifies ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ as

| | | | | | |
|---|---|---|---|---|---|
| $n=3$ | 1 | $3 = {}^3C_1$ | | $3 = {}^3C_2$ | 1 |
| $n=4$ | 1 | 4 | $6 = {}^4C_2$ | 4 | 1 |

### Some Standard Expansions

We know that

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n, \text{ where } n \in N$$

Putting $y = 1$, we have

$$(1+x)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \cdots + {}^nC_n$$
$$= {}^nC_nx^n + {}^nC_{n-1}x^{n-1} + {}^nC_{n-1}x^{n-2} + \cdots + {}^nC_0$$
$$= {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + \cdots + {}^nC_nx^n$$
$$= \sum_{r=0}^{n} {}^nC_rx^r$$

In the above expansion, replacing $x$ by $-x$, we have

$$(1-x)^n = {}^nC_0 - {}^nC_1x^1 + {}^nC_2x^2 - \cdots + (-1)^r\,{}^nC_rx^r + \cdots + (-1)^n\,{}^nC_nx^n$$

$$= \sum_{r=0}^{n} {}^nC_r(-x)^r$$

$$= \sum_{r=0}^{n} (-1)^r\,{}^nC_rx^r$$

#### *Tips for finding coefficients of required term*

- Coefficient of $x^r$ in the expansion of $(1+x)^n$ is ${}^nC_r$
  e.g. coefficient of $x^6$ in the expansion of $(1+x)^{10}$ is ${}^{10}C_6$

- Coefficient of $x^r$ in the expansion of $(1 + ax)^n$ is ${}^nC_r a^r$ as term containing $x^r$ is ${}^nC_r a^r x^r$
  e.g. coefficient of $x^5$ in the expansion of $(1 + 2x)^{12}$ is ${}^{12}C_5 2^5$
  coefficient of $x^6$ in the expansion of $(1 - 3x)^{15}$ is ${}^{15}C_6 3^6$
  coefficient of $x^7$ in the expansion of $(1 - 4x)^{13}$ is $-{}^{13}C_7 4^7$
- Coefficient of $x^r$ in the expansion of $(a + bx)^n$ is ${}^nC_r b^r a^{n-r}$ as term containing $x^r$ is ${}^nC_r a^{n-r}(bx)^r$
  e.g. coefficient of $x^4$ in the expansion of $(3 + 2x)^9$ is ${}^9C_4 2^4 3^5$ (understand the distribution of exponent '9' when exponent 4 is taken by $2x$ the remaining exponent 5 will be taken by 3)
  Coefficient of $x^3$ in the expansion of $(4 - 5x)^{10}$ is $-{}^{10}C_3 5^3 4^7$
- Coefficient of $x^r$ in the expansion of $(1 + x^p)^n$ is ${}^nC_{r/p}$ if $r$ is multiple of $p$
  e.g. coefficient of $x^{10}$ in the expansion of $(1 + x^2)^{15}$ is ${}^{15}C_5$
  coefficients of $x^{12}$ in the expansion of $(1 + x^3)^{10}$ is ${}^{10}C_4$
  coefficient of $x^{10}$ in the expansion of $(1 + 3x^2)^{12}$ is ${}^{12}C_5 3^5$
  coefficient of $x^{15}$ in the expansion of $(2 + 5x^3)^{20}$ is ${}^{20}C_5 5^5 2^{15}$

**Example 6.1** **Simplify: $x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$.**

**Sol.** We have,

$$(x + a)^n = {}^nC_0 + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + \cdots$$

$$(x + 2a)^5 = {}^5C_0 x^5 + {}^5C_1 x^4(2a) + {}^5C_2 x^3(2a)^2 + {}^5C_3 x^2(2a)^3 + {}^5C_4 x(2a)^4 + {}^5C_5(2a)^5$$

$$= x^5 + 10x^4a + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$$

**Example 6.2** **Find the value of**

$$\frac{18^3+7^3+3\times18\times7\times25}{3^6+6\times243\times2+15\times81\times4+20\times27\times8+15\times9\times16+6\times3\times32+64}$$

**Sol.** The form of the numerator is

$$a^3 + b^3 + 3ab(a + b) = (a + b)^3$$

$\therefore\ N' = (18 + 7)^3 = 25^3$

$\therefore\ D' = 3^6 + {}^6C_1 3^5 \times 2^1 + {}^6C_2 3^4 \times 2^2 + {}^6C_3 3^3 \times 2^3 + {}^6C_4 3^2 \times 2^4 + {}^6C_5 3 \times 2^5 + {}^6C_6 2^6$

This is clearly the expansion of $(3 + 2)^6 = 5^6 = (25)^3$.

$$\therefore\ \frac{N'}{D'} = \frac{(25)^3}{(25)^3} = 1$$

**Example 6.3** **Evaluate $(1.0025)^{10}$, correct to six decimal places.**

**Sol.** $(1.0025)^{10} = (1 + 0.0025)^{10}$

$$= 1 + {}^{10}C_1(0.0025) + {}^{10}C_2(0.0025)^2 + {}^{10}C_3(0.0025)^3 + \cdots + (0.0025)^{10}$$

$$= 1 + 10 \times (0.0025) + 45(0.00000625)$$

Leaving other terms, as we require the value up to five places of decimals, we have

$$(1.0025)^{10} = 1 + 0.025 + 0.000281 = 1.025281$$

**Example 6.4** **Find the 6th term in the expansion of $(2x^2 - 1/3x^2)^{10}$.**

**Sol.** $T_{r+1} = {}^nC_r x^{n-r}y^r$ for $(x + y)^n$

Hence for $(2x^2 - 1/3x^2)^{10}$,

$$T_6 = {}^{10}C_5(2x^2)^5\left(-\frac{1}{3x^2}\right)^5$$

$$= -\frac{10!}{5!5!}32 \times \frac{1}{243}$$

$$= -\frac{896}{27}$$

**Example 6.5** **If the 21st and 22nd terms in the expansion of $(1 - x)^{44}$ are equal, then find the value of $x$**

**Sol.** $T_{22} = T_{21} \Rightarrow {}^{44}C_{21}(-x)^{21} = {}^{44}C_{20}(-x)^{20}$

$$\therefore\ \frac{{}^{44}C_{21}}{{}^{44}C_{20}} = \frac{1}{-x}\ or\ \frac{{}^nC_r}{{}^nC_{r-1}} = -\frac{1}{x} = \frac{n-r+1}{r}$$

Put $n = 44$, $r = 21$

$$\therefore\ -\frac{1}{x} = \frac{44-21+1}{21} = \frac{24}{21} = \frac{8}{7}$$

$\therefore\ x = -7/8$

**Example 6.6** **Find the coefficient of $x^4$ in the expansion of $(x/2 - 3/x^2)^{10}$.**

**Sol.** In the expansion of $(x/2 - 3/x^2)^{10}$, the general term is

$$T_{r+1} = {}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(-\frac{3}{x^2}\right)^r$$

$$= {}^{10}C_r(-1)^r\frac{3^r}{2^{10-r}}x^{10-3r}$$

Here, the exponent of $x$ is

$$10 - 3r = 4 \Rightarrow r = 2$$

$$\therefore\ T_{2+1} = {}^{10}C_2\left(\frac{x}{2}\right)^8\left(-\frac{3}{x^2}\right)^2$$

$$= \frac{10\times9}{1\times2}\times\frac{1}{2^8}\times3^2\times x^4$$

$$= \frac{405}{256}x^4$$

Therefore, the required coefficient is 405/256.

**Example 6.7** **Find the term in $\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ which has the same power of $a$ and $b$.**

**Sol.** We have,

$$T_{r+1} = {}^{21}C_r\left(\sqrt[3]{\frac{a}{\sqrt{b}}}\right)^{21-r}\left(\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{r}$$

$$= {}^{21}C_r a^{7-(r/2)}b^{(2/3)r-(7/2)}$$

Since the powers of $a$ and $b$ are the same, therefore $7-\frac{r}{2}=\frac{2}{3}r-\frac{7}{2} \Rightarrow r=9$

**Example 6.8** **Find the term independent of $x$ in the expansion of $(1+x+2x^3)\,[(3x^2/2)-(1/3x)]^9$.**

**Sol.** We have,

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$$

$$= (1+x+2x^3)\left[\left(\frac{3}{2}x^2\right)^9 - {}^9C_1\left(\frac{3}{2}x^2\right)^8(1/3x) + \cdots + (-1)^9(1/3x)^9\right] \quad (1)$$

Therefore, the term independent of $x$ in the expansion is

$$1a_0 + 1a_1 + 2a_3 \quad (2)$$

where $a_m$ is the coefficient of $x^m$ in the second bracket [ ] of (1). Now, $(r+1)^{\text{th}}$ term in [ ] of (1) is

$${}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}(-1/3x)^r$$

$$= (-1)^r\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}(1/3^r)(x^{18-3r})$$

$\therefore \quad a_{18-3r}$ = coefficient of $x^{18-3r}$

$$= (-1)^r\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}(1/3^r)$$

Now for $a_0$,

$18-3r=0$

$$\Rightarrow r=6 \Rightarrow a_0 = (-1)^6\,{}^9C_6\left(\frac{3}{2}\right)^{9-6}\times\frac{1}{3^6}=\frac{7}{18}$$

For $a_1$,

$18-3r=-1$

$\Rightarrow r = 19/3$, which is fractional

$\therefore \quad a_1 = 0$

For $a_3$,

$18-3r=-3$

$$\Rightarrow r=7 \Rightarrow a_3 = (-1)^7\,{}^9C_7\left(\frac{3}{2}\right)^{9-7}\frac{1}{3^7}=-\frac{1}{27}$$

Hence from (2), the required term is

$$1\times\frac{7}{18}+0+2\times\left(-\frac{1}{27}\right)=\frac{17}{52}$$

**Example 6.9** **Find the coefficient of $x^k$ in $1+(1+x)+(1+x)^2+\cdots+(1+x)^n$ $(0\le k\le n)$.**

**Sol.** The expression being in G.P., we have

$$E = 1+(1+x)+(1+x)^2+\cdots+(1+x)^n$$

$$= \frac{(1+x)^{n+1}-1}{(1+x)-1} = x^{-1}[(1+x)^{n+1}-1]$$

Therefore, the coefficient of $x^k$ in $E$ is equal to the coefficient of $x^{k+1}$ in $[(1+x)^{n+1}-1]$, which is given by ${}^{n+1}C_{k+1}$.

**Example 6.10** **Find the coefficient of $x^4$ in the expansion of $(2-x+3x^2)^6$.**

**Sol.** $(2-x+3x^2)^6 = [2-x(1-3x)]^6$

$$= 2^6 - {}^6C_1\,2^5x(1-3x) + {}^6C_2\,2^4x^2 \times (1-3x)^2 - {}^6C_3\,2^3x^3(1-3x)^3 + {}^6C_4\,2^2x^4(1-3x)^4 - {}^6C_5\,2x^5(1-3x)^5 + {}^6C_6\,2^6$$

Obviously, $x^4$ occurs in 3rd, 4th and 5th terms. Now, 3rd term is $15\times16x^2(1-6x+9x^2)$. Here, the coefficient of $x^4$ is $15\times16\times9=2160$. The 4th term is $-20\times8x^3(1-9x+27x^2-27x^3)$. Here, the coefficient of $x^4$ is $20\times8\times9=1440$. The 5th term is $15\times4x^4[1-4\times3x+\cdots+(3x)^4]$. Here, the coefficient of $x^4$ is $15\times4=60$. Hence, the required coefficient of $x^4$ is $2160+1440+60=3660$.

**Example 6.11** **Find the coefficient of $x^{50}$ in $(1+x)^{101}\times(1-x+x^2)^{100}$.**

**Sol.** $(1+x)^{101}(1-x+x^2)^{100}$

$$= (1+x)[(1+x)^{100}(1-x+x^2)^{100}]$$

$$= (1+x)(1-x^3)^{100}$$

$$= (1-x^3)^{100} + x(1-x^3)^{100}$$

Now, coefficient of $x^{50}$ in $[(1-x^3)^{100}+x(1-x^3)^{100}]$ is Coefficient of $x^{50}$ in $(1-x^3)^{100}$ + coefficient of $x^{49}$ in $(1-x^3)^{100}$

= 0 (as 49 and 50 are not a multiple of 3)

**Example 6.12** **If sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2+\frac{1}{x}\right)^m$ is 46, then find the coefficient of the term that does not contain $x$**

**Sol.** We are given

$${}^mC_0 + {}^mC_1 + {}^mC_2 = 46$$

$\Rightarrow \quad 2m + m(m-1) = 90$

$\Rightarrow \quad m^2+m-90=0$

$\Rightarrow \quad m = 9$ as $m>0$

Now, $(r+1)$th term of $\left(x^2+\frac{1}{x}\right)^m$ is ${}^mC_r(x^2)^{m-r}\left(\frac{1}{x}\right)^r$

$$= {}^{m}C_r\, x^{2m-3r}$$

For this to be independent of $x$, $2m - 3r = 0 \Rightarrow r = 6$

$\therefore$ term independent of $x$ is ${}^9C_6 = 84$.

**Example 6.13** **Find the coefficient of $x^{20}$ in $\left(x^2 + 2 + \frac{1}{x^2}\right)^{-5} (1 + x^2)^{40}$.**

**Sol.** $E = (1 + x^2)^{40} \left(x + \frac{1}{x}\right)^{-10} = x^{10}(1 + x^2)^{30}$

Hence we have to choose the term of $x^{20}$ in $x^{10}(1 + x^2)^{30}$ or the term of $(x^2)^5$ in $(1 + x^2)^{30}$ which will be ${}^{30}C_5$.

**Example 6.14** **Find the coefficient of $x^5$ in the expansion of $(1 + x^2)^5 . (1 + x)^4$ is 60.**

**Sol.** Coefficient of $x^5$ in $(1 + x^2)^5 . (1 + x)^4$

$= \text{Coefficient of } x^5 \text{ in } ({}^5C_0 + {}^5C_1x^2 + {}^5C_2x^4 + {}^5C_3x^6 + \cdots)$
$({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + \cdots)$

$= {}^5C_1{}^4C_3 + {}^5C_2{}^4C_1$

$= 20 + 40$

$= 60$

**Example 6.15** **Find the coefficient of $x^{13}$ in the $(1 - x)^5 \times (1 + x + x^2 + x^3)^4$.**

**Sol.** $E = (1 - x)^5 (1 + x)^4 (1 + x^2)^4$

$= (1 - x)(1 - x^2)^4 (1 + x^2)^4$

$= (1 - x)(1 - x^4)^4$

$= (1 - x)[1 - 4(x^4) + 6(x^4)^2 - 4(x^4)^3 + (x^4)^4]$

Coefficient of $x^{13}$ is $(-1)(-4) = 4$

**Example 6.16** Find the coefficient of $x^4$ in the expansion of $(1 + x + x^2 + x^3)^{11}$

**Sol.** $(1 + x)^{11}(1 + x^2)^{11}$

$= (1 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3x^3 + {}^{11}C_4x^4 + \cdots)$

$\times (1 + {}^{11}C_1 (x^2) + {}^{11}C_2(x^2)^2 + \cdots)$

Coefficient of $x^4$ is ${}^{11}C_2 .1 + {}^{11}C_1.{}^{11}C_2 + {}^{11}C_4 = 990$

**Example 6.17** **Find the number of terms which are free from radical signs in the expansion of $(y^{1/5} + x^{1/10})^{55}$.**

**Sol.** In the expansion of $(y^{1/5} + x^{1/10})^{55}$,

$$T_{r+1} = {}^{55}C_r (y^{1/5})^{55-r} (x^{1/10})^r = {}^{55}C_r\, y^{11-r/5}x^{r/10}$$

Thus $T_{r+1}$ will be independent of radicals if the exponents $r/5$ and $r/10$ are integers for $0 \le r \le 55$, which is possible only when $r = 0, 10, 20, 30, 40, 50$.

Therefore, there are six terms, i.e., $T_1, T_{11}, T_{21}, T_{31}, T_{41}, T_{51}$ which are independent of radicals.

**Concept Application Exercise 6.1**

1. Find the constant term in the expansion of $(x - 1/x)^6$.
2. Find the coefficient of $x^{-10}$ in the expansion of $\left(\frac{a}{x} + bx\right)^{12}$.
3. If $x^4$ occurs in the $r^{\text{th}}$ term in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$, then find the value of $r$.
4. The first three terms in the expansion of $(1 + ax)^n$ $(n \neq 0)$ are 1, $6x$ and $16x^2$. Then find the value of $a$ and $n$.
5. If $p$ and $q$ be positive, then prove that the coefficients of $x^p$ and $x^q$ in the expansion of $(1 + x)^{p+q}$ will be equal.
6. If the coefficient of $4^{\text{th}}$ term in the expansion of $(a + b)^n$ is 56, then find the value of $n$.
7. In $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of $7^{\text{th}}$ term from the beginning to the $7^{\text{th}}$ term from the end is 1/6, then find the value of $n$.
8. If the coefficient of $(2r + 3)^{\text{th}}$ and $(r - 1)^{\text{th}}$ terms in the expansion of $(1 + x)^{15}$ are equal, then find the value of $r$.
9. If $x^p$ occurs in the expansion of $(x^2 + 1/x)^{2n}$, prove that its coefficient is
$$\frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}.$$
10. Find the number of irrational terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$.

## Multinomial Expansions

Consider the expansion of $(x + y + z)^{10}$. In the expansion, each term has different powers of $x$, $y$ and $z$ and sum of these powers is always 10.

One of the terms is $\lambda x^2y^3z^5$. Now, the coefficient of this term $\lambda$ is equal to the number of ways 2 $x$'s, 3 $y$'s and 5 $z$'s are arranged, i.e., $10!/(2!3!5!)$. Thus,

$$(x + y + z)^{10} = \sum \frac{10!}{P_1!P_2!P_3!} x^{P_1} y^{P_2} z^{P_3}$$

where $P_1 + P_2 + P_3 = 10$ and $0 \le P_1, P_2, P_3 \le 10$. In general,

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \frac{n!}{P_1!P_2!\cdots P_r!} x_1^{P_1} x_2^{P_2} \ldots x_r^{P_r}$$

where $P_1 + P_2 + P_3 + \cdots + P_r = n$ and $0 \le P_1, P_2, \ldots, P_r \le n$.

### *Number of Terms in the Expansion of $(x_1 + x_2 + \cdots + x_r)^n$*

From the general term of the above expansion, we can conclude that number of terms is equal to the number of ways different powers can be distributed to $x_1, x_2, x_3, \ldots, x_n$ such that sum of powers is always $n$.

Number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_r = n$ is ${}^{n+r-1}C_{r-1}$.

For example, number of terms in the expansion of $(x+y+z)^3$ is $^{3+3-1}C_{3-1} = {}^5C_2 = 10$.

As in the expansion, we have terms like $x^0y^0z^3, x^0y^1z^2, x^0y^2z^1, x^0y^3z^0, x^1y^0z^2, x^1y^1z^1, x^1y^2z^0, x^2y^0z^1, x^2y^1z^0, x^3y^0z^0$.

Number of terms in $(x+y+z)^n$ is $^{n+3-1}C_{3-1} = {}^{n+2}C_2$.

Number of terms in $(x+y+z+w)^n$ is $^{n+4-1}C_{4-1} = {}^{n+3}C_3$ and so on.

**Example 6.18** **If the number of terms in the expansion of $(x+y+z)^n$ are 36, then find the value of $n$.**

**Sol.** Number of terms in the expansion of $(x+y+z)^n$ is $(n+1)(n+2)/2 = 36$

$\Rightarrow \quad (n+1)(n+2) = 72$

$\Rightarrow \quad n = 7$

**Example 6.19** **Find the coefficient of $a^3b^4c$ in the expansion of $(1+a-b+c)^9$.**

**Sol.** $$(1+a-b+c)^9 = \Sigma \frac{9!}{x_1!x_2!x_3!x_4!}(1)^{x_1}(a)^{x_2}(-b)^{x_3}(c)^{x_4}$$

(for $x_1 = 1, x_2 = 3, x_3 = 4, x_4 = 1$)

Hence, the coefficient of $a^3b^4c$ is $\frac{9!}{1!3!4!1!} = \frac{9!}{3!4!}$.

**Example 6.20** **Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$.**

**Sol.** In this case, write $a^3b^4c^5 = (ab)^x(bc)^y(ca)^z$ (say). Then,

$a^3b^4c^5 = a^{z+x}\, b^{x+y}\, c^{y+z}$

$\Rightarrow \quad z+x = 3, x+y = 4, y+z = 5$

Adding all,

$2(x+y+z) = 12$

$\Rightarrow \quad x+y+z = 6$

Then, $x = 1, y = 3, z = 2$.

Therefore, the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$ or the coefficient of $(ab)^1(bc)^3(ca)^2$ in the expansion of $(bc+ca+ab)^6$ is $6!/(1!3!2!) = 60$.

**Example 6.21** **Find the coefficient of $x^7$ in the expansion of $(1+3x-2x^3)^{10}$.**

**Sol.** Coefficient of $x^7$ in the expansion of $(1+3x-2x^3)^{10}$ is

$$\sum \frac{10!}{n!\,n_2!\,n_3!}(1)^{n_1}(3x)^{n_2}(-2x^3)^{n_3}$$

where $n_1+n_2+n_3 = 10$ and $n_2+3n_3 = 7$, the possible values of $n_1$, $n_2$ and $n_3$ are shown in the margin.

| $n_1$ | $n_2$ | $n_3$ |
|---|---|---|
| 3 | 7 | 0 |
| 5 | 4 | 1 |
| 7 | 1 | 2 |

Therefore the coefficient of $x^7$ is

$$\frac{10!}{3!\,7!\,0!}(1)^3(3)^7(-2)^0 + \frac{10!}{5!\,4!\,1!}(1)^5(3)^4(-2)^1 + \frac{10}{7!\,1!\,2!}(1)^7(3)^1(-2)^2$$

$= 262440 - 204120 + 4320$

$= 62640$

## ANALYSIS OF BINOMIAL EXPANSION

### Sum of Binomial Coefficients

For the sake of convenience, the coefficients $^nC_0, {}^nC_1, \ldots, {}^nC_r, \ldots, {}^nC_n$ are usually denoted by $C_0, C_1, \ldots, C_r, \ldots, C_n$, respectively.

- $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$

**Proof:**

$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$

Putting $x = y = 1$, we have

$(1+1)^n = 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$

Similarly,

$^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} = (1+1)^{n+1} = 2^{n+1}$

- $C_0 - C_1 + C_2 - \cdots + C_n = 0$

**Proof:**

$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \cdots + {}^nC_ny^n$

Putting $x = 1$ and $y = -1$, we have

$(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - \cdots = 0$

$\Rightarrow 0 = ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) - ({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots)$

$\Rightarrow ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) = ({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots)$

$\Rightarrow C_0 + C_1 + C_2 + \cdots + C_n = 2^n = ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) + ({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots)$

$= 2({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots)$

$\Rightarrow ({}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots) = 2^{n-1}$

Similarly,

$({}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots) = 2^{n-1}$

- $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$

See above proof.

e.g. $^{n+1}C_0 + {}^{n+1}C_2 + {}^{n+1}C_4 + {}^{n+1}C_6 + \cdots = 2^n$

$^{n-1}C_1 + {}^{n-1}C_3 + {}^{n-1}C_5 + {}^{n-1}C_7 + \cdots = 2^{n-2}$

$^{20}C_0 + {}^{20}C_2 + {}^{20}C_4 + {}^{20}C_6 + \cdots = 2^{19}$

$^{15}C_1 + {}^{15}C_3 + {}^{15}C_5 + \cdots = 2^{14}$

etc.

**Example 6.22** **Find the sum $^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9$.**

**Sol.** We know that

$$2^{n-1} = {}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$$

So,

$$^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + \cdots + {}^{10}C_9 = 2^{10-1} = 2^9$$

**Example 6.23** **Find the sum**

$$\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)}+\frac{1}{5!(n-5)}+\cdots.$$

**Sol.** $S=\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+\cdots$

Multiplying each term by $n!/n!$, $S$ reduces to

$$S=\frac{1}{n!}\left[\frac{n!}{1!(n-1)!}+\frac{1}{3!}\frac{n!}{(n-3)!}+\frac{1}{5!}\frac{n!}{(n-5)!}+\cdots\right]$$

$$=\frac{1}{n!}\left[{}^nC_1+{}^nC_3+{}^nC_5+\cdots\right]$$

$$=\frac{2^{n-1}}{n!}$$

**Example 6.24** **Find the sum** $\sum_{k=0}^{10}{}^{20}C_k$.

**Sol.** $S=\sum_{k=0}^{10}{}^{20}C_k={}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_{10}$

Now,

$${}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_9+{}^{20}C_{10}+{}^{20}C_{11}+\cdots+{}^{20}C_{20}=2^{20}$$

$$\Rightarrow\ ({}^{20}C_0+{}^{20}C_{20})+({}^{20}C_1+{}^{20}C_{19})+\cdots+({}^{20}C_9+{}^{20}C_{11})+{}^{20}C_{10}=2^{20}$$

$$\Rightarrow\ 2[{}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_{10}]+{}^{20}C_{10}=2^{20}\ (\because {}^nC_r={}^nC_{n-r})$$

$$\therefore\ S={}^{20}C_0+{}^{20}C_1+\cdots+{}^{20}C_{10}=2^{19}-\frac{1}{2}\,{}^{20}C_{10}$$

**Example 6.25** **Find the sum of the series** ${}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots\ldots+{}^{15}C_7$

**Sol.** Let $S={}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots+{}^{15}C_7$

Here the series is exactly half series(8 terms)

As the full series (16 terms) will be ${}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots+{}^{15}C_7+{}^{15}C_8+\ldots+{}^{15}C_{15}$

So the sum of S is exactly half of the full series that is half of $2^{15}$ which is $2^{14}$

Hence, ${}^{15}C_0+{}^{15}C_1+{}^{15}C_2+\ldots+{}^{15}C_7=2^{14}$.

**Example 6.26** **If the sum of coefficient of first half terms in the expansion of $(x+y)^n$ is 256 then find the greatest coefficient in the expansion**

**Sol.** Sum of coefficient of first half of the terms $=2^{n-1}$ (half series) $=256=2^8$

$\Rightarrow\ n=9$

$\Rightarrow$ greatest coefficient $={}^9C_4=126$

## Sum of Coefficients in Binomial Expansion

For $(x+y)^n={}^nC_0x^n+{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2+\cdots+{}^nC_ny^n$, we get the sum of coefficients by putting $x=y=1$, which is $2^n$.

Similarly, in the expansion of $(x+y+z)^n$, we get the sum of coefficients by putting $x=y=z=1$.

For expansion of the type $(x^2+x+1)^n=a_0+a_1x+a_2x^2+\cdots+a_{2n}x^{2n}$, we get the sum of coefficients by putting $x=1$ or $3^n=a_0+a_1+a_2+\cdots+a_{2n}$ which is the required sum of coefficients.

Infact to find sum of coefficients we put the value of all variables as 1. (where variables are in $x^n$ form)

In the above expansion, to get the sum of coefficients of even powers of $x$ and odd powers of $x$, put $x=1$ and $x=-1$ alternatively and then add or subtract the two results.

**Example 6.27** **Find the sum of all the coefficients in the binomial expansion of $(x^2+x-3)^{319}$.**

**Sol.** Putting $x=1$ in $(x^2+x-3)^{319}$, we get the sum of coefficients equal to $(1+1-3)^{319}=-1$.

**Example 6.28** **If the sum of the coefficients in the expansion of $(a^2x^2-2ax+1)^{51}$ vanishes, then find the value of $a$.**

**Sol.** The sum of the coefficients of the polynomial $(\alpha^2x^2-2\alpha x+1)^{51}$ is obtained by putting $x=1$ in $(\alpha^2x^2-2\alpha x+1)^{51}$. Hence, from the given condition,

$$(\alpha^2-2\alpha+1)^{51}=0\Rightarrow\alpha=1$$

**Example 6.29** **If $(1+x-2x^2)^{20}=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_{40}x^{40}$, then find the value of $a_1+a_3+a_5+\cdots+a_{39}$.**

**Sol.** $(1+x-2x^2)^{20}=a_0+a_1x+a_2x^2+\cdots+a_{40}x^{40}$

Putting $x=1$, we get

$$a_0+a_1+a_2+a_3+\cdots+a_{40}=0 \qquad (1)$$

Putting $x=-1$, we get

$$a_0-a_1+a_2-a_3+\cdots-a_{39}+a_{40}=2^{20} \qquad (2)$$

Subtracting (1) from (2), we get

$$2[a_1+a_3+\cdots+a_{39}]=-2^{20}$$

$$\Rightarrow\ a_1+a_3+\cdots+a_{39}=-2^{19}$$

## Middle Term in Binomial Expansion

Consider

$$(x+y)^n={}^nC_0x^n+{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2+\cdots+{}^nC_ny^n$$

- The middle term depends upon the value of $n$.
  - (a) If $n$ is even, then total number of terms in the expansion is odd. So, there is only one middle term, i.e., $(n/2+1)^{\text{th}}$ term is the middle term.
  - (b) If $n$ is odd, then total number of terms in the expansion is even. So, there are two middle terms, i.e., $[(n+1)/2]^{\text{th}}$ and $[(n+3)/2]^{\text{th}}$ terms are the two middle terms.
- Middle term always carries the greatest binomial coefficient. As when $n$ is an even middle term, $T_{n/2+1}$ has the greatest binomial coefficient ${}^nC_{n/2}$.

  And when $n$ is an odd middle term, $T_{(n+1)/2}$ and $T_{(n+3)/2}$ or $T_{\left(\frac{n-1}{2}\right)+1}$ and $T_{\left(\frac{n+1}{2}\right)+1}$ have the greatest binomial coefficients ${}^nC_{\left(\frac{n-1}{2}\right)}$ and ${}^nC_{\left(\frac{n+1}{2}\right)}$.

**Example 6.30** **If the middle term in the expansion of $(x^2+1/x)^n$ is $924x^6$, then find the value of $n$.**

**Sol.** Since there is only one middle term here, $n$ is even, therefore $(n/2 + 1)^{th}$ term is the middle term. Hence,

$$^nC_{n/2}\,(x^2)^{n/2}\left(\frac{1}{x}\right)^{n/2} = 924x^6$$

$\Rightarrow \quad x^{n/2} = x^6$

$\Rightarrow \quad n = 12$ (also $^{12}C_6 = 924$)

**Example 6.31** **If the coefficient of the middle term in the expansion of $(1+x)^{2n+2}$ is $\alpha$ and the coefficients of middle terms in the expansion of $(1+x)^{2n+1}$ are $\beta$ and $\gamma$, then relate $\alpha, \beta$ and $\gamma$.**

**Sol.** Since $(n+2)^{th}$ term is the middle term in the expansion of $(1+x)^{2n+2}$, therefore $\alpha = {}^{2n+2}C_{n+1}$. Since $(n+1)^{th}$ and $(n+2)^{th}$ terms are middle terms in the expansion of $(1+x)^{2n+1}$, therefore,

$$\beta = {}^{2n+1}C_n \text{ and } \gamma = {}^{2n+1}C_{n+1}$$

But

$$^{2n+1}C_n + {}^{2n+1}C_{n+1} = {}^{2n+2}C_{n+1} \Rightarrow \beta + \gamma = \alpha$$

**Concept Application Exercise 6.2**

1. Find the sum of coefficients in $(1 + x - 3x^2)^{4163}$.
2. If the sum of coefficients in the expansion of $(x - 2y + 3z)^n$ is 128, then find the greatest coefficient in the expansion of $(1+x)^n$.
3. If $(1 + x - 2x^2)^6 = 1 + a_1x + a_2x^2 + \cdots + a_{12}x^{12}$, then find the value of $a_2 + a_4 + a_6 + \cdots + a_{12}$.
4. Find the middle term in the expansion of $\left(x^2 + \frac{1}{x^2} + 2\right)^n$.
5. In the expansion of $(1+x)^{50}$, find the sum of coefficients of odd powers of $x$.
6. Find the following sum:
   $\frac{1}{n!} + \frac{1}{2!\,(n-2)!} + \frac{1}{4!\,(n-4)!} + \cdots$.
7. Find the sum of the last 30 coefficients in the expansion of $(1+x)^{59}$, when expanded in ascending powers of $x$.
8. Find the sum $\sum_{j=0}^{n}({}^{4n+1}C_j + {}^{4n+1}C_{2n-j})$.

## RATIO OF CONSECUTIVE TERMS/COEFFICIENTS

Coefficients of $x^r$ and $x^{r+1}$ are $^nC_{r-1}$ and $^nC_r$, respectively. Also, we know that

$$\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$$

Similarly,

$$\frac{^nC_{r+1}}{^nC_r} = \frac{n-r}{r+1} \text{ (replacing } r \text{ by } r+1)$$

$$\frac{^{n+1}C_{r+1}}{^{n+1}C_r} = \frac{n-r+1}{r+1} \text{ (replacing } r \text{ by } r+1 \text{ and } n \text{ by } n+1) \text{ and so on.}$$

**Example 6.32** **If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are 165, 330 and 462, respectively, then find the value of $n$.**

**Sol.** The coefficients of three consecutive terms, i.e. $(r+1)^{th}$, $(r+2)^{th}$, $(r+3)^{th}$ in expansion of $(1+x)^n$ are 165, 330 and 462 respectively, Then, coefficient of $(r+1)^{th}$ term is $^nC_r = 165$, coefficient of $(r+2)^{th}$ term is $^nC_{r+1} = 330$ and coefficient of $(r+3)^{th}$ term is $^nC_{r+2} = 462$.

$$\therefore \quad \frac{^nC_{r+1}}{^nC_r} = \frac{n-r}{r+1} = 2 \Rightarrow n - r = 2(r+1) \Rightarrow r = \frac{1}{3}(n-2)$$

and

$$\frac{^nC_{r+2}}{^nC_{r+1}} = \frac{n-r-1}{r+2} = \frac{231}{165}$$

$\Rightarrow \quad 165(n-r-1) = 231(r+2)$ or $165n - 627 = 396r$

$\Rightarrow \quad 165n - 627 = 396 \times \frac{1}{3} \times (n-2)$

$\Rightarrow \quad 165n - 627 = 132(n-2)$ or $n = 11$

**Example 6.33** **If $C_r = {}^nC_r$, then prove that $(C_0 + C_1)(C_1 + C_2)\cdots(C_{n-1} + C_n) = (C_1C_2\cdots C_{n-1}C_n)(n+1)^n/n!$**

**Sol.** We have,

$$(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\cdots(C_{n-1} + C_n)$$

$$= C_1C_2\cdots C_{n-1}C_n\left(1+\frac{C_0}{C_1}\right)\left(1+\frac{C_1}{C_2}\right)\left(1+\frac{C_2}{C_3}\right)\cdots\left(1+\frac{C_{n-1}}{C_n}\right)$$

$$= C_1C_2\cdots C_{n-1}C_n\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n-1}\right)\left(1+\frac{3}{n-2}\right)\cdots\left(1+\frac{n}{1}\right)$$

$$= C_1C_2\cdots C_{n-1}C_n\frac{(n+1)^n}{n!}$$

**Example 6.34** **If $a_1, a_2, a_3, a_4$ are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then prove that $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$.**

**Sol.** Let the coefficients of $T_r, T_{r+1}, T_{r+2}, T_{r+3}$ be $a_1, a_2, a_3, a_4$, respectively, in the expansion of $(1+x)^n$. Then,

$$\frac{a_2}{a_1} = \frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$$

$$\Rightarrow \quad 1 + \frac{a_2}{a_1} = \frac{n+1}{r}$$

Similarly,

$$1 + \frac{a_3}{a_2} = \frac{n+1}{r+1}, \quad 1 + \frac{a_4}{a_3} = \frac{n+1}{r+2}$$

Now,

$$\text{L.H.S.} = \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4}$$

$$= \frac{1}{1+\frac{a_2}{a_1}} + \frac{1}{1+\frac{a_4}{a_3}}$$

$$= \frac{r}{n+1} + \frac{r+2}{n+1} = \frac{2(r+1)}{n+1} = 2\frac{1}{1+\frac{a_3}{a_2}} = \frac{2a_2}{a_2+a_3}$$

= R.H.S.

**Example 6.35** **Find the sum $\sum_{r=1}^{n} \frac{r\,{}^nC_r}{{}^nC_{r-1}}$.**

**Sol.** $\sum_{r=1}^{n} \frac{r\,{}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^{n} r \cdot \frac{n-r+1}{r}$

$$= (n+1)\sum_{r=1}^{n} - \sum_{r=1}^{n} r$$

$$= n(n+1) - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}$$

**Concept Application Exercise 6.3**

1. In the expansion of $(1 + x)^n$, 7th and 8th terms are equal. Find the value of $(7/x + 6)^2$.
2. Find the sum $\sum_{r=1}^{n} r^2 \frac{{}^nC_r}{{}^nC_{r-1}}$.
3. Show that no three consecutive binomial coefficients can be in (i) G.P., (ii) H.P.
4. If the 3rd, 4th, 5th and 6th terms in the expansion of $(x + a)^n$ be, respectively, $a$, $b$, $c$ and $d$, prove that $\frac{b^2-ac}{c^2-bd} = \frac{5a}{3c}$.

## APPLICATIONS OF BINOMIAL EXPANSION

### Important Result

$$2 \le \left(1+\frac{1}{n}\right)^n < 3, n \ge 1, n \in N.$$

**Proof:**

By the use of binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}$$

$$\times \frac{1}{n^3} + \cdots + \frac{n(n-1)(n-2)\cdots[n-(n-1)]}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+ \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \quad (1)$$

$$= 1 + 1\frac{\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}}$$

$$= 1 + 2\left\{1-\left(\frac{1}{2}\right)^n\right\}$$

$$= 3 - \frac{1}{2^{n-1}} \quad (2)$$

Hence, from (1) and (2),

$$2 \le \left(1+\frac{1}{n}\right)^n < 3, n \ge 1$$

**Example 6.36** **Find the positive integer just greater than $(1 + 0.0001)^{10000}$.**

**Sol.** $(1+0.0001)^{10000} = \left(1+\frac{1}{10000}\right)^{10000}$

Now we know that

$$2 \le \left(1+\frac{1}{n}\right)^n < 3, n \ge 1, n \in N$$

Hence positive integer just greater than $(1 + 0.0001)^{10000}$ is 3.

**Example 6.37** **Find (i) the last digit, (ii) the last two digits and (iii) the last three digits of $17^{256}$.**

**Sol.** We have,

$17^{256} = (17)^{128} = (289)^{128} = (290 - 1)^{128}$

$\therefore\ 17^{256} = {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126} - \cdots$
$- {}^{128}C_{125}(290)^3 + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

$= [{}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126} - \cdots -$
${}^{128}C_{125}(290)^3] + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

$= 1000\,m + {}^{128}C_2(290)^2 - {}^{128}C_1(290) + 1 \quad (m \in I)$

$= 1000\,m + \frac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$

$= 1000\,m + (128)(127)(290)(145) - 128 \times 290 + 1$

$= 1000\,m + (128)(290)(127 \times 145 - 1) + 1$

$= 1000\,m + (128)(290)(18414) + 1$

$= 1000\,m + 683527680 + 1$

$= 1000m + 683527000 + 680 + 1$

$= 1000(m + 683527) + 681$

Hence, the last three digits of $17^{256}$ must be 681. As a result, the last two digits of $17^{256}$ are 81 and the last digit of $17^{256}$ is 1.

**Example 6.38** **If $10^m$ divides the number $101^{100} - 1$, then find the greatest value of $m$.**

**Sol.** $(1 + 100)^{100} = 1 + 100 \times 100 + \frac{100 \times 99}{1 \times 2} \times (100)^2$
$+ \frac{100 \times 99 \times 98}{1 \times 2 \times 3}(100)^3 + \cdots$

$\Rightarrow \quad (101)^{100} - 1 = 100$

$$\times 100\left[1 + \frac{100\times 99}{1\times 2} + \frac{100\times 9\times 98}{1\times 2\times 3}\times 100 + \cdots\right]$$

From above, it is clear that $(101)^{100} - 1$ is divisible by $(100)^2 = 10000$. So greatest value of $m$ is 4.

**Important Results**

$(1 + x)^n - 1 = {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n$ is always divsible by $x$.

Also, $(1 + x)^n - 1 - nx = {}^nC_2x^2 + \cdots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n$ is always divisible by $x^2$.

**Example 6.39** **Prove that for each $n \in N$, $2^{3n} - 1$ is divisible by 7.**

**Sol.** $2^{3n} - 1 = (2^3)^n - 1 = (1 + 7)^n - 1$

$= [1 + {}^nC_1(7) + {}^nC_2(7)^2 + \cdots + {}^nC_n(7)^n] - 1$

$= 7[{}^nC_1 + {}^nC_2(7) + \cdots + {}^nC_n(7)^{n-1}]$

$\Rightarrow$ $2^{3n} - 1$ is divisible by 7 for all $n \in N$

**Example 6.40** **Find the remainder when $6^n - 5n$ is divided by 25**

**Sol.** $6^n - 5n = (1 + 5)^n - 5n$

$= (1 + 5n + {}^nC_2 \cdot 5^2 + {}^nC_3 5^3 + \cdots) - 5n$

$= 25({}^nC_2 + {}^nC_3 + \cdots) + 1$

Hence $6^n - 5n$ when divided by 25 leaves 1 as remainder.

**Example 6.41** **Using binomial theorem, show that $2^{3n} - 7n - 1$ is divisible by 49. Hence, show that $2^{3n+3} - 7n - 8$ is divisible by 49, $n \in N$.**

**Sol.** $2^{3n} - 7n - 1 = (2^3)^n - 7n - 1$

$= (1 + 7)^n - 7n - 1$

$= 1 + 7n + {}^nC_2 7^2 + {}^nC_3 7^3 + \cdots + {}^nC_n 7^n - 7n - 1$

$= 7^2[{}^nC_2 + {}^nC_3 7 + \cdots + {}^nC_n 7^{n-2}] = 49K \quad (1)$

where $K$ is an integer.

Therefore, $2^{3n} - 7n - 1$ is divisible by 49. Now,

$2^{3n+3} - 7n - 8 = 2^3\, 2^{3n} - 7n - 8$

$= 8(2^{3n} - 7n - 1) + 49n$

$= 8 \times 49K + 49n \quad$ [From (1)]

$= 49(8K + n)$

Therefore, $2^{3n+3} - 7n - 8$ is divisible by 49.

## Finding Remainder Using Binomial Theorem

To find the remainder when $a^n$ is divided by $b$, we adjust the power of $a$ to $a^m$ which is very close to $b$ say with difference 1. Also, the remainder is always positive. When number of the type $3k - 1$ is divided by 3, we have

$$\frac{3k-1}{3} = \frac{3k-3+2}{3} = k - 1 + \frac{2}{3}$$

Hence, the remainder is 2.

Following illustrations will explain the exact procedure.

**Example 6.42** **Find the remainder when $5^{99}$ is divided by 8 is**

**Sol.** $5^{99} = 5(5^2)^{49}$

$= 5(24 + 1)^{49}$

$= 5({}^{49}C_0 24^{49} + {}^{49}C_1 24^{48} + \ldots + {}^{49}C_{48} 24 + 1)$

Hence remainder when $5^{99}$ is divided by 8 is 5

**Example 6.43** **Find the remainder when $5^{99}$ is divided by 13.**

**Sol.** Here $5^2 = 25$ which is close to $26 = 13 \times 2$. Hence,

$E = 5^{99} = 5 \times 5^{98} = 5 \times (5^2)^{49} = 5(26 - 1)^{49}$

$\Rightarrow \quad E = 5[{}^{49}C_0\, 26^{49} - {}^{49}C_1\, 26^{48} + {}^{49}C_2\, 26^{47} - \cdots + {}^{49}C_{48}\, 26 - {}^{49}C_{49}]$

$= 5 \times 26k - 5$

Now,

$$\frac{E}{13} = 10k - \frac{5}{13} = 10k - 1 + \frac{8}{13}$$

Hence, the remainder is 8.

**Example 6.44** **Find the value of $\{3^{2003}/28\}$, where $\{\cdot\}$ denotes the fractional part.**

**Sol.** $E = 3^{2003} = 3^{2001} \times 3^2 = 9(27)^{667} = 9(28 - 1)^{667}$

$\Rightarrow \quad E = 9[{}^{667}C_0\, 28^{667} - {}^{667}C_1\, (28)^{666} + \cdots - {}^{667}C_{667}]$

$\Rightarrow \quad E = 9 \times 28k - 9$

$\Rightarrow \quad \frac{E}{28} = 9k - \frac{9}{28} = 9k - 1 + \frac{19}{28}$

That means if we divide $3^{2003}$ by 28, the remainder is 19. Thus,

$$\left\{\frac{3^{2003}}{28}\right\} = \frac{19}{28}$$

**Example 6.45** **Find the remainder when $1690^{2608} + 2608^{1690}$ is divided by 7.**

**Sol.** Here base (1690 and 2608) is too big, so first let us reduce it.

$1690 = 7 \times 241 + 3$ and $2608 = 7 \times 372 + 4$

Let,

$S = 1690^{2608} + 2608^{1690}$

$= (7 \times 241 + 3)^{2608} + (7 \times 372 + 4)^{1690}$

$= 7k + 3^{2608} + 4^{1690}$ (where $k$ is some positive integer)

Let,

$S' = 3^{2608} + 4^{1690}$

Clearly, the remainder in $S$ and $S'$ will be the same when divided by 7.

$S' = 3 \times 3^{3\times 867} + 4 \times 4^{3\times 563}$

$= 3 \times 27^{867} + 4 \times 64^{563}$

$= 3(28 - 1)^{867} + 4(63 + 1)^{563}$

$= 3[7n - 1] + 4[7m + 1]$ $(m, n \in I)$

$= 7p + 1$ (where $p$ is some positive integer)

Hence, the remainder is 1.

**Example 6.46** **Find the remainder when $x = 5^{5^{5^{\cdot^{\cdot^{5}}}}}$ (24 times 5) is divided by 24.**

**Sol.** Here exponent is $5^{5^{5^{\cdot^{\cdot^{5}}}}}$ (23 times 5) is an odd natural number. Therefore, $x = 5^{2m+1} = 5 \times (25^m)$, where $m$ is a natural number. Thus,

$$x = 5 \times (24 + 1)^m$$
$$= 5 + \text{a multiple of } 24$$

Hence, the remainder is 5.

## Expansion: $(x + y)^n \pm (x - y)^n$

We know that

$$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2\,x^{n-2}\,y^2 + \cdots + {}^nC_ny^n \quad (1)$$

and

$$(x - y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2\,x^{n-2}\,y^2 + \cdots + (-1)^n\,{}^nC_ny^n \quad (2)$$

Adding (1) and (2), we have

$$(x + y)^n + (x - y)^n = 2[{}^nC_0x^n + {}^nC_2\,x^{n-2}\,y^2 + {}^nC_4\,x^{n-4}\,y^4 + \cdots]$$

Subtracting (2) from (1), we have

$$(x + y)^n - (x - y)^n = 2[{}^nC_1x^{n-1}y + {}^nC_3x^{n-3}y^3 + {}^nC_5x^{n-5}y^5 + \cdots]$$

Above results are best used in the following illustrations.

**Example 6.47** **Prove that $\sqrt{10}[(\sqrt{10}+1)^{100} - (\sqrt{10}-1)^{100}]$ is an even integer.**

**Sol.** $\sqrt{10}[(\sqrt{10}+1)^{100} - (\sqrt{10}-1)^{100}]$

$$= 2\sqrt{10}[{}^{100}C_1(\sqrt{10})^{99} + {}^{100}C_3(\sqrt{10})^{97} + {}^{100}C_5(\sqrt{10})^{95} + \cdots]$$
$$= 2[{}^{100}C_1(\sqrt{10})^{100} + {}^{100}C_3(\sqrt{10})^{98} + {}^{100}C_5(\sqrt{10})^{96} + \cdots]$$
$$= 2[{}^{100}C_1(10)^{50} + {}^{100}C_3(10)^{49} + {}^{100}C_5(10)^{48} + \cdots]$$

which is an even number.

**Example 6.48** **If $9^7 - 7^9$ is divisible by $2^n$, then find the greatest value of $n$, where $n \in N$.**

**Sol.** We have,

$$9^7 - 7^9 = (1 + 8)^7 - (1 - 8)^9$$
$$= (1 + {}^7C_1\,8^1 + {}^7C_2\,8^2 + \cdots + {}^7C_7\,8^7) - (1 - {}^9C_1\,8^1 + {}^9C_2\,8^2 - \cdots - {}^9C_9\,8^9)$$
$$= 16 \times 8 + 64[({}^7C_2 + \cdots + {}^7C_7\,8^5) - ({}^9C_2 - \cdots - {}^9C_9\,8^7)]$$
$$= 64k \text{ (where } k \text{ is some integer)}$$

Therefore, $9^7 - 7^9$ is divisible by 64.

**Example 6.49** **Find the degree of the polynomial**

$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^7 - \left(\frac{1-\sqrt{4x+1}}{2}\right)^7\right\}$$

**Sol.** $\dfrac{1}{\sqrt{4x+1}}\left\{\left(\dfrac{1+\sqrt{4x+1}}{2}\right)^7 - \left(\dfrac{1-\sqrt{4x+1}}{2}\right)^7\right\}$

$$= \frac{2}{2^7\sqrt{4x+1}}[{}^7C_1\sqrt{4x+1} + {}^7C_3(\sqrt{4x+1})^3 \;\; C_5(\sqrt{4x+1})^5 + {}^7C_7(\sqrt{4x+1})^7]$$
$$= \frac{1}{2^6}[{}^7C_1 + {}^7C_3(4x+1) + {}^7C_5(4x+1)^2 + {}^7C_7(4x+1)^3]$$

Clearly, the degree of the polynomial is 3.

**Example 6.50** **If $(2+\sqrt{3})^n = I + f$ where $I$ and $n$ are +ive integers and $0 < f < 1$, show that $I$ is an odd integer and $(1-f)(I+f) = 1$.**

**Sol.** $(2 + \sqrt{3})^n = I + f$

or

$$I + f = 2^n + {}^nC_1\,2^{n-1}\sqrt{3} + {}^nC_2\,2^{n-2}(\sqrt{3})^2 + {}^nC_3\,2^{n-3}(\sqrt{3})^3 + \cdots \quad (1)$$

Now,

$$0 < 2 - \sqrt{3} < 1 \Rightarrow 0 < (2 - \sqrt{3})^n < 1$$

Let $(2 - \sqrt{3})^n = f'$ where $0 < f' < 1$.

$$\therefore\; f' = 2^n - {}^nC_1\,2^{n-1}\sqrt{3} + {}^nC_2\,2^{n-2}(\sqrt{3})^2 - {}^nC_3 2^{n-3}(\sqrt{3})^3 + \quad (2)$$

Adding (1) and (2), we get

$$I + f + f' = 2[2^n + {}^nC_2\,2^{n-2} \times 3 + \ldots]$$

or

$$I + f + f' = \text{even integer} \quad (3)$$

Now, $0 < f < 1$ and $0 < f' < 1$.

$$\therefore\; 0 < f + f' < 2$$

Hence from (3), we conclude that $f + f'$ is an integer between 0 and 2.

$$\therefore\; f + f' = 1 \Rightarrow f' = 1 - f \quad (4)$$

From (3) and (4), we get $I + 1$ is an even integer. Therefore, $I$ is an odd integer. Now,

$$I + f = (2 + \sqrt{3})^n, f' = 1 - f = (2 - \sqrt{3})^n$$
$$\therefore\; (I + f)(1 - f) = [(2 + \sqrt{3})(2 - \sqrt{3})]^n = (4 - 3)^n = 1$$
$$\therefore\; (I + f)(1 - f) = 1$$

### Concept Application Exercise 6.4

1. Using binomial theorem, show that $3^{2n+2} - 8n - 9$ is divisible by 64, $\forall\, n \in N$.
2. For each $n \in N$, prove that $49^n + 16n - 1$ is divisible by 64.
3. Find the remainder when $27^{40}$ is divided by 12.
4. Let $n$ be an odd natural number greater than 1. Then, find the number of zeros at the end of the sum $99^n + 1$.
5. Find the last two digits of the number $(23)^{14}$.
6. The number of non-zero terms in the expansion of $(1 + 3\sqrt{2}x)^9 + (1 - 3\sqrt{2}x)^9$ is______.
7. Find the value of $(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6$.
8. If
$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^n - \left(\frac{1-\sqrt{4x+1}}{2}\right)^n\right\} = a_0 + a_1x + \cdots + a_5x^5$$
then find the possible values of $n$.
9. Show that the integer next above $(\sqrt{3} + 1)^{2m}$ contains $2^{m+1}$ as a factor.

## USE OF COMPLEX NUMBERS IN BINOMIAL THEOREM

- Writing the binomial expression of $(\cos\theta + i\sin\theta)^n$ and equating the real part to $\cos n\theta$ and the imaginary part to $\sin n\theta$, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \cdots$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta + \cdots$$

$$\Rightarrow \tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - {}^nC_7\tan^7\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta + \cdots}$$

- We get a very interesting results if in binomial expansion, any variable is replaced with $i = \sqrt{-1}$ or $\omega$ (cube roots of unity).

**Example 6.51** **Find the sum $C_0 - C_2 + C_4 - C_6 + \cdots$ where $C_r = {}^nC_r$.**

**Sol.** Consider

$$(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$$

Put $x = i$ where $i = \sqrt{-1}$. Then,

$$(1 + i)^n = C_0 + C_1i + C_2i^2 + C_3i^3 + C_4i^4 + \cdots$$
$$= (C_0 - C_2 + C_4 - C_6 + \cdots) + i(C_1 - C_3 + C_5 + \cdots)$$

$$\Rightarrow C_0 - C_2 + C_4 - C_6 + \cdots = \text{Real part of } (1 + i)^n$$

$$= \text{Re}\left[(\sqrt{2})^n\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^n\right]$$

$$= \text{Re}\left[2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right]$$

$$= \text{Re}\left[2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right] \text{ [De' Moivre's theorem]}$$

$$= 2^{n/2}\cos\frac{n\pi}{4}$$

Also,

$$C_1 - C_3 + C_5 - \cdots = \text{Im}[(1 + i)^n] = 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$$

**Example 6.52** **Prove that**

$$ {}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$$

**Sol.** We have $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_nx^n$

Putting $x = 1$, $\omega$ and $\omega^2$, we have $2^n = C_0 + C_1 + C_2 + \cdots + C_n$

$$(1 + \omega)^n = C_0 + C_1\omega + C_2\omega^2 + \cdots + C_n\omega^n$$
$$(1 + \omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + \cdots + C_n\omega^{2n}$$

Adding the above three equations,

$$2^n + (1 + \omega)^n + (1 + \omega^2)^n = 3(C_0 + C_3 + C_6 + \cdots)$$

because

$$1 + \omega^k + \omega^{2k} = 0 \text{ if } k \neq 3m \qquad m \in N$$
$$= 3 \text{ if } k = 3m$$

Now,

$$1 + \omega = -\omega^2 = -\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$1 + \omega^2 = -\omega = -\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

Hence by De' Moivre's theorem,

$$2^n + (1 + \omega)^n + (1 + \omega^2)^n$$
$$= 2^n + \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$
$$= 2^n + 2\cos\frac{n\pi}{3}$$

**Example 6.53** **If $T_0, T_1, T_2, \ldots, T_n$ represent the terms in the expansion of $(x + a)^n$, then find the value of $(T_0 - T_2 + T_4 - \cdots)^2 + (T_1 - T_3 + T_5 - \cdots)^2$, $n \in N$.**

**Sol.** $(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \cdots$

$$= T_0 + T_1 + T_2 + T_3 + \cdots$$

Replacing $a$ by $ai$, we have

$$(x + ai)^n = {}^nC_0x^n + {}^nC_1x^{n-1}ai + {}^nC_2x^{n-2}(ai)^2 + {}^nC_3x^{n-3}(ai)^3 + \cdots$$
$$= ({}^nC_0x^n - {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 - \cdots)$$
$$+ i({}^nC_1x^na - {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 - \cdots)$$
$$= (T_0 - T_2 + T_4 - \cdots) + i(T_1 - T_3 + T_5 - \cdots)$$

Taking modulus of both sides and squaring

$$|x + ai|^{2n} = |(T_0 - T_2 + T_4 - \cdots) + i(T_1 - T_3 + T_5 - \cdots)|^2$$
$$\Rightarrow (x^2 + a^2)^n = (T_0 - T_2 + T_4 - \cdots)^2 + (T_1 - T_3 + T_5 - \cdots)^2$$

**Example 6.54** **If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, find the value of $a_0 + a_3 + a_6 + \cdots$, $n \in N$.**

**Sol.** $(1 + x + x^2) = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$ (1)

In (1), putting $x = 1$, we get

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$ (2)

In (1), putting $x = \omega$, we get

$$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + \cdots$$

$\Rightarrow 0 = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + \cdots$ (3)

In (1), putting $x = \omega^2$, we get

$$(1 + \omega^2 + \omega^4)^n = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\omega^{10} + a_6\omega^{12} + \cdots$$

$\Rightarrow 0 = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + \cdots$ (4)

Adding (2), (3) and (4), we have

$$3^n = 3(a_0 + a_3 + a_6 + \cdots)$$
$$\Rightarrow a_0 + a_3 + a_6 + \cdots = 3^{n-1}$$

### Concept Application Exercise 6.5

1. Find the sum ${}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots$.
2. Find the value of ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \cdots + {}^{4n}C_{4n}$.

## GREATEST TERM IN BINOMIAL EXPANSION

Consider the expansion of $(1 + x)^7$ when $x = 1/2$. Terms of the expansion are given by the following table.

| Term | Value |
|---|---|
| $T_1$ | ${}^7C_0 = 1$ |
| $T_2$ | ${}^7C_1\ (1/2) = 7/2$ |
| $T_3$ | ${}^7C_2\ (1/2)^2 = 21/4$ |
| $T_4$ | ${}^7C_3\ (1/2)^3 = 35/8$ |
| $T_5$ | ${}^7C_4\ (1/2)^4 = 35/16$ |
| $T_6$ | ${}^7C_5\ (1/2)^5 = 21/32$ |
| $T_7$ | ${}^7C_6\ (1/2)^6 = 7/64$ |
| $T_8$ | ${}^7C_7\ (1/2)^7 = 1/128$ |

Here, we can observe that value of the term increases till the 3rd term and then the value of the term decreases. Then, here $T_3$ is the greatest term.

$T_3$ is the greatest term, hence $T_3/T_2 > 1$ and also $T_4/T_3 < 1$.

In any binomial expansion, the values of the terms increase, reach a maximum and then decreases.

So in general to locate the maximum term in the expansion $(1 + x)^n$ we find the value of $r$ till the ratio $T_{r+1}/T_r$ is greater than 1 as for this value of $r$, any term is always greater than its previous term. The value of $r$ till this occurs gives the greatest term.

So for the greatest term, let

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}x \geq 1 \Rightarrow r \leq \frac{(n+1)x}{1+x}$$

Then the greatest term occurs for $r = [(n + 1)x/(1 + x)]$, which is the integral part of $[(n + 1)x/(1 + x)]$.

If $[(n + 1)x/(1 + x)]$ is an exact integer, then $T_r$ and $T_{r+1}$ both are the greatest terms.

When we have an expansion in which positive and negative sign occurs alternatively, we find the numerically greatest term (ignoring the negative value), for which we find the value of $r$ considering ratio $|T_{r+1}/T_r|$.

**Note:** *The greatest coefficient in the binomial expansion is equivalent to the greatest term when $x = 1$.*

**Example 6.55** **Find the greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$.**

**Sol.** Let $(r + 1)^{th}$ term be the greatest term in the expansion of $\left(1+\frac{1}{\sqrt{3}}\right)^{20}$. Now,

$$\frac{T_{r+1}}{T_r} = \frac{20-r+1}{r}\left(\frac{1}{\sqrt{3}}\right)$$

Let

$$T_{r+1} \geq T_r$$

$$\Rightarrow 20 - r + 1 \geq \sqrt{3}r$$

$$\Rightarrow 21 \geq r(\sqrt{3} + 1)$$

$$\Rightarrow r \leq \frac{21}{\sqrt{3}+1}$$

$$\Rightarrow r \leq 7.686$$

$$\Rightarrow r = 7$$

for which the greatest term occurs.

Hence, the greatest term in $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

$$T_8 = \sqrt{3}\ {}^{20}C_7\left(\frac{1}{\sqrt{3}}\right)^7 = \frac{25840}{9}$$

**Example 6.56** **Find the numerically greatest term in the expansion of $(3 - 5x)^{15}$ when $x = 1/5$.**

**Sol.** $(3 - 5x)^{15} = 3^{15}(1 - 5x/3)^{15} = 3^{15}(1 - 1/3)^{15}$ (for $x = 1/5$)

Now consider $(1 - 1/3)^{15}$.

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{15-r+1}{r}\left|-\frac{1}{3}\right| \geq 1$$

$$\Rightarrow 16 - r \geq 3r$$

$$\Rightarrow r \leq 4$$

Hence, $T_4$ and $T_5$ are the numerically greatest terms.

$$T_4 = {}^{15}C_3\ 3^{15-3}(-5x)^3 = -455 \times (3^{12})$$

and

$$T_5 = {}^{15}C_4\ 3^{15-4}(-5x)^4 = 455 \times (3^{12})$$

Also, $|T_4| = |T_5| = 455 \times (3^{12})$.

**Example 6.57** **Given that the 4th term in the expansion of $[2 + (3/8)x]^{10}$ has the maximum numerical value. Then find the range of value of $x$.**

**Sol.** Let $T_4$ be numerically the greatest term in the expansion of $2^{10}\left[1+\left(\frac{3}{16}\right)x\right]^{10}$. Therefore,

$$\left|\frac{T_4}{T_3}\right| \geq 1 \text{ and } \left|\frac{T_4}{T_5}\right| \geq 1$$

$$\Rightarrow \left|\frac{T_4}{T_3}\right| \geq 1 \text{ and } \left|\frac{T_5}{T_4}\right| \leq 1$$

Now,

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}x$$

Taking $r = 3$ and $r = 4$ and replacing $x$ by $3x/16$ and putting $n = 10$ in the above two relations, we get

$$\left|\frac{11-3}{3}\frac{3x}{16}\right| \geq 1 \text{ and } \left|\frac{11-4}{4}\frac{3x}{16}\right| \leq 1$$

$$\Rightarrow |x| \geq 2 \text{ and } |x| \leq \frac{64}{21} \qquad (1)$$

$\Rightarrow \quad x^2 \geq 4$ and $x^2 \leq \left(\frac{64}{21}\right)^2$

$\Rightarrow \quad 4 \leq x^2 \leq \left(\frac{64}{21}\right)^2$

$\Rightarrow \quad 2 \leq x \leq \frac{64}{21}$ or $-\frac{64}{21} \leq x \leq -2$

**Example 6.58** **Find the greatest coefficient in the expansion of $(1 + 2x/3)^{15}$.**

**Sol.** The greatest coefficient is equal to the greatest term when $x = 1$.

For $x = 1$, $\frac{T_{r+1}}{T_r} = \frac{15 - r + 1}{r}\frac{2}{3}$

Let $\frac{T_{r+1}}{T_r} \geq 1$

$\Rightarrow \quad \frac{15 - r + 1}{r}\frac{2}{3} \geq 1$

$\Rightarrow \quad 32 - 2r \geq 3r$

$\Rightarrow \quad r \leq 32/5$

$\Rightarrow \quad r = 6$

Hence, 7th term has the greatest coefficient and its value is $T_{6+1} = {}^{15}C_6(2/3)^6$.

**Concept Application Exercise 6.6**

1. Find the largest term in the expansion of $(3 + 2x)^{50}$ where $x = 1/5$.
2. If $x = 1/3$, find the greatest term in the expansion of $(1 + 4x)^8$.
3. If $n$ is an even positive integer, then find the values of $x$ if the greatest term in the expansion of $(1 + x)^n$ may have the greatest coefficient also.
4. If in the expansion of $(2x + 5)^{10}$, the numerically greatest term is equal to the middle term, then find the values of $x$.

## SUM OF SERIES

### Important Facts and Formulas for Finding Sum of Series

- $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ (1)

$$r\,{}^nC_r = r\frac{n!}{(n-r)!r!}$$

$$= \frac{n!}{(n-r)!(r-1)!}$$

$$= n\frac{(n-1)!}{(n-r)!(r-1)!}$$

$$= n\,{}^{n-1}C_{r-1}$$

Similarly,

$$r\,{}^{n+1}C_r = (n+1)\,{}^nC_{r-1}$$

$$r\,{}^nC_{r-1} = [(r-1)+1]\,{}^nC_{r-1}$$

$$= (r-1)\,{}^nC_{r-1} + {}^nC_{r-1}$$

$$= n\,{}^{n-1}C_{r-2} + {}^nC_{r-1}$$

$$r^2\,{}^nC_r = r n\,{}^{n-1}C_{r-1}$$

$$= n[(r-1)+1]\,{}^{n-1}C_{r-1}$$

$$= n[(r-1)\,{}^{n-1}C_{r-1} + {}^{n-1}C_{r-1}]$$

$$= n[(n-1)\,{}^{n-2}C_{r-2} + {}^{n-1}C_{r-1}]$$

and so on.

In the problems, we must remove any factor which is in terms of $r$ which is multiplied by binomial coefficient.

- $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

  In $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ replace $n$ by $n + 1$ and $r$ by $r + 1$.

  $\therefore \quad (r+1)\,{}^{n+1}C_{r+1} = (n+1)\,{}^nC_r$

  $\Rightarrow \quad \frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$ (2)

  Similarly,

  $\frac{{}^{n-1}C_{r-1}}{r} = \frac{{}^nC_r}{n}$ [replacing $r$ by $r - 1$ and $n$ by $n - 1$ in (2)]

  Also,

  $$\frac{{}^nC_r}{(r+1)(r+2)} = \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)},$$

  $$\frac{{}^nC_r}{(r+1)(r+2)(r+3)} = \frac{{}^{n+3}C_{r+3}}{(n+1)(n+2)(n+3)},$$

  and so on.

- Always adjust the power of variable to suffix $r$ of binomial coefficient ${}^nC_r$.

  Consider the following example:

  $(-1)^r r\,{}^nC_r = (-1)^r n\,{}^{n-1}C_{r-1} = -n\,{}^{n-1}C_{r-1}(-1)^{r-1}$. Here ${}^{n-1}C_{r-1}$ $(-1)^{r-1}$ is standard general term of binomial series $(1 - x)^{n-1}$. Similaraly,

  $$\frac{{}^nC_r 2^r}{(r+1)} = \frac{{}^{n+1}C_{r+1}2^r}{(n+1)} = \frac{{}^{n+1}C_{r+1}2^{r+1}}{2(n+1)}$$

**Example 6.59** **Find the sum $C_0 + 3\,C_1 + 3^2\,C_2 + \cdots + 3^nC_n$.**

**Sol.** $S = C_0 + 3\,C_1 + 3^2\,C_2 + \cdots + 3^n\,C_4$

$= (1 + 3)^n = 4^n$

**Example 6.60** **If $(1 + x)^n = \sum_{r=0}^{n} C_r x^r$, then prove that $C_1 + 2C_2 + 3C_3 + \cdots + nC_n = n2^{n-1}$.**

**Sol. Method (i): By summation**

$r^{th}$ term of the given series, is

$t_r = r\,{}^nC_r \Rightarrow t_r = n\,{}^{n-1}C_{r-1}$

Sum of the series is

$$\sum_{r=1}^{n} t_r = n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$= n\,({}^{n-1}C_0 + {}^{n-1}C_1 + \cdots + {}^{n-1}C_{n-1})$
$= n\,2^{n-1}$

**Method (ii): By calculus**

We have,

$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n \quad (1)$

Differentiating (1) with respect to $x$, we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \cdots + nC_nx^{n-1} \quad (2)$

Putting $x = 1$ in (2),

$n\,2^{n-1} = C_1 + 2C_2 + \cdots + {}^nC_n$

**Example 6.61** **Find the sum $1\,C_0 + 2\,C_1 + 3\,C_2 + \cdots + (n+1)\,C_n$ where $C_r = {}^nC_r$.**

**Sol.** $1C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n$

$$= \sum_{r=0}^{n}(r+1)\,{}^nC_r$$

$$= \sum_{r=0}^{n}\left[r\,{}^nC_r + {}^nC_r\right]$$

$$= n\sum_{r=1}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$$

$$= ({}^{n-1}C_0 + {}^{n-1}C_1 + \cdots + {}^{n-1}C_{n-1}) + ({}^nC_0 + {}^nC_1 + \cdots + {}^nC_n)$$

$$= n\,2^{n-1} + 2^n = 2^{n-1}(n+2)$$

**Example 6.62** **Find the sum $1 \times 2 \times C_1 + 2 \times 3\,C_2 + \cdots + n(n+1)C_n$, where $C_r = {}^nC_r$.**

**Sol.** $S = 1 \times 2 \times C_1 + 2 \times 3 \times C_2 + \cdots + n\,(n+1)\,C_n$

$$= \sum_{r=1}^{n} r(r+1)\,{}^nC_r$$

$$= \sum_{r=0}^{n}(r+1)[r\,{}^nC_r]$$

$$= \sum_{r=1}^{n}(r+1)[n\,{}^{n-1}C_{r-1}]$$

$$= n\sum_{r=1}^{n}[(r-1)+2]\,{}^{n-1}C_{r-1}$$

$$= n\sum_{r=1}^{n}\left[(n-1)\,{}^{n-2}C_{r-2} + 2\,{}^{n-1}C_{r-1}\right]$$

$$= n(n-1)\sum_{r=2}^{n}{}^{n-2}C_{r-2} + 2n\sum_{r=1}^{n}{}^{n-1}C_{r-1}$$

$= n\,(n-1)\,2^{n-2} + 2n\,2^{n-1}$
$= 2^{n-2}\,n\,[n - 1 + 4]$
$= n(n+3)\,2^{n-2}$

**Example 6.63** **If $n > 2$, then prove that $C_1\,(a-1) - C_2 \times (a-2) + \cdots + (-1)^{n-1}\,C_n(a-n) = a$, where $C_r = {}^nC_r$.**

**Sol.** $C_1\,(a-1) - C_2\,(a-2) + \cdots + (-1)^{n-1}\,C_n(a-n)$

$T_r = (-1)^{r-1}\,(a-r)\,{}^nC_r$
$= (-1)^{r-1}\,(a\,{}^nC_r - r\,{}^nC_r)$
$= (-1)^{r-1}\,(a\,{}^nC_r - n\,{}^{n-1}C_{r-1})$
$= -a\,(-1)^r\,{}^nC_r - n\,(-1)^{r-1} \times {}^{n-1}C_{r-1}$

Now,

$$S = \sum_{r=1}^{n} T_r$$

$= -a\,[(1-1)^n - {}^nC_0] - n(1-1)^{n-1}$
$= an$

**Example 6.64** **Find the sum $3\,{}^nC_0 - 8\,{}^nC_1 + 13\,{}^nC_2 - 18 \times {}^nC_3 + \cdots$.**

**Sol.** The general term of the series is $T_r = (-1)^r\,(3+5r)\,{}^nC_r$ where $r = 0, 1, 2, \ldots, n$. Therefore, sum of the series is given by

$$S = \sum_{r=0}^{n}(-1)^r(3+5r)\,{}^nC_r$$

$$= 3\left(\sum_{r=0}^{n}(-1)^r\,{}^nC_r\right) + 5\left(\sum_{r=1}^{n}(-1)^r\,n\,{}^{n-1}C_{r-1}\right)$$

$$= 3\left(\sum_{r=0}^{n}(-1)^r\,{}^nC_r\right) - 5n\left(\sum_{r=1}^{n}(-1)^{r-1}\,{}^{n-1}C_{r-1}\right)$$

$= 3(1-1)^n - 5n(1-1)^{n-1}$
$= 0$

**Example 6.65** **If $x + y = 1$, prove that $\sum_{r=0}^{n} r\,{}^nC_r\,x^r\,y^{n-r} = nx$.**

**Sol.** We have,

$$\sum_{r=0}^{n} r\,{}^nC_r x^r y^{n-r} = \sum_{r=1}^{n} n\,{}^{n-1}C_{r-1}x^{r-1}x^1 y^{n-r}$$

$$= nx\sum_{r=1}^{n}{}^{n-1}C_{r-1}x^{r-1}y^{(n-1)-(r-1)}$$

$= nx(x+y)^{n-1}$
$= nx \quad [\because x + y = 1]$

**Example 6.66** **If $(1+x)^n = \sum_{r=0}^{n} C_r x^r$, show that**

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

**Sol. Method (i): By summation**

$r^{\text{th}}$ term of the given series is

$$t_r = \frac{{}^nC_{r-1}}{r} = \frac{{}^{n+1}C_r}{n+1}$$

Required sum is

$$\sum_{r=1}^{n+1} t_r = \sum_{r=1}^{n+1}\frac{{}^{n+1}C_r}{n+1}$$

$$= \frac{1}{n+1}\,({}^{n+1}C_1 + {}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1})$$

$$=\frac{2^{n+1}-1}{n+1}$$

**Method (ii): By calculus**

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n \quad (1)$$

Integrating both the sides of (1) with respect to $x$ between the limits 0 and $x$, we have

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left[C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots + \frac{C_nx^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \cdots + \frac{C_nx^{n+1}}{n+1} \quad (2)$$

Substituting $x = 1$ in (2), we get

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$$

**Example 6.67** **Prove that $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots = \frac{2^n-1}{n+1}$ where $C_r = {}^nC_r$.**

**Sol.** $S = \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \cdots$

General term of the series is

$$\frac{{}^nC_{2r-1}}{2r} = \frac{{}^{n+1}C_{2r}}{n+1}$$

where $r = 1, 2, 3, \ldots$

$$\therefore \quad S = \frac{1}{n+1}\left[{}^{n+1}C_2 + {}^{n+1}C_4 + {}^{n+1}C_6 + \cdots\right]$$

$$= \frac{1}{n+1}\left[({}^{n+1}C_0 + {}^{n+1}C_4 + {}^{n+1}C_6 + \cdots) - {}^{n+1}C_0\right]$$

$$= \frac{1}{n+1}[2^n - 1]$$

**Example 6.68** **Find the sum $2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \frac{2^4}{4}C_3 + \cdots + \frac{2^{11}}{11}C_{10}$.**

**Sol.** We have,

$$2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \cdots + \frac{2^{11}}{11}C_{10}$$

$$= \sum_{r=0}^{10} {}^{10}C_r \frac{2^{r+1}}{r+1}$$

$$= \frac{1}{11}\sum_{r=0}^{10} \frac{11}{r+1}\,{}^{10}C_r\, 2^{r+1}$$

$$= \frac{1}{11}\sum_{r=0}^{10} {}^{11}C_{r+1}\, 2^{r+1}$$

$$= \frac{1}{11}({}^{11}C_1 2^1 + \cdots + {}^{11}C_{11} 2^{11})$$

$$= \frac{1}{11}({}^{11}C_0 2^0 + {}^{11}C_1 2^1 + \cdots + {}^{11}C_{11} 2^{11} - {}^{11}C_0 2^0)$$

$$= \frac{1}{11}[(1+2)^{11} - 1] = \frac{3^{11}-1}{11}$$

**Example 6.69** **Prove that**

$$\frac{1}{n+1} = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + \frac{3({}^nC_3)}{4} - \quad + (-1)^{n+1}\frac{n({}^nC_n)}{n+1}.$$

**Sol.** $S = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + \frac{3({}^nC_3)}{4} \cdots + (-1)^{n+1}\frac{n({}^nC_n)}{n+1}$

$$= \sum_{r=1}^{n} \frac{r\,{}^nC_r}{(r+1)}(-1)^{r+1}$$

$$= \sum_{r=1}^{n} r\frac{{}^{n+1}C_{r+1}}{n+1}(-1)^{r+1}$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}[(r+1)-1]\,{}^{n+1}C_{r+1}(-1)^{r+1}$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}[(r+1)\,{}^{n+1}C_{r+1}(-1)^{r+1} - {}^{n+1}C_{r+1}(-1)^{r+1}]$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}[-(n+1)\,{}^nC_r(-1)^r - {}^{n+1}C_{r+1}(-1)^{r+1}]$$

$$= -[-{}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + (-1)^n\,{}^nC_n]$$

$$-\frac{1}{n+1}[{}^{n+1}C_2 - {}^{n+1}C_3 + \cdots (-1)^{n+1}\,{}^{n+1}C_{n+1}]$$

$$= -[(1-1)^n - 1] - \frac{1}{n+1}[(1-1)^{n+1} - {}^{n+1}C_0 + {}^{n+1}C_1]$$

$$= 1 - \frac{1}{n+1}[(n+1)-1]$$

$$= 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

**Example 6.70** **If $k$ and $n$ be +ive integers and $s_k = 1^k + 2^k + 3^k + \cdots + n^k$, then prove that**

$$\sum_{r=1}^{m} {}^{m+1}C_r s_r = (n+1)^{m+1} - (n+1)$$

**Sol.** $S = \sum_{r=1}^{m} {}^{m+1}C_r\, s_r = [{}^{m+1}C_1 s_1 + {}^{m+1}C_2 s_2 + \cdots + {}^{m+1}C_m s_m]$

$$= {}^{m+1}C_1(1 + 2 + 3 + \cdots + n)$$
$$+ {}^{m+1}C_2(1^2 + 2^2 + 3^2 + \cdots + n^2)$$
$$+ {}^{m+1}C_3(1^3 + 2^3 + 3^3 + \cdots + n^3)$$
$$+ {}^{m+1}C_m(1^m + 2^m + 3^m + \cdots + n^m)$$
$$= ({}^{m+1}C_1\, 1 + {}^{m+1}C_2\, 1^2 + {}^{m+1}C_3\, 1^3 + \cdots + {}^{m+1}C_m\, 1^m)$$
$$+ ({}^{m+1}C_1\, 2 + {}^{m+1}C_2\, 2^2 + {}^{m+1}C_3\, 2^3 + \cdots + {}^{m+1}C_m 2^m)$$
$$+ \cdots + ({}^{m+1}C_1\, n + {}^{m+1}C_2\, n^2 + \cdots + {}^{m+1}C_m\, n^m)$$
$$= [(1+1)^{m+1} - 1 - {}^{m+1}C_{m+1}\, 1^{m+1}]$$
$$+ [(1+2)^{m+1} - 1 - {}^{m+1}C_{m+1}\, 2^{m+1}]$$
$$+ [(1+3)^{m+1} - 1 - {}^{m+1}C_{m+1}\, 3^{m+1}] + \cdots$$
$$= (2^{m+1} - 1^{m+1}) + (3^{m+1} - 2^{m+1}) + (4^{m+1} - 3^{m+1})$$
$$+ \cdots + [(1+n)^{m+1} - n^{m+1}] - n$$
$$= (1+n)^{m+1} - 1 - n = (1+n)^{m+1} - (n+1)$$

**Example 6.71** **Prove that $\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+\cdots+\frac{(-1)^{n-1}}{n}C_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$**

**Sol.** $S=\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+\cdots+\frac{(-1)^{n-1}}{n}C_n$

Now,

$$(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$$

Dividing by $x$,

$$\frac{(1+x)^n}{x}-\frac{C_0}{x}=C_1+C_2x+\cdots+C_nx^{n-1}$$

or

$$C_1+C_2x+C_2x^2+\cdots+C_nx^{n-1}$$

$$=\frac{(1+x)^n-1}{(1+x)-1}$$

$$=1+(1+x)+(1+x)^2+\cdots(1+x)^{n-1}$$

Integrating both sides between limits –1 and 0, we have

$$\int_{-1}^{0}(C_1+C_2x+C_3x^2+\cdots+C_nx^{n-1})dx$$

$$=\int_{-1}^{0}[1+(1+x)+\cdots+(1+x)^{n-1}]dx$$

$$\Rightarrow \left[C_1x+\frac{C_2x^2}{2}+\frac{C_3x^3}{3}+\cdots+\frac{C_nx^n}{n}\right]_{-1}^{0}$$

$$=\left[x+\frac{(1+x)^2}{2}+\frac{(1+x)^3}{3}+\cdots+\frac{(1+x)^n}{n}\right]_{-1}^{0}$$

$$\Rightarrow C_1-\frac{C_2}{2}+\frac{C_3}{3}-\cdots+(-1)^n\frac{C_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

***Replacing r by n – r is equivalent to writing the series in the reverse order.***

**Example 6.72** **Prove that $\sum_{r=0}^{n}{}^nC_r\sin rx\cos(n-r)x=2^{n-1}\sin(nx)$.**

**Sol.** Here sum is given by

$$S=\sum_{r=0}^{n}{}^nC_r\sin rx\cos(n-r)x$$

$$\Rightarrow S=\sum_{r=0}^{n}{}^nC_{n-r}\sin(n-r)x\cos rx \text{ (replacing } r \text{ by } n-r)$$

$$\Rightarrow 2S=\sum_{r=0}^{n}{}^nC_r\sin nx=\sin nx\times 2^n$$

$$\Rightarrow S=2^{n-1}\sin nx$$

**Example 6.73** **If for $n\in N$, $\sum_{k=0}^{2n}(-1)^k\left({}^{2n}C_k\right)^2=A$, then find the value of $\sum_{k=0}^{2n}(-1)^k(k-2n)\left({}^{2n}C_k\right)^2$.**

**Sol.** Let,

$$S=\sum_{k=0}^{2n}(-1)^k(k-2n)\left({}^{2n}C_k\right)^2 \qquad (1)$$

$$\Rightarrow S=-\sum_{k=0}^{2n}(-1)^{2n-k}(2n-k)\left({}^{2n}C_{2n-k}\right)^2$$

Writing the terms in $S$ in the reverse order, we get

$$S=-\sum_{k=0}^{2n}(-1)^k\left({}^{2n}C_k\right)^2 \qquad (2)$$

Adding (1) and (2), we get

$$2S=-2n\sum_{k=0}^{2n}(-1)^k({}^{2n}C_k)^2=-2nA$$

$$\Rightarrow S=-nA$$

**Concept Application Exercise 6.7**

1. Prove that
$$\frac{1^2}{3}{}^nC_1+\frac{1^2+2^2}{5}{}^nC_2+\frac{1^2+2^2+3^2}{7}{}^nC_3+\cdots+\frac{1^2+2^2+\cdots+n^2}{2n+1}{}^nC_n=\frac{n(n+3)}{6}2^{n-2}.$$
2. If $p+q=1$, then show that $\sum_{r=0}^{n}r^2\,{}^nC_rp^rq^{n-r}=npq+n^2p^2$.
3. Prove that $1-{}^nC_1\frac{1+x}{1+nx}+{}^nC_2\frac{1+2x}{(1+nx)^2}-{}^nC_3\frac{1+3x}{(1+nx)^3}+\cdots$ up to $(n+1)$ terms $=0$.
4. Prove that $\frac{{}^nC_0}{1}+\frac{{}^nC_2}{3}+\frac{{}^nC_4}{5}+\frac{{}^nC_6}{7}+\cdots=\frac{2^n}{n+1}$.
5. If $(1+x)^{15}=C_0+C_1x+C_2x^2+\cdots+C_{15}x^{15}$, then find the sum of $C_2+2C_3+3C_4+\cdots+14C_{15}$.
6. Find the coefficient of $x^n$ in the polynomial $(x+{}^nC_0)(x+3\,{}^nC_1)\times(x+5\,{}^nC_2)\cdots[x+(2n+1)\,{}^nC_n]$.
7. Find the value of ${}^{20}C_0-\frac{{}^{20}C_1}{2}+\frac{{}^{20}C_2}{3}-\frac{{}^{20}C_3}{4}+\cdots$.
8. Find the value of
$$\frac{1}{81^n}-\frac{10}{81^n}{}^{2n}C_1+\frac{10^2}{81^n}{}^{2n}C_2-\frac{10^3}{81^n}{}^{2n}C_3+\cdots+\frac{10^{2n}}{81^n}$$

## MISCELLANEOUS SERIES

### Series from Multiplication of Two Series

*Standard Results*

**1.** ${}^mC_r+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\cdots+{}^nC_r={}^{m+n}C_r$, where $r<m$, $r<n$ and $m, n, r$ are +ve integers.

**Proof:**

L.H.S. $={}^mC_r+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\cdots+{}^nC_r$

$={}^mC_r\,{}^nC_0+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\cdots+{}^mC_0\,{}^nC_r$

= coefficient of $x^r$ in $[(1+x)^m(1+x)^n]$

= coefficient of $x^r$ in $(1+x)^{m+n}$

$={}^{m+n}C_r={}^{(\text{sum of prefixes } m \text{ and } n)}C_{\text{constant sum of suffixes}}$

**2.** ${}^nC_0^2+{}^nC_1^2+\cdots+{}^nC_n^2={}^{2n}C_n$

**Proof:**

${}^nC_0^2+{}^nC_1^2+{}^nC_2^2+\cdots+{}^nC_n^2$

$= ({}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \cdots + {}^nC_n\,{}^nC_0)$
$=$ Coefficient of $x^n$ in $(1 + x)^n (1 + x)^n$
$=$ Coefficient of $x^n$ in $(1 + x)^{2n}$
$= {}^{2n}C_n$

3. ${}^nC_0^2 - {}^nC_1^2 + {}^nC_2^2 + \cdots + (-1)^n\,{}^nC_n^2$

**Proof:**

${}^nC_0^2 - {}^nC_1^2 + {}^nC_2^2 - \cdots + (-1)^n\,{}^nC_n^2$
$= {}^nC_0\,{}^nC_n - {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + \cdots + (-1)^n\,{}^nC_n\,{}^nC_0$
$=$ Coefficient of $x^n$ in $(1 + x)^n (1 - x)^n$
$=$ Coefficient of $x^n$ in $(1 - x^2)^n$

Now, general term in $(1 - x^2)^n$ is $T_{r+1} = {}^nC_r(-x^2)^r$.

For $x^n$, $2r = n \Rightarrow r = n/2 \Rightarrow n$ must be even.

If $n$ is odd, $r$ is not an integer or we can say, when $n$ is odd, $x^n$ term does not occur in $(1 - x^2)^n$ which is obvious. When $r$ is even, $r = n/2$. Hence,

$$T_{n/2+1} = {}^nC_{n/2}\,(-x^2)^{n/2} = (-1)^n\,{}^nC_{n/2}x^n$$

$$\Rightarrow\ C_0^2 - C_1^2 + C_2^2 + \cdots + (-1)^n\,C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n\,{}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

**Example 6.74** **Find the sum $\sum_{i=0}^{r} {}^{n_1}C_{(r-i)}\ {}^{n_2}C_i$.**

**Sol.** $S = \sum_{i=0}^{r} ({}^{n_1}C_{r-i})({}^{n_2}C_i)$

$=$ Coefficient of $x^r$ in $(1+x)^{n_1}\ (1+x)^{n_2}$
$= {}^{n_1+n_2}C_r$

**Example 6.75** **Prove that $\sum_{i=0}^{2n} r({}^{2n}C_r)^2 = {}^{4n-1}C_{2n+1}$.**

**Sol.** $S = \sum_{r=0}^{2n} r({}^{2n}C_r)^2$

$$= \sum_{r=0}^{2n} (r\,{}^{2n}C_r)({}^{2n}C_r)$$

$$= \sum_{r=0}^{2n} (2n)\,{}^{2n-1}C_{r-1}\ {}^{2n}C_r$$

$$= 2n\sum_{r=0}^{2n} {}^{2n-1}C_{r-1}\ {}^{2n}C_{2n-r}$$

Here, the sum of suffixes is $r - 1 + (2n - r) = 2n - 1$ which is constant.

$\therefore\ S =$ Coefficient of $x^{2n-1}$ in the expansion of $(1 + x)^{2n-1} (1 + x)^{2n}$
$=$ Coefficient of $x^{2n-1}$ in the expansion of $(1 + x)^{4n-1}$
$= {}^{4n-1}C_{2n-1}$

**Example 6.76** **Using binomial theorem (without using the formula for ${}^nC_r$), prove that**

$${}^nC_4 + {}^mC_2 - {}^mC_1\,{}^nC_2 = {}^mC_4 - {}^{m+n}C_1\,{}^mC_3 + {}^{m+n}C_2\,{}^mC_2 - {}^{m+n}C_3\,{}^mC_1 + {}^{m+n}C_4$$

**Sol.** ${}^mC_4 - {}^{m+n}C_1\,{}^mC_3 + {}^{m+n}C_2\,{}^mC_2 - {}^{m+n}C_3\,{}^mC_1 + {}^{m+n}C_4$
$= {}^{m+n}C_0\,{}^mC_4 - {}^{m+n}C_1\,{}^mC_3 + {}^{m+n}C_2\,{}^mC_2 - {}^{m+n}C_3\,{}^mC_1 + {}^{m+n}C_4\,{}^mC_0$
$=$ Coefficient of $x^4$ in $(1 + x)^{m+n} (1 - x)^m$
$=$ Coefficient of $x^4$ in $(1 - x^2)^m (1 + x)^n$
$=$ Coefficient of $x^4$ in $[1 - {}^mC_1\,x^2 + {}^mC_2\,x^4 - \cdots][1 + {}^nC_1x + {}^nC_2\,x^2 + \cdots + {}^nC_n\,x^n]$
$= {}^nC_4 - {}^mC_1 \times {}^nC_2 + {}^mC_2$

## Binomial Inside Binomial

**Example 6.77** **Find the sum $\sum_{r=0}^{n} {}^{n+r}C_r$.**

**Sol.** $\sum_{r=0}^{n} {}^{n+r}C_r = {}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{n+n}C_n$

$= {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \cdots + {}^{2n}C_n$
$=$ Coefficient of $x^n$ in $\{(1 + x)^n + (1 + x)^{n+1} + \cdots + (1 + x)^{2n}\}$

$$= \text{Coefficient of } x^n \text{ in } \frac{(1+x)^n\left\{1-(1+x)^{n+1}\right\}}{1-(1+x)}$$

$=$ Coefficient of $x^{n+1}$ in $\{(1 + x)^{2n+1} - (1 + x)^n\}$
$= {}^{2n+1}C_{n+1}$

**Example 6.78** **Prove that ${}^nC_0\ {}^{2n}C_n - {}^nC_1\ {}^{2n-1}C_n + {}^nC_2 \times {}^{2n-2}C_n + \cdots + (-1)^n\,{}^nC_n\,{}^nC_n = 1$.**

**Sol.** We know that

${}^{2n}C_n =$ Coefficient of $x^n$ in $(1 + x)^{2n}$
${}^{2n-1}C_n =$ Coefficient of $x^n$ in $(1 + x)^{2n-1}$
${}^{2n-2}C_n =$ Coefficient of $x^n$ in $(1 + x)^{2n-2}$
${}^{2n-3}C_n =$ Coefficient of $x^n$ in $(1 + x)^{2n-3}$
$\vdots$
${}^nC_n =$ Coefficient of $x^n$ in $(1 + x)^n$

Thus we have,

${}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-1}C_n + {}^nC_2\,{}^{2n-2}C_n + \cdots + (-1)^n\,{}^nC_n\,{}^nC_n$
$=$ Coefficient of $x^n$ in $[C_0\,(1 + x)^{2n} - C_1\,(1 + x)^{2n-1} + C_2\,(1 + x)^{2n-2} - C_3(1 + x)^{2n-3} + \cdots + (-1)^n\,C_n\,(1 + x)^n]$
$=$ Coefficient of $x^n$ in $(1 + x)^n\,[C_0\,(1 + x)^n - C_1\,(1 + x)^{n-1} + C_2(1 + x)^{n-2} - C_3(1 + x)^{n-3} + \cdots + (-1)^nC_n]$
$=$ Coefficient of $x^n$ in $(1 + x)^n\,[(1 + x) - 1]^n$
$=$ Coefficient of $x^n$ in $(1 + x)^n\,x^n$
$= 1$

**Example 6.79** **Prove that ${}^mC_1\,{}^nC_m - {}^mC_2\,{}^{2n}C_m + {}^mC_3\,{}^{3n}C_m - \cdots = (-1)^{m-1}\,n^m$.**

**Sol.** ${}^mC_1\,{}^nC_m - {}^mC_2\,{}^{2n}C_m + {}^mC_3\,{}^{3n}C_m - \cdots + (-1)^{m-1}\,{}^mC_m\,{}^{mn}C_m$
$=$ Coefficient of $x^m$ in

$^mC_1\,(1+x)^n - {}^mC_2\,(1+x)^{2n} + {}^mC_3(1+x)^{3n} - \cdots + (-1)^{m-1}\,{}^mC_m\,(1+x)^{mn}$

$=$ Coefficient of $x^m$ in

$^mC_0 - [{}^mC_0 - {}^mC_1\,(1+x)^n + {}^mC_2\,(1+x)^{2n} - \cdots + (-1)^m\,{}^mC_m(1+x)^{mn}]$

$=$ Coefficient of $x^m$ in $[1 - \{1-(1+x)^n\}^m]$

$=$ Coefficient of $x^m$ in $[1 - \{-nx - {}^nC_2\,x^2 - {}^nC_3x^3 - \cdots - {}^nC_n\,x^n\}^m]$

$= -(-n)^m = -(-1)^m n^m = -(-1)^{m-1}\,(-1)^2 n^m$

$= -(-1)^{m-1} n^m$

**Example 6.80** **If $(18x^2 + 12x + 4)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, prove that**

$$a_r = 2^n\,3^r\,({}^{2n}C_r + {}^nC_1\,{}^{2n-2}C_r + {}^nC_2\,{}^{2n-4}C_r + \cdots)$$

**Sol.** $(18x^2 + 12x + 4)^n = 2^n\,[2 + 9x^2 + 6x]^n$

Now, $a_r$ is coefficient of $x^r$ in $2^n\,[(3x+1)^2 + 1]^n$. Hence,

$a_r =$ Coefficient of $x^r$ in $2^n\,[{}^nC_0(3x+1)^{2n} + {}^nC_1(3x+1)^{2n-2} + {}^nC_2(3x+1)^{2n-4} + \cdots + {}^nC_r(3x+1)^{2n-2r} + \cdots]$

$\Rightarrow\ a_r = 2^n\,[{}^nC_0\,3^r\,{}^{2n}C_r + {}^nC_1\,3^r\,{}^{2n-2}C_r + {}^nC_2\,3^r\,{}^{2n-4}C_r + \cdots]$

$= 2^n\,3^r\,[{}^nC_0\,{}^{2n}C_r + {}^nC_1\,{}^{2n-2}C_r + {}^nC_2\,{}^{2n-4}C_r + \cdots]$

### Concept Application Exercise 6.8

1. Prove that $\sum_{r=0}^{n} r(n-r)C_r^2 = n^2({}^{2n-2}C_n)$.
2. Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \cdots + ({}^{2n}C_{2n})^2 = (-1)^n\,{}^{2n}C_n$.
3. Prove that ${}^nC_0\,{}^nC_0 - {}^{n+1}C_1\,{}^nC_1 + {}^{n+2}C_2\,{}^nC_2 - \cdots = (-1)^n$.
4. Prove that ${}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-2}C_n + {}^nC_2\,{}^{2n-4}C_n - \cdots = 2^n$.
5. Find the value of $\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\,{}^mC_p\right)$. And hence, find the value of $\lim_{n\to\infty}\frac{1}{3^n}\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\,{}^mC_p\right)$.

## Sum of the Series when *i* and *j* are Dependent

Consider, sum of the series $\sum\sum_{0\le i<j\le n} f(i)f(j)$

In the given summation, $i$ and $j$ are not independent.

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)\,f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$, $i$ and $j$ are independent. In this summation, three types of terms occur, those when $i<j$, $i>j$ and $i=j$.

Also, sum of terms when $i<j$ is equal to the sum of the terms when $i>j$ if $f(i)$ and $f(j)$ are symmetrical. So, in that case

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) = \sum_{0\le i<j\le n}\sum f(i)f(j) + \sum_{0\le j<i\le n}\sum f(i)f(j) + \sum_{i=j}\sum f(i)f(j)$$

$$= 2\sum_{0\le i<j\le n}\sum f(i)f(j) + \sum_{i=j}\sum f(i)f(j)$$

$$\Rightarrow \sum_{0\le i<j\le n}\sum f(i)f(j) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} f(i)f(j) - \sum_{i=j}\sum f(i)f(j)}{2}$$

When $f(i)$ and $f(j)$ are not symmetrical, we find the sum by listing all the terms.

**Example 6.81** **Find the sum**

a. $\sum\sum_{0\le i<j\le n} {}^nC_i\,{}^nC_j$

b. $\sum\sum_{0\le i\le j\le n} {}^nC_i\,{}^nC_j$

c. $\sum\sum_{i\ne j} {}^nC_i\,{}^nC_j$

**Sol. a.** $\sum_{0\le i<j\le n}\sum {}^nC_i\,{}^nC_j$

$$= \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\,{}^nC_j\right) - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{n}{}^nC_i\,2^n\right) - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^n2^n - \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^{2n} - {}^{2n}C_n}{2}$$

**b.** $\sum_{0\le i\le j\le n}\sum {}^nC_i\,{}^nC_j$

$$= \frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\,{}^nC_j\right) + \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{n}{}^nC_i\,2^n\right) + \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^n2^n + \sum_{i=0}^{n}({}^nC_i)^2}{2}$$

$$= \frac{2^{2n} + {}^{2n}C_n}{2}$$

**c.** $\sum_{i\ne j}\sum {}^nC_i\,{}^nC_j$

$$= \left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\,{}^nC_j\right) - \sum_{i=0}^{n}({}^nC_i)^2$$

$$= \left(\sum_{i=0}^{n}{}^nC_i\,2^n\right) - \sum_{i=0}^{n}({}^nC_i)^2$$

$$= 2^{2n} - {}^{2n}C_n$$

**Example 6.82** **Find the value of $\sum\sum_{0\le i<j\le n} ({}^nC_i + {}^nC_j)$.**

**Sol.** $$\sum\sum_{0\le i<j\le n}({}^nC_i+{}^nC_j)=\frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}({}^nC_i+{}^nC_j)\right)-\sum_{i=0}^{n}2\,{}^nC_i}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n}{}^nC_i+\sum_{j=0}^{n}{}^nC_j\right)\right)-2\times 2^n}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}\left({}^nC_i\sum_{j=0}^{n}1+2^n\right)\right)-2^{n+1}}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}({}^nC_i(n+1)+2^n)\right)-2^{n+1}}{2}$$

$$=\frac{(n+1)\sum_{i=1}^{n}{}^nC_i+2^n\sum_{i=0}^{n}1-2^{n+1}}{2}$$

$$=\frac{(n+1)2^n+2^n(n+1)-2^{n+1}}{2}$$

$$=(n+1)2^n-2^n=n2^n$$

**Example 6.83** **Find the value of** $\sum\sum_{1\le i<j\le n-1}(i\,j)\,{}^nC_i\,{}^nC_j$.

**Sol.** $$S=\sum\sum_{1\le i<j\le n-1}(i\,{}^nC_i)(j\,{}^nC_j)$$

$$=n^2\sum\sum_{1\le i<j\le n-1}{}^{n-1}C_{i-1}\,{}^{n-1}C_{j-1}$$

$$=n^2\left(\frac{2^{2(n-1)}-{}^{2(n-1)}C_{n-1}}{2}\right)$$

**Example 6.84** **Find the value of** $\sum_{0\le i<j\le n}(i+j)({}^nC_i+{}^nC_j)$.

**Sol.** Here sum does not change if we replace $i$ by $n-1$ and $j$ by $n-j$

By doing so, in fact we are writing the series in the reverse order.

$$\therefore\quad S=\sum\sum_{0\le i<j\le n}(i+j)({}^nC_i+{}^nC_j)\qquad(1)$$

$$=\sum\sum_{0\le i<j\le n}(n-i+n-j)({}^nC_{n-i}+{}^nC_{n-j})$$

$$\therefore\quad S=\sum\sum_{0\le i<j\le n}(2n-(i+j))\,({}^nC_i+{}^nC_n)\qquad(2)$$

Adding (1) and (2), we have

$$2S=2n\sum\sum_{0\le i<j\le n}({}^nC_i+{}^nC_j)$$

$$\Rightarrow\quad S=n\times n2^n=n^2\,2^n$$

**Example 6.85** **Find the sum** $\sum\sum_{0\le i<j\le n}{}^nC_i$.

**Sol.** $$\sum\sum_{0\le i<j\le n}{}^nC_i=\frac{\left(\sum_{i=0}^{n}\sum_{j=0}^{n}{}^nC_i\right)-\sum_{i=0}^{n}{}^nC_i}{2}$$

$$=\frac{\left(\sum_{i=0}^{n}(n+1)^n C_i\right)-\sum_{i=0}^{n}{}^nC_i}{2}$$

$$=\frac{(n+1)2^n-2^n}{2}$$

$$=n\times 2^{n-1}$$

**Example 6.86** **Find the sum** $\sum\sum_{0\le i<j\le n}j\,{}^nC_i$.

**Sol.** $$\sum\sum_{0\le i<j\le n}j\,{}^nC_i$$

$$=\sum_{r=0}^{n-1}{}^nC_r[(r+1)+(r+2)+\cdots+(n)]$$

$$=\sum_{r=0}^{n-1}{}^nC_r[(r+1)+(r+2)+\cdots+(n)]$$

$$=\sum_{r=0}^{n}{}^nC_r\left(\frac{n+1}{2}(n-r)-\frac{r(n-r)}{2}\right)$$

$$=\frac{n+1}{2}\sum_{r=0}^{n}(n-r)\,{}^nC_r\quad-\frac{n}{2}\sum_{r=0}^{n}r\,{}^nC_r+\frac{1}{2}\sum_{r=0}^{n}r^2\,{}^nC_r$$

$$=\frac{n+1}{2}\sum_{r=0}^{n}r\,{}^nC_r-\frac{n}{2}\sum_{r=0}^{n}r\,{}^nC_r+\frac{1}{2}\sum_{r=0}^{n}r^2\,{}^nC_r$$

$$=\frac{1}{2}\left(\sum_{r=0}^{n}r\,{}^nC_r+\sum_{r=0}^{n}r^2\,{}^nC_r\right)$$

$$=\frac{1}{2}(n\,2^{n-1}+n(n-1)\,2^{n-2}+n\,2^{n-1})=n(n+3)\,2^{n-3}$$

## BINOMIAL THEOREM FOR ANY INDEX

Let $n$ be a rational number and $x$ be a real number such that $|x|<1$. Then

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$
$$+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r+\cdots\infty$$

**Note:**

- *The condition $|x|<1$ is unnecessary if n is a whole number, while the same condition is essential if n is a rational number other than a whole number.*
- *Note that there are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.*
- *In the above expansion, the first term is unity. If the first term is not unity and the index of the binomial is either a negative integer or a fraction, then we expand as follows:*

$$(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$$

$$= a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\cdots\right\}$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2+\cdots$$

*The expansion is valid when* $|x/a| < 1$ *or equivalently* $|x| < |a|$.

- *Expansion of* $(x + a)^n$ *for any rational index:*

***Case I: When*** $|x| > |a|$, ***i.e.,*** $|a/x| < 1$

$$(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n$$

$$= x^n\left\{1+n\frac{a}{x}+\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3+\cdots\right\}$$

***Case II: When*** $|x| < |a|$, ***i.e.,*** $|x/a| < 1$

$$(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n$$

$$= a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3+\cdots\right\}$$

- *If n is a positive integer, the above expansion contains* $(n + 1)$ *terms and coincides with* $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_nx^n$, *because*

$${}^nC_0 = 1,\ {}^nC_1 = n,\ {}^nC_2 = \frac{n(n-1)}{2!},\ {}^nC_3 = \frac{n(n-1)(n-2)}{3!}$$

- *The general term in the expansion of* $(1 + x)^n$ *is given by*

$$T_{r+1} = \frac{n(n-1)(n-2)\cdots[n-(r-1)]}{r!}x^r$$

- *Let n be a positive integer, then by replacing n by –n in the expansion for* $(1 + x)^n$, *we get*

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

$$+(-1)^r\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r+\cdots$$

$$= 1 - {}^nC_1x + {}^{n+1}C_2x^2 - {}^{n+2}C_3x^3 + \cdots + {}^{n+r-1}C_r(-x)^r + \cdots$$

*Now replacing x by –x and n by –n in the expression of* $(1 + x)^n$, *we get*

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

$$+\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}x^r+\cdots$$

*If it is –ve integer then* $(1 - x)^{-n} = 1 + {}^nC_1x + {}^{n+1}C_2x^2 + {}^{n+2}C_3x^3 + \cdots + {}^{n+r-1}C_rx^r + \cdots$

## Important Expansions

**(i)** $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^r x^r + \cdots$

**(ii)** $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$

**(iii)** $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

**(iv)** $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

**Example 6.87** **Find the condition for which the formula** $(a+b)^m = a^m + ma^{m-1}b + \dfrac{m(m-1)}{1\times 2}a^{m-2}b^2+\cdots$ **holds.**

**Sol.** The expression can be written as $a^m\left\{\left(1+\dfrac{b}{a}\right)^m\right\}$. Hence, it is valid only when

$$|b/a| < 1$$

$$\Rightarrow |b| < |a|$$

**Example 6.88** **Find the values of** $x$**, for which** $1/(\sqrt{5+4x})$ **can be expanded as infinite series.**

**Sol.** The given expression can be written as $5^{1/2}\left(1+\dfrac{4x}{5}\right)^{-1/2}$ and is valid only when

$$\left|\frac{4}{5}x\right| < 1 \Rightarrow |x| < \frac{5}{4}$$

**Example 6.89** **Find the fourth term in the expansion of** $(1 - 2x)^{3/2}$**.**

**Sol.** We have,

$$(1-2x)^{3/2} = 1 + \frac{3}{2}(-2x) + \frac{\frac{3}{2}\times\frac{1}{2}}{2!}(-2x)^2 + \frac{\frac{3}{2}\times\frac{1}{2}\left(-\frac{1}{2}\right)}{3!}(-2x)^3+\cdots$$

Hence, the 4th term is $x^3/2$.

**Example 6.90** **Prove that the coefficient of** $x^r$ **in the expansion of** $(1 - 2x)^{-1/2}$ **is** $(2r)!/[2^r(r!)^2]$**.**

**Sol.** Coefficient of $x^r$ is

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-r+1\right)}{r!}(-2)^r$$

$$= \frac{1\times3\times5\times\cdots\times(2r-1)}{2^r}\,\frac{(-1)^r\,(-1)^r\,2^r}{r!}$$

$$= \frac{1\times3\times5\times\cdots\times(2r-1)}{r!}$$

$$= \frac{1\times2\times3\times4\times5\times\cdots\times(2r-1)(2r)}{(2\times4\times6\times8\times\cdots\times(2r))r!}$$

$$= \frac{(2r)!}{2^r(r!)^2}$$

**Example 6.91** **Find the sum**

$$1-\frac{1}{8}+\frac{1}{8}\times\frac{3}{16}-\frac{1\times3\times5}{8\times16\times24}+\ldots$$

**Sol.** Comparing the given series to

$$1+nx+\frac{n(n-1)}{2!}x^2+\cdots=(1+x)^n$$

we get

$$nx=-\frac{1}{8} \text{ and } \frac{n(n-1)}{2!}x^2=\frac{3}{128}$$

$$\Rightarrow \quad x=\frac{1}{4},\ n=-\frac{1}{2}$$

Hence,

$$1-\frac{1}{8}+\frac{1}{8}\frac{3}{16}-\ldots=\left(1+\frac{1}{4}\right)^{-1/2}=\frac{2}{\sqrt{5}}$$

**Example 6.92** **Assuming $x$ to be so small that $x^2$ and higher powers of $x$ can be neglected, prove that**

$$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}=1-\frac{305}{96}x$$

**Sol.** We have,

$$\frac{\left(1+\frac{3}{4}x\right)^{-4}(16-3x)^{1/2}}{(8+x)^{2/3}}=\frac{\left(1+\frac{3}{4}x\right)^{-4}(16)^{1/2}\left(1-\frac{3x}{16}\right)^{1/2}}{8^{2/3}\left(1+\frac{x}{8}\right)^{2/3}}$$

$$=\left(1+\frac{3}{4}x\right)^{-4}\left(1-\frac{3x}{16}\right)^{1/2}\left(1+\frac{x}{8}\right)^{-2/3}$$

$$=\left\{1+(-4)\left(\frac{3}{4}x\right)\right\}\left\{1+\frac{1}{2}\left(\frac{-3x}{16}\right)\right\}\left\{1+\left(-\frac{2}{3}\right)\left(\frac{x}{8}\right)\right\}$$

$$=(1-3x)\left(1-\frac{3}{32}x\right)\left(1-\frac{x}{12}\right)$$

$$=\left(1-3x-\frac{3}{32}x\right)\left(1-\frac{x}{12}\right) \quad [\text{neglecting } x^2]$$

$$=\left(1-\frac{99}{32}x\right)\left(1-\frac{x}{12}\right)=1-\frac{99}{32}x-\frac{x}{12} \quad [\text{neglecting } x^2]$$

$$=1-\frac{305}{96}x$$

**Example 6.93** **Find the coefficient of $x^n$ in the expansion of $(1-9x+20x^2)^{-1}$.**

**Sol.** We have,

$$(1-9x+20x^2)^{-1}=[(1-5x)(1-4x)]^{-1}$$

$$=\frac{1}{(1-5x)(1-4x)}=\frac{5}{1-5x}-\frac{4}{1-4x}$$

$$=5(1-5x)^{-1}-4(1-4x)^{-1}$$

$$=5[1+5x+(5x)^2+\cdots+(5x)^n+\cdots]-4[1+4x+(4x)^2+\cdots+(4x)^n+\cdots]$$

Therefore the coefficient of $x^n$ is $5^{n+1}-4^{n+1}$.

## Concept Application Exercise 6.9

1. If the third term in the expansion of $(1+x)^m$ is $-\frac{1}{8}x^2$, then find the value of $m$.
2. Find the cube root of 217, correct to two decimal places.
3. Find the coefficient of $x^2$ in $\left(\frac{a}{a+x}\right)^{1/2}+\left(\frac{a}{a-x}\right)^{1/2}$.
4. If $|x|<1$, then find the coefficient of $x^n$ in the expansion of $(1+x+x^2+\cdots)^2$.
5. If $|x|>1$, then expand $(1+x)^{-2}$.
6. If $|x|<1$, then find the coefficient of $x^n$ in the expansion of $(1+2x+3x^2+4x^3+\cdots)^{1/2}$.
7. If $(r+1)$th term is the first negative term in the expansion of $(1+x)^{7/2}$, then find the value of $r$.

# EXERCISES

## Subjective Type

Solutions on page 6.34

1. Find the coefficients of $x^n$ in $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)^2$.

2. If $\sum_{r=0}^{n}\{a_r(x-\alpha+2)^r - b_r(\alpha-x-1)^r\}=0$, then prove that $b_n-(-1)^n a_n=0$.

3. Evaluate ${}^nC_0\,{}^nC_2+2\,{}^nC_1\,{}^nC_3+3\,{}^nC_2\,{}^nC_4+\cdots+(n-1)\,{}^nC_{n-2}\,{}^nC_n$.

4. Prove that $\sum_{r=0}^{n}{}^nC_r(-1)^r\left[i^r+i^{2r}+i^{3r}+i^{4r}\right]$

$= 2^n + 2^{n/2+1}\cos(n\pi/4)$ where $i=\sqrt{-1}$.

5. Let $a=(4^{1/401}-1)$ and for each $n\ge 2$, let $b_n={}^nC_1+{}^nC_2\cdot a+{}^nC_3\cdot a^2+\ldots+{}^nC_n\cdot a^{n-1}$. Find the value of $(b_{2006}-b_{2005})$.

6. If $n$ be a positive integer, prove that

$$1-2n+\frac{2n(2n-1)}{2!}-\frac{2n(2n-1)(2n-2)}{3!}+\cdots+(-1)^{n-1}\frac{2n(2n-1)\cdots(n+2)}{(n-1)!}$$

$= (-1)^{n+1}(2n)!/2(n!)^2$

7. Prove that $\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\cdots+(-1)^n\frac{{}^nC_n}{x+n}$

$=\frac{n!}{x(x+1)\cdots(x+n)}$,

where $n$ is any positive integer and $x$ is not a negative integer.

8. Prove that

${}^{100}C_0\,{}^{100}C_2+{}^{100}C_2\,{}^{100}C_4+{}^{100}C_4\,{}^{100}C_6+\cdots+{}^{100}C_{98}\,{}^{100}C_{100}$

$=\frac{1}{2}\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$.

9. Prove that $\sum_{r=1}^{m-1}\frac{2r^2-r(m-2)+1}{(m-r)\,{}^mC_r}=m-\frac{1}{m}$.

10. Find the coefficient of $x^{50}$ in the expression

$(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\cdots+1001x^{1000}$

11. Prove that

$${}^nC_1-\left(1+\frac{1}{2}\right){}^nC_2+\left(1+\frac{1}{2}+\frac{1}{3}\right){}^nC_3+\cdots+(-1)^{n-1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right){}^nC_n=\frac{1}{n}$$

12. Prove that ${}^nC_1({}^nC_2)^2({}^nC_3)^3\cdots({}^nC_n)^n\le\left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$, $\forall\, n\in N$.

13. Prove that

$$\frac{1}{m!}{}^nC_0+\frac{n}{(m+1)!}{}^nC_1+\frac{n(n-1)}{(m+2)!}{}^nC_2+\cdots+\frac{n(n-1)\cdots2\times1}{(m+n)!}{}^nC_n$$

$$=\frac{(m+n+1)(m+n+2)\cdots(m+2n)}{(m+n)!}$$

14. If $n=12\,m$ ($m\in N$), prove that

$${}^nC_0-\frac{{}^nC_2}{(2+\sqrt{3})^2}+\frac{{}^nC_4}{(2+\sqrt{3})^4}-\frac{{}^nC_6}{(2+\sqrt{3})^6}+\cdots=(-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n$$

15. Prove that in the expansion of $(1+x)^n(1+y)^n(1+z)^n$, the sum of the coefficients of the terms of degree $r$ is ${}^{3n}C_r$.

## Objective Type

Solutions on page 6.36

**Each question has four choices a, b, c and d, out of which *only one* is correct.**

1. If the coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ be in A.P., then $n=$
   **a.** 7 only **b.** 14 only
   **c.** 7 or 14 **d.** none of these

2. If $x^m$ occurs in the expansion of $(x+1/x^2)^{2n}$, then the coefficient of $x^m$ is
   **a.** $\frac{(2n)!}{(m)!(2n-m)!}$ **b.** $\frac{(2n)!3!3!}{(2n-m)!}$
   **c.** $\frac{(2n)!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$ **d.** none of these

3. The coefficient of $x^5$ in the expansion of $(1+x)^{21}+(1+x)^{22}+\cdots+(1+x)^{30}$ is
   **a.** ${}^{51}C_5$ **b.** ${}^9C_5$
   **c.** ${}^{31}C_6-{}^{21}C_6$ **d.** ${}^{30}C_5+{}^{20}C_5$

4. The coefficient of $1/x$ in the expansion of $(1+x)^n(1+1/x)^n$ is
   **a.** $\frac{n!}{(n-1)!(n+1)!}$ **b.** $\frac{(2n)!}{(n-1)!(n+1)!}$
   **c.** $\frac{(2n)!}{(2n-1)!(2n+1)!}$ **d.** none of these

5. In the expansion of $(1+3x+2x^2)^6$, the coefficient of $x^{11}$ is
   **a.** 144 **b.** 288
   **c.** 216 **d.** 576

6. If the coefficients of $r^{th}$ and $(r+1)^{th}$ terms in the expansion of $(3+7x)^{29}$ are equal, then $r$ equals
   **a.** 15 **b.** 21
   **c.** 14 **d.** none of these

7. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1:7:42, then the value of $n$ is
   **a.** 60 **b.** 70.
   **c.** 55 **d.** none of these

8. If $(1+2x+x^2)^n=\sum_{r=0}^{2n}a_rx^r$, then $a_r=$
   **a.** $({}^nC_r)^2$ **b.** ${}^nC_r\cdot{}^nC_{r+1}$
   **c.** ${}^{2n}C_r$ **d.** ${}^{2n}C_{r+1}$

**9.** If in the expansion of $(1 + x)^n$, $a, b, c$ are three consecutive coefficients, then $n =$

**a.** $\frac{ac+ab+bc}{b^2+ac}$ **b.** $\frac{2ac+ab+bc}{b^2-ac}$

**c.** $\frac{ab+ac}{b^2-ac}$ **d.** none of these

**10.** The coefficient of the middle term in the binomial expansion in powers of $x$ of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same, if $\alpha$ equals

**a.** $-\frac{5}{3}$ **b.** $\frac{10}{3}$

**c.** $-\frac{3}{10}$ **d.** $\frac{3}{5}$

**11.** The coefficient of $x^n$ in the expansion of $(1 - x)(1 - x)^n$ is

**a.** $n - 1$ **b.** $(-1)^n (1 + n)$

**c.** $(-1)^{n-1} (n - 1)^2$ **d.** $(-1)^{n-1} n$

**12.** If the coefficients of $r^{th}$, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion of $(1 + y)^m$ are in A.P., then $m$ and $r$ satisfy the equation

**a.** $m^2 - m(4r + 1) + 4r^2 + 2 = 0$

**b.** $m^2 - m(4r - 1) + 4r^2 - 2 = 0$

**c.** $m^2 - m(4r - 1) + 4r^2 + 2 = 0$

**d.** $m^2 - m(4r + 1) + 4r^2 - 2 = 0$

**13.** If the coefficient of $x^7$ in $\left[ax^2 + \left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of $x^{-7}$ in $\left[ax - \left(\frac{1}{bx^2}\right)\right]^{11}$, then $a$ and $b$ satisfy the relation

**a.** $a + b = 1$ **b.** $a - b = 1$

**c.** $ab = 1$ **d.** $\frac{a}{b} = 1$

**14.** The coefficient of $a^8b^4c^9d^9$ in $(abc + abd + acd + bcd)^{10}$ is

**a.** $10!$ **b.** $\frac{10!}{8!4!9!9!}$

**c.** 2520 **d.** none of these

**15.** The coefficient of $x^5$ in the expansion of $(x^2 - x - 2)^5$ is

**a.** –83 **b.** –82

**c.** –86 **d.** –81

**16.** The coefficient of $x^2y^3$ in the expansion of $(1 - x + y)^{20}$ is

**a.** $\frac{20!}{2!3!}$ **b.** $-\frac{20!}{2!3!}$

**c.** $\frac{20!}{5!2!3!}$ **d.** none of these

**17.** If the term independent of $x$ in the $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then $k$ equals

**a.** 2, –2 **b.** 3, –3

**c.** 4, –4 **d.** 1, –1

**18.** The coefficient of $x^{10}$ in the expansion of $(1 + x^2 - x^3)^8$ is

**a.** 476 **b.** 496

**c.** 506 **d.** 528

**19.** If the coefficient of $x^n$ in $(1 + x)^{101} (1 - x + x^2)^{100}$ is non-zero, then $n$ cannot be of the form

**a.** $3r + 1$ **b.** $3r$

**c.** $3r + 2$ **d.** none of these

**20.** The coefficient of $x^{28}$ in the expansion of $(1 + x^3 - x^6)^{30}$ is

**a.** 1 **b.** 0

**c.** $^{30}C_6$ **d.** $^{30}C_3$

**21.** The term independent of $a$ in the expansion of

$$\left(1 + \sqrt{a} + \frac{1}{\sqrt{a}-1}\right)^{-30}$$ is

**a.** $^{30}C_{20}$ **b.** 0

**c.** $^{30}C_{10}$ **d.** none of these

**22.** The coefficient of $x^{53}$ in the expansion

$$\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} 2^m$$ is

**a.** $^{100}C_{47}$ **b.** $^{100}C_{53}$

**c.** $-^{100}C_{53}$ **d.** $-^{100}C_{100}$

**23.** The coefficient of the term independent of $x$ in the expansion of

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$$

is

**a.** 210 **b.** 105

**c.** 70 **d.** 112

**24.** In the expansion of $(1 + x + x^3 + x^4)^{10}$, the coefficient of $x^4$ is

**a.** $^{40}C_4$ **b.** $^{10}C_4$

**c.** 210 **d.** 310

**25.** The approximate value of $(1.0002)^{3000}$ is

**a.** 1.6 **b.** 1.4

**c.** 1.8 **d.** 1.2

**26.** The last two digits of the number $3^{400}$ are

**a.** 81 **b.** 43

**c.** 29 **d.** 01

**27.** The expression

$$\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\frac{2}{\sqrt{2x^2+1}+\sqrt{2x^2-1}}\right)^6$$

is a polynomial of degree

**a.** 6 **b.** 8

**c.** 10 **d.** 12

**28.** The coefficient of $x^r [0 \le r \le (n - 1)]$ in the expansion of $(x + 3)^{n-1} + (x + 3)^{n-2} (x + 2) + (x + 3)^{n-3} (x + 2)^2 + \cdots + (x + 2)^{n-1}$ are

**a.** $^nC_r (3^r - 2^n)$ **b.** $^nC_r (3^{n-r} - 2^{n-r})$

**c.** $^nC_r(3^r + 2^{n-r})$ **d.** none of these

**29.** If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \cdots + a_{20}x^{20}$, then $a_1$ equals

**a.** 10 **b.** 20

**c.** 210 **d.** none of these

**30.** In the expansion of $(5^{1/2} + 7^{1/8})^{1024}$, the number of integral terms is

**a.** 128 **b.** 129

**c.** 130 **d.** 131

**31.** The total number of terms which are dependent on the value of $x$, in the expansion of $\left(x^2-2+\frac{1}{x^2}\right)^n$ is equal to

**a.** $2n+1$ **b.** $2n$
**c.** $n$ **d.** $n+1$

**32.** If the 6th term in the expansion of $\left(\frac{1}{x^{8/3}}+x^2\log_{10}x\right)^8$ is 5600, then $x$ equals

**a.** 1 **b.** $\log_e 10$
**c.** 10 **d.** $x$ does not exist

**33.** If $n$ is an integer between 0 and 21, then the minimum value of $n!(21-n)!$ is attained for $n=$

**a.** 1 **b.** 10
**c.** 12 **d.** 20

**34.** In the expansion of $(3^{-x/4}+3^{5x/4})^n$ the sum of binomial coefficient is 64 and term with the greatest binomial coefficient exceeds the third by $(n-1)$, the value of $x$ must be

**a.** 0 **b.** 1
**c.** 2 **d.** 3

**35.** If the last term in the binomial expansion of $\left(2^{1/3}-\frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{3^{5/3}}\right)^{\log_3 8}$, then the 5th term from the beginning is

**a.** 210 **b.** 420
**c.** 105 **d.** none of these

**36.** If ${}^{n+1}C_{r+1}:{}^{n}C_{r}:{}^{n-1}C_{r-1}=11:6:3$, then $nr=$

**a.** 20 **b.** 30
**c.** 40 **d.** 50

**37.** The value of $x$ for which the sixth term in the expansion of

$$\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$$

is 84 is

**a.** 4 **b.** 1 or 2
**c.** 0 or 1 **d.** 3

**38.** The number of integral terms in the expansion of $(\sqrt{3}+\sqrt[8]{5})^{256}$ is

**a.** 33 **b.** 34
**c.** 35 **d.** none of these

**39.** The number of real negative terms in the binomial expansion of $(1+ix)^{4n-2}$, $n\in N$, $x>0$ is

**a.** $n$ **b.** $n+1$
**c.** $n-1$ **d.** $2n$

**40.** If in the expansion of $(a-2b)^n$, the sum of 5th and 6th terms is 0, then the values of $a/b=$

**a.** $\frac{n-4}{5}$ **b.** $\frac{2(n-4)}{5}$
**c.** $\frac{5}{n-4}$ **d.** $\frac{5}{2(n-4)}$

**41.** The number of distinct terms in the expansion of $\left(x+\frac{1}{x}+x^2+\frac{1}{x^2}\right)^{15}$ is/are (with respect to different power of $x$)

**a.** 255 **b.** 61
**c.** 127 **d.** none of these

**42.** The sum of the coefficients of even power of $x$ in the expansion of $(1+x+x^2+x^3)^5$ is

**a.** 256 **b.** 128
**c.** 512 **d.** 64

**43.** Maximum sum of coefficient in the expansion of $(1-x\sin\theta+x^2)^n$ is

**a.** 1 **b.** $2^n$
**c.** $3^n$ **d.** 0

**44.** If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coefficient in the expansion is

**a.** 924 **b.** 792
**c.** 1594 **d.** none of these

**45.** If the sum of the coefficients in the expansion of $(1-3x+10x^2)^n$ is $a$ and if the sum of the coefficients in the expansion of $(1+x^2)^n$ is $b$, then

**a.** $a=3b$ **b.** $a=b^3$
**c.** $b=a^3$ **d.** none of these

**46.** If $(1+x-2x^2)^6=1+a_1x+a_2x^2+a_3x^3+\cdots$, then the value of $a_2+a_4+a_6+\cdots+a_{12}$ will be

**a.** 32 **b.** 31
**c.** 64 **d.** 1024

**47.** The fractional part of $2^{4n}/15$ is $(n\in N)$

**a.** $\frac{1}{15}$ **b.** $\frac{2}{15}$
**c.** $\frac{4}{15}$ **d.** none of these

**48.** The value of ${}^{15}C_0^2-{}^{15}C_1^2+{}^{15}C_2^2-\cdots-{}^{15}C_{15}^2$ is

**a.** 15 **b.** $-15$
**c.** 0 **d.** 51

**49.** The value of

$$\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}+\binom{30}{2}\binom{30}{12}+\cdots+\binom{30}{20}\binom{30}{30}=$$

**a.** ${}^{60}C_{20}$ **b.** ${}^{30}C_{10}$
**c.** ${}^{60}C_{30}$ **d.** ${}^{40}C_{30}$

**50.** If $f(x)=x^n$, then the value of $f(1)+\frac{f^1(1)}{1}+\frac{f^2(1)}{2!}+\cdots+\frac{f^n(1)}{n!}$, where $f^r(x)$ denotes the $r^{th}$ order derivative of $f(x)$ with respect to $x$, is

**a.** $n$ **b.** $2^n$
**c.** $2^{n-1}$ **d.** none of these

**51.** The value of ${}^{20}C_0+{}^{20}C_1+{}^{20}C_2+{}^{20}C_3+{}^{20}C_4+{}^{20}C_{12}+{}^{20}C_{13}+{}^{20}C_{14}+{}^{20}C_{15}$ is

**a.** $2^{19}-\frac{({}^{20}C_{10}+{}^{20}C_9)}{2}$ **b.** $2^{19}-\frac{({}^{20}C_{10}+2\times{}^{20}C_9)}{2}$
**c.** $2^{19}-\frac{{}^{20}C_{10}}{2}$ **d.** none of these

52. The value of $\frac{{}^nC_0}{n}+\frac{{}^nC_1}{n+1}+\frac{{}^nC_2}{n+2}+\cdots+\frac{{}^nC_n}{2n}$ is equal to

a. $\int_0^1 x^{n-1}(1-x)^n dx$ b. $\int_1^2 x^n(x-1)^{n-1}dx$

c. $\int_1^2 x^{n-1}(1+x)^n dx$ d. $\int_0^1 (1-x)^n x^{n-1}dx$

53. If $C_0, C_1, C_2, \ldots, C_n$ are the binomial coefficients, then $2\times C_1+2^3\times C^3+2^5\times C_5+\cdots$ equals

a. $\frac{3^n+(-1)^n}{2}$ b. $\frac{3^n-(-1)^n}{2}$

c. $\frac{3^n+1}{2}$ d. $\frac{3^n-1}{2}$

54. The value of $\sum_{r=1}^{n}(-1)^{r+1}\frac{{}^nC_r}{r+1}$ is equal to

a. $-\frac{1}{n+1}$ b. $-\frac{1}{n}$

c. $\frac{1}{n+1}$ d. $\frac{n}{n+1}$

55. $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$ then $C_0C_2+C_1C_3+C_2C_4+\cdots+C_{n-2}C_n=$

a. $\frac{(2n)!}{(n!)^2}$ b. $\frac{(2n)!}{(n-1)!(n+1)!}$

c. $\frac{(2n)!}{(n-2)!(n+2)!}$ d. none of these

56. The sum of series ${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3+\cdots+{}^{20}C_{10}$ is

a. $\frac{1}{2}\,{}^{20}C_{10}$ b. 0

c. ${}^{20}C_{10}$ d. $-{}^{20}C_{10}$

57. ${}^{404}C_4-{}^4C_1{}^{303}C_4+{}^4C_2{}^{202}C_4-{}^4C_3{}^{101}C_4$ is equal to

a. $(401)^4$ b. $(101)^4$

c. 0 d. $(201)^4$

58. If $(3+x^{2008}+x^{2009})^{2010}=a_0+a_1x+a_2x^2+\cdots+a_nx^n$, then the value of $a_0-\frac{1}{2}a_1-\frac{1}{2}a_2+a_3-\frac{1}{2}a_4-\frac{1}{2}a_5+a_6-\cdots$ is

a. $3^{2010}$ b. 1

c. $2^{2010}$ d. none of these

59. The value of $\sum_{r=0}^{10} r\,{}^{10}C_r\,3^r(-2)^{10-r}$ is

a. 20 b. 10

c. 300 d. 30

60. The value of $\sum_{r=0}^{40} r\,{}^{40}C_r\,{}^{30}C_r$ is

a. $40\,{}^{69}C_{29}$ b. $40\,{}^{70}C_{30}$

c. ${}^{69}C_{29}$ d. ${}^{70}C_{30}$

61. The value of $\sum_{r=1}^{15}\frac{r2^r}{(r+2)!}$ is equal to

a. $\frac{(17)!-12^{16}}{(17)!}$ b. $\frac{(18)!-2^{17}}{(18)!}$

c. $\frac{(16)!-2^{15}}{(16)!}$ d. $\frac{(15)!-2^{14}}{(15)!}$

62. $(n+2)\,{}^nC_0 2^{n+1}-(n+1)\,{}^nC_1 2^n+n\,{}^nC_2 2^{n-1}-\cdots$ is equal to

a. 4 b. $4n$

c. $4(n+1)$ d. $2(n+2)$

63. The value of $\sum_{r=0}^{50}(-1)^r\frac{{}^{50}C_r}{r+2}$ is equal to

a. $\frac{1}{50\times 51}$ b. $\frac{1}{52\times 50}$

c. $\frac{1}{52\times 51}$ d. none of these

64. In the expansion of $[(1+x)/(1-x)]^2$, the coefficient of $x^n$ will be

a. $4n$ b. $4n-3$

c. $4n+1$ d. none of these

65. The sum of $1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2+\cdots\infty$ will be

a. $x^n$ b. $x^{-n}$

c. $\left(1-\frac{1}{x}\right)^n$ d. none of these

66. $\sum_{k=1}^{\infty}k\left(1-\frac{1}{n}\right)^{k-1}=$

a. $n(n-1)$ b. $n(n+1)$

c. $n^2$ d. $(n+1)^2$

67. The coefficient of $x^4$ in the expansion of $\{\sqrt{1+x^2}-x\}^{-1}$ in ascending powers of $x$, when $|x|<1$, is

a. 0 b. $\frac{1}{2}$

c. $-\frac{1}{2}$ d. $-\frac{1}{8}$

68. $1+\frac{1}{3}x+\frac{1\times 4}{3\times 6}x^2+\frac{1\times 4\times 7}{3\times 6\times 9}x^3+\cdots$ is equal to

a. $x$ b. $(1+x)^{1/3}$

c. $(1-x)^{1/3}$ d. $(1-x)^{-1/3}$

69. $1+\frac{1}{4}+\frac{1\times 3}{4\times 8}+\frac{1\times 3\times 5}{4\times 8\times 12}+\cdots=$

a. $\sqrt{2}$ b. $\frac{1}{\sqrt{2}}$

c. $\sqrt{3}$ d. $\frac{1}{\sqrt{3}}$

70. If $|x|<1$, then

$$1+n\left(\frac{2x}{1+x}\right)+\frac{n(n+1)}{2!}\left(\frac{2x}{1+x}\right)^2+\cdots$$

is equal to

a. $\left(\frac{2x}{1+x}\right)^n$ b. $\left(\frac{1+x}{2x}\right)^n$

c. $\left(\frac{1-x}{1+x}\right)^n$ d. $\left(\frac{1+x}{1-x}\right)^n$

**71.** The coefficient of $x^5$ in $(1+2x+3x^2+\cdots)^{-3/2}$ is $(|x|<1)$

a. 21 b. 25

c. 26 d. none of these

**72.** If $|x|<1$, then the coefficient of $x^n$ in expansion of $(1+x+x^2+x^3+\cdots)^2$ is

a. $n$ b. $n-1$

c. $n+2$ d. $n+1$

**73.** If $x$ is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is $(|x|<1)$

a. 5th term b. 8th term

c. 6th term d. 7th term

**74.** If $x$ is so small that $x^3$ and higher powers of $x$ may be neglected, then

$$\frac{(1+x)^{3/2}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}}$$

may be approximated as

a. $3x+\frac{3}{8}x^2$ b. $1-\frac{3}{8}x^2$

c. $\frac{x}{2}-\frac{3}{x}x^2$ d. $-\frac{3}{8}x^2$

**75.** If the expansion in powers of $x$ of the function $1/[(1-ax)(1-bx)]$ is $a_0+a_1x+a_2x^2+a_3x^3+\cdots$, then $a_n$ is

a. $\frac{b^n-a^n}{b-a}$ b. $\frac{a^n-b^n}{b-a}$

c. $\frac{a^{n+1}-b^{n+1}}{b-a}$ d. $\frac{b^{n+1}-a^{n+1}}{b-a}$

**76.** Value of $\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}({}^kC_r)$ is

a. $\frac{2}{3}$ b. $\frac{4}{3}$

c. 2 d. 1

**77.** The sum of rational term in $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}$ is equal to

a. 12632 b. 1260

c. 126 d. none of these

**78.** If $f(x)=1-x+x^2-x^3+\cdots-x^{15}+x^{16}-x^{17}$, then the coefficient of $x^2$ in $f(x-1)$ is

a. 826 b. 816

c. 822 d. none of these

**79.** If $p=(8+3\sqrt{7})^n$ and $f=p-[p]$, where $[\cdot]$ denotes the greatest integer function, then the value of $p(1-f)$ is equal to

a. 1 b. 2

c. $2^n$ d. $2^{2n}$

**80.** The value of $\sum_{r=0}^{10}(r)\,{}^{20}C_r$ is equal to

a. $20(2^{18}+{}^{19}C_{10})$ b. $10(2^{18}+{}^{19}C_{10})$

c. $20(2^{18}+{}^{19}C_{11})$ d. $10(2^{18}+{}^{19}C_{11})$

**81.** The last two digits of the number $(23)^{14}$ are

a. 01 b. 03

c. 09 d. none of these

**82.** Let $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$ and $\frac{f(x)}{1-x}=b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots$, then

a. $b_n+b_{n-1}=a_n$ b. $b_n-b_{n-1}=a_n$

c. $b_n/b_{n-1}=a_n$ d. none of these

**83.** If $(1-x^2)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{2n-r}$, then $a_r$ is equal to

a. ${}^nC_r$ b. ${}^nC_r3^r$

c. ${}^{2n}C_r$ d. ${}^nC_r2^r$

**84.** $[({}^nC_0+{}^nC_3+\cdots)-1/2({}^nC_1+{}^nC_2+{}^nC_4+{}^nC_5+\cdots)]^2 + 3/4({}^nC_1-{}^nC_2+{}^nC_4-{}^nC_5+\cdots)^2=$

a. 3 b. 4

c. 2 d. 1

**85.** If $\frac{x^2+x+1}{1-x}=a_0+a_1x+a_2x^2+\cdots$, then $\sum_{r=1}^{50}a_r$ is equal to

a. 148 b. 146

c. 149 d. none of these

**86.** '$p$' is a prime number and $n<p<2n$. If $N={}^{2n}C_n$, then

a. $p$ divides $N$ b. $p^2$ divides $N$

c. $p$ cannot divide $N$ d. none of these

**87.** $\sum_{r=0}^{300}a_rx^r=(1+x+x^2+x^3)^{100}$. If $a=\sum_{r=0}^{300}a_r$, then $\sum_{r=0}^{300}ra_r$ is equal to

a. $300a$ b. $100a$

c. $150a$ d. $75a$

**88.** The value of $\sum_{r=1}^{n+1}\left(\sum_{k=1}^{n}{}^kC_{r-1}\right)$ (where $r,k,n\in N$) is equal to

a. $2^{n+1}-2$ b. $2^{n+1}-1$

c. $2^{n+1}$ d. none of these

**89.** If $(1-x)^{-n}=a_0+a_1x+a_2x^2+\cdots+a_rx^r+\cdots$, then $a_0+a_1+a_2+\cdots+a_r$ is equal to

a. $\frac{n(n+1)(n+2)\cdots(n+r)}{r!}$ b. $\frac{(n+1)(n+2)\cdots(n+r)}{r!}$

c. $\frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}$ d. none of these

**90.** The value $\sum_{r=0}^{20}r(20-r)({}^{20}C_r)^2$ is equal to

a. $400\,{}^{39}C_{20}$ b. $400\,{}^{40}C_{19}$

c. $400\,{}^{39}C_{19}$ d. $400\,{}^{38}C_{20}$

## Multiple Correct Answers Type

Solutions on page 6.44

**Each question has four choices a, b, c and d, out of which *one or more* answers are correct.**

1. The value/values of $x$ in the expression $\left(x + x^{\log_{10} x}\right)^5$ if the third term in the expansion is 10,00,000 is/are
   **a.** 10 **b.** 100
   **c.** $10^{-5/2}$ **d.** $10^{-3/2}$
2. If the coefficients of $r^{th}$, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the expansion of $(1 + x)^{14}$ are in A.P., then $r$ is/are
   **a.** 5 **b.** 12
   **c.** 10 **d.** 9
3. In the expansion of $(x + a)^n$ if the sum of odd terms be $P$ and sum of even terms be $Q$, then
   **a.** $P^2 - Q^2 = (x^2 - a^2)^n$
   **b.** $4PQ = (x + a)^{2n} - (x - a)^{2n}$
   **c.** $2(P^2 + Q^2) = (x + a)^{2n} + (x - a)^{2n}$
   **d.** none of these
4. If $\left(4 + \sqrt{15}\right)^n = I + f$, where $n$ is an odd natural number, $I$ is an integer and $0 < f < 1$, then
   **a.** $I$ is an odd integer **b.** $I$ is an even integer
   **c.** $(I + f)(1 - f) = 1$ **d.** $1 - f = \left(4 - \sqrt{5}\right)^n$
5. The number of values of $r$ satisfying the equation ${}^{69}C_{3r-1} - {}^{69}C_{r^2} = {}^{69}C_{r^2-1} - {}^{69}C_{3r}$ is
   **a.** 1 **b.** 2
   **c.** 3 **d.** 7
6. If the $4^{th}$ term in the expansion of $(ax + 1/x)^n$ is 5/2, then
   **a.** $a = \frac{1}{2}$ **b.** $n = 8$
   **c.** $a = \frac{2}{3}$ **d.** $n = 6$
7. The sum of the coefficient in the expansion of $(1 + ax - 2x^2)^n$ is
   **a.** positive, when $a < 1$ and $n = 2k$, $k \in N$
   **b.** negative, when $a < 1$ and $n = 2k + 1$, $k \in N$
   **c.** positive, when $a > 1$ and $n \in N$
   **d.** zero, when $a = 1$
8. If
   $$f(m) = \sum_{i=0}^{m} \binom{30}{30-i}\binom{20}{m-i}$$
   where
   $\binom{p}{q} = {}^pC_q$, then
   **a.** maximum value of $f(m)$ is ${}^{50}C_{25}$
   **b.** $f(0) + f(1) + \cdots + f(50) = 2^{50}$
   **c.** $f(m)$ is always divisible by 50 $(1 \le m \le 49)$
   **d.** The value of $\sum_{m=0}^{50} (f(m))^2 = {}^{100}C_{50}$
9. If for $z$ as real or complex, $(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \cdots + C_{16} z^{32}$, then
   **a.** $C_0 - C_1 + C_2 - C_3 + \cdots + C_{16} = 1$
   **b.** $C_0 + C_3 + C_6 + C_9 + C_{12} + C_{15} = 3^7$
   **c.** $C_2 + C_5 + C_8 + C_{11} + C_{14} = 3^6$
   **d.** $C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16} = 3^7$
10. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$,
    **a.** there are exactly 730 rational terms
    **b.** there are exactly 5831 irrational terms
    **c.** the term which involves greatest binomial coefficients is irrational
    **d.** the term which involves greatest binomial coefficients is rational
11. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, then $C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots (-1)^{n-1}(C_0 + C_1 + \cdots + C_{n-1})$, where $n$ is even integer is
    **a.** a positive value
    **b.** a negative value
    **c.** divisible by $2^{n-1}$
    **d.** divisible by $2^n$
12. In the expansion of $\left(x^2 + 1 + \frac{1}{x^2}\right)^n$, $n \in N$,
    **a.** number of terms is $2n + 1$
    **b.** coefficient of constant term is $2^{n-1}$
    **c.** coefficient of $x^{2n-2}$ is $n$
    **d.** coefficient of $x^2$ in $n$
13. The value of ${}^nC_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \cdots + {}^{n+m-1}C_m$ is equal to
    **a.** ${}^{m+n}C_{n-1}$
    **b.** ${}^{m+n}C_{n-1}$
    **c.** ${}^mC_1 + {}^{m+1}C_2 + {}^{m+2}C_3 + \cdots + {}^{m+n-1}C_n$
    **d.** ${}^{m+n}C_{m-1}$
14. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, $n \in N$, then $C_0 - C_1 + C_2 - \cdots + (-1)^{m-1} C_{m-1}$ is equal to $(m < n)$
    **a.** $\frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}(-1)^{m-1}$
    **b.** ${}^{n-1}C_{m-1}(-1)^{m-1}$
    **c.** $\frac{(n-1)(n-2)\cdots(n-m)}{(m-1)!}(-1)^{m-1}$
    **d.** ${}^{n-1}C_{n-m}(-1)^{m-1}$
15. $10^{th}$ term of $\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{20}$
    **a.** an irrational number
    **b.** a rational number
    **c.** a positive integer
    **d.** a negative integer

16. For the expansion $(x \sin p + x^{-1} \cos p)^{10}$, $(p \in R)$,

a. the greatest value of the term independent of $x$ is $10!/2^5(5!)^2$

b. the least value of sum of coefficient is zero

c. the greatest value of sum of coefficient is 32

d. the least value of the term independent of $x$ occurs when $p = (2n+1)\frac{\pi}{4}, n \in Z$

17. Let $(1 + x^2)^2 (1 + x)^n = \sum_{k=0}^{n+4} a_k x^k$. If $a_1$, $a_2$ and $a_3$ are in arithmetic progression, then the possible value/values of $n$ is/are

a. 5 b. 4

c. 3 d. 2

18. The middle term in the expansion of $(x/2 + 2)^8$ is 1120; then $x \in R$ is equal to

a. –2 b. 3

c. –3 d. 2

19. For which of the following values of $x$, $5^{th}$ term is the numerically greatest term in the expansion of $(1 + x/3)^{10}$:

a. –2 b. 1.8

c. 2 d. –1.9

20. For natural numbers $m$, $n$, if $(1 - y)^m (1 + y)^n = 1 + a_1 y + a_2 y^2 + \cdots$, and $a_1 = a_2 = 10$, then

a. $m < n$ b. $m > n$

c. $m + n = 80$ d. $m - n = 20$

## Reasoning Type

*Solutions on page 6.47*

**Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.**

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** The value of $({}^{10}C_0) + ({}^{10}C_0 + {}^{10}C_1) + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2) + \cdots + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_9)$ is $10 \cdots 2^9$.
**Statement 2:** ${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n = n2^{n-1}$.

2. **Statement 1:** Greatest term in the expansion of $(1 + x)^{12}$, when $x = 11/10$ is $7^{th}$.
**Statement 2:** $7^{th}$ term in the expansion of $(1 + x)^{12}$ has the factor ${}^{12}C_6$ which is greatest value of ${}^{12}C_r$.

3. **Statement 1:** Remainder when $3456^{2222}$ is divided by 7 is 4.
**Statement 2:** Remainder when $5^{2222}$ is divided 7 is 4.

4. **Statement 1:** Three consecutive binomial coefficients are always in A.P.
**Statement 2:** Three consecutive binomial coefficients are not in H.P. or G.P.

5. **Statement 1:** If $n \in N$ and '$n$' is not a multiple of 3 and $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then the value of $\sum_{r=0}^{n} (-1)^r a_r\, {}^nC_r$ is zero.
**Statement 2:** The coefficient of $x^n$ in the expansion of $(1 - x^3)$ is zero, if $n = 3k + 1$ or $n = 3k + 2$.

6. **Statement 1:** $3^{2n+2} - 8n - 9$ is divisible by 64, $\forall\, n \in N$.
**Statement 2:** $(1 + x)^n - nx - 1$ is divisible by $x^2$, $\forall\, n \in N$.

7. **Statement 1:** The number of distinct terms in $(1 + x + x^2 + x^3 + x^4)^{1000}$ is 4001.
**Statement 2:** The number of distinct terms in the expansion $(a_1 + a_2 + \ldots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

8. **Statement 1:** The coefficient of $x^n$ in

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)^3 \text{ is } \frac{3^n}{n!}.$$

**Statement 2:** The coefficient of $x^n$ in $e^{3x}$ is $\frac{3^n}{n!}$.

9. **Statement 1:** In the expansion of $(1 + x)^{41}(1 - x + x^2)^{40}$, the coefficient of $x^{85}$ is zero.
**Statement 2:** In the expansion of $(1 + x)^{41}$ and $(1 - x + x^2)^{40}$, $x^{85}$ term does not occur.

10. **Statement 1:** The total number of dissimilar terms in the expansion of $(x_1 + x_2 + \cdots + x_n)^3$ is $\frac{n(n+1)(n+2)}{6}$.
**Statement 2:** The total number of dissimilar terms in the expansion of $(x_1 + x_2 + x_3)^n$ is $\frac{n(n+1)(n+2)}{6}$.

11. **Statement 1:** If $p$ is a prime number $(p \neq 2)$, then $\left[\left(2 + \sqrt{5}\right)^p\right] - 2^{p+1}$ is always divisible by $p$ (where $[\cdot]$ denotes the greatest integer function).
**Statement 2:** If $n$ is prime, then ${}^nC_1, {}^nC_2, {}^nC_3, \ldots, {}^nC_{n-1}$ must be divisible by $n$.

12. Let $n$ be a positive integer and $k$ be a whole number, $k \le 2n$.
**Statement 1:** The maximum value of ${}^{2n}C_k$ is ${}^{2n}C_n$.
**Statement 2:** $\frac{{}^{2n}C_{k+1}}{{}^{2n}C_k} < 1$, for $k = 0, 1, 2, \ldots, n-1$ and $\frac{{}^{2n}C_k}{{}^{2n}C_{k-1}} > 1$ for $k = n + 1, n + 2, \ldots, 2n$

13. **Statement 1:** The sum of coefficients in the expansion of $(3^{-x/4} + 3^{5x/4})^n$ is $2^n$.
**Statement 2:** The sum of coefficients in the expansion of $(x + y)^n$ is $2^n$ when we put $x = y = 1$.

14. **Statement 1:** ${}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \cdots + {}^nC_r = 0$, if $m + n < r$.
**Statement 2:** ${}^nC_r = 0$ if $n < r$.

15. **Statement 1:** $\sum\sum_{0 \le i < j \le n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$ is equal to $\frac{n^2}{2}a$, where $a = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$.

**Statement 2:** $\sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$.

## Linked Comprehension Type

*Solutions on page 6.48*

**Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.**

### For Problems 1–3

The sixth term in the expansion of $\left[\sqrt{\left\{2^{\log\left(10-3^x\right)}\right\}}+\sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^m$ is equal to 21, if it is known that the binomial coefficient of the 2nd, 3rd and 4th terms in the expansion represents, respectively, the first, third and fifth terms of an A.P. (the symbol log stands for logarithm to the base 10).

**1.** The value of $m$ is

**a.** 6 **b.** 7
**c.** 8 **d.** 9

**2.** The sum of possible values of $x$ is

**a.** 1 **b.** 3
**c.** 4 **d.** none of these

**3.** The minimum value of expression is

**a.** 64 **b.** 32
**c.** 128 **d.** none of these

### For Problems 4–6

The 2nd, 3rd and 4th terms in the expansion of $(x + a)^n$ are 240, 720 and 1080, respectively.

**4.** The value of $(x - a)^n$ can be

**a.** 64 **b.** –1
**c.** –32 **d.** none of these

**5.** The value of least term in the expansion is

**a.** 16 **b.** 160
**c.** 32 **d.** 81

**6.** The sum of odd-numbered terms is

**a.** 1664 **b.** 2376
**c.** 1562 **d.** 1486

### For Problems 7–9

If $(1 + x + x^2)^{20} = a_0 + a_1x + a_2x^2 \cdots + a_{40}x^{40}$, then answer the following questions.

**7.** The value of $a_0 + a_1 + a_2 + \cdots + a_{19}$ is

**a.** $\frac{1}{2}(9^{10} + a_{20})$ **b.** $\frac{1}{2}(9^{10} - a_{20})$
**c.** $\frac{9^{10}}{2}$ **d.** none of these

**8.** The value of $a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2$ is

**a.** $\frac{1}{2}a_{20}(1 - a_{20})$ **b.** $\frac{1}{2}a_{20}(1 + a_{20})$
**c.** $\frac{1}{2}a_{20}^2$ **d.** none of these

**9.** The value of $a_0 + 3a_1 + 5a_2 + \cdots + 81a_{40}$ is

**a.** $161 \times 3^{20}$ **b.** $41 \times 3^{40}$
**c.** $41 \times 3^{20}$ **d.** none of these

### For Problems 10–12

An equation $a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = 0$ has roots ${}^{99}C_0$, ${}^{99}C_1$, ${}^{99}C_2$, …, ${}^{99}C_{99}$.

**10.** The value of $a_{99}$ is equal to

**a.** $2^{98}$ **b.** $2^{99}$
**c.** $-2^{99}$ **d.** none of these

**11.** The value of $a_{98}$ is

**a.** $\frac{2^{198} - {}^{198}C_{99}}{2}$ **b.** $\frac{2^{198} + {}^{198}C_{99}}{2}$
**c.** $2^{99} - {}^{99}C_{49}$ **d.** none of these

**12.** The value of $({}^{99}C_0)^2 + ({}^{99}C_1)^2 + \cdots + ({}^{99}C_{99})^2$ is equal to

**a.** $2a_{98} - a_{99}^2$ **b.** $a_{99}^2 - a_{98}$
**c.** $a_{99}^2 - 2a_{98}$ **d.** none of these

### For Problems 13–15

Any complex number in polar form can be an expression in Euler's form as $\cos\theta + i\sin\theta = e^{i\theta}$. This form of the complex number is useful in finding the sum of series $\sum_{r=0}^{n} {}^nC_r(\cos\theta + i\sin\theta)^r$.

$$\sum_{r=0}^{n} {}^nC_r(\cos r\theta + i\sin r\theta) = \sum_{r=0}^{n} {}^nC_r e^{ir\theta}$$

$$= \sum_{r=0}^{n} {}^nC_r\left(e^{i\theta}\right)^r$$

$$= \left(1 + e^{i\theta}\right)^n$$

Also, we know that the sum of binomial series does not change if $r$ is replaced by $n - r$.

Using these facts, answer the following questions.

**13.** The value of $\sum_{r=0}^{100} {}^{100}C_r\ (\sin rx)$ is equal to

**a.** $2^{100}\cos^{100}\frac{x}{2}\sin 50x$ **b.** $2^{100}\sin(50x)\cos\frac{x}{2}$
**c.** $2^{101}\cos^{100}(50x)\sin\frac{x}{2}$ **d.** $2^{101}\sin^{100}(50x)\cos(50x)$

**14.** In triangle $ABC$, the value of $\sum_{r=0}^{50} {}^{50}C_r a^r b^{n-r}\cos(rB - (50 - r)A)$ is equal to (where $a$, $b$, $c$ are sides of triangle opposite to angle $A$, $B$, $C$, respectively, and $s$ is semi-perimeter)

**a.** $c^{49}$ **b.** $(a + b)^{50}$
**c.** $(2s - a - b)^{50}$ **d.** none of these

**15.** If

$$f(x) = \frac{\sum_{r=0}^{50} {}^{50}C_r \sin 2rx}{\sum_{r=0}^{50} {}^{50}C_r \cos 2rx}$$

then the value of $f(\pi/8)$ is equal to

**a.** 1 **b.** –1
**c.** irrational value **d.** none of these

**For Problems 16–18**

Let

$$P=\sum_{r=1}^{50}\frac{{}^{50+r}C_r(2r-1)}{{}^{50}C_r(50+r)},\ Q=\sum_{r=0}^{50}\left({}^{50}C_r\right)^2,\ R=\sum_{r=0}^{100}(-1)^r\left({}^{100}C_r\right)^2$$

**16.** The value of $P-Q$ is equal to

**a.** 1 **b.** $-1$
**c.** $2^{50}$ **d.** $2^{100}$

**17.** The value of $P-R$ is equal to

**a.** 1 **b.** $-1$
**c.** $2^{50}$ **d.** $2^{100}$

**18.** $Q+R$ is equal to

**a.** $2P+1$ **b.** $2P-1$
**c.** $2P+2$ **d.** $2P-2$

**For Problems 19–21**

$P$ is a set containing $n$ elements. A subset $A$ of $P$ is chosen and the set $P$ is reconstructed by replacing the elements of $A$. A subset $B$ of $P$ is chosen again.

**19.** The number of ways of choosing $A$ and $B$ such that $A$ and $B$ have no common elements is

**a.** $3^n$ **b.** $2^n$
**c.** $4^n$ **d.** none of these

**20.** The number of ways of choosing $A$ and $B$ such that $B$ contains just one element more than $A$ is

**a.** $2^n$ **b.** ${}^{2n}C_{n-1}$
**c.** ${}^{2n}C_n$ **d.** $(3^n)^2$

**21.** The number of ways of choosing $A$ and $B$ such that $B$ is a subset of $A$ is

**a.** ${}^{2n}C_n$ **b.** $4^n$
**c.** $3^n$ **d.** none of these

## Matrix-Match Type

*Solutions on page 6.51*

**Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:**

| | p | q | r | s |
|---|---|---|---|---|
| a | (p) | (q) | (r) | (s) |
| b | (p) | (q) | (r) | (s) |
| c | (p) | (q) | (r) | (s) |
| d | (p) | (q) | (r) | (s) |

**1.**

| Column I | Column II |
|---|---|
| **a.** If ${}^{(n+1)}C_4+{}^{(n+1)}C_3+{}^{(n+2)}C_3>{}^{(n+3)}C_3$, then possible value/values of $n$ is/are | **p.** 4 |
| **b.** The remainder when $(3053)^{456}-(2417)^{333}$ is divided by 9 is less than | **q.** 5 |
| **c.** The digit in the unit place of the number $183!+3^{183}$ is greater than | **r.** 6 |
| **d.** If sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2+\frac{1}{x}\right)^m$ is 46, then the index of the term that does not contain $x$ is greater than | **s.** 7 |

**2.**

| Column I | Column II |
|---|---|
| **a.** ${}^{32}C_0^2-{}^{32}C_1^2+{}^{32}C_2^2-\cdots+{}^{32}C_{32}^2=$ | **p.** ${}^{63}C_{32}$ |
| **b.** ${}^{32}C_0^2+{}^{32}C_1^2+{}^{32}C_2^2-\cdots+{}^{32}C_{32}^2=$ | **q.** ${}^{32}C_{16}$ |
| **c.** $\frac{1}{32}\left(1\times{}^{32}C_1^{\,2}+2\times{}^{32}C_2^{\,2}+\cdots+32\times{}^{32}C_{32}\right)=$ | **r.** 0 |
| **d.** ${}^{31}C_0^2-{}^{31}C_1^2+{}^{31}C_2^2-\cdots-{}^{31}C_{31}^2=$ | **s.** ${}^{64}C_{32}$ |

**3.**

| Column I | Column II |
|---|---|
| **a.** $\sum\sum_{i\neq j}{}^{10}C_i\,{}^{10}C_j$ | **p.** $\frac{2^{20}-{}^{20}C_{10}}{2}$ |
| **b.** $\sum\sum_{0\le i\le j\le n}{}^{10}C_i\,{}^{10}C_j$ | **q.** $2^{20}-{}^{20}C_{10}$ |
| **c.** $\sum\sum_{0\le i<j\le n}{}^{10}C_i\,{}^{10}C_j$ | **r.** $2^{20}$ |
| **d.** $\sum_{i=0}^{10}\sum_{j=0}^{10}{}^{10}C_i\,{}^{10}C_j$ | **s.** $\frac{2^{20}+{}^{20}C_{10}}{2}$ |

**4**

| Column I | Column II |
|---|---|
| **a.** The sum of binomial coefficients of terms containing power of $x$ more than $x^{20}$ in $(1+x)^{41}$ is divisible by | **p.** $2^{39}$ |
| **b.** The sum of binomial coefficients of rational terms in the expansion of $(1+\sqrt{2})^{42}$ is divisible by | **q.** $2^{40}$ |
| **c.** If $\left(x+\frac{1}{x}+x^2+\frac{1}{x^2}\right)^{21}=a_0x^{-42}+a_1x^{-41}+a_2x^{-40}+\cdots+a_{84}x^{42}$, then $a_0+a_2+\cdots+a_{84}$ is divisible by | **r.** $2^{41}$ |
| **d.** The sum of binomial coefficients of positive real terms in the expansion of $(1+ix)^{42}$ $(x>0)$ is divisible by | **s.** $2^{38}$ |

**5**

| Column I | Column II |
|---|---|
| **a.** The coefficient of two consecutive terms in the expansion of $(1+x)^n$ will be equal, then $n$ can be | **p.** 9 |
| **b.** If $15^n+23^n$ is divided by 19, then $n$ can be | **q.** 10 |
| **c.** ${}^{10}C_0\,{}^{20}C_{10}-{}^{10}C_1\,{}^{18}C_{10}+{}^{10}C_2\,{}^{16}C_{10}-\cdots$ is divisible by $2^n$, then $n$ can be | **r.** 11 |
| **d.** If the coefficients of $T_r$, $T_{r+1}$, $T_{r+2}$ terms of $(1+x)^{14}$ are in A.P., then $r$ is less than | **s.** 12 |

## Integer Type

Solutions on page 6.52

1. If the coefficients $x^7$ in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and coefficient of $x^{-7}$ in $\left(ax-\frac{1}{bx^2}\right)^{11}$ are equal then the value of $ab$ is.
2. If the coefficients of the $(2r+4)^{th}$, $(r+2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, then the value of $r$ is.
3. If the coefficients of the $r^{th}$, $(r+1)^{th}$, $(r-2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, then the largest value of $r$ is.
4. If the three consecutive coefficient in the expansion of $(1+x)^n$ are 28, 56 and 70, then the value of $n$ is.
5. Degree of the polynomial $[\sqrt{x^2+1}+\sqrt{x^2-1}]^8+\left[\frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}}\right]^8$ is.
6. Least positive integer just greater than $(1+0.00002)^{50000}$ is.
7. If the middle term in the expansion of $\left(\frac{x}{2}+2\right)^8$ is 1120; then the sum of possible real values of $x$ is.
8. If the second term of the expansion $\left[a^{1/13}+\frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$, then the value of $\frac{^nC_3}{^nC_2}$ is.
9. Number of values in set of values of '$r$' for which $^{23}C_r+2.{}^{23}C_{r+1}+{}^{23}C_{r+2}\geq{}^{25}C_{15}$ is.
10. The largest value of $x$ for which the fourth term in the expansion, $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336 is.
11. If the constant term in the binomial expansion of $\left(x^2-\frac{1}{x}\right)^n$, $n\in N$ is 15, then the value of $n$ is equal to.
12. Let $a=3^{\frac{1}{223}}+1$ and for all $n\geq 3$, let $f(n)={}^nC_0.a^{n-1}-{}^nC_1.a^{n-2}+{}^nC_2.a^{n-3}-\cdots+(-1)^{n-1}.{}^nC_{n-1}.a^0$. If the value of $f(2007)+f(2008)=3^k$ where $k\in N$, then the value of $k$ is.
13. Let $1+\sum_{r=1}^{10}(3^r.{}^{10}C_r+r.{}^{10}C_r)=2^{10}(\alpha\cdot 4^5+\beta)$ where $\alpha,\beta\in N$ and $f(x)=x^2-2x-k^2+1$. If $\alpha,\beta$ lies between the roots of $f(x)=0$, then find the smallest positive integral value of $k$.
14. Let $a$ and $b$ be the coefficient of $x^3$ in $(1+x+2x^2+3x^3)^4$ and $(1+x+2x^2+3x^3+4x^4)^4$ respectively. Then the value of $4a/b$ is
15. If $R$ is remainder when $6^{83}+8^{83}$ is divided by 49, then the value of $R/5$ is
16. Sum of last three digits of the number $N=7^{100}-3^{100}$ is
17. Given $(1-2x+5x^2-10x^3)(1+x)^n=1+a_1x+a_2x^2+\cdots$ and that $a_1^2=2a_2$ then the value of $n$ is.
18. The remainder, if $1+2+2^2+2^3+\cdots+2^{1999}$ is divided by 5 is.
19. The largest real value for $x$ such that $\sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{32}{3}$ is.
20. The value of $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\sum_{t=0}^{r-1}\frac{1}{5^n}.{}^nC_r.{}^rC_t.3^t\right)$ is equal to.

## Archives

Solutions on page 6.54

### Subjective Type

1. Given that $C_1+2C_2x+3C_3x^2+\cdots+2n\,C_{2n}x^{2n-1}=2n(1+x)^{2n-1}$, where $C_r=(2n)!/[r!(2n-r)!]$; $r=0,1,2,\ldots,2n$, then prove that $C_1^2-2C_2^2+3C_3^2-\cdots-2nC_{2n}^2=(-1)^n nC_n$. **(IIT-JEE, 1979)**
2. If $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$, then show that the sum of the products of the coefficient taken two at a time, represented by $\sum\sum_{0\leq i<j\leq n}{}^nC_i\,{}^nC_j$ is equal to $2^{2n-1}-\frac{(2n)!}{2(n!)^2}$. **(IIT-JEE, 1983)**
3. Given, $s_n=1+q+q^2+\cdots+q^n$, $S_n=1+\frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+\cdots+\left(\frac{q+1}{2}\right)^n$, $q\neq 1$

   Prove that ${}^{n+1}C_1+{}^{n+1}C_2\,s_1+{}^{n+1}C_3\,s_2+\cdots+{}^{n+1}C_{n+1}\,s_n=2^nS_n$. **(IIT-JEE, 1984)**
4. Prove that

   $$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\cdots\text{up to } m \text{ terms}\right]=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$ **(IIT-JEE, 1985)**
5. Let $R=(5\sqrt{5}+11)^{2n+1}$ and $f=R-[R]$ where [ ] denotes the greatest integer function, prove that $Rf=4^{2n+1}$. **(IIT-JEE, 1988)**
6. Prove that $C_0-2^2C_1+3^2C_2-4^2C_3+\cdots+(-1)^n(n+1)^2\times C_n=0$ where $C_r={}^nC_r$. **(IIT-JEE, 1989)**
7. If $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ and $a_k=1$ for all $k\geq n$, then show that $b_n={}^{2n+1}C_{n+1}$. **(IIT-JEE, 1992)**
8. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\,{}^{3n}C_{2r-1}=0$, where $k=3n/2$ and $n$ is an even integer. **(IIT-JEE, 1993)**
9. Let $n$ be a positive integer and $(1+x+x^2)^n=a_0+a_1x+\cdots+a_{2n}x^{2n}$. Show that $a_0^2-a_1^2+a_2^2+\cdots+a_{2n}^2=a_n$. **(IIT-JEE, 1994)**

**10.** Prove that

$$\frac{3!}{2(n+3)}=\sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$$ **(IIT-JEE, 1997)**

**11.** For any positive integer $m, n$ (with $n \geq m$), let

$$\binom{n}{m}={}^nC_m$$

Prove that

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\cdots+\binom{m}{m}=\binom{n+1}{m+1}.$$

Hence or otherwise, prove that

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\cdots+(n-m+1)\binom{m}{m}=\binom{n+2}{m+2}$$

**(IIT-JEE, 2000)**

**12.** Prove that $(25)^{n+1}-24n+5735$ is divisible by $(24)^2$ for all $n = 1, 2, \ldots$ **(IIT-JEE, 2002)**

**13.** If $n$ and $k$ are positive integers, show that

$$2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n}{2}\binom{n-2}{k-2}$$

$$-\cdots+(-1)^k\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}$$ where $\binom{n}{k}$ stands for ${}^nC_k$. **(IIT-JEE, 2003)**

## Objective Type

***Fill in the blanks***

1. The larger of $99^{50}+100^{50}$ and $101^{50}$ is ______. **(IIT-JEE, 1982)**
2. The sum of the coefficients of the polynomial $(1+x-3x^2)^{2163}$ is ______. **(IIT-JEE, 1982)**
3. If $(1+ax)^n = 1+8x+24x^2+\cdots$, then $a =$ ______ and $n =$ ______. **(IIT-JEE, 1983)**
4. The sum of the rational terms in the expansion of $(\sqrt{2}+3^{1/5})^{10}$ is ______. **(IIT-JEE, 1997)**

***Multiple choice questions with one correct answer***

**1.** Given positive integers $r > 1$, $n > 2$ and that the coefficient of $(3r)^{\text{th}}$ and $(r+2)^{\text{th}}$ terms in the binomial expansion of $(1+x)^{2n}$ are equal. Then **(IIT-JEE, 1983)**

**a.** $n = 2r$ **b.** $n = 2r+1$
**c.** $n = 3r$ **d.** none of these

**2.** The coefficient of $x^4$ in $(x/2-3/x^2)^{10}$ is **(IIT-JEE, 1983)**

**a.** $\frac{405}{256}$ **b.** $\frac{504}{259}$
**c.** $\frac{450}{263}$ **d.** none of these

**3.** If $C_r$ stands for ${}^nC_r$, then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2-2C_1^2+3C_2^2-\cdots+(-1)^n(n+1)C_n^2],$$

where $n$ is an even positive integer is equal to

**a.** 0 **b.** $(-1)^{n/2}(n+1)$
**c.** $(-1)^n(n+2)$ **d.** $(-1)^n n$ **(IIT-JEE, 1986)**

**4.** If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals

**a.** $(n-1)a_n$ **b.** $na_n$
**c.** $(1/2)na_n$ **d.** none of the above

**(IIT-JEE, 1988)**

**5.** The expression $\left(x+(x^3-1)^{\frac{1}{2}}\right)^5+\left(x-(x^3-1)^{\frac{1}{2}}\right)^5$ is a polynomial of degree

**a.** 5 **b.** 6
**c.** 7 **d.** 8 **(IIT-JEE, 1992)**

**6.** For $2 \leq r \leq n$,

$$\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}=$$

**a.** $\binom{n+1}{r-1}$ **b.** $2\binom{n+1}{r+1}$
**c.** $2\binom{n+2}{r}$ **d.** $\binom{n+2}{r}$ **(IIT-JEE, 2000)**

**7.** In the binomial expansion of $(a-b)^n$ $n \geq 5$, the sum of the $5^{\text{th}}$ and $6^{\text{th}}$ terms is zero. Then $a/b$ equals

**a.** $(n-5)/6$ **b.** $(n-4)/5$
**c.** $n/(n-4)$ **d.** $6/(n-5)$ **(IIT JEE, 2001)**

**8.** The sum

$$\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$$

where $\binom{p}{q}=0$

if $p < q$ is maximum when $m$ is

**a.** 5 **b.** 10
**c.** 15 **d.** 20 **(IIT-JEE, 2002)**

**9.** The coefficient of $t^{24}$ in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

**a.** ${}^{12}C_6+3$ **b.** ${}^{12}C_6+1$
**c.** ${}^{12}C_6$ **d.** ${}^{12}C_6+2$ **(IIT-JEE, 2003)**

**10.** If ${}^{n-1}C_r=(k^2-3)\,{}^nC_{r+1}$, then $k \in$

**a.** $(-\infty,-2]$ **b.** $[2,\infty)$
**c.** $[-\sqrt{3},\sqrt{3}]$ **d.** $(\sqrt{3},2]$ **(IIT-JEE, 2004)**

**11.** The value of

$$\binom{30}{0}\binom{30}{10}-\binom{30}{1}\binom{30}{11}+\binom{30}{2}\binom{30}{12}-\cdots+\binom{30}{20}\binom{30}{30},$$

where

$\binom{n}{r}={}^nC_r$ is

**a.** $\binom{30}{10}$ **b.** $\binom{30}{15}$
**c.** $\binom{60}{30}$ **d.** $\binom{31}{10}$

**(IIT-JEE, 2005)**

# ANSWERS AND SOLUTIONS

## Subjective Type

1. We have,

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)^2$$

$$=\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}\right)$$

Now, $x^n$ term is generated if terms of the two brackets are multiplied as shown by loops above.

Hence, the coefficient of $x^n$ is

$$1\times\frac{1}{n!}+\frac{1}{1!}\times\frac{1}{(n-1)!}+\frac{1}{2!}\times\frac{1}{(n-2)!}+\cdots+\frac{1}{n}$$

$$=\frac{1}{n!}\left(\frac{n!}{n!}+\frac{n!}{(n-1)!1!}+\frac{n!}{(n-2)!2!}+\cdots+\frac{n!}{n!}\right)$$

$$=\frac{1}{n!}({}^nC_0+{}^nC_1+{}^nC_2+\cdots+{}^nC_n)$$

$$=\frac{2^n}{n!}$$

2. Let $a-x-1=t$, so that

$$\sum_{r=0}^{n}a_r(1-t)^r=\sum_{r=0}^{n}b_rt^r$$

$\Rightarrow$ $b_n$ = Coefficient of $t^n$ in $\sum_{r=0}^{n}a_r(1-t)^r$

= Coefficient of $t^n$ in $a_n(1-t)^n$

$=(-1)^na_n$

3. $S={}^nC_0{}^nC_2+2{}^nC_1{}^nC_3+3{}^nC_2{}^nC_4+\cdots+(n-1)\,{}^nC_{n-2}{}^nC_n$

$={}^nC_0{}^nC_{n-2}+2\,{}^nC_1{}^nC_{n-3}+3\,{}^nC_2{}^nC_{n-4}+\cdots+(n-1){}^nC_{n-2}{}^nC_0$

$$=\sum_{r=1}^{n}r\,{}^nC_{r-1}{}^nC_{n-r-1}$$

$$=\sum_{r=1}^{n}((r-1)+1){}^nC_{r-1}{}^nC_{n-r-1}$$

$$=\sum_{r=1}^{n}\left[(r-1){}^nC_{r-1}{}^nC_{n-r-1}+{}^nC_{r-1}{}^nC_{n-r-1}\right]$$

$$=\sum_{r=1}^{n}\left[n^{n-1}C_{r-2}{}^nC_{n-r-1}+{}^nC_{r-1}{}^nC_{n-r-1}\right]$$

$$=n^{2n-1}C_{n-3}+{}^{2n}C_{n-2}$$

4. $$\sum_{r=0}^{n}{}^nC_r(-1)^r\left[i^r+i^{2r}+i^{3r}+i^{4r}\right]$$

$$=\sum_{r=0}^{n}{}^nC_r(-1)^r\left[i^r+(-1)^r+(-i)^r+1\right]$$

$$=\sum_{r=0}^{n}\left[{}^nC_ri^r+{}^nC_r+{}^nC_r(-i)^r+{}^nC_r(-1)^r\right]$$

$$=(1+i)^n+(1+1)^n+(1-i)^n+(1-1)^n$$

$$=(\sqrt2)^n\left(\frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right)^n+(\sqrt2)^n\left(\frac{1}{\sqrt2}-\frac{i}{\sqrt2}\right)^n+2^n$$

$$=(\sqrt2)^n\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^n+(\sqrt2)^n\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^n+2^n$$

$$=(\sqrt2)^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)+(\sqrt2)^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)+2^n$$

$$=2(\sqrt2)^n\cos\frac{n\pi}{4}+2^n$$

5. $b_n={}^nC_1+{}^nC_2\,a+{}^nC_3\,a^2+\cdots+{}^nC_na^{n-1}$

$$=\frac{1}{a}[a^nC_1+{}^nC_2a^2+{}^nC_3a^3+\cdots+{}^nC_na^n]$$

$$=\frac{1}{a}[{}^nC_0+{}^nC_1a+{}^nC_2\,a^2+\cdots+{}^nC_na^n-1]$$

$$=\frac{1}{a}((1+a)^n-1)$$

$$b_{2006}-b_{2005}=\frac{1}{a}[(1+a)^{2006}-1)]-\frac{1}{a}[(1+a)^{2005}-1)]$$

$$=\frac{1}{a}[(1+a)^{2006}-1-(1+a)^{2005}+1]$$

$$=\frac{1}{a}(1+a)^{2005}(1+a-1)$$

$$=(1+a)^{2005}$$

$$=\left(1+4^{\frac{1}{401}}-1\right)^{2005}$$

$$=4^{\frac{2005}{401}}$$

6. Let $A=1-2n+\dfrac{2n(2n-1)}{2!}-\cdots+(-1)^{n-1}\dfrac{2n(2n-1)\cdots(n+2)}{(n-1)!}$

$$={}^{2n}C_0-{}^{2n}C_1+{}^{2n}C_2-\cdots+(-1)^{n-1}\,{}^{2n}C_{n-1}$$

$$=\frac{1}{2}\left[2^{2n}C_0-2^{2n}C_1+2^{2n}C_2-\cdots+(-1)^n2^{2n}C_{n-1}\right]$$

$$=\frac{1}{2}\left[\left({}^{2n}C_0+{}^{2n}C_{2n}\right)-\left({}^{2n}C_1+{}^{2n}C_{2n-1}\right)+\left({}^{2n}C_2+{}^{2n}C_{2n-2}\right)-\cdots+\left((-1)^{n-1}\,{}^{2n}C_{n-1}+(-1)^{n+1}\,{}^{2n}C_{n+1}\right)\right]$$

$$=\frac{1}{2}\left[{}^{2n}C_0-{}^{2n}C_1+{}^{2n}C_2-{}^{2n}C_3+\cdots(-1)^n\,{}^{2n}C_n+\cdots+{}^{2n}C_{2n}-(-1)^n\,{}^{2n}C_n\right]$$

$$=\frac{1}{2}\left[(1-1)^{2n}-(-1)^n\,{}^{2n}C_n\right]$$

$$=\frac{1}{2}\left[(-1)^{n+1}\,{}^{2n}C_n\right]$$

$$=(-1)^{n+1}\frac{(2n)!}{2n!n!}$$

7. Let $f(x)=\dfrac{n!}{x(x+1)(x+2)\cdots(x+n)}$

$$=\frac{A_0}{x}+\frac{A_1}{x+1}+\frac{A_2}{x+2}+\cdots+\frac{A_n}{x+n}\text{ (by partial fractions)}$$

Then,

$$A_0=[x\,f(x)]_{x=0}=\frac{n!}{1\times2\times3\times\cdots\times n}=1={}^nC_0$$

$$A_1=[(x+1)f(x)]_{x=-1}$$

$$=\frac{n!}{(-1)\{1\times2\times\cdots\times(n-1)\}}$$

$$=\frac{-(n!)}{(n-1)!}=-{}^nC_1$$

$$A_2=[(x+2)f(x)]_{x=-2}$$

$$=\frac{n!}{(-2)\times(-1)\times1\times2\times\cdots\times(n-2)}$$

$$=\frac{n!}{2!(n-2)!}={}^nC_2 \text{ and so on}$$

Thus,

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)}$$

$$=\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\cdots+(-1)^n\frac{{}^nC_n}{x+n} \quad (1)$$

**8.** To find $S={}^{100}C_0{}^{100}C_2+{}^{100}C_2{}^{100}C_4+{}^{100}C_4{}^{100}C_6+\cdots+{}^{100}C_{98}\,{}^{100}C_{100}$

Consider,

$${}^{100}C_0{}^{100}C_2+{}^{100}C_1{}^{100}C_3+{}^{100}C_2{}^{100}C_4+{}^{100}C_3{}^{100}C_5+\cdots+{}^{100}C_{98}{}^{100}C_{100}$$

$$={}^{100}C_0{}^{100}C_{98}+{}^{100}C_1{}^{100}C_{97}+{}^{100}C_2{}^{100}C_{96}+{}^{100}C_3{}^{100}C_{95}+\cdots+{}^{100}C_{98}{}^{100}C_0$$

= Coefficient of $x^{98}$ in $(1+x)^{100}(1+x)^{100}$

= Coefficient of $x^{98}$ in $(1+x)^{200}$

$$={}^{200}C_{98} \quad (1)$$

Also,

$${}^{100}C_0{}^{100}C_2-{}^{100}C_1{}^{100}C_3+{}^{100}C_2{}^{100}C_4-{}^{100}C_3{}^{100}C_5+\cdots+{}^{100}C_{98}{}^{100}C_{100}$$

= Coefficient of $x^{98}$ in $(1+x)^{100}(1-x)^{100}$

= Coefficient of $x^{98}$ in $(1-x^2)^{100}$

$$=-{}^{100}C_{49} \quad (2)$$

Adding (1) and (2), we have

$$2({}^{100}C_0{}^{100}C_2+{}^{100}C_2{}^{100}C_4+{}^{100}C_4{}^{100}C_6+\cdots+{}^{100}C_{98}{}^{100}C_{100})=\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$$

$$\Rightarrow\ {}^{100}C_0{}^{100}C_2+{}^{100}C_2{}^{100}C_4+{}^{100}C_4{}^{100}C_6+\cdots+{}^{100}C_{98}{}^{100}C_{100}=\frac{1}{2}\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$$

**9.** $$\sum_{r=1}^{m-1}\frac{2r^2-r(m-2)+1}{(m-r)\,{}^mC_r}$$

$$=\sum_{r=1}^{m-1}\frac{(r+1)^2-r(m-r)}{(m-r)\,{}^mC_2}$$

$$=\sum_{r=1}^{m-1}\left(\frac{(r+1)^2}{(m-r)\,{}^mC_r}-\frac{r}{{}^mC_r}\right)$$

$$=\sum_{r=1}^{m-1}\left(\frac{(r+1)}{(m-r)\left(\frac{{}^mC_r}{r+1}\right)}-\frac{r}{{}^mC_r}\right)$$

$$=\sum_{r=1}^{m-1}\left(\frac{(r+1)}{{}^mC_{r+1}}-\frac{r}{{}^mC_r}\right)$$

$$=\sum_{r=1}^{m-1}(t_{r+1}-t_r),\text{ where } t_r=\frac{r}{{}^mC_r}$$

$$=t_m-t_1$$

$$=\frac{m}{{}^mC_m}-\frac{1}{{}^mC_1}$$

$$=m-\frac{1}{m}$$

**10.** The given series is an A.G.P. Let us first find its sum.

Writing $t$ for $1+x$, let

$$S=t^{1000}+2xt^{999}+3x^2t^{998}+\cdots+1001x^{1000}$$

$$\therefore\ (x/t)S=xt^{999}+2x^2t^{998}+\cdots+1001x^{1001}/t$$

Subtracting, we get

$$S(1-x/t)=t^{1000}+xt^{999}+x^2t^{998}+\cdots+x^{1000}-1001x^{1001}/t$$

$$=\frac{t^{1000}[1-(x/t)^{1001}]}{1-x/t}-1001x^{1001}/t$$

or $S=(1+x)^{1002}-x^{1001}(1+x)-1001x^{1001}$

(putting $t=1+x$ and simplifying)

Therefore, coefficient of $x^{50}$ in the expansion is equal to the coefficient of $x^{50}$ in the expansion of $(1+x)^{1002}$, which is equal to ${}^{1002}C_{50}$.

**11.** Let $S={}^nC_1-\left(1+\frac{1}{2}\right){}^nC_2+\left(1+\frac{1}{2}+\frac{1}{3}\right){}^nC_3+\cdots+(-1)^{n-1}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right){}^nC_n$.

The $r^{th}$ term of the series is

$$T(r)=(-1)^{r-1}\,{}^nC_r\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r}\right)$$

Let us consider a series whose general term is

$$T_1(r)=(-1)^{r-1}C_r(1+x+x^2+\cdots+x^{r-1})=(-1)^{r-1}C_r\left(\frac{1-x^r}{1-x}\right)$$

$$=\frac{(-1)^{r-1}C_r}{1-x}+\frac{(-1)^r x^r C_r}{1-x}$$

$$\Rightarrow\ \sum_{r=1}^{n}T_1(r)=\frac{1}{(x-1)}\sum_{r=1}^{n}(-1)^rC_r+\frac{1}{(1-x)}\sum_{r=1}^{n}(-1)^rC_rx^r$$

$$\Rightarrow\ \sum_{r=1}^{n}T_1(r)=\frac{1}{(x-1)}(0-1)+\frac{1}{(1-x)}((1-x)^n-1)=(1-x)^{n-1}=L\text{ (say)}$$

Clearly, $S=\int_0^1(1-x)^{n-1}dx=\frac{1}{n}$

**12.** Consider the series $S={}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\cdots+n\,{}^nC_n$

For the series $T_r=r\,{}^nC_r=n\,{}^{n-1}C_{r-1}$

$$\therefore\ S=\sum_{r=1}^{n}T_r=\sum_{r=1}^{n}n\,{}^{n-1}C_{r-1}=n\cdot2^{n-1} \quad (1)$$

Now, we have

A.M. ≥ G.M.

$$\Rightarrow\ \frac{{}^nC_1+2\,{}^nC_2+3\,{}^nC_3+\cdots+n\,{}^nC_n}{\left(\frac{n(n+1)}{2}\right)}\ge\left[({}^nC_1)({}^nC_2)^2\times({}^nC_3)^3\cdots({}^nC_n)^n\right]^{\frac{1}{\frac{n(n+1)}{2}}}$$

$$\Rightarrow \frac{n \times 2^{n-1}}{\left(\frac{n(n+1)}{2}\right)} \geq \left[({}^nC_1)({}^nC_2)^2({}^nC_3)^2 \cdots ({}^nC_n)^n\right]^{\frac{2}{n(n+1)}}$$

$$\Rightarrow ({}^nC_1)({}^nC_2)^2({}^nC_3)^3 \cdots ({}^nC_n)^n \leq \left(\frac{2^n}{n+1}\right)^{\frac{n(n+1)}{2}}$$

or $$({}^nC_1)({}^nC_2)^2({}^nC_3)^3 \cdots ({}^nC_n)^n \leq \left(\frac{2^n}{n+1}\right)^{{}^{n+1}C_2}$$

**13.** $$\frac{1}{m!}{}^nC_0 + \frac{n}{(m+1)!}{}^nC_1 + \frac{n(n-1)}{(m+2)!}{}^nC_2 + \cdots + \frac{n(n-1)\times\cdots\times 2\times 1}{(m+n)!}{}^nC_n$$

$$= \frac{n!}{(m+n)!}\left(\frac{(m+n)!}{m!n!}{}^nC_0 + \frac{(m+n)!n}{(m+1)!n!}{}^nC_1 + \frac{(m+n)!n(n-1)}{(m+2)!n!}{}^nC_2 + \cdots + \frac{(m+n)!}{n!}\frac{n(n-1)\times\cdots\times 2\times 1}{(m+n)!}{}^nC_n\right)$$

$$= \frac{n!}{(m+n)!}\left({}^{m+n}C_n\,{}^nC_0 + \frac{(m+n)!}{(m+1)!(n-1)!}{}^nC_1 + \frac{(m+n)!}{(m+2)!(n-2)!} \times {}^nC_2 + \cdots + \frac{(m+n)!}{1}\frac{1}{(m+n)!}{}^nC_n\right)$$

$$= \frac{n!}{(m+n)!}\left({}^{m+n}C_n\,{}^nC_0 + {}^{m+n}C_{n-1}\,{}^nC_1 + {}^{m+n}C_{n-2}\,{}^nC_2 + \cdots + {}^{m+n}C_0\,{}^nC_n\right)$$

$$= \frac{n!}{(m+n)!}\text{[coefficient of } x^n \text{ in } (1+x)^{m+n}(1+x)^n]$$

$$= \frac{n!}{(m+n)!}\text{[coefficient of } x^n \text{ in } (1+x)^{m+2n}]$$

$$= \frac{n!}{(m+n)!}{}^{m+2n}C_n$$

$$= \frac{n!}{(m+n)!}\frac{(m+2n)!}{(m+n)!n!}$$

$$= \frac{(m+2n)!}{(m+n)!(m+n)!}$$

$$= \frac{(m+n+1)(m+n+2)(m+n+3)\cdots(m+2n)}{(m+n)!}$$

**14.** $${}^nC_0 - \frac{{}^nC_2}{(2+\sqrt{3})^2} + \frac{{}^nC_4}{(2+\sqrt{3})^4} - \frac{{}^nC_6}{(2+\sqrt{3})^6} + \cdots = (-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n$$

$$= \text{Real part of } \left(1 + \frac{i}{2+\sqrt{3}}\right)^n$$

$$= \text{Real part of } \left(1 + i(2-\sqrt{3})\right)^n$$

$$= \text{Real part of } \left(1 + i\tan\frac{\pi}{12}\right)^n$$

$$= \text{Real part of } \frac{\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^n}{\cos^n\frac{\pi}{12}}$$

$$= \text{Real part of } \frac{\left(\cos\frac{n\pi}{12} + i\sin\frac{n\pi}{12}\right)}{\cos^n\frac{\pi}{12}}$$

$$= \frac{\cos\frac{n\pi}{12}}{\cos^n\frac{\pi}{12}}$$

$$= \frac{\cos m\pi}{\cos^n\frac{\pi}{12}}$$

$$= (-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n \quad \left[\because \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$

**15.** The given expansion can be written as

$$\{\underbrace{(1+x)(1+x)(1+x)\cdots(1+x)}_{n\text{ factors}}\}\{\underbrace{(1+y)(1+y)(1+y)\cdots(1+y)}_{n\text{ factors}}\}\{\underbrace{(1+z)(1+z)(1+z)\cdots(1+z)}_{n\text{-factors}}\}$$

There are $3n$ factors in this product. To get a term of degree $r$, we choose $r$ brackets out of these $3n$ brackets and then multiply second terms in each bracket. There are ${}^{3n}C_r$ such terms each having the coefficient 1. Hence, the sum of the coefficients is ${}^{3n}C_r$.

## Objective Type

**1. c.** Coefficient of $T_5$ is ${}^nC_4$, that of $T_6$ is ${}^nC_5$ and that of $T_7$ is ${}^nC_6$. According to the condition, $2\,{}^nC_5 = {}^nC_4 + {}^nC_6$. Hence,

$$2\left[\frac{n!}{(n-5)!5!}\right] = \left[\frac{n!}{(n-4)!4!} + \frac{n!}{(n-6)!6!}\right]$$

$$\Rightarrow 2\left[\frac{1}{(n-5)5}\right] = \left[\frac{1}{(n-4)(n-5)} + \frac{1}{6\times 5}\right]$$

After solving, we get $n = 7$ or $14$.

**2. c.** $T_{r+1} = {}^{2n}C_r x^{2n-r}\left(\frac{1}{x^2}\right)^r = {}^{2n}C_r x^{2n-3r}$

This contains $x^m$. If $2n - 3r = m$, then

$$r = \frac{2n-m}{3}$$

$\Rightarrow$ Coefficient of $x^m = {}^{2n}C_r$, $r = \frac{2n-m}{3}$

$$= \frac{2n!}{(2n-r)!r!} = \frac{2n!}{\left(2n - \frac{2n-m}{3}\right)!\left(\frac{2n-m}{3}\right)!}$$

$$= \frac{2n!}{\left(\frac{4n+m}{3}\right)!\left(\frac{2n-m}{3}\right)!}$$

**3. c.** $(1+x)^{21} + (1+x)^{22} + \cdots + (1+x)^{30}$

$$= (1+x)^{21}\left[\frac{(1+x)^{10}-1}{(1+x)-1}\right] = \frac{1}{x}\left[(1+x)^{31} - (1+x)^{21}\right]$$

$\Rightarrow$ Coefficient of $x^5$ in the given expression

$$= \text{Coefficient of } x^5 \text{ in } \left\{\frac{1}{x}\left[(1+x)^{31} - (1+x)^{21}\right]\right\}$$

$$= \text{Coefficient of } x^6 \text{ in } [(1+x)^{31} - (1+x)^{21}]$$

$= {}^{31}C_6 - {}^{21}C_6$

**4. b.** c.e. of $x^{-1}$ in $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$= $ c.e. of $x^{-1}$ in $\frac{(1+x)^{2n}}{x^n}$

$= $ c.e. of $x^{n-1}$ in $(1+x)^{2n}$

$= {}^{2n}C_{n-1}$

$= \frac{(2n)!}{(n-1)!(n+1)!}$

**5. d.** $(1+3x+2x^2)^6 = [1+x(3+2x)]^6$

$= 1 + {}^6C_1 x(3+2x) + {}^6C_2 x^2(3+2x)^2 + {}^6C_3 x^3(3+2x)^3 + {}^6C_4 x^4(3+2x)^4 + {}^6C_5 x^5(3+2x)^5 + {}^6C_6 x^6(3+2x)^6$

We get $x^{11}$ only from ${}^6C_6 x^6(3+2x)^6$. Hence, coefficient of $x^{11}$ is ${}^6C_5 \times 3 \times 2^5 = 576$.

**6. b.** We have $T_{r+1} = {}^{29}C_r 3^{29-r}(7x)^r = ({}^{29}C_r \times 3^{29-r} \times 7^r)\, x^r$

Coefficient of $(r+1)^{\text{th}}$ term is ${}^{29}C_r \times 3^{29-r} \times 7^r$

and coefficient of $r^{\text{th}}$ term is ${}^{29}C_{r-1} \times 3^{30-r} \times 7^{r-1}$

From given condition,

${}^{29}C_r \times 3^{29-r} \times 7^r = {}^{29}C_{r-1} \times 3^{30-r} \times 7^{r-1}$

$\Rightarrow \frac{{}^{29}C_r}{{}^{29}C_{r-1}} = \frac{3}{7} \Rightarrow \frac{30-r}{r} = \frac{3}{7} \Rightarrow r = 21$

**7. c.** Let $(r+1)^{\text{th}}$, $(r+2)^{\text{th}}$ and $(r+3)^{\text{th}}$ be three consecutive terms. Then,

${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1:7:42$

Now,

$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{1}{7} \Rightarrow \frac{r+1}{n-r} = \frac{1}{7} \Rightarrow n - 8r = 7$ (i)

$\frac{{}^nC_{r+1}}{{}^nC_{r+2}} = \frac{7}{42} \Rightarrow \frac{r+2}{n-r-1} = \frac{1}{6} \Rightarrow n - 7r = 13$ (ii)

Solving (i) and (ii), we get $n = 55$.

**8. c.** $(1+2x+x^2)^n = \sum_{r=0}^{2n} a_r x^r \Rightarrow [(1+x)^2]^n = \sum_{r=0}^{2n} a_r x^r$

$\Rightarrow (1+x)^{2n} = \sum_{r=0}^{2n} a_r x^r$

$\Rightarrow \sum_{r=0}^{2n} {}^{2n}C_r x^r = \sum_{r=0}^{2n} a_r x^r$

$\Rightarrow a_r = {}^{2n}C_r$

**9. b.** Here $a = {}^nC_r$, $b = {}^nC_{r+1}$ and $c = {}^nC_{r+2}$.

Put $n = 2$, $r = 0$, then option (b) holds the condition, i.e.,

$n = \frac{2ac + ab + bc}{b^2 - ac}$

**10. c.** Middle term of $(1+\alpha x)^4$ is $T_3$.

Its coefficient is ${}^4C_2 (\alpha)^2 = 6\alpha^2$.

Middle term of $(1-\alpha x)^6$ is $T_4$.

Its coefficient is ${}^6C_3(-\alpha)^3 = -20\alpha^3$.

According to question,

$6\alpha^2 = -20\alpha^3$

$\Rightarrow 3\alpha^2 + 10\alpha^3 = 0$

$\Rightarrow \alpha^2(3+10\alpha) = 0$

$\Rightarrow \alpha = -\frac{3}{10}$

**11. b.** $(1-x)(1-x)^n$

$= (1-x)[1 + n(-x) + \cdots + {}^nC_{n-1}(-x)^{n-1} + {}^nC_n(-x)^n]$

Therefore, coefficient of $x^n$ is

${}^nC_n(-1)^n - {}^nC_{n-1}(-1)^{n-1} = (-1)^n + (-1)^n n$

$= (-1)^n(1+n)$

**12. d.** Here, the coefficients of $T_r$, $T_{r+1}$ and $T_{r+2}$ in $(1+y)^m$ are in A.P.

$\Rightarrow$ ${}^mC_{r-1}$ ${}^mC_r$ and ${}^mC_{r+1}$ are in A.P.

$\Rightarrow 2\,{}^mC_r = {}^mC_{r-1} + {}^mC_{r+1}$

$\Rightarrow 2\frac{m!}{r!(m-r)!} = \frac{m!}{(r-1)!(m-r+1)!} + \frac{m!}{(r+1)!(m-r-1)!}$

$\Rightarrow \frac{2}{r(m-r)} = \frac{1}{(m-r+1)(m-r)} + \frac{1}{(r+1)r}$

$\Rightarrow m^2 - m(4r+1) + 4r^2 - 2 = 0$

**13. c.** For $\left(ax^2 + \left(\frac{1}{bx}\right)\right)^{11}$, $T_{r+1} = {}^{11}C_r (ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$

$= {}^{11}C_r a^{11-r}\frac{1}{b^r}x^{22-3r}$

For $x^7$,

$22 - 3r = 7$

$\Rightarrow 3r = 15$

$\Rightarrow r = 5$

$\Rightarrow T_6 = {}^{11}C_5 a^6 \frac{1}{b^5} x^7$

$\Rightarrow$ Coefficient of $x^7$ is ${}^{11}C_5 \frac{a^6}{b^5}$

Similarly, coefficient of $x^{-7}$ in $\left(ax - \left(\frac{1}{bx^2}\right)\right)^{11}$ is ${}^{11}C_6 \frac{a^5}{b^6}$.

Given that

${}^{11}C_5 \frac{a^6}{b^5} = {}^{11}C_6 \frac{a^5}{b_6}$

$\Rightarrow a = \frac{1}{b}$

$\Rightarrow ab = 1$

**14. c.** $a^{10}b^{10}c^{10}d^{10}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^{10}$

Therefore the required coefficient is equal to the coefficient of $a^{-2}b^{-6}c^{-1}d^{-1}$ in $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^{10}$, which is given by

$\frac{10!}{2!6!1!1!} = \frac{10\times 9\times 8\times 7}{2} = 2520$

**15. d.** $(x-2)^5(x+1)^5$

$= [{}^5C_0 x^5 - {}^5C_1 x^4 \times 2 + \cdots][{}^5C_0 + {}^5C_1 x + \cdots]$

$\Rightarrow$ Coefficient of $x^5$

$= {}^5C_0{}^5C_5 - {}^5C_1 \times 2 \times {}^5C_4 + {}^5C_2 \times 2^2 \times {}^5C_3 - {}^5C_3 \times 2^3 \times {}^5C_2 + {}^5C_4 \times 2^4 \times {}^5C_1 - {}^5C_5 \times 2^5 \times {}^5C_0$

$= 1 - 5\times 5\times 2 + 10\times 10\times 4 - 10\times 10\times 8 + 5\times 5\times 16 - 32$

$= -81$

**16. d.** The general term in the expansion of $(1-x+y)^{20}$ is

$\frac{20!}{r!s!t!}1^r(-x)^s(y)^t$, where $r+s+t = 20$

For $x^2y^3$, we have the term

$$\frac{20!}{15!2!3!}1^{15}(-x)^2(y)^3$$

Hence, the coefficient of $x^2y^3$ is

$$\frac{20!}{15!2!3!}$$

**17. b.** $t_{r+1} = {}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r = {}^{10}C_r x^{5-5r/2}(-k)^r$

For this to be independent of $x$, $r$ must be 2, so that

${}^{10}C_2 k^2 = 405 \Rightarrow k = \pm 3$

**18. a.** We rewrite the given expression as $[1 + x^2(1-x)]^8$ and expand by using the binomial theorem. We have,

$[1 + x^2(1-x)]^8$

$= {}^8C_0 + {}^8C_1 x^2(1-x) + {}^8C_2 x^4(1-x)^2 + {}^8C_3 x^6(1-x)^3 + {}^8C_4 x^8(1-x)^4 + {}^8C_5 x^{10}(1-x)^5 + \cdots$

The two terms which contain $x^{10}$ are ${}^8C_4 x^8(1-x)^8$ and ${}^8C_5 x^{10}(1-x)^5$.

Thus, the coefficient of $x^{10}$ in the given expression is given by ${}^8C_4$ [coefficient of $x^2$ in the expansion of $(1-x)^4$] + ${}^8C_5$

$= {}^8C_4(6) + {}^8C_5 = \frac{8!}{4!4!}(6) + \frac{8!}{3!5!}$

$= (70)(6) + 56 = 476$

**19. c.** We have,

$(1+x)^{101}(1-x+x^2)^{100} = (1+x)((1+x)(1-x+x^2))^{100}$

$= (1+x)(1+x^3)^{100} = (1+x)\{C_0 + C_1x^3 + C_2x^6 + \cdots + C_{100}x^{300}\}$

$= (1+x)\sum_{r=0}^{n} {}^nC_r x^{3r} = \sum_{r=0}^{n} {}^nC_r x^{3r} + \sum_{r=0}^{n} {}^nC_r x^{3r+1}$

Hence, there will be no term containing $3r + 2$.

**20. b.** $(1 + x^3 - x^6)^{30}$

$= \{1 + x^3(1-x^3)\}^{30}$

$= {}^{30}C_0 + {}^{30}C_1 x^3(1-x^3) + {}^{30}C_2 x^6(1-x^3)^2 + \cdots$

Obviously, each term will contain $x^{3m}$, $m \in N$. But 28 is not divisible by 3. Therefore, there will be no term containing $x^{28}$.

**21. b.** $\left(1 + \sqrt{a} + \frac{1}{\sqrt{a}-1}\right)^{-30}$

$= \left(\frac{a}{\sqrt{a}-1}\right)^{-30}$

$= \left(\frac{\sqrt{a}-1}{a}\right)^{30}$

$= \frac{1}{a^{30}}(1-\sqrt{a})^{30}$

$= \frac{1}{a^{30}}\{{}^{30}C_0 - {}^{30}C_1\sqrt{a} + \cdots + {}^{30}C_{30}(\sqrt{a})^{30}\}$

There is no term independent of $a$.

**22. c.** The given sigma is the expansion of

$[(x-3)+2]^{100} = (x-1)^{100} = (1-x)^{100}$

Therefore, $x^{53}$ will occur in $T_{54}$.

$T_{54} = {}^{100}C_{53}(-x)^{53}$

Therefore, the coefficient is $-{}^{100}C_{53}$.

**23. a.** We have,

$\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}$

$= \frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x^{1/2}(x^{1/2}-1)}$

$= \frac{(x^{1/3}+1)(x^{2/3} - x^{1/3} + 1)}{x^{2/3} - x^{1/3} + 1} - \frac{x^{1/2}+1}{x^{1/2}}$

$= x^{1/3} + 1 - 1 - x^{-1/2} = x^{1/3} - x^{-1/2}$

$\therefore \left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10} = (x^{1/3} - x^{-1/2})^{10}$

Let $T_{r+1}$ be the general term in $(x^{1/3} - x^{-1/2})^{10}$. Then,

$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r}(-1)^r(x^{-1/2})^r$

For this term to be independent of $x$, we must have

$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0 \Rightarrow r = 4$

So, the required coefficient is ${}^{10}C_4(-1)^4 = 210$.

**24. d.** $(1 + x + x^3 + x^4)^{10} = (1+x)^{10}(1+x^3)^{10}$

$= (1 + {}^{10}C_1x + {}^{10}C_2x^2 + {}^{10}C_3x^3 + {}^{10}C_4x^4 \cdots)(1 + {}^{10}C_1x^3 + {}^{10}C_2x^6 + \cdots)$

Therefore, coefficient of $x^4$ is ${}^{10}C_1{}^{10}C_1 + {}^{10}C_4 = 310$.

**25. a.** $(1.0002)^{3000} = (1 + 0.0002)^{3000}$

$= 1 + (3000)(0.0002) + \frac{(3000)(2999)}{1.2}(0.0002)^2 + \cdots$

$= 1 + (3000)(0.0002) = 1.6$

**26. d.** $3^{400} = 81100 = (1 + 80)^{100}$

$= {}^{100}C_0 + {}^{100}C_1\, 80 + \cdots + {}^{100}C_{100}\, 80^{100}$.

$\Rightarrow$ Last two digits are 01

**27. a.** We have,

$\frac{2}{\sqrt{2x^2+1} + \sqrt{2x^2-1}}$

$= \frac{2\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)}{(2x^2+1) - (2x^2-1)} = \sqrt{2x^2+1} - \sqrt{2x^2-1}$

Thus, the given expression can be written as

$\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$

But $(a+b)^6 + (a-b)^6 = 2[a^6 + {}^6C_2a^4b^2 + {}^6C_4a^2b^4 + b^6]$

Therefore, $\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$

$= 2[(2x^2+1)^3 + 15(2x^2+1)^2(2x^2-1) + 15(2x^2+1) \times (2x^2-1)^2 + (2x^2-1)^3]$

which is a polynomial of degree 6.

**28. b.** We have,

$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \cdots + (x+2)^{n-1}$

$= \frac{(x+3)^n - (x+2)^n}{(x+3) - (x+2)} = (x+3)^n - (x+2)^n$

$\left(\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + \cdots + a^{n-1}\right)$

Therefore, coefficient of $x^r$ in the given expression is equal to coefficient of $x^r$ in $[(x+3)^n - (x+2)^n]$, which is given by ${}^nC_r 3^{n-r} - {}^nC_r 2^{n-r} = {}^nC_r (3^{n-r} - 2^{n-r})$.

**29. b.** $a_1$ = coefficient of $x$ in $(1 + 2x + 3x^2)^{10}$

= coefficient of $x$ in $((1 + 2x) + 3x^2)^{10}$

= coefficient of $x$ in

$({}^{10}C_0(1 + 2x)^{10} + {}^{10}C_1 (1 + 2x)^9(3x^2) + \cdots)$

= coefficient of $x$ in ${}^{10}C_0(1 + 2x)^{10}$

$= {}^{10}C_0 2.{}^{10}C_1 = 20$

**30. b.** $T_{r+1} = {}^{1024}C_r(5^{1/2})^{1024-r}(7^{1/8})^r$

Now this term is an integer if $1024 - r$ is an even integer, for which $r = 0, 2, 4, 6, \ldots, 1022, 1024$ of which $r = 0, 8, 16, 2424, \ldots, 1024$ are divisible by 8 which makes $r/8$ an integer.

For A.P., $r = 0, 8, 16, 24, \ldots, 1024$,

$1024 = 0 + (n-1)8 \Rightarrow n = 129$

**31. c.** $\left(x^2 - 2 + \frac{1}{x^2}\right)^n = \frac{1}{x^{2n}}(x^4 - 2x^2 + 1)^n = \frac{(x^2-1)^{2n}}{x^{2n}}$

Total number of terms that are dependent on $x$ is equal to number of terms in the expansion of $(x^2 - 1)^{2n}$ that have degree of $x$ different from $2n$, which is given by $(2n + 1) - 1 = 2n$.

**32. c.** It is given that 6th term in the expansion of $\left(\frac{1}{x^{8/3}} + x^2 \log_{10} x\right)^8$ is 5600, therefore

$${}^8C_5\left(x^2 \log_{10} x\right)^5\left(\frac{1}{x^{8/3}}\right)^3 = 5600$$

$\Rightarrow$ $56x^{10}(\log_{10} x)^5 \frac{1}{x^8} = 5600$

$\Rightarrow$ $x^2(\log_{10} x)^5 = 100$

$\Rightarrow$ $x^2 (\log_{10} x)^5 = 10^2 (\log_{10} 10)^5$

$\Rightarrow$ $x = 10$

**33. b.** $n!\,(21-n)! = 21!\frac{n!\,(21-n)!}{21!} = \frac{21!}{{}^{21}C_n}$ which is minimum when ${}^{21}C_n$ is maximum which occurs when $n = 10$

**34. a.** To get sum of coefficients put $x = 0$. Given that sum of coefficients is

$2^n = 64$

$\Rightarrow$ $n = 6$

The greatest binomial coefficient is ${}^6C_3$.

Now given that

$T_4 - T_3 = 6 - 1 = 5$

$\Rightarrow$ ${}^6C_3(3^{-x/4})^3(3^{5x/4})^3 - {}^6C_2(3^{-x/4})^2(3^{5x/4})^4 = 5$

which is satisfied by $x = 0$.

**35. a.** Last term of $\left(2^{1/3} - \frac{1}{\sqrt{2}}\right)^n$ is

$$T_{n+1} = {}^nC_n(2^{1/3})^{n-n}\left(-\frac{1}{\sqrt{2}}\right)^n = {}^nC_n(-1)^n\frac{1}{2^{n/2}} = \frac{(-1)^n}{2^{n/2}}$$

Also, we have

$$\left(\frac{1}{3^{5/3}}\right)^{\log_3 8} = \frac{1}{(3^{5/3})^{3\log_3 2}} = 3^{-(5/3)\log_3 2^3} = 2^{-5}$$

Thus,

$$\frac{(-1)^n}{2^{n/2}} = 2^{-5}$$

$\Rightarrow$ $\frac{(-1)^n}{2^{n/2}} = \frac{(-1)^{10}}{2^5}$

$\Rightarrow$ $n/2 = 5$

$\Rightarrow$ $n = 10$

Now,

$$T_5 = T_{4+1} = {}^{10}C_4(2^{1/3})^{10-4}\left(-\frac{1}{\sqrt{2}}\right)^4$$

$$= \frac{10!}{4!6!}(2^{1/3})^6(-1)^4(2^{-1/2})^4$$

$= 210\,(2^2)\,(1)\,(2^{-2}) = 210$

**36. d.** Given,

$$\frac{{}^{n+1}C_{r+1}}{{}^nC_r} = \frac{11}{6} \Rightarrow \frac{\frac{n+1}{r+1} \times {}^nC_r}{{}^nC_r} = \frac{11}{6}$$

$\Rightarrow$ $6n + 6 = 11r + 11 \Rightarrow 6n - 11r = 5$ (1)

Also,

$$\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{6}{3} \Rightarrow \frac{\frac{n}{r} \times {}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{6}{3} \Rightarrow n = 2r \quad (2)$$

From (1) and (2), $r = 5$ and $n = 10$,

$\therefore$ $nr = 50$

**37. b.** By the given condition,

$84 = T_6 = T_{5+1}$

$$= {}^7C_5\left(2^{\log_2\sqrt{9^{x-1}+7}}\right)^2\left(\frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right)^5$$

$= 21\ 2^{\log_2(9^{x-1}+7)}\,2^{-\log_2(3^{x-1}+1)}$

$\Rightarrow$ $4 = 2^{\log_2\frac{9^{x-1}+7}{3^{x-1}+1}} = \frac{9^{x-1}+7}{3^{x-1}+1}$

$\Rightarrow$ $(3^{x-1})^2 - 4 \times 3^{x-1} + 3 = 0$

$\Rightarrow$ $(3^{x-1} - 1)(3^{x-1} - 3) = 0$

$\Rightarrow$ $3^{x-1} = 1$ or $3$

$\Rightarrow$ $3^{x-1} = 3^0$ or $3^1$

$\Rightarrow$ $x - 1 = 0$ or $1$

$\Rightarrow$ $x = 1, 2$

**38. a.** General term,

$T_{r+1} = {}^{256}C_r(\sqrt{3})^{256-r}(\sqrt[8]{5})^r$

$= {}^{256}C_r\, 3^{\frac{256-r}{2}}\, 5^{\frac{r}{8}}$

The terms are integral if $\frac{256-r}{2}$ and $\frac{r}{8}$ are both positive integers.

$\therefore$ $r = 0, 8, 16, 24, \ldots, 256$

Hence, there are 33 integral terms.

**39. a.** $T_{r+1} = {}^{4n-2}C_r(ix)^r$

$T_{r+1}$ is negative, if $i^r$ is negative and real.

$i^r = -1 \Rightarrow r = 2, 6, 10, \ldots$ which form an A.P.

$0 \le r \le 4n - 2$

$4n - 2 = 2 + (r - 1)\,4 \Rightarrow r = n$

The required number of terms is $n$.

**40. b.** $T_5 = {}^nC_4\, a^{n-4}\,(-2b)^4$

and $T_6 = {}^nC_5\, a^{n-5}(-2b)^5$

As $T_5 + T_6 = 0$, we get

$${}^nC_4\, 2^4 a^{n-4} b^4 = {}^nC_5\, 2^5 a^{n-5} b^5$$

$$\Rightarrow \quad \frac{a^{n-4}b^4}{a^{n-5}\ b^5} = \frac{n!2^5}{5!(n-5)!}\cdot\frac{4!\,(n-4)!}{n!2^4}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{2(n-4)}{5}$$

**41. b.** $\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{15}$

$$= \left(\frac{x^3 + x + x^4 + 1}{x^2}\right)^{15}$$

$$= \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_{60}x^{60}}{x^{30}}$$

Hence, the total number of terms is 61.

**42. c.** $(1 + x + x^2 + x^3)^5 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{15}x^{15}$

Putting $x = 1$ and $x = -1$ alternatively, we have

$a_0 + a_1 + a_2 + a_3 + \cdots + a_{15} = 4^5$ (1)

$a_0 - a_1 + a_2 - a_3 + \cdots - a_{15} = 0$ (2)

Adding (1) and (2), we have

$2(a_0 + a_2 + a_4 + \cdots + a_{14}) = 4^5$

$\Rightarrow \quad a_0 + a_2 + a_4 + \cdots + a_{14} = 2^9 = 512$

**43. c.** Sum of coefficients in $(1 - x\sin\theta + x^2)^n$ is $(1 - \sin\theta + 1)^n$ (putting $x = 1$)

This sum is greatest when $\sin\theta = -1$, then maximum sum is $3^n$.

**44. a.** We know that the sum of the coefficients in a binomial expansion is obtained by replacing each variable by unit in the given expression. Therefore, sum of the coefficients in $(a + b)^n$ is given by $(1 + 1)^n$.

$\therefore \quad 4096 = 2^n \Rightarrow 2^n = 2^{12} \Rightarrow n = 12$

Hence, $n$ is even. So, the greatest coefficient is ${}^nC_{n/2}$, i.e., ${}^{12}C_6 = 924$.

**45. b.** We have, $a$ = sum of the coefficients in the expansion of $(1 - 3x + 10x^2)^n = (1 - 3 + 10)^n = (8)^n = (2)^{3n}$ (putting $x = 1$)

Now, $b$ = sum of the coefficients in the expansion of $(1 + x^2)^n = (1 + 1)^n = 2^n$. Clearly, $a = b^3$.

**46. b.** $(1 + x - 2x^2)^6 = 1 + a_1 x + a_2 x^2 + \cdots$

Putting $x = 1$, we get

$0 = 1 + a_1 + a_2 + a_3 + \cdots + a_{12}$ (1)

Putting $x = -1$, we get

$64 = 1 - a_1 + a_2 - a_3 + \cdots + a_{12}$ (2)

(1) + (2) gives

$64 = 2[1 + a_2 + a_4 + \cdots a_{12}]$

$\Rightarrow \quad 1 + a_2 + a_4 + \cdots + a_{12} = 32$

$\Rightarrow \quad a_2 + a_4 + \cdots + a_{32} = 31$

**47. a.** $\frac{2^{4n}}{15} = \frac{(15+1)^n}{15}$

$$= \frac{({}^nC_0 15^n + {}^nC_1 15^{n-1} + \cdots + {}^nC_{n-1}15 + {}^nC_n)}{15}$$

$$= \text{Integer} + \frac{1}{15}$$

Hence, the fractional part of $\frac{2^{4n}}{15}$ is $\frac{1}{15}$.

**48. c.** As we know that ${}^nC_0 - {}^nC_1^2 + {}^nC_2^2 - {}^nC_3^2 + \cdots + (-1)^n\, {}^nC_n^2 = 0$ (if $n$ is odd) and in the question $n = 15$ (odd). Hence, sum of given series is 0.

**49. b.** $(1-x)^{30} = {}^{30}C_0 x^0 - {}^{30}C_1 x^1 + {}^{30}C_2 x^2 + \cdots + (-1)^{30}\,{}^{30}C_{30}x^{30}$ (1)

$(x + 1)^{30} = {}^{30}C_0 x^{30} + {}^{30}C_1 x^{29} + {}^{30}C_2 x^{28} + \cdots + {}^{30}C_{10}x^{20} + \cdots + {}^{30}C_{30}x^0$ (2)

Multiplying (1) and (2) and equating the coefficient of $x^{20}$ on both sides, we get required sum is equal to coefficient of $x^{20}$ in $(1 - x^2)^{30}$, which is given by ${}^{30}C_{10}$.

**50. b.** We have, $f(x) = x^n$. So,

$f^1(x) = nx^{n-1} \Rightarrow f'(1) = n$

$f^2(x) = n(n-1)x^{n-2} \Rightarrow f^2(1) = n(n-1)$

$f^3(x) = n(n-1)(n-2)x^{n-3} \Rightarrow f^3(1) = n(n-1)(n-2)$

$\vdots$

$f^n(x) = n(n-1)(n-2)\cdots 1 \Rightarrow f^n(1) = n(n-1)(n-2)\cdots 1$

$$\Rightarrow \quad f(1) + \frac{f^1(1)}{1} + \frac{f^2(1)}{2!} + \cdots + \frac{f^n(1)}{n!}$$

$$= 1 + \frac{n}{1} + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}$$

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$$

$$= 2^n$$

**51. b.** Given series is ${}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \cdots + {}^{20}C_8$

$$= \frac{1}{2}(2\cdot{}^{20}C_0 + 2\,{}^{20}C_1 + \cdots 2\cdot{}^{20}C_8)$$

$$= \frac{1}{2}\Big[\big({}^{20}C_0 + {}^{20}C_1 + \cdots + {}^{20}C_8 + {}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11} + \cdots + {}^{20}C_{20}\big) - ({}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11})\Big]$$

$$= \frac{1}{2}\left[2^{20} - 2\cdot{}^{20}C_9 - {}^{20}C_{10}\right]$$

$$= 2^{19} - \frac{(2\cdot{}^{20}C_9 + {}^{10}C_{10})}{2}$$

$$= \frac{(2^{20} - {}^{20}C_{10})}{2} - {}^{20}C_9 = 2^{19} - \frac{({}^{20}C_{10} + 2\times{}^{20}C_9)}{2}$$

**52. b.** Let,

$$S = \frac{{}^nC_0}{n} + \frac{{}^nC_1}{n+1} + \frac{{}^nC_2}{n+2} + \cdots + \frac{{}^nC_n}{2n}$$

$$= {}^nC_0\int_0^1 x^{n-1}dx + {}^nC_1\int_0^1 x^n dx + \cdots + {}^nC_n\int_0^1 x^{2n-1}dx$$

$$= \int_0^1 [{}^nC_0 x^{n-1} + {}^nC_1 x^n + \cdots + {}^nC_n x^{2n-1}]dx$$

$$= \int_0^1 x^{n-1}(1+x)^n\,dx$$

$$= \int_1^2 x^n(x-1)^{n-1}dx$$

**53. b.** $(1 + x)^n = C_0 + C_1 x + C_2 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n$

$(1 - x)^n = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \cdots + (-1)^n C_n x^n$

$\Rightarrow \quad [(1 + x)^n - (1 - x)^n] = 2\,[C_1 x + C_3 x^3 + C_5 x^5 + \cdots]$

$\Rightarrow \quad \frac{1}{2}[(1 + x)^n - (1 - x)^n] = C_1 x + C_3 x^3 + C_5 x^5 + \cdots$

Putting $x = 2$, we have

$$2\ C_1 + 2^3\ C_3 + 2^5\ C_5 + \cdots = \frac{3^n - (-1)^n}{2}$$

**54. d.** $\sum_{r=1}^{n} (-1)^{r+1} \frac{{}^nC_r}{(r+1)} = \frac{1}{n+1} \sum_{r=1}^{n} (-1)^{r+1}\ {}^{n+1}C_{r+1}$

$$= \frac{1}{n+1}(0 - 1 + (n+1)) = \frac{n}{n+1}$$

**55. c.** $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_{n-1}x^{n-1} + C_nx^n \quad (1)$

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_{n-1}x + C_n \quad (2)$

Multiplying Eqs. (1) and (2) and equating the coefficient of $x^{n-2}$, we get

$C_0C_2 + C_1C_3 + C_2C_4 + \cdots + C_{n-2}C_n$

= Coefficient of $x^{n-2}$ in $(1+x)^{2n}$

$= {}^{2n}C_{n-2}$

$= \frac{(2n)!}{(n-2)!(n+2)!}$

**56. a.** We know that

$(1-1)^{20} = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots + {}^{20}C_{10} - {}^{20}C_{11} + {}^{20}C_{12} - \cdots + {}^{20}C_{20} = 0$

$2\ ({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots - {}^{20}C_9) + {}^{20}C_{10} = 0$

$[\because\ {}^{20}C_{20} = {}^{20}C_0,\ {}^{20}C_{19} = {}^{20}C_1$, etc.]

$\Rightarrow\ {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots - {}^{20}C_9 + {}^{20}C_{10}$

$= -\frac{1}{2}\,{}^{20}C_{10} + {}^{20}C_{10} = \frac{1}{2}\,{}^{20}C_{10}$

**57. b.** The given expression is the coefficient of $x^4$ in

${}^4C_0(1+x)^{404} - {}^4C_1(1+x)^{303} + {}^4C_2(1+x)^{202} - {}^4C_3(1+x)^{101} + {}^4C_4$

= Coefficient of $x^4$ in $[(1+x)^{101} - 1]^4$

= Coefficient of $x^4$ in $({}^{101}C_1\,x + {}^{101}C_2x^2 + \cdots)^4$

$= (101)^4$

**58. c.** Put $x = \omega, \omega^2$

$(3 + \omega + \omega^2)^{2010} = a_0 + a_1\omega + a_2\omega^2 + \cdots$

$\Rightarrow\ 2^{2010} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega + \cdots \quad (1)$

and

$2^{2010} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega + \cdots \quad (2)$

Adding (1) and (2), we have

$2 \times 2^{2010} = 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 - \cdots$

$\Rightarrow\ 2^{2010} = a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 + a_6 \cdots$

**59. d.** $\sum_{r=0}^{10} r\,{}^{10}C_r 3^r(-2)^{10-r}$

$= 10\sum_{r=0}^{10} {}^9C_{r-1}3^r(-2)^{10-r}$

$= 10 \times 3\sum_{r=0}^{10} {}^9C_{r-1}3^{r-1}(-2)^{10-r}$

$= 30(3-2)^{10}$

$= 30$

**60. a.** $\sum_{r=0}^{40} r\,{}^{40}C_r\,{}^{30}C_r$

$= 40\sum_{r=0}^{40} {}^{39}C_{r-1}\,{}^{30}C_r$

$= 40\sum_{r=0}^{40} {}^{39}C_{r-1}\,{}^{30}C_{30-r}$

$= 40\,{}^{39+30}C_{r-1+30-r}$

$= 40\,{}^{69}C_{29}$

**61. a.** $\frac{r \times 2^r}{(r+2)!} = \frac{(r+2-2)2^r}{(r+2)!}$

$= \frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!}$

$= -\left(\frac{2^{r+1}}{(r+2)!} - \frac{2^r}{(r+1)!}\right)$

$= -(V(r) - V(r-1))$

$\Rightarrow\ \sum_{r=1}^{15} \frac{r \times 2^r}{(r+2)!} = -(V(15) - V(0))$

$= -\left(\frac{2^{16}}{17!} - \frac{2}{2!}\right)$

$= 1 - \frac{2^{16}}{(17)!}$

**62. c.** $t_{r+1} = (-1)^r(n-r+2)\,{}^nC_r\,2^{n-r+1}$

$= (n+2)\,2^{n+1}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r - 2^{n+1}(-1)^r\,r\,{}^nC_r\left(\frac{1}{2}\right)^r$

$= (n+2)\,2^{n+1}\,{}^nC_r\left(-\frac{1}{2}\right)^r + 2^n\,n\,{}^{n-1}C_{r-1}\left(-\frac{1}{2}\right)^{r-1}$

$\therefore$ Sum $= (n+2)\,2^{n+1}\left\{{}^nC_0 - {}^nC_1 \times \frac{1}{2} + {}^nC_2 \times \left(\frac{1}{2}\right)^2 - \cdots\right\}$

$+ n\,2^n\left\{{}^{n-1}C_0 - {}^{n-1}C_1 \times \frac{1}{2} + {}^{n-1}C_2 \times \left(\frac{1}{2}\right)^2 + \cdots\right\}$

$= (n+2)\,2^{n+1}\left(1 - \frac{1}{2}\right)^n + n\,2^n\left(1 - \frac{1}{2}\right)^{n-1}$

$= 2(n+2) + 2n$

$= 4n + 4$

**63. c.** Here,

$T_r = (-1)^r \frac{{}^{50}C_r}{r+2}$

$= (-1)^r(r+1)\frac{{}^{50}C_r}{(r+1)(r+2)}$

$= (-1)^r(r+1)\frac{{}^{52}C_{r+2}}{51 \times 52}$

$= (-1)^r \frac{[(r+2)-1]\,{}^{52}C_{r+2}}{51 \times 52}$

$= (-1)^r \frac{[52\,{}^{51}C_{r+1} - {}^{52}C_{r+2}]}{51 \times 52}$

$= \frac{[-52\,{}^{51}C_{r+1}(-1)^{r+1} - {}^{52}C_{r+2}(-1)^{r+2}]}{51 \times 52}$

$\sum_{r=0}^{50} (-1)^r \frac{{}^{50}C_r}{r+2}$

$= \sum_{r=0}^{50} \frac{[-52\,{}^{51}C_{r+1}(-1)^{r+1} - {}^{52}C_{r+2}(-1)^{r+2}]}{51 \times 52}$

$$= -52\,\frac{(1-1)^{51} - {}^{51}C_0}{51\times 52} - \frac{(1-1)^{52} - {}^{52}C_0 + {}^{52}C_1}{51\times 52}$$

$$= \frac{1}{51} - \frac{1}{52}$$

$$= \frac{1}{51\times 52}$$

**Alternative solution:**

$$(1-x)^n = \sum_{r=0}^{n} {}^nC_r(-1)^r x^r$$

$$\Rightarrow \quad x(1-x)^n = \sum_{r=0}^{n}(-1)^r\, {}^nC_r x^{r+1}$$

Integrating both sides within the limits 0 to 1, we get

$$\int_0^1 x(1-x)^n dx = \sum_{r=0}^{n}(-1)^r \frac{{}^nC_r}{r+2}$$

$$\Rightarrow \quad \sum_{r=0}^{n}(-1)^r \frac{{}^nC_r}{r+2} = \int_0^1 x(1-x)^n dx$$

$$= \int_0^1 (1-x)x^n dx \quad \text{(Replace } x \text{ by } 1-x)$$

$$= \left.\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right|_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{1}{(n+1)(n+2)}$$

Now put $n = 50$.

**64. a.** Given term can be written as

$$(1+x)^2(1-x)^{-2} = (1+2x+x^2)\,[1+2x+3x^2+\cdots+(n-1)\times x^{n-2} + nx^{n-1} + (n+1)x^n + \cdots]$$

Coefficient of $x^n$ is $(n+1+2n+n-1) = 4n$.

**65. a.** $1 + n\left(1-\frac{1}{x}\right) + \frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2 + \cdots\infty$

$$= 1 - n\left[-\left(1-\frac{1}{x}\right)\right] + \frac{-n(-n-1)}{2!}\left[-\left(1-\frac{1}{x}\right)\right]^2 + \cdots\infty$$

$$= \left[1-\left(1-\frac{1}{x}\right)\right]^{-n}$$

$$= x^n$$

**66. c.** $\sum_{k=1}^{n} k\left(1-\frac{1}{n}\right)^{k-1}$

$$= 1 + 2\left(1-\frac{1}{n}\right)^1 + 3\left(1+\frac{1}{n}\right)^2 + \cdots$$

$$= 1 + 2t + 3t^2 + \cdots$$

$$= (1-t)^{-2}$$

$$\left[1-\left(1-\frac{1}{n}\right)\right]^{-2} = \left(\frac{1}{n}\right)^{-2} = n^2$$

**67. d.** $\left[\sqrt{1+x^2} - x\right]^{-1} = \frac{1}{\sqrt{1+x^2}-x} \times \frac{\left(\sqrt{1+x^2}+x\right)}{\left(\sqrt{1+x^2}+x\right)}$

$$= \frac{\sqrt{1+x^2}+x}{1+x^2-x^2} = x + \sqrt{1+x^2} = x + \left(1+x^2\right)^{1/2}$$

$$= x + 1 + \frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^4}{2!} + \cdots$$

Obviously, the coefficient of $x^4$ is $-1/8$.

**68. d.** Let,

$$(1+y)^n = 1 + \frac{1}{3}x + \frac{1\times 4}{3\times 6}x^2 + \frac{1\times 4\times 7}{3\times 6\times 9}x^3 + \cdots$$

$$= 1 + ny + \frac{n(n-1)}{2!}y^2 + \cdots$$

Comparing the terms, we get

$$ny = \frac{1}{3}x,\ \frac{n(n-1)}{2!}y^2 = \frac{1\times 4}{3\times 6}x^2$$

Solving, $n = -1/3$, $y = -x$. Hence, the given series is $(1-x)^{-1/3}$.

**69. a.** Let the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \cdots\infty$$

$$\Rightarrow \quad nx = \frac{1}{4} \Rightarrow n^2x^2 = \frac{1}{16}$$

Also,

$$\frac{n(n-1)}{2}x^2 = \frac{3}{32} \Rightarrow \frac{2n}{n-1} = \frac{\frac{1}{16}}{\frac{3}{32}} = \frac{2}{3}$$

$$\Rightarrow \quad 3n = n-1$$

$$\Rightarrow \quad 2n = -1$$

$$\Rightarrow \quad n = -\frac{1}{2}$$

$$\Rightarrow \quad x = -\frac{1}{2}$$

$$\Rightarrow \quad \text{Required sum} = \left(1-\frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$= (2)^{\frac{1}{2}} = \sqrt{2}$$

**70. d.** Required value is

$$\left(1-\frac{2x}{1+x}\right)^{-n} = \left(\frac{1+x-2x}{1+x}\right)^{-n} = \left(\frac{1-x}{1+x}\right)^{-n} = \left(\frac{1+x}{1-x}\right)^{n}$$

**71. d.** $(1+2x+3x^2+\cdots)^{-3/2} = [(1-x)^{-2}]^{-3/2}$

$$= (1-x)^3 = 1 - 3x + 3x^2 - x^3$$

Therefore, coefficient of $x^5$ is 0.

**72. d.** $(1+x+x^2+\cdots)^2 = ((1-x)^{-1})^2 = (1-x)^{-2}$

$$= 1 + 2x + 3x^2 + \cdots$$

Therefore, coefficient of $x^n$ is $n+1$.

**73. b.** $T_{r+1}$ in $(1+x)^n$ is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$$

For first negative term,

$n-r+1<0$

$\Rightarrow \frac{27}{5}-r+1<0$

$\Rightarrow r>\frac{32}{5}$

Thus, first negative term occurs when $r=7$.

**74. d.** $\frac{(1+x)^{3/2}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}}$

$=\frac{\left(1+\frac{3}{2}x+\frac{3}{8}x^2\right)-\left(1+\frac{3}{2}x+3\frac{x^2}{4}\right)}{(1-x)^{1/2}}$

$=\frac{-3}{8}x^2(1-x)^{-1/2}$

$=-\frac{3}{8}x^2\left(1+\frac{x}{2}\right)$

$=-\frac{3}{8}x^2$

**75. d.** $\frac{1}{(1-ax)(1-bx)}=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$

But $(1-ax)^{-1}(1-bx)^{-1}=(1+ax+a^2x^2+\cdots)(1+bx+b^2x^2+\cdots)$

$\Rightarrow$ Coefficient of $x^n$ is $b^n+ab^{n-1}+a^2b^{n-2}+\cdots+a^{n-1}b+a^n$

$=\frac{b^{n+1}-a^{n+1}}{b-a}$

$\Rightarrow a_n=\frac{b^{n+1}-a^{n+1}}{b-a}$

**76. c.** $\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}({}^kC_r)$

$=\sum_{k=1}^{\infty}\left(\frac{1}{3^k}\left(\sum_{r=0}^{k}{}^kC_r\right)\right)$

$=\sum_{k=1}^{\infty}\left(\frac{2^k}{3^k}\right)$

$=\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots\infty$

$=\frac{2/3}{1-\frac{2}{3}}=2$

**77. d.** General term in the expansion of $(\sqrt{2}+\sqrt[3]{3}+\sqrt[6]{5})^{10}$ is

$\frac{10!}{a!b!c!}(\sqrt{2})^a(\sqrt[3]{3})^b(\sqrt[6]{5})^c$ where $a+b+c=10$.

For rational term, we have the following:

| **Value of $a, b, c$** | **Value of term** |
|---|---|
| $a=4, b=0, c=6$ | $\frac{10!}{4!0!6!}(\sqrt{2})^4(\sqrt[3]{3})^0(\sqrt[6]{5})^6=4200$ |
| $a=10, b=0, c=0$ | $\frac{10!}{10!0!0!}(\sqrt{2})^{10}(\sqrt[3]{3})^0(\sqrt[6]{5})^0=32$ |
| $a=4, b=6, c=0$ | $\frac{10!}{4!6!0!}(\sqrt{2})^4(\sqrt[3]{3})^6(\sqrt[6]{5})^0=7560$ |

**78. b.** $f(x)=1-x+x^2-x^3+\cdots-x^{15}+x^{16}-x^{17}=\frac{1-x^{18}}{1+x}$

$\Rightarrow f(x-1)=\frac{1-(x-1)^{18}}{x}$

Therefore, required coefficient of $x^2$ is equal to coefficient of $x^3$ in $1-(x-1)^{18}$, which is given by ${}^{18}C_3=816$.

**79. a.** $p=(8+3\sqrt{7})^n={}^nC_0 8^n+{}^nC_1 8^{n-1}(3\sqrt{7})+\cdots$

Let, $p_1=(8-3\sqrt{7})^n={}^nC_0 8^n-{}^7C_1 8^{n-1}(3\sqrt{7})+\cdots$

$p_1+p_2=2({}^nC_0 8^n+{}^nC_2 8^{n-2}(3\sqrt{7})^2+\cdots)$ = even integer

$p_1$ clearly belongs to (0, 1)

$\Rightarrow [p]+f+p_1$ = even integer

$\Rightarrow f+p_1$ = integer

$f\in(0,1), p_1\in(0,1)$

$\Rightarrow f+p\in(0,2)$

$\Rightarrow f+p_1=1$

$\Rightarrow p_1=1-f$

Now, $p(1-f)=pp_1=[(8+3\sqrt{7})^n(8-3\sqrt{7})]^n=1$

**80. a.** $\sum_{r=0}^{10}(r)\,{}^{20}C_r=\sum_{r=1}^{10}20\times{}^{19}C_{r-1}$

$=20({}^{19}C_0+{}^{19}C_1+\cdots+{}^{19}C_{10})$

$=20({}^{19}C_0+{}^{19}C_1+\cdots+{}^{19}C_{10})$

$=20\left(\frac{1}{2}\times2^{19}+{}^{19}C_{10}\right)$

$=20(2^{18}+{}^{19}C_{10})$

**81. c.** $(23)^{14}=(529)^7=(530-1)^7$

$={}^7C_0(530)^7-{}^7C_1(530)^6+\cdots-{}^7C_5(530)^2+{}^7C_6\,530-1$

$={}^7C_0(530)^7-{}^7C_1(530)^6+\cdots+3710-1=100m+3709$

Therefore, last two digits are 09.

**82. b.** $\frac{f(x)}{1-x}=b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots$

$\Rightarrow a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$

$=(1-x)(b_0+b_1x+b_2x^2+\cdots+b_nx^n+\cdots)$

Comparing the coefficient of $x^n$ on both the sides,

$a_n=b_n-b_{n-1}$.

**83. d.** $(1-x)^n(1+x)^n=\sum_{r=0}^{n}a_rx^r(1-x)^n(1-x)^{n-r}$

$\Rightarrow (1-x+2x)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{n-r}$

$\Rightarrow \sum_{r=0}^{n}{}^nC_r(1-x)^{n-r}(2x)^r=\sum_{r=0}^{n}a_rx^r(1-x)^{n-r}$

Comparing general term, we get $a_r={}^nC_r2^r$.

**84. d.** $(1+\omega)^n = {}^nC_0 + {}^nC_1\,\omega + \cdots$

$$= ({}^nC_0 + {}^nC_3 + \cdots) + ({}^nC_1 + {}^nC_4 + \cdots)\left(\frac{-1+\sqrt{3}i}{2}\right)$$

$$+ ({}^nC_2 + {}^nC_5 + \cdots)\left(\frac{-1-\sqrt{3}i}{2}\right)$$

$$= ({}^nC_0 + {}^nC_3 + \cdots) - \frac{1}{2}({}^nC_1 + {}^nC_2 + {}^nC_4 + {}^nC_5 \ldots)$$

$$+\frac{i\sqrt{3}}{2}({}^nC_1 - {}^nC_2 + {}^nC_4 - {}^nC_5 + \cdots)$$

Equating the modulus, we get $|(-\omega^2)^n| = 1$.

**85. c.** $\dfrac{(x^2+x+1)(1-x)}{(1-x)^2} = (1-x^3)(1-x)^{-2}$

$$= (1-x^3)(1+2x+3x^2+\cdots)$$

Now, $a_r = (r+1)-(r-2) = 3$

But $a_1 = 2$

So,

$$\sum_{r=1}^{50} a_r = 2 + 49\times 3 = 149$$

**86. a.** $N = {}^{2n}C_n = \dfrac{(2n)!}{(n!)^2} = \dfrac{(n+1)(n+2)\cdots(n+n)}{(n!)}$

$\Rightarrow$ $(n!)N = (n+1)(n+2)\cdots(n+n)$

Since $n < p < 2n$, so $p$ divides $(n+1)(n+2)\cdots(n+n)$.

**87. c.** $\sum_{r=0}^{300} a_r \times x^r = (1+x+x^2+x^3)^{100}$

Clearly, '$a_r$' is the coefficient of $x^r$ in the expansion of $(1+x+x^2+x^3)^{100}$.

Replacing $x$ by $1/x$ in the given equation, we get

$$\sum_{r=0}^{300} a_r\left(\frac{1}{x}\right)^r = \frac{1}{x^{300}}(x^3+x^2+x+1)^{100}$$

$$\Rightarrow \sum_{r=0}^{300} a_r x^{300-r} = (1+x+x^2+x^3)^{100}$$

Here, $a_r$ represents the coefficient of $x^{300-r}$ in $(1+x+x^2+x^3)^{100}$.

Thus, $a_r = a_{300-r}$

Let $I = \sum_{r=0}^{300} r\times a_r$

$$= \sum_{r=0}^{300}(300-r)a_{300-r}$$

$$= \sum_{r=0}^{300}(300-r)a_r$$

$$= 300\sum_{r=0}^{300} a_r - \sum_{r=0}^{300} r a_r$$

$\Rightarrow$ $2I = 300a$

$\Rightarrow$ $I = 150a$

**88. a.** $\displaystyle\sum_{r=1}^{n+1}\left(\sum_{k=1}^{n} {}^kC_{r-1}\right)$

$$= \sum_{r=1}^{n+1}\left(\sum_{k=1}^{n}\left({}^{k+1}C_r - {}^kC_r\right)\right)$$

$$= \sum_{r=1}^{n+1}\left({}^{n+1}C_r - {}^1C_r\right)$$

$$= 2^{n+1} - 2$$

**89. b.** We have,

$$(1-x)^{-n} = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots$$

and

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$$

Hence,

$a_0 + a_1 + a_2 + \cdots + a_r$

= Coefficient of $x^r$ in the product of the two series

= Coefficient of $x^r$ in $(1-x)^{-n}(1-x)^{-1}$

= Coefficient of $x^r$ in $(1-x)^{-(n+1)}$

$$= \frac{(n+1)(n+2)\cdots(n+r)}{r!}$$

$$= {}^{r+n+1-1}C_{n+1-1} = {}^{n+r}C_n$$

**90. d.** $\displaystyle\sum_{r=0}^{20} r(20-r)\times({}^{20}C_r)^2 = \sum_{r=0}^{20} r\times{}^{20}C_r(20-r)\times{}^{20}C_{20-r}$

$$\Rightarrow \sum_{r=0}^{20} 20\,{}^{19}C_{r-1}\times 20\times{}^{19}C_{19-r}$$

$$= 400\sum_{r=0}^{20}{}^{19}C_{r-1}\times{}^{19}C_{19-r}$$

= 400 × coefficient of $x^{18}$ in $(1+x)^{19}(1+x)^{19}$

$= 400\times{}^{38}C_{18}$

$= 400\times{}^{38}C_{20}$

## Multiple Correct Answers Type

**1. a, c.**

Inclusion of $\log x$ implies $x > 0$.

Now, 3rd term in the expansion is

$$T_{2+1} = {}^5C_2\,x^{5-2}\,(x^{\log_{10}x})^2 = 1000000 \text{ (given)}$$

or

$$x^{3+2\log_{10}x} = 10^5$$

Taking logarithm of both sides, we get

$$(3+2\log_{10}x)\log_{10}x = 5$$

or

$$2y^2 + 3y - 5 = 0, \text{ where } \log_{10}x = y$$

or

$$(y-1)(2y+5) = 0 \text{ or } y = 1 \text{ or } -5/2$$

or

$$\log_{10}x = 1 \text{ or } -5/2$$

$\therefore$ $x = 10^1 = 10$ or $10^{-5/2}$

**2. a, d.**

Coefficients of $r^{th}$, $(r+1)^{th}$ and $(r+2)^{th}$ terms are ${}^{14}C_{r-1}$, ${}^{14}C_r$ and ${}^{14}C_{r+1}$.

If these coefficients are in A.P., then

$$2({}^{14}C_r) = {}^{14}C_{r-1} + {}^{14}C_{r+1}$$

$$\Rightarrow \frac{2(14)!}{r!(14-r)!} = \frac{(14)!}{(r-1)!(15-r)!} + \frac{(14)!}{(r+1)!(13-r)}$$

$$\Rightarrow \frac{2(14)!}{r!(14-r)!} = \frac{(14)![(r+1)r + (15-r)(14-r)]}{(r+1)!(15-r)!}$$

$\Rightarrow$ $2(15-r)(r+1) = 2r^2 - 28r + 210$

$\Rightarrow$ $r^2 - 14r + 45 = 0$ or $(r-5)(r-9) = 0$

$\Rightarrow$ $r = 5$ or $9$

**3. a, b, c.**

We have,

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_n a^n$$

$$= [{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \cdots] + [{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots]$$

or

$$(x + a)^n = P + Q \quad (1)$$

Similarly,

$$(x - a)^n = P - Q \quad (2)$$

(i) (1) × (2) ⇒ $P^2 - Q^2 = (x^2 - a^2)^n$

(ii) Squaring (1) and (2) and subtracting (2) from (1), we get
$4PQ = (x + a)^{2n} - (x - a)^{2n}$

(iii) Squaring (1) and (2) and adding,
$2(P^2 + Q^2) = (x + a)^{2n} + (x - a)^{2n}$

**4. a, c, d.**

$$I + f = \left(4 + \sqrt{15}\right)^n$$

Let $f' = \left(4 - \sqrt{15}\right)^n$. Then $0 < f' < 1$

$$I + f = {}^nC_0 4^n + {}^nC_1 4^{n-1}\sqrt{15} + {}^nC_2 4^{n-2}\, 15 + {}^nC_3 4^{n-3}\left(\sqrt{15}\right)^3 + \cdots$$

$$f' = {}^nC_0 4^n - {}^nC_1 4^{n-1}\sqrt{15} + {}^nC_2 4^{n-2}\cdot 15 - {}^nC_3 4^{n-3}\left(\sqrt{15}\right)^3 + \cdots$$

∴ $I + f + f' = 2\,({}^nC_0 4^n + {}^nC_2 4^{n-2} \times 15 + \cdots) =$ even integer

∵ $0 < f + f' < 2 \Rightarrow f + f' = 1 \Rightarrow 1 - f = f'$

Thus, $I$ is an odd integer. Now,

$$1 - f = f' = (4 - \sqrt{15})^n$$
$$(I + f)(1 - f) = (I + f) f' = 1$$

**5. c, d.**

$${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r^2-1} + {}^{69}C_{r^2}$$

$$\Rightarrow \quad {}^{70}C_{3r} = {}^{70}C_{r^2}$$

Thus, $r^2 = 3r$ or $70 - 3r = r^2$ so that $r = 0, 3$ or $7, -10$.

Hence, $r = 3$ and $7$ (as the given equation is not defined for $r = 0$ and $-10$).

**6. a, d.**

It is given that the fourth term in the expansion of

$\left(ax + \dfrac{1}{x}\right)^n$ is $\dfrac{5}{2}$, therefore

$${}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^nC_3\, a^{n-3}\, x^{n-6} = \frac{5}{2} \quad \text{(i)}$$

[∵ R.H.S. is independent of $x$]

Putting $n = 6$ in (i), we get ${}^6C_3\, a^3 = \dfrac{5}{2} \Rightarrow a^3 = \dfrac{1}{8} \Rightarrow a = \dfrac{1}{2}$

**7. a, b, c, d.**

We know that to get the sum of coefficients, we put $x = 1$.

Then, sum of coefficients is $(1 + ax - 2x^2)^n$ is $(a - 1)^n$.

Obviously, when $a > 1$, sum is positive for any $n$.

**8. a, b, d.**

$$f(m) = \sum_{i=0}^{m}\binom{30}{30-i}\binom{20}{m-i} = \sum_{i=0}^{m}\binom{30}{i}\binom{20}{m-i} = {}^{50}C_m$$

$f(m)$ is greatest when $m = 25$. Also,

$$f(0) + f(1) + \cdots + f(50)$$
$$= {}^{50}C_0 + {}^{50}C_1 + {}^{50}C_2 + \cdots + {}^{50}C_{50} = 2^{50}$$

Also, ${}^{50}C_m$ is not divisible by 50 for any $m$ as 50 is not a prime number

$$\sum_{m=0}^{50}(f(m))^2 = ({}^{50}C_0)^2 + ({}^{50}C_1)^2 + ({}^{50}C_2)^2 + \cdots + ({}^{50}C_{50})^2 = {}^{100}C_{50}$$

**9. a, b, d.**

$$(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \cdots + C_{16} z^{32} \quad (1)$$

Putting $z = i$, where $i = \sqrt{-1}$,

$$(1 - 1 + 1)^8 = C_0 - C_1 + C_2 - C_3 + \cdots + C_{16}$$
$$\Rightarrow \quad C_0 - C_1 + C_2 - C_3 + \cdots + C_{16} = 1$$

Also, putting $z = \omega$,

$$(1 + \omega^2 + \omega^4)^8 = C_0 + C_1\omega^2 + C_2\omega^4 + \cdots + C_{16}\omega^{32}$$
$$\Rightarrow \quad C_0 + C_1\omega^2 + C_2\omega + C_3 + \cdots + C_{16}\omega^2 = 0 \quad (2)$$

Putting $x = \omega^2$,

$$(1 + \omega^4 + \omega^8)^8 = C_0 + C_1\omega^4 + C_2\omega^8 + \cdots + C_{16}\omega^{64}$$
$$\Rightarrow \quad C_0 + C_1\omega + C_2\omega^2 + \cdots + C_{16}\omega = 0 \quad (3)$$

Putting $x = 1$,

$$3^8 = C_0 + C_1 + C_2 + \cdots + C_{16} \quad (4)$$

Adding (2), (3) and (4), we have

$$3(C_0 + C_3 + \cdots + C_{15}) = 3^8$$
$$\Rightarrow \quad C_0 + C_3 + \cdots + C_{15} = 3^7$$

Similarly, first multiplying (1) by $z$ and then putting $1, \omega, \omega^2$ and adding, we get

$$C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16} = 3^7$$

Multiplying (1) by $z^2$ and then putting $1, \omega, \omega^2$ and adding, we get

$$C_2 + C_5 + C_8 + C_{11} + C_{14} = 3^7$$

**10. a, b, c.**

General term is ${}^{6561}C_r\, 7^{\frac{6561-r}{3}}\, 11^{\frac{r}{9}}$.

To make the term free of radical sign, $r$ should be a multiple of 9.

∴ $r = 0, 9, 18, 27, \ldots, 6561$.

Hence, there are 730 terms. The greatest binomial coefficients are

${}^{6561}C_{\frac{6561-1}{2}}$ and ${}^{6561}C_{\frac{6561-3}{2}}$ or ${}^{6561}C_{3280}$ and ${}^{6561}C_{3279}$.

Now, 3280 are 3279 are not a multiple of 3; hence, both terms involving greatest binomial coefficients are irrational.

**11. b, c.**

For $n = 2m$, the given expression is

$$C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots (-1)^{n-1}(C_0 + C_1 + \cdots + C_{n-1})$$
$$= C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots - (C_0 + C_1 + \cdots + C_{2m-1})$$
$$= -(C_1 + C_3 + C_5 + \cdots + C_{2m-1})$$
$$= -(C_1 + C_3 + C_5 + \cdots + C_{n-1}) = -2^{n-1}$$

**12. a, c,** $\left(x^2 + 1 + \dfrac{1}{x^2}\right)$

$$= {}^nC_0 + {}^nC_1\left(x^2 + \frac{1}{x^2}\right) + {}^nC_2\left(x^2 + \frac{1}{x^2}\right)^2 + \ldots\ldots + {}^nC_n\left(x^2 + \frac{1}{x^2}\right)^n$$

This contains each of the term $x^0, x^2, x^4, \ldots x^2n, x^{-2}, x^{-4}, \ldots, x^{-2n}$

coefficient of constant term $= nC_0 + (nC_2)(2) + (nC_4)(4C_2) + (nC_6)(6C_3) + \ldots \neq 2^{n-1}$ coefficient of $x^{2n-2}$ in $nC_{n-1} = n$

coefficient of $x^2$ is ${}^nC_1 + ({}^nC_3)({}^3C_1) + (nC_5)({}^5C_2) + \ldots > n$

**13. a, c, d.**

$${}^nC_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \cdots + {}^{n+m-1}C_m$$
$$= {}^nC_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + \cdots + {}^{n+m-1}C_{n-1}$$

= Coefficient of $x^{n-1}$ in $(1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n+2} + \cdots + (1 + x)^{n+m-1}$

$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^n\left[\frac{(1+x)^m-1}{(1+x)-1}\right]$

$= \text{Coefficient of } x^{n-1} \text{ in } \frac{(1+x)^{m+n}-(1+x)^n}{x}$

$= \text{Coefficient of } x^n \text{ in } \left[(1+x)^{m+n}-(1+x)^n\right]$

$= {}^{m+n}C_n - 1$

Similarly, we can prove

$${}^mC_1 + {}^{m+1}C_2 + {}^{m+2}C_3 + \cdots + {}^{m+n-1}C_n = {}^{m+n}C_m - 1$$

**14. a, b, d.**

$\frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}$

$= \frac{(n-1)(n-2)\cdots(n-m+1)(n-m)\cdots 2\cdot 1}{(n-m)!(m-1)!}$

$= {}^{n-1}C_{m-1}$

$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1}$

$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^n(1+x)^{-1}$

Now,

$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_{m-1}x^{m-1} + \cdots + C_nx^n$ (1)

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^{m-1}x^{m-1} + \cdots$ (2)

Collecting the coefficients of $x^{m-1}$ in the product of (1) and (2), we get

$(-1)^{m-1}C_0 + (-1)^{m-2}C_1 + \cdots + C_{m-1}$

$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1}$

$= {}^{n-1}C_{m-1}$

$\therefore \quad C_0 - C_1 + C_2 - \cdots + (-1)^{m-1}C_{m-1}$

$= {}^{n-1}C_{m-1}(-1)^{m-1}$

$= \frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}(-1)^{m-1}$

**15. a**

We have,

$\frac{17}{4} + 3\sqrt{2} = \frac{1}{4}(9 + 8 + 12\sqrt{2})$

$= \frac{1}{4}(3 + 2\sqrt{2})^2$

$\therefore \quad 3 - \sqrt{\frac{17}{4} + 3\sqrt{2}} = 3 - \frac{1}{2}(3 + 2\sqrt{2})$

$= \frac{3}{2} - \sqrt{2}$

Hence, the 10th term of $\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{20} = \left(\frac{3}{2} - \sqrt{2}\right)^{20}$ is

$${}^{20}C_9\left(\frac{3}{2}\right)^{20-9}(-\sqrt{2})^9$$

which is an irrational number.

**16. a, b, c.**

$(x\sin p + x^{-1}\cos p)^{10}$

The general term in the expansion is

$T_{r+1} = {}^{10}C_r(x\sin p)^{10-r}(x^{-1}\cos p)^r$

For the term independent of $x$, we have $10 - 2r = 0$ or $r = 5$.

Hence, the independent term is

$${}^{10}C_5 \sin^5 p \cos^5 p = {}^{10}C_5\frac{\sin^5 2p}{32}$$

which is the greatest when $\sin 2p = 1$.

The least value of ${}^{10}C_5\frac{\sin^5 2p}{32}$ is $-\frac{10!}{2^5(5!)^2}$ when $\sin 2p$

$= -1$ or $p = (4n-1)\frac{\pi}{4}, n \in Z$.

Sum of coefficient is $(\sin p + \cos p)^{10}$, when $x = 1$ or $(1 + \sin 2p)^5$, which is least when $\sin 2p = -1$.

Hence, least sum of coefficients is zero. Greatest sum of coefficient occurs when $\sin 2p = 1$. Hence, greatest sum is $2^5 = 32$.

**17. b, c, d.**

L.H.S. $= (1 + 2x^2 + x^4)(1 + C_1x + C_2x^2 + C_3x^3 + \cdots)$

R.H.S. $= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

Comparing the coefficients of $x, x^2, x^3, \ldots$

$a_1 = C_1, a_2 = C_2 + 2, a_3 = C_3 + 2C_1$ (1)

Now, $2a_2 = a_1 + a_3$ (A.P.)

$\Rightarrow \quad 2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2{}^nC_1)$ [Using (1)]

$\Rightarrow \quad 2\frac{n(n-1)}{2} + 4 = 3n + \frac{n(n-1)(n-2)}{6}$

$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow \quad (n-2)(n^2 - 7n + 12) = 0$

$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$

$\Rightarrow \quad n = 2, 3, 4$

**18. a, d.**

Middle term is $\left(\frac{n}{2} + 1\right)^{\text{th}}$ or $(4+1)^{\text{th}}$ or $T_5$

$\Rightarrow \quad T_5 = {}^8C_4\left(\frac{x}{2}\right)^4 \times 2^4 = 1120$

$\Rightarrow \quad \frac{8\times7\times6\times5}{1\times2\times3\times4}x^4 = 1120$

$\Rightarrow \quad x^4 = \frac{1120}{70} = 16$

$\Rightarrow \quad (x^2 + 4)(x^2 - 4) = 0$

$\Rightarrow \quad x = \pm 2 \ (\because \ x \in R)$

**19. a, b, c, d.**

Let $T_5$ be numerically the greatest term in the expansion of $(1 + x/3)^{10}$. Then,

$\left|\frac{T_5}{T_4}\right| \geq 1$ and $\left|\frac{T_6}{T_5}\right| \leq 1$

Now,

$\frac{T_{r+1}}{T_r} = \frac{10 - r + 1}{r}\frac{x}{3}$

$\Rightarrow \quad \left|\frac{7}{4}\times\frac{x}{3}\right| \geq 1$ and $\left|\frac{6}{5}\times\frac{x}{3}\right| \leq 1$

$\Rightarrow \quad |x| \geq \frac{12}{7}$ and $|x| \leq \frac{5}{2}$ (1)

$\Rightarrow \quad \frac{12}{7} \leq |x| \leq \frac{5}{2}$

$\Rightarrow \quad x \in \left[-\frac{5}{2}, -\frac{12}{7}\right] \cup \left[\frac{12}{7}, \frac{5}{2}\right]$

**20. a, c.**

$(1-y)^m(1+y)^n$

$= (1-{}^mC_1y+{}^mC_2y^2-\cdots)(1+{}^nC_1y+{}^nC_2y^2+\cdots)$

$= 1+(n-m)y+\left\{\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-mn\right\}y^2+\cdots$

Given,

$a_1 = 10$

$\Rightarrow a_1 = n-m = 10$ (1)

$a_2 = \frac{m^2+n^2-m-n-2mn}{2} = 10$

$(m-n)^2-(m+n) = 20$

$\Rightarrow m+n = 80$ (2)

Solving (1) and (2), we get $m = 35$, $n = 45$.

## Reasoning Type

**1. a.** $({}^{10}C_0)+({}^{10}C_0+{}^{10}C_1)+({}^{10}C_0+{}^{10}C_1+{}^{10}C_2)+\cdots+({}^{10}C_0+{}^{10}C_1+{}^{10}C_2+\cdots+{}^{10}C_9)$

$= 10\,{}^{10}C_0+9\,{}^{10}C_1+8\,{}^{10}C_2+\cdots+{}^{10}C_9$

$= {}^{10}C_1+2\,{}^{10}C_2+3\,{}^{10}C_3+\cdots+10\,{}^{10}C_{10}$

$= \sum_{r=1}^{10} r\,{}^{10}C_r = 10\sum_{r=1}^{10} {}^9C_{r-1} = 10\times 2^9$

**2. b.** $\frac{T_{r+1}}{T_r} = \frac{12-r+1}{r}\frac{11}{10}$

Let,

$T_{r+1} \ge T_r \Rightarrow 13-r \ge 1.1x$

$\Rightarrow 13 \ge 2.1r$

$\Rightarrow r \le 6.19$

Hence, the greatest term occurs for $r = 6$. Hence, 7th term is the greatest term. Also, the binomial coefficient of 7th term is ${}^{12}C_6$ which is the greatest binomial coefficient.

But this is not the reason for which $T_7$ is the greatest. Here, it is co-incident that the greatest term has the greatest binomial coefficient

Hence, statement 1 is true, statement 2 is true; but statement 2 is not a correct explanation of statement 1.

**3. a.** $3456^{2222} = (7\times 493+5)^{2222}$

$= (7k+5)^{2222}$

$= 7m+5^{2222}$

Now,

$5^{2222} = 5^2(5^3)^{740}$

$= 25(125)^{740}$

$= 25(126-1)^{740}$

$= 25[7n+1]$

$= 175n+25$

Remainder when $175n+25$ is divided by 7 is 4.

Hence, both the statements are correct and statement 2 is a correct explanation of statement 1.

**4. d.** Statement 2 is true as it is the property of binomial coefficients. But statement 1 is false as three consecutive binomial coefficients may be in A.P. but not always.

**5. a.** $(1+x+x^2)^n = \sum_{r=0}^{2n} a_rx^r$ (1)

We know that

$(1-x)^n = \sum_{r=0}^{n}(-1)^{n-r}\,{}^nC_rx^r = \sum_{r=0}^{n}(-1)^{n-r}\,{}^nC_rx^{n-r}$ (2)

Multiplying (1) and (2), we get

$\sum_{r=0}^{n}(-1)^{n-r}\,{}^nC_ra_r =$ coefficient of $x^n$ in $(1-x^3)^n$

Since $n \ne 3k$, therefore

$\sum_{r=0}^{n}(-1)^{n-r}a_r\,{}^nC_r = 0$

$\Rightarrow \sum_{r=0}^{n}(-1)^r a_r\,{}^nC_r = 0$

Hence, both the statements are correct and statement 2 is a correct explanation of statement 1.

**6. a.** $(1+x)^n-nx-1 = (1+{}^nC_1x+{}^nC_2x^2+\cdots+{}^nC_nx^n)-nx-1$ (1)

$= {}^nC_2x^2+\cdots+{}^nC_nx^n$

$= x^2({}^nC_2+{}^nC_3x+\cdots+{}^nC_nx^{n-2})$

Hence, $(1+x)^n-nx-1$ is divisible by $x^2$.

Now in (1), replace $x$ by 8 and $n$ by $n+1$. Then, we have

$(1+8)^{n+1}-(n+1)8-1 = 8^2({}^nC_2+{}^nC_38+\cdots+{}^nC_n8^{n-2})$

$\Rightarrow 9^{2n+2}-8n-9 = 8^2({}^nC_2+{}^nC_38+\cdots+{}^nC_n8^{n-2})$

which is divisible by 64.

Hence, both the statements are correct and statement 2 is a correct explanation of statement 1.

**7. b.** $(1+x+x^2+x^3+x^4)^{1000} = a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_{4000}x^{4000}$

Clearly, there are 4001 terms. Also, number of term in the expansion $(a_1+a_2+\cdots+a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

Clearly, statement 2 has nothing to do with statement 1.

**8. a.** Coefficient of $x^n$ in $\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\frac{x^n}{n!}\right)^3$

$=$ Coefficient of $x^n$ in $\left(1+x+\frac{x^2}{2!}+\cdots\frac{x^n}{n!}+\cdots\right)^3$

as higher powers of $x$ are not counted while calculating the coefficient of $x^n$)

$=$ Coefficient of $x^n$ in $e^{3x} = \frac{3^n}{n!}$

**9. b.** $(1+x)^{41}(1-x+x^2)^{40}$

$= (1+x)(1+x)^{40}(1-x+x^2)^{40}$

$= (1+x)(1+x^3)^{40}$

$= (1+x^3)^{40}+x(1+x^3)^{40}$

$= (1+{}^{40}C_1x^3+{}^{40}C_2x^6+\cdots+{}^{40}C_{40}x^{120})+({}^{40}C_0+{}^{40}C_1x^4+{}^{40}C_2x^7+\cdots+{}^{40}C_{40}x^{121})$

Hence, the coefficient of $x^{85}$ is zero as there is no term in the above expansion which has $x^{85}$.

Also, statement 2 is correct but it is not a correct explanation of statement 1.

**10. b.** We know that the total number of terms in $(x_1+x_2+\cdots+x_r)^n$ is ${}^{n+r-1}C_{r-1}$. So, the total number of term in $(x_1+x_2+\cdots+x_n)^3$ is

${}^{3+n-1}C_{n-1} = {}^{n+2}C_{n-1} = {}^{n+2}C_3 = \frac{(n+2)(n+1)n}{6}$

and the total number of terms in $(x_1+x_2+x_3)^n$ is

${}^{n+3-1}C_{n-1} = {}^{n+2}C_3 = \frac{(n+2)(n+1)n}{6}$

**11. a.** We have,

$$\left(2+\sqrt{5}\right)^p+\left(2-\sqrt{5}\right)^p = 2[2^p + {}^pC_2\, 2^{p-5}\, 5$$
$$+ {}^pC_4\, 2^{p-4}\, 5^2 + \cdots + {}^pC_{p-1}\, 2\times 5^{(p-1)/2}] \qquad (1)$$

From (1), $\left(2+\sqrt{5}\right)^p+\left(2-\sqrt{5}\right)^p$ is an integer and

$$-1<\left(2-\sqrt{5}\right)^p<0 \quad (\because p \text{ is odd})$$

So,

$$\left[\left(2+\sqrt{5}\right)^p\right] = (2+\sqrt{5})^p + (2-\sqrt{5})^p$$
$$= 2^{p+1} + {}^pC_2\, 2^{p-1}\, 5 + \cdots + {}^pC_{p-1}\, 2^2\, 5^{(p-1)/2}$$

$$\therefore \quad \left[\left(2+\sqrt{5}\right)^p\right] - 2^{p+1} = 2[{}^pC_2\, 2^{p-2}\, 5 + {}^pC_4 2^{p-4}\, 5^2 + \cdots + {}^pC_{p-1}\, 2\times 5^{(p-1)/2}]$$

Now, all the binomial coefficients

$${}^pC_2 = \frac{p(p-1)}{1\times 2},$$

$${}^pC_4 = \frac{p(p-1)(p-2)(p-3)}{1\times2\times3\times4}, \ldots, {}^pC_{p-1} = p$$

are divisible by the prime $p$. Thus, R.H.S. is divisible by $p$.

**12. a.** Statement 2 is true (can be checked easily) and that is why ${}^{2n}C_0 < {}^{2n}C_1 < {}^{2n}C_2 < \cdots < {}^{2n}C_{n-1} < {}^{2n}C_n > {}^{2n}C_{n+1} \cdots > {}^{2n}C_{2n}$.

**13. b.** Obviously, statement 2 is true. But to get the sum of coefficients in the expansion of $(3^{-x/4} + 3^{5x/4})^n$, we must put $x = 0$.

**14. a.** We know that

${}^mC_r + {}^mC_{r-1}\, {}^nC_1 + {}^mC_{r-2}\, {}^nC_2 + \cdots + {}^nC_r$
$=$ Coefficient of $x^r$ in $(1+x)^m(1+x)^n$
$=$ Coefficient of $x^r$ in $(1+x)^{m+n}$
$= {}^{m+n}C_r$
$= 0$ as $m + n < r$

**15. a.** $$S = \sum\sum_{0\le i<j\le n}\left(\frac{i}{{}^nC_i}+\frac{j}{{}^nC_j}\right)$$

$$= \sum\sum_{0\le i<j\le n}\left(\frac{n-i}{{}^nC_{n-i}}+\frac{n-j}{{}^nC_{n-j}}\right)$$

$$= n\sum\sum_{0\le i<j\le n}\left(\frac{1}{{}^nC_i}+\frac{1}{{}^nC_j}\right) - S$$

$$\Rightarrow \quad S = \frac{n}{2}\sum\sum_{0\le i<j\le n}\left(\frac{1}{{}^nC_i}+\frac{1}{{}^nC_j}\right)$$

$$= \frac{n}{2}\left(\sum_{r=0}^{n-1}\frac{n-r}{{}^nC_r}+\sum_{r=1}^{n}\frac{r}{{}^nC_r}\right)$$

$$= \frac{n}{2}\left(\sum_{r=0}^{n}\frac{n}{{}^nC_r}\right)$$

$$= \frac{n^2}{2}a$$

## Linked Comprehension Type

**For Problems 1–3**

**1. b, 2. d, 3. c.**

**Sol.** The coefficient of the 2nd, 3rd and 4th terms in the expansion are ${}^mC_1$, ${}^mC_2$ and ${}^mC_3$, which are given in A.P. Hence,

$$2\,{}^mC_2 = {}^mC_1 + {}^mC_3$$

$$\Rightarrow \quad \frac{2m(m-1)}{2!} = m + \frac{m(m-1)(m-2)}{3!}$$

$$\Rightarrow \quad m(m^2 - 9m + 14) = 0$$

$$\Rightarrow \quad m(m-2)(m-7) = 0$$

$$\Rightarrow \quad m = 7 \ (\because m \ne 0 \text{ or } 2 \text{ as } 6^{\text{th}} \text{ term is given equal to } 21)$$

Now, 6th term in the expansion, when $m = 7$, is

$${}^7C_5\left[\sqrt{\{2^{\log(10-3^x)}\}}\right]^{7-5}\times\left[\sqrt[5]{\{2^{(x-3)\log 3}\}}\right]^5 = 21$$

$$\Rightarrow \quad \frac{7\times 6}{2!}\,2^{\log(10-3^x)}\times 2^{(x-2)\log 3} = 21$$

$$\Rightarrow \quad 2^{\log(10-3^x)+(x-2)\log 3} = 1 = 2^0$$

$$\Rightarrow \quad \log(10-3^x) + (x-2)\log 3 = 0$$

$$\Rightarrow \quad \log(10-3^x)(3)^{(x-2)} = 0$$

$$\Rightarrow \quad (10-3^x)\times 3^x \times 3^{-2} = 1$$

$$\Rightarrow \quad 10\times 3^x - (3^x)^2 = 9$$

$$\Rightarrow \quad (3^x)^2 - 10\times 3^x + 9 = 0$$

$$\Rightarrow \quad (3^x - 1)(3^x - 9) = 0$$

$$\Rightarrow \quad 3^x - 1 = 0 \Rightarrow 3^x = 1 = 3^0 \Rightarrow x = 0$$

$$\Rightarrow \quad 3^x - 9 = 0 \Rightarrow 3^x = 3^2 \Rightarrow x = 2$$

Hence, $x = 0$ or $2$. When $x = 2$,

$$\left[\sqrt{\{2^{\log(10-3^x)}\}}+\sqrt[5]{\{2^{(x-2)\log 3}\}}\right]^m$$

$$= [1+1]^7 = 128$$

When $x = 0$,

$$\left[\sqrt{\{2^{\log(10-3^x)}\}}+\sqrt[5]{\{2^{(x-2)\log 3}\}}\right]^m$$

$$= \left[\sqrt{\{2^{\log 9}\}}+\sqrt[5]{\{2^{-2\log 3}\}}\right]^7$$

$$= \left[2^{\frac{\log 9}{2}}+\frac{1}{2^{\frac{\log 9}{5}}}\right]^7 > 2^7$$

Hence, the minimum value is 128.

**For Problems 4–6**

**4. b, 5. c, 6. c.**

**Sol.** 2nd term is ${}^nC_1 x^{n-1} a = 240$ (1)

3rd term is ${}^nC_2\, x^{n-2}\, a^2 = 720$ (2)

4th term is ${}^nC_3\, x^{n-3}\, a^3 = 1080$ (3)

Multiplying (1) and (3) and dividing by the square of (2), we get

$$\frac{{}^nC_1\times{}^nC_3}{({}^nC_2)^2} = \frac{240\times1080}{(720)^2}$$

$$\Rightarrow \quad \frac{n\times n(n-1)(n-2)(2!)^2}{n^2(n-1)^2\times 3!} = \frac{1}{2}$$

$\Rightarrow\ 4(n-2) = 3(n-1)\quad (\because n \neq 1)$

$\Rightarrow\ n = 5$

Putting $n = 5$, from (1) and (2), we get

$5x^4a = 240$ and $10x^3a^2 = 720$

$$\Rightarrow \frac{(5x^4a)^2}{10x^3a^2} = \frac{(240)^2}{720}$$

or

$x^5 = 32$

$\therefore\ x = 2$

$$\therefore\ a = \frac{240}{5x^4} = \frac{48}{2^4} = 3$$

Hence, $x = 2$, $a = 3$ and $n = 5$.

$(x-a)^n = (2-3)^5 = -1$

Also,

$(2+3)^5 = 2^5 + {}^5C_1 2^4\times 3 + {}^5C_2 2^3 \times 3^2 + {}^5C_3 2^2\times 3^3 + {}^5C_4 2\times 3^4 + {}^5C_5 3^5$

$= 32 + 240 + 720 + 1080 + 810 + 243$

Hence, least value of the term is 32.

Sum of odd-numbered terms is $32 + 720 + 810 = 1562$.

**For Problems 7–9**

**7. b, 8. a, 9. c.**

**Sol.** Let,

$$(1 + x + x^2)^{20} = \sum_{r=0}^{40} a_r x^r \tag{1}$$

Replacing $x$ by $1/x$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^{20} = \sum_{r=0}^{40} a_r \left(\frac{1}{x}\right)^r$$

$$\Rightarrow\ (1 + x + x^2)^{20} = \sum_{r=0}^{40} a_r x^{40-r} \tag{2}$$

Since (1) and (2) are same series, coefficient of $x^r$ in (1) = coefficient of $x^r$ in (2)

$\Rightarrow\ a_r = a_{40-r}$

In (1), putting $x = 1$, we get

$3^{20} = a_0 + a_1 + a_2 + \cdots + a_{40}$

$= (a_0 + a_1 + a_2 + \cdots + a_{19}) + a_{20} + (a_{21} + a_{n+2} + \cdots + a_{40})$

$= 2(a_0 + a_1 + a_2 + \cdots + a_{19}) + a_{20}\ (\because\ a_r = a_{40-r})$

$$\Rightarrow\ a_0 + a_1 + a_2 + \cdots + a_{19} = \frac{1}{2}(3^{20} - a_{20}) = \frac{1}{2}(9^{10} - a_{20})$$

Also,

$a_0 + 3a_1 + 5a_2 + \cdots + 81a_{40}$

$= (a_0 + 81a_{40}) + (3a_1 + 79a_{39}) + \cdots + (39a_{19} + 43a_{21}) + 41a_{20}$

$= 82(a_0 + a_1 + a_2 + \cdots + a_{19}) + 41a_{20}$

$= 41(9^{10} - a_{20}) + 41a_{20}$

$= 41 \times 3^{20}$

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots$ suggests that we have to multiply the two expansions.

Replacing $x$ by $-1/x$ in (1), we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^{20} = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \cdots + \frac{a_{40}}{x^{40}}$$

$$\Rightarrow\ (1 - x + x^2)^{20} = a_0x^{40} - a_1x^{39} + a_2x^{38} - \cdots + a_{40} \tag{3}$$

Clearly,

$a_0^2 - a_1^2 + a_2^2 + \cdots + a_{40}^2$ is the coefficient of $x^{40}$ in

$(1 + x + x^2)^{20}(1 - x + x^2)^{20}$

$=$ Coefficient of $x^{40}$ in $(1 + x^2 + x^4)^{20}$

In $(1 + x^2 + x^4)^{20}$, replace $x^2$ by $y$, then the coefficient of $y^{20}$ in $(1 + y + y^2)^{20}$ is $a_{20}$. Hence,

$a_0^2 - a_1^2 + a_2^2 - \cdots + a_{40}^2 = a_{20}$

$\Rightarrow\ (a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2) + a_{20}^2 + (-a_{21}^2 + \cdots + a_{40}^2) = a_{20}$

$\Rightarrow\ 2(a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2) + a_{20}^2 = a_{20}$

$$\Rightarrow\ a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2 = \frac{a_{20}}{2}[1 - a_{20}]$$

**For Problems 10–12**

**10. c, 11. a, 12. c.**

**Sol.** $a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = 0$ has roots ${}^{99}C_0, {}^{99}C_1, {}^{99}C_2, \ldots, {}^{99}C_{99}$.

$\Rightarrow\ a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = (x - {}^{99}C_0)(x - {}^{99}C_1)(x - {}^{99}C_2)\cdots(x - {}^{99}C_{99})$

Now, sum of roots is

$${}^{99}C_0 + {}^{99}C_1 + {}^{99}C_2 + \cdots + {}^{99}C_{99} = -\frac{a_{99}}{\text{coefficient of } x^{100}}$$

$\Rightarrow\ a_{99} = -2^{99}$

Also, sum of product of roots taken two at a time is

$$\frac{a_{99}}{\text{coefficient of } x^{100}}$$

$$\therefore \sum\sum_{0 \le i < j \le 99} {}^{99}C_i\, {}^{99}C_j = \frac{\left(\sum_{i=0}^{99}\sum_{j=0}^{99} {}^{99}C_i {}^{99}C_j\right) - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{99} {}^{99}C_i 2^{99}\right) - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{2^{99}2^{99} - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{2^{198} - {}^{198}C_{99}}{2}$$

$({}^{99}C_0)^2 + ({}^{99}C_1)^2 + \cdots + ({}^{99}C_{99})^2$

$= ({}^{99}C_0 + {}^{99}C_1 + {}^{99}C_2 \cdots + {}^{99}C_{99})^2 - 2\sum\sum_{0 \le i < j \le 99} {}^{99}C_i\,{}^{99}C_j$

$= (-a_{99})^2 - 2a_{98}$

$= a_{99}^2 - 2a_{98}$

**For Problems 13–15**

**13. a, 14. c, 15. a.**

**Sol.** a. $$\sum_{r=0}^{100} {}^{100}C_r \sin rx = \text{Im}\left(\sum_{r=0}^{100} {}^{100}C_r e^{irx}\right) \quad (\text{Im} = \text{imaginary part})$$

$$= \text{Im}\left(\sum_{r=0}^{100} {}^{100}C_r (e^{ix})^r\right)$$

$= \text{Im}((1 + e^{ix})^{100})$

$= \text{Im}\,(1 + \cos x + i \sin x)^{100}$

$= \text{Im}\left(2\cos^2\frac{x}{2} + 2i\sin\frac{x}{2}\times\cos\frac{x}{2}\right)^{100}$

$= \text{Im}\left(2\cos\frac{x}{2}\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)\right)^{100}$

$= 2^{100}\cos^{100}\frac{x}{2}\,\sin(50x)$

$\sum_{r=0}^{50} {}^{50}C_r a^r \times b^{50-r} \times \cos(rB - (50 - r)A)$

$= \text{Re}\left(\sum_{r=0}^{50} {}^{50}C_r a^r \times b^{50-r} \times e^{i(rB-(50-r)A)}\right)$

$= \text{Re}\left(\sum_{r=0}^{50} {}^{50}C_r (a\times e^{iB})^r \times (b \times e^{-iA})^{50-r}\right)$

$= \text{Re}(ae^{iB} + b\,e^{-iA})^{50}$

$= \text{Re}(a\cos B + ia\sin B + b\cos A - ib\sin A)^{50}$

$= \text{Re}(a\cos B + b\cos A)^{50} = c^{50}$ ($\because$ $a\sin B = b\sin A$)

$$\frac{\sum_{r=0}^{50} {}^{50}C_r \sin 2rx}{\sum_{r=0}^{50} {}^{50}C_r \cos 2rx} = \frac{\sum_{r=0}^{50} {}^{50}C_{50-r} \sin 2(50-r)x}{\sum_{r=0}^{50} {}^{50}C_{50-r} \cos 2(50-r)x}$$

$$= \frac{\sum_{r=0}^{50} {}^{50}C_r[\sin 2rx + \sin 2(50-r)x]}{\sum_{r=0}^{50} {}^{50}C_r[\cos 2rx + \cos 2(50-r)x]}\quad\left(\because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right)$$

$$= \frac{\sum_{r=0}^{50} {}^{50}C_r 2\sin(50x)\cos(2r-50)x}{\sum_{r=0}^{50} {}^{50}C_r 2\cos(50x)\cos(2r-50)x}$$

$= \tan(50x)$

$\Rightarrow f(\pi/8) = \tan(25\pi/4) = \tan(6\pi + \pi/4) = 1$

**For Problems 16–18**

**16. b, 17. b, 18. c.**

**Sol.**

General term of the series is

$$T(r) = \sum_{r=1}^{50} \frac{{}^{50+r}C_r(2r-1)}{{}^{50}C_r(50+r)}$$

$$= \frac{{}^{50+r}C_r}{{}^{50}C_r}\left(1 - \frac{50-r+1}{50+r}\right)$$

$$= \frac{{}^{50+r}C_r}{{}^{50}C_r} - \frac{{}^{50+r}C_r}{{}^{50}C_r}\left(\frac{50-r+1}{50+r}\right)$$

Now,

$$\frac{{}^{50+r}C_r}{{}^{50}C_r}\left(\frac{50-r+1}{50+r}\right)$$

$$= \frac{(50-r+1)(50+r)!r!(50-r)!}{r!50!(50+r)50!}$$

$$= \frac{(50-r+1)!(50+r-1)!}{50!50!}$$

$$= \frac{{}^{50+r-1}C_{r-1}}{{}^{50}C_{r-1}}$$

$$\Rightarrow T(r) = \frac{{}^{50+r}C_r}{{}^{50}C_r} - \frac{{}^{50+r-1}C_{r-1}}{{}^{50}C_{r-1}} = V(r) - V(r-1)$$

where $V(r) = \dfrac{{}^{50+r}C_r}{{}^{50}C_r}$

Now, sum of the given series

$$P = \sum_{r=1}^{50} T(r) = V(50) - V(0)$$

$$= \frac{{}^{100}C_{50}}{{}^{50}C_{50}} - \frac{{}^{50}C_0}{{}^{50}C_0} = {}^{100}C_{50} - 1$$

Also,

$$Q = \sum_{r=0}^{50}({}^{50}C_r)^2 = {}^{50}C_0^2 + {}^{50}C_1^2 + {}^{50}C_2^2 + \cdots + {}^{50}C_{50}^2 = {}^{100}C_{50}$$

$\Rightarrow P - Q = -1$

We know that

$$C_0^2 - C_1^2 + C_2^2 + \cdots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n\,{}^nC_{n/2}, & \text{if } n \text{ is even}\end{cases}$$

$$\Rightarrow \sum_{r=0}^{100}(-1)^r\left({}^{100}C_r\right)^2 = (-1)^{100}\,{}^{100}C_{50} = {}^{100}C_{50}$$

$\Rightarrow P - R = -1$

$Q + R = 2^{100}C_{50} = 2P + 2$

**For Problems 19–21**

**19. a, 20. b, 21. c.**

**Sol.** Suppose $A$ contains $r$ $(0 \le r \le n)$ elements.

Then, $B$ is constructed by selecting some elements from the remaining $n - r$ elements. Here, A can be chosen in ${}^nC_r$ ways and $B$ in ${}^{n-r}C_0 + {}^{n-r}C_1 + \cdots + {}^{n-r}C_{n-r} = 2^{n-r}$ ways.

So, the total number of ways of choosing $A$ and $B$ is ${}^nC_r \times 2^{n-r}$.

But $r$ can vary from 0 to $n$. So, total number of ways is

$$\sum_{r=0}^{n} {}^nC_r \times 2^{n-r} = (1+2)^n = 3^n$$

If $A$ contains $r$ elements, then $B$ contains $(r+1)$ elements. Then, the number of ways of choosing $A$ and $B$ is ${}^nC_r \times {}^nC_{r+1} = C_r C_{r+1}$.

But $r$ can vary from 0 to $(n-1)$.

So, the number of ways is

$$\sum_{r=0}^{n-1} C_r C_{r+1} = C_0C_1 + C_1C_2 + \cdots + C_{n-1}C_n = {}^{2n}C_{n-1}$$

Let $A$ contains $r$ $(0 \le r \le n)$ elements.

Then, $A$ can be chosen in ${}^nC_r$ ways. The subset $B$ of $A$ can have at most $r$ elements, and the number of ways of choosing $B$ is $2^r$.

Therefore, the number of ways of choosing $A$ and $B$ is ${}^nC_r \times 2^r$.

But $r$ can vary from 0 to $n$.

So, the total number of ways is

$$\sum_{r=0}^{n} {}^nC_r \times 2^r = (1+2)^n = 3^n$$

## Matrix-Match Type

**1. a → q, r, s; b → p, q, r, s; c → p, q, r; d → p, q.**

**a.** ${}^{(n+1)}C_4 + {}^{(n+1)}C_3 + {}^{(n+2)}C_3 = {}^{(n+3)}C_4$

$\Rightarrow {}^{(n+3)}C_4 > {}^{(n+3)}C_3 \Rightarrow \frac{{}^{n+3}C_4}{{}^{n+3}C_3} > 1$

$\Rightarrow n > 4$ or $n \ge 5$

**b.** $(3053)^{456} - (2417)^{333}$

$= (339 \times 9 + 2)^{456} - (269 \times 9 - 4)^{333}$

Remainder of given number is same as remainder of $2^{456} + 4^{333}$

and

$2^{456} + 4^{333} = (64)^{76} + (64)^{111}$

$= (1 + 63)^{76} + (1 + 63)^{111}$

$= (1 + 9 \times 7)^{76} + (1 + 9 \times 7)^{111}$

Hence, the remainder is 2.

**c.** We know that $n!$ terminates in 0 for $n \ge 5$ and $3^{4n}$ terminates in 1 ($\because\ 3^4 = 81$).

Therefore, $3^{180} = (3^4)^{45}$ terminates in 1.

Also, $3^3 = 27$ terminates in 7.

Hence, $183! + 3^{183}$ terminates in 7.

That is, the digit in the unit place is 7.

**d.** We are given

${}^mC_0 + {}^mC_1 + {}^mC_2 = 46$

$\Rightarrow 2m + m(m-1) = 90$

$\Rightarrow m^2 + m - 90 = 0$

$\Rightarrow m = 9$ as $m > 0$

Now, $(r+1)^{\text{th}}$ term of $\left(x^2 + \frac{1}{x}\right)^m$ is

$${}^mC_r(x^2)^{m-r}\left(\frac{1}{x}\right)^r = {}^mC_r\, x^{2m-3r}$$

For this to be independent of $x$, $2m - 3r = 0 \Rightarrow r = 6$.

**2. a → q; b → s; c → p; d → r.**

We know that

$${}^nC_0^2 + {}^nC_1^2 + \cdots + {}^nC_n^2 = {}^{2n}C_n$$

and

$${}^nC_0^2 - {}^nC_1^2 + \cdots + (-1)^n\,{}^nC_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ {}^nC_{n/2}(-1)^n, & \text{if } n \text{ is even} \end{cases}$$

From this, ${}^{31}C_0^2 - {}^{31}C_1^2 + {}^{31}C_2^2 - \cdots - {}^{31}C_{31}^2 = 0$

${}^{32}C_0^2 - {}^{32}C_1^2 + {}^{32}C_2^2 - \cdots + {}^{32}C_{32}^2 = {}^{32}C_{16}$

${}^{32}C_0^2 + {}^{32}C_1^2 + {}^{32}C_2^2 - \cdots + {}^{32}C_{32}^2 = {}^{64}C_{32}$

Also, $(1/32)(1 \times {}^{32}C_1^2 + 2 \times {}^{32}C_2^2 - \cdots + 32 \times {}^{32}C_{32}^2)$

$$= \frac{1}{32}\sum_{r=1}^{32} r\left({}^{32}C_r\right)^2$$

$$= \frac{1}{32}\sum_{r=1}^{32} r\,{}^{32}C_r\,{}^{32}C_{32-r}$$

$$= \frac{1}{32}\sum_{r=1}^{32} 32\,{}^{31}C_{r-1}\,{}^{32}C_{32-r}$$

$$= {}^{63}C_{31} = {}^{63}C_{32}$$

**3. a → q; b → s; c → p; d → r.**

In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} f(i) \times f(j) = \sum_{i=1}^{n}\left(f(i)\left(\sum_{j=1}^{n} f(j)\right)\right)$ $i$ and $j$ are independent. In this summation, three types of terms occur, for which $i < j$, $i > j$ and $i = j$. Also, the sum of terms when $i < j$ is equal to the sum of the terms when $i > j$ if $f(i)$ and $f(j)$ are symmetrical. So, in that case

$$\sum_{i=0}^{n}\sum_{j=0}^{n} f(i) \times f(j) = \sum\sum_{0 \le i < j \le n} f(i) \times f(j) + \sum\sum_{0 \le j < i \le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$= 2\sum\sum_{0 \le i < j \le n} f(i)f(j) + \sum\sum_{i=j} f(i)f(j)$$

$$\Rightarrow \sum\sum_{0 \le i < j \le n} f(i)f(j) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} f(i) \times f(j) - \sum\sum_{i=j} f(i)f(j)}{2}$$

**a.** $\sum\sum_{i \ne j} {}^{10}C_i\,{}^{10}C_j = \sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\,{}^{10}C_j - \sum_{i=0}^{10} {}^{10}C_i^2 = 2^{20} - {}^{20}C_{10}$

**b.** $\sum\sum_{0 \le i \le j \le 10} {}^{10}C_i\,{}^{10}C_j = \frac{\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\,{}^{10}C_j + \sum_{i=0}^{10} {}^{10}C_i^2}{2} = \frac{2^{20} + {}^{20}C_{10}}{2}$

**c.** $\sum\sum_{0 \le i < j \le 10} {}^{10}C_i\,{}^{10}C_j = \frac{\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\,{}^{10}C_j - \sum_{i=0}^{10} {}^{10}C_i^2}{2}$

$$= \frac{2^{20} - {}^{20}C_{10}}{2}$$

**d.** $\sum_{i=0}^{10}\sum_{j=0}^{10} {}^{10}C_i\,{}^{10}C_j = \sum_{i=0}^{10} {}^{10}C_i \sum_{j=0}^{10} {}^{10}C_j = 2^{20}$

**4. a → p, q, s; b → p, q, r, s; c → p, q, r, s; d → p, q, s.**

**a.** In $(1+x)^{41} = {}^{41}C_0 + {}^{41}C_1x + {}^{41}C_2x^2 + \cdots + {}^{41}C_{20}x^{20} + {}^{41}C_{21}x^{21} + \cdots + {}^{41}C_{41}x^{41}$

$\Rightarrow {}^{41}C_{21} + {}^{41}C_{22} + \cdots + {}^{41}C_{41} = 2^{40}$

**b.** $(1 + \sqrt{2})^{42} = {}^{42}C_0 + {}^{42}(C_1\sqrt{2}) + {}^{42}C_2(\sqrt{2})^2 + {}^{42}C_1(\sqrt{2})^3 + \cdots + {}^{42}C_{42}(\sqrt{2})^{42}$

Sum of binomial coefficients of rational terms is

${}^{42}C_0 + {}^{42}C_2 + {}^{42}C_4 + \cdots + {}^{42}C_{42} = 2^{41}$

**c.** $\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{21} = \left(\frac{x^3 + x + x^4 + 1}{x^2}\right)^{21}$

$$= \frac{a_0 + a_1x + a_2x^2 + \cdots + a_{82}x^{82}}{x^{42}} \quad (1)$$

Now, putting $x = 1$, we get

$4^{21} = a_0 + a_1 + a_2 + \cdots + a_{82}$

Putting $x = -1$, we get

$0 = a_0 - a_1 + a_2 - a_3 + \cdots + a_{82}$

Adding, we get

$4^{21} = 2(a_0 + a_2 + \cdots + a_{82})$

$\Rightarrow a_0 + a_2 + \cdots + a_{82} = 2^{41}$

**d.** We know that

$${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots = 2^{n/2}\cos\frac{n\pi}{4} \quad (1)$$

and

$${}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \cdots = 2^{n-1} \quad (2)$$

$$\Rightarrow \quad {}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots = \frac{1}{2}\left(2^{n/2}\times\cos\frac{n\pi}{4} + 2^{n-1}\right)$$

For $n = 42$,

$${}^{42}C_0 + {}^{42}C_4 + {}^{42}C_8 + \cdots = \frac{1}{2}\left(2^{21}\times\cos\frac{21\pi}{2} + 2^{41}\right) = 2^{40}$$

**5. a → p, r; b → p, r. → p, q; d → q, r, s.**

**a.** Let consecutive coefficients be ${}^nC_r$ and ${}^nC_{r+1}$. Then,

$$\frac{n!}{(n-r)!r!} = \frac{n!}{(n-r-1)!(r+1)!}$$

$$\Rightarrow \quad \frac{1}{(n-r)(n-r-1)!r!} = \frac{1}{(n-r-1)!(r+1)r!}$$

$$\Rightarrow \quad r + 1 = n - r$$

$$\Rightarrow \quad n = 2r + 1$$

Hence, $n$ is odd.

**b.** $E = (19-4)^n + (19+4)^n$

$$2\left[{}^nC_0 19^n + {}^nC_2 19^{n-2}4^2 + \cdots + {}^nC_n 4^n\right] \text{ when } n \text{ is even}$$

**or**

$$2\left[{}^nC_0 19^n + {}^nC_2\cdot 19^{n-2}\cdot 4^2 + \cdots + {}^nC_{n-1}19\cdot 4^{n-1}\right] \text{ then } n \text{ is odd}$$

$\Rightarrow$ $E$ is divisible by 19 when $n$ is odd

**c.** ${}^{10}C_0\,{}^{20}C_{10} - {}^{10}C_1\,{}^{18}C_{10} + {}^{10}C_2\,{}^{16}C_{10} - \cdots$

= Coefficient of $x^{10}$ in $[{}^{10}C_0(1+x)^{20} - {}^{10}C_1 \times (1+x)^{18} + {}^{10}C_2(1+x)^{16} - \cdots]$

= Coefficient of $x^{10}$ in $[{}^{10}C_0((1+x)^2)^{10} - {}^{10}C_1 \times ((1+x)^2)^9 + {}^{10}C_2((1+x)^2)^8 - \cdots]$

= Coefficient of $x^{10}$ in $[(1+x)^2 - 1]^{10}$

= Coefficient of $x^{10}$ in $[2x + x^2]^{10}$

$= 2^{10}$

**d.** $T_r = {}^{14}C_{r-1}x^{r-1}$; $T_{r+1} = {}^{14}C_r x^r$; $T_{r+2} = {}^{14}C_{r+1}x^{r+1}$

By the given condition,

$$2\,{}^{14}C_r = {}^{14}C_{r-1} + {}^{14}C_{r+1} \quad (1)$$

$$\Rightarrow \quad 2 = \frac{{}^{14}C_{r-1}}{{}^{14}C_r} + \frac{{}^{14}C_{r+1}}{{}^{14}C_r}$$

$$\Rightarrow \quad 2 = \frac{r}{14-r+1} + \frac{14-(r+1)+1}{r+1}$$

$$\Rightarrow \quad 2 = \frac{r}{15-r} + \frac{14-r}{r+1}$$

$$\Rightarrow \quad r = 9$$

## Integer Type

**1.(1)** Let $x^7$ occurs in $T_{r+1}$ term, then

$$T_{r+1} = {}^nC_r(ax^2)^{n-r}\left(\frac{1}{bx}\right)^r$$

$$= {}^{11}C_r\frac{a^{11-r}}{b^r}.x^{22-2r-r}$$

for $x^7 \Rightarrow 22 - 3r = 7 \Rightarrow r = 5$

Hence, coefficients of $x^7$ is ${}^{11}C_5\frac{a^6}{b^5}$

Let $x^{-7}$ occur in $T_{r+1}$ term, then

$$T_{r+1} = {}^{11}C_r(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r$$

$$= {}^{11}C_r\frac{a^{11-r}}{(-b)^r}x^{11-3r}$$

For $x^{-7} \Rightarrow 11 - 3r = -7 \Rightarrow r = 6$

Hence, coefficient of $x^{-7}$ is ${}^{11}C_6\frac{a^5}{b^6}$

Now ${}^{11}C_5\frac{a^5}{b^6} = {}^{11}C_6\frac{a^6}{b^5}$

$$\Rightarrow \quad {}^{11}C_5\,a = {}^{11}C_6\frac{a^5}{b^6}$$

$$\Rightarrow \quad {}^{11}C_5\,a = {}^{11}C_{11-6}\frac{1}{b}$$

$$\Rightarrow \quad {}^{11}C_5\,a = {}^{11}C_5\frac{1}{b}$$

$$\Rightarrow \quad ab = 1$$

**2.(6)** Coefficients of $(2r+4)^{\text{th}}$ and $(r-2)^{\text{th}}$ terms are equal.

$\Rightarrow$ ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$ (when ${}^nC_x = {}^nC_y$, then $x = y$ or $x + y = n$)

$\Rightarrow$ $2r + 3 + r - 3 = 18 \quad \Rightarrow \quad r = 6$

**3.(9)** According to the question,

${}^{14}C_{r-1}, {}^{14}C_r, {}^{14}C_{r+1}$ are in A.P., so $\left\{b = \frac{a+c}{2}\right\}$

$$\Rightarrow \quad 2.\,{}^{14}C_r = {}^{14}C_{r-1} + {}^{14}C_{r+1}$$

$$\Rightarrow \quad \frac{2.14!}{(14-r)!r!} = \frac{14!}{(14-r+1)!(r-1)!} + \frac{14!}{(14-r-1)!(r+1)!}$$

$$\Rightarrow \quad \frac{2}{(14-r)(13-r)!r(r-1)!} = \frac{1}{(15-r)(14-r)(13-r)!(r-1)!} + \frac{1}{(13-r)!(r+1)r(r-1)!}$$

$$\Rightarrow \quad \frac{2}{(14-r)r} = \frac{1}{(15-r)(14-r)} + \frac{1}{r(r+1)}$$

$$\Rightarrow \quad \frac{2}{(14-r)r} - \frac{1}{r(r+1)} = \frac{1}{(15-r)(14-r)}$$

$$\Rightarrow \quad \frac{3r-12}{r(r+1)} = \frac{1}{(15-r)}$$

$$\Rightarrow \quad r = 5 \text{ or } 9$$

**4.(8)** Let the three consecutive coefficients be ${}^nC_{r-1} = 28$, ${}^nC_r = 56$ and ${}^nC_{r+1} = 70$,

so that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{56}{28} = 2$ and

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1} = \frac{70}{56} = \frac{5}{4}$$

This gives $n + 1 = 3r$ and $4n - 5 = 9r$

$$\therefore \quad \frac{4n-5}{n+1} = 3 \Rightarrow n = 8$$

**5.(8)** $= [\sqrt{x^2+1} + \sqrt{x^2-1}]^8 + \left[\sqrt{x^2+1} - \sqrt{x^2-1}\right]^8$

$$= 2\left[{}^8C_0\left(\sqrt{x^2+1}\right)^8 + {}^8C_2\left(\sqrt{x^2+1}\right)^6\left(\sqrt{x^2-1}\right)^2\right.$$

$$+{}^8C_4\left(\sqrt{x^2+1}\right)^4\left(\sqrt{x^2-1}\right)^4$$

$$\left.{}^8C_6\left(\sqrt{x^2+1}\right)^2\left(\sqrt{x^2-1}\right)^6+{}^8C_8\left(\sqrt{x^2-1}\right)^8\right]$$

which has degree 8.

**6.(3)** $(1+0.00002)^{50000}=\left(1+\dfrac{1}{50000}\right)^{50000}$.

Now we know that $2\le\left(1+\dfrac{1}{n}\right)^n<3\ \forall n\ge 1 \Rightarrow$ Least integer is 3

**7.(0)** Middle term is $\left(\dfrac{n}{2}+1\right)^{\text{th}}$, i.e., $(4+1)^{\text{th}}$, i.e., $T_5$

$\therefore\ T_5={}^8C_4\left(\dfrac{x}{2}\right)^4.2^4=1120 \Rightarrow x^4=\dfrac{8.7.6.5}{1.2.3.4}x^4=1120$

$\Rightarrow x^4=\dfrac{1120}{70}=16$

$\Rightarrow (x^2+4)(x^2-4)=0$

$\therefore\ x=\pm 2$ only as $x\in R$

**8.(4)** $T_2={}^nC_1\,(a^{1/13})^{n-1}\cdot a\sqrt{a}=14a^{5/2}$

$\Rightarrow n\cdot a^{\frac{n-1}{13}}=14a$

$\Rightarrow n\cdot a^{\frac{n-14}{13}}=14$

$\Rightarrow \dfrac{n-14}{13}=0$

$\Rightarrow n=14$

$\Rightarrow \dfrac{{}^{14}C_3}{{}^{14}C_2}=\dfrac{14!}{3!\cdot 11!}\ \dfrac{2!\cdot 12!}{14!}=\dfrac{12}{3}=4$

**9.(5)** ${}^{23}C_r+2.\,{}^{23}C_{r+1}+{}^{23}C_{r+2}={}^{24}C_{r+1}+{}^{24}C_{r+2}={}^{25}C_{r+2}\ge{}^{25}C_{15}$

$\therefore$ $(r+2)$ can be 10, 11, 12, 13 and 15

so 5 elements.

**10.(4)**

$$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$$

$$=\left((\sqrt{4^x+44})^{2/5}+\left(\frac{1}{\sqrt[3]{2^{x-1}+7}}\right)\right)^8$$

$$=\left((4^x+44)^{1/5}+\frac{1}{(2^{x-1}+7)^{1/3}}\right)^8$$

Now $T_4=T_{3+1}={}^8C_3\,((4^x+44)^{1/5})^{8-3}\dfrac{1}{((2^{x-1}+7)^{1/3})^3}$

Given $336={}^8C_3\left(\dfrac{4^x+44}{2^{x-1}+7}\right)$

Let $2^x=y$

$\Rightarrow\ 336={}^8C_3\left(\dfrac{y^2+44}{(y/2)+7}\right)$

$\Rightarrow\ 336=\dfrac{8\times7\times6}{3\times2\times1}\left(\dfrac{2(y^2+44)}{y+14}\right)$

$\Rightarrow\ y^2-3y+2=0\ \Rightarrow\ y=0,2$

**11.(6)** $T_{r+1}={}^nC_r\,(x^2)^{n-r}\,(-1)^r\,x^{-r}$

$={}^nC_r\,x^{2n-3r}(-1)^r$

Constant term $={}^nC_r\,(-1)^r$ if $2n=3r$

i.e., coefficient of $x=0$

hence, ${}^nC_{2n/3}\,(-1)^{2n/3}=15={}^6C_4\ n=6$

**12.(9)** $f(n)={}^nC_0a^{n-1}-{}^nC_1\,a^{n-2}+{}^nC_2a^{n-3}+\cdots+(-1)^{n-1}\,{}^nC_{n-1}a^0$

$=\dfrac{1}{a}\,({}^nC_0\,a^n-{}^nC_1\,a^{n-1}+{}^nC_2\,a^{n-2}+\cdots+(-1)^{n-1}\,{}^nC_{n-1}\,a')$

$=\dfrac{1}{a}\,((a-1)^n-(-1)^n\,{}^nC_n)$

$=\dfrac{1}{a}\left(\left(\dfrac{1}{3^{223}}-(-1)^n\right)\right)$

$f(x)=\dfrac{3^{\frac{n}{223}}-(-1)^n}{\left(3^{\frac{1}{223}}+1\right)}$

$\Rightarrow f(2007)=\dfrac{3^{\frac{2007}{223}}+1}{3^{\frac{1}{223}}+1}$

$\Rightarrow f(2008)=\dfrac{3^{\frac{2008}{223}}-1}{3^{\frac{1}{223}}+1}$

$\Rightarrow f(2007)+f(2008)=\dfrac{3^{\frac{2007}{223}}+3^{\frac{2008}{223}}}{3^{\frac{1}{223}}+1}$

$=\dfrac{3^9+3^{9+\frac{1}{223}}}{3^{\frac{1}{223}}+1}$

$=3^9\dfrac{\left(1+3^{\frac{1}{223}}\right)}{\left(1+3^{\frac{1}{223}}\right)}=3^9$

$\Rightarrow\ 3^9=3^k$ then $k=9$

**13.(5)** We have $1+\sum\limits_{r=1}^{10}(3^r.{}^{10}C_r+r.{}^{10}C_r)$

$=1+\sum\limits_{r=1}^{10}3^r.{}^{10}C_r+10\sum\limits_{r=1}^{10}{}^9C_{r-1}$

$=1+4^{10}-1+10\cdot 2^9$

$=4^{10}+5.2^{10}=2^{10}\,(4^5+5)$

$=2^{10}\,(\alpha\cdot 4^5+\beta)$, so $\alpha=1$ and $\beta=5$

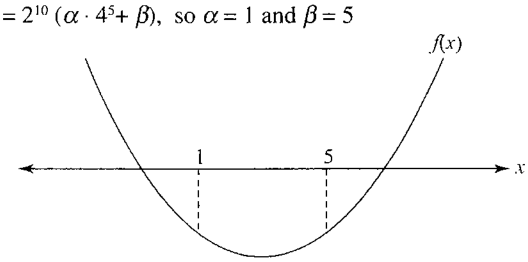


**Fig. 6.1**

Now $f(1)<0$ and $f(5)<0$

$f(1) < 0 \Rightarrow -k^2 < 0 \Rightarrow k \neq 0$

and $f(5) < 0$

$\Rightarrow 16 - k^2 < 0$

$\Rightarrow k^2 - 16 > 0$

$\Rightarrow k \in (-\infty, 4) \cup (4, \infty)$

Hence, the smallest positive integral value of $k = 5$.

**14.(4)** We have $b$ = coefficient of $x^3$ in

$$\left((1 + x + 2x^2 + 3x^3) + 4x^4\right)^4$$

= coefficient of $x^3$ in $[{}^4C_0(1 + x + 2x^2 + 3x^3)^4 (4x^4)^0 + {}^4C_1(1 + x + 2x^2 + 3x^3)^3 (4x^4)^1 + \cdots]$

= coefficient of $x^3$ in $(1 + x + 2x^2 + 3x^3)^4$ =

Hence, $4a/b = 4$.

**15.(7)** $(1 + 7)^{83} + (7 - 1)^{83} = (1 + 7)^{83} - (1 - 7)^{83}$

$= 2[{}^{83}C_1 \cdot 7 + {}^{83}C_3 \cdot 7^3 + \cdots + {}^{83}C_{83} \cdot 7^{83}] = (2 \cdot 7 \cdot 83) + 49I$

where $I$ is an integer

Now $14 \times 83 = 1162$

$\therefore \frac{1162}{49} = 23\frac{35}{49}$

$\therefore$ Remainder is 35

**16.(0)** Consider $(5 + 2)^{100} - (5 - 2)^{100}$

$= 2\,[{}^{100}C_1\, 5^{99} \cdot 2 + {}^{100}C_3\, 5^{97} \cdot 2^3 + \cdots + {}^{100}C_{99} 5 \cdot 2^{99}]$

$= 2\,[1000 \cdot 5^{98} + 1000.{}^{100}C_3 \cdot 5^{94} + \cdots + 1000 \cdot 2^{98}]$

$\Rightarrow$ Minimum 000 as last three digits.

**17.(6)** $(1 - 2x + 5x^2 - 10x^3)\,[{}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots] = 1 + a_1x + a_2x^2 + \cdots$

$\Rightarrow a_1 = n - 2$ and $a_2 = \frac{n(n-1)}{2} - 2n + 5$

Given that $a_1^2 = 2a_2$

$\Rightarrow (n - 2)^2 = n(n - 1) - 4n + 10$

$\Rightarrow n^2 - 4n + 4 = n^2 - 5n + 10$

$\Rightarrow n = 6$

**18.(0)** $1 + 2 + 2^2 + 2^3 + \cdots + 2^{1999}$

$= \frac{1(2^{2000} - 1)}{1}$

$= 2^{2000} - 1$

$= (1 - 5)^{1000} - 1$

$= 1 - {}^{1000}C_1 \cdot 5 + {}^{1000}C_2 \cdot 5^2 + \cdots + {}^{1000}C_{1000} \cdot 5^{1000} - 1$

which is divisible by 5.

**19.(1)** $= \sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)$

$= \sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)\frac{4!}{4!}$

$= \sum_{k=0}^{4}\frac{{}^4C_k \cdot 3^{4-k} \cdot x^k}{4!} = \frac{(3+x)^4}{4!}$

According to the question,

$\frac{(3+x)^4}{4!} = \frac{32}{3}$

$\Rightarrow (3 + x)^4 = 256$

$\Rightarrow x + 3 = 4 \Rightarrow x = 1$

**20.(1)** $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{5^n} \cdot {}^nC_r \left(\sum_{t=0}^{r-1} {}^rC_t \cdot 3^t\right) = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{5^n} \cdot {}^nC_r\,(4^r - 3^r)$

$= \lim_{n\to\infty} \frac{1}{5^n}\left(\sum_{r=1}^{n} {}^nC_r\, 4^r - \sum_{r=1}^{n} {}^nC_r\, 3^r\right) = \lim_{n\to\infty} \frac{1}{5^n}\,(5^n - 4^n) = 1$

## Archives

### Subjective Type

1. Given that

$C_1 + 2C_2\,x + 3C_3\,x^2 + \cdots + 2n\, C_{2n}x^{2n-1} = 2n(1 + x)^{2n-1}$

where

$$C_r = \frac{2n!}{r!(2n - r)!} \quad (1)$$

Integrating both sides with respect to $x$, under the limits 0 to $x$, we get

$[C_1x + C_2x^2 + C_3x^3 + \cdots + C_{2n}x^{2n}]_0^x = \left[(1+x)^{2n}\right]_0^x$

$\Rightarrow C_1x + C_2x^2 + C_3x^3 + \cdots + C_{2n}x^{2n} = (1 + x)^{2n} - 1$

$\Rightarrow C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_{2n}x^{2n} = (1 + x)^{2n} \quad (2)$

Changing $x$ by $-1/x$, we get

$$C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \frac{C_3}{x^3} + \cdots + (-1)^{2n}\frac{C_{2n}}{x^{2n}} = \left(1 - \frac{1}{x}\right)^{2n}$$

$\Rightarrow C_0x^{2n} - C_1x^{2n-1} + C_2x^{2n-2} - C_3x^{2n-3} + \cdots + C_{2n} = (x - 1)^{2n} \quad (3)$

Multiplying Eqs. (1) and (3) and equating the coefficient of $x^{2n-1}$ on both sides, we get

$-C_1^2 + 2\,C_2^2 - 3C_3^2 + \cdots + 2n\,C_{2n}^2$

= Coefficient of $x^{2n-1}$ in $(x^2 - 1)^{2n-1}(x - 1)$

$= 2n$[coefficient of $x^{2n-2}$ in $(x^2 - 1)^{2n-1}$ – coefficient of $x^{2n-1}$ in $(x^2 - 1)^{2n-1}$]

$= 2n[{}^{2n-1}C_{n-1}(-1)^{n-1} - 0]$

$= (-1)^{n-1}\, 2n^{2n-1}C_{n-1}$

$\Rightarrow C_1^2 - 2\,C_2^2 + 3C_3^2 + \cdots + 2nC_{2n}^2$

$= (-1)^n\, 2n^{2n-1}C_{n-1}$

$= (-1)^n\, n \times \left(\frac{2n}{n} \times {}^{2n-1}C_{n-1}\right)$

$= (-1)^n\, n\, {}^{2n}C_n$

$= (-1)^n\, n\, C_n \quad (\because {}^{2n}C_n = C_n)$

2. See solved example 6.81

3. $s_n$ is in geometric progression, hence

$s_n = \frac{q^{n+1} - 1}{q - 1},\ q \neq 1$

$$S_n = \frac{\left(\frac{q+1}{2}\right)^{n+1} - 1}{\left(\frac{q+1}{2}\right) - 1} = \frac{(q+1)^{n+1} \cdot - 2^{n+1}}{2^n(q - 1)} \quad (A)$$

Consider

${}^{(n+1)}C_1 + {}^{(n+1)}C_2 s_1 + {}^{(n+1)}C_3 s_3 + \cdots + {}^{(n+1)}C_{n+1} s_n$

$$= {}^{(n+1)}C_1\left(\frac{q-1}{q-1}\right) + {}^{(n+1)}C_2\,\frac{q^2-1}{q-1} + \cdots + {}^{(n+1)}C_{n+1}\,\frac{q^{n+1}-1}{q-1}$$

$$= \left(\frac{1}{q-1}\right)\left[\left\{{}^{(n+1)}C_1 q + {}^{(n+1)}C_2 q^2 + \cdots + {}^{(n+1)}C_{n+1} q^{n+1}\right\}\right.$$

$$\left. - \left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + \cdots + {}^{(n+1)}C_{n+1}\right\}\right]$$

$$= \left(\frac{1}{q-1}\right)\left[\left\{(1+q)^{n+1}-1\right\} - \left\{2^{n+1}-1\right\}\right] \quad \text{(B)}$$

$$= \frac{(1+q)^{n+1} - 2^{n+1}}{q-1}$$

Thus,

$${}^{(n+1)}C_1 + {}^{(n+1)}C_2 s_1 + \cdots + {}^{(n+1)}C_{n+1} s_n = \frac{(1+q)^{n+1} - 2^{n+1}}{q-1}$$

But from (A), we have

$${}^{n+1}C_1 + {}^{(n+1)}C_2 S_1 + \cdots + {}^{(n+1)}C_{n+1} S_{n+1} = 2^n S_n$$

**4.** $\displaystyle\sum_{r=0}^{n} (-1)^r \times {}^nC_r\left(\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \cdots m \text{ terms}\right)$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \cdots m \text{ terms}$$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \cdots m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left[1-\left(\frac{1}{2^n}\right)^m\right]}{1-\frac{1}{2^n}} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

**5.** Here $f = R - [R]$ is the fractional part of $R$. Thus, if $I$ is the integral part of $R$, then

$R = I + f = (5\sqrt{5} + 11)^{2n+1}$, and $0 < f < 1$

Let $f' = (5\sqrt{5} - 11)^{2n+1}$. Then $0 < f' < 1$ (as $5\sqrt{5} - 11 < 1$)

Now, $I + f - f' = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2[{}^{2n+1}C_1 (5\sqrt{5})^{2n} \times 11 + {}^{2n+1}C_3 (5\sqrt{5})^{2n-2} \times 11^3 + \cdots]$

= an even integer (1)

$\Rightarrow$ $f - f'$ must also be an integer

$\Rightarrow$ $f - f' = 0$, $\because$ $0 < f < 1, 0 < f' < 1$

$\Rightarrow$ $f = f'$

$\therefore$ $Rf = Rf' = (5\sqrt{5} + 11)^{2n+1}(5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

**6.** $S = C_0 - 2^2C_1 + 3^2C_2 - \cdots + (-1)^n (n+1)^2 C_n$

$T_r = (-1)^r r^2\, {}^nC_r$

$= (-1)^r r(r\, {}^nC_r)$

$= (-1)^r r(n\, {}^{n-1}C_{r-1})$

$= n(-1)^r ((r-1)+1)({}^{n-1}C_{r-1})$

$= n(-1)^r ((r-1)\,{}^{n-1}C_{r-1} + {}^{n-1}C_{r-1})$

$= n(-1)^r ((n-1)\,{}^{n-2}C_{r-2} + {}^{n-1}C_{r-1})$

$= n(n-1)\,{}^{n-2}C_{r-2}(-1)^{r-2} - n\,{}^{n-1}C_{r-1}(-1)^{r-1}$

$\Rightarrow$ $S = \displaystyle\sum_{r=0}^{n} T_r$

$= n(n-1)(1-1)^{n-2} - n(1-1)^{n-1}$

$= 0$

**7.** Given that

$$\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r \quad (1)$$

and

$a_k = 1, \forall k \ge n$

In Eq. (1), let us put $x - 3 = y$ or $x - 3 = y$ or $x - 2 = y + 1$ and we get

$$\sum_{r=0}^{2n} a_r(1+y)^r = \sum_{r=0}^{2n} b_r(y)^r$$

$\Rightarrow$ $a_0 + a_1(1+y) + \cdots + a_{n-1}(1+y)^{n-1}$

$$+ (1+y)^n + (1+y)^{n+1} + \cdots + (1+y)^{2n} = \sum_{r=0}^{2n} b_r y^r$$

[Using $a_k = 1, \forall k \ge n$]

Equating the coefficients of $y^n$ on both the sides, we get

${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \cdots + {}^{2n}C_n = b_n$

$\Rightarrow$ $({}^{n+1}C_{n+1} + {}^{n+1}C_n) + {}^{n+2}C_n + \cdots + {}^{2n}C_n = b_n$

[Using ${}^nC_n = {}^{n+1}C_{n+1} = 1$]

$\Rightarrow$ $b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + \cdots + {}^{2n}C_n$ [Using ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$]

Combining the terms in similar way, we get

$b_n = {}^{2n}C_{n+1} + {}^{2n}C_n$

$\Rightarrow$ $b_n = {}^{2n+1}C_{n+1}$

**8.** $S = \displaystyle\sum_{r=1}^{k} (-3)^{r-1}\, {}^{3n}C_{2r-1}$, $k = \dfrac{3n}{2}$ and $n$ is even

$\Rightarrow k = \dfrac{3(2m)}{2} = 3m$

$$\Rightarrow S = \sum_{r=1}^{3m} (-3)^{r-1} \times {}^{6m}C_{2r-1} = {}^{6m}C_1 - 3\,{}^{6m}C_3 + 3^2\,{}^{6m}C_5 - \cdots (-3)^{3m-1}\,{}^{6m}C_{6m-1}$$

$$= \frac{1}{\sqrt{3}}\left[\sqrt{3}\,{}^{6m}C_1 - (\sqrt{3})^3\,{}^{6m}C_3 + (\sqrt{3})^5\,{}^{6m}C_5 - \cdots + (-1)^{3m-1}(\sqrt{3})^{6m-1}\,{}^{6m}C_{6m-1}\right]$$

There is an alternate sign series with odd binomial coefficients. Hence, we should replace $x$ by $\sqrt{3}i$ in $(1+x)^{6m}$. Therefore,

$$(1+\sqrt{3}i)^{6m} = {}^{6m}C_0 + {}^{6m}C_1(\sqrt{3}i) + {}^{6m}C_2(\sqrt{3}i)^2 + {}^{6m}C_3(\sqrt{3}i)^3 + \cdots + {}^{6m}C_{6m}(\sqrt{3}i)^{6m}$$

$\Rightarrow$ $\sqrt{3} \times {}^{6m}C_1 - (\sqrt{3})^3\,{}^{6m}C_3 + (\sqrt{3})^5\,{}^{6m}C_5 + \cdots$

= Imaginary part in $(1+\sqrt{3}i)^{6m}$

$$= \text{Im}\left[2^{6m}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{6m}\right]$$

$$= \text{Im}\left[2^{6m}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{6m}$$

$$= \text{Im}\left[2^{6m}(\cos 2m\pi + i\sin 2m\pi)\right] = \text{Im}[2^{6m}] = 0$$

$\Rightarrow$ $S = 0$

**9.** $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots$ suggests that we have to multiply two expansions

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n} \quad (1)$$

Replacing $x$ by $-1/x$, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \dots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow \quad (1 - x + x^2)^n = a_0x^{2n} - a_1x^{2n-1} + a_2x^{2n-2} - \cdots + a_{2n} \quad (2)$$

Clearly,

$a_0^2 - a_1^2 + a_2^2 \cdots + a_{2n}^2$ is the coefficient of $x^{2n}$ in

$$(1 + x + x^2)^n (1 - x + x^2)^n$$

$$\Rightarrow \quad a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2$$

$= $ Coefficient of $x^{2n}$ in $(1 + x^2 + x^4)^n$

In $(1 + x^2 + x^4)^n$, replace $x^2$ by $y$, then the coefficient of $y^n$ in $(1 + y + y^2)^n$ is $a_n$. Hence,

$$a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2 = a_n$$

**10.** $\displaystyle\sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_3}\right)$

$$= \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!r!}\frac{3!r!}{(r+3)!}$$

$$= 3!\sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!(r+3)!}$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^r\, {}^{n+3}C_{r+3}$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(-1)^{r+3}\, {}^{n+3}C_{r+3}$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[-{}^{n+3}C_3 + {}^{n+3}C_4 - \cdots (-1)^{n+3}\, {}^{n+3}C_{n+3}\right]$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2 - {}^{n+3}C_3 + \cdots + (-1)^{n+3}\, {}^{n+3}C_{n+3}) - ({}^{n+3}C_0 - {}^{n+3}C_1 + {}^{n+3}C_2)\right]$$

$$= -\frac{3!}{(n+1)(n+2)(n+3)}\left[(1-1)^{n+3} - (1 - (n+3) - \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\left[(1 - n - 3 + \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!}{(n+1)(n+2)(n+3)}\frac{(n^2+3n+2)}{2} = \frac{3!}{2(n+3)}$$

**11.** We know that the coefficient of $x^r$ in the binomial expansion of $(1 + x)^n$ is ${}^nC_r$.

$\therefore \quad {}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m$

$= $ Coefficient of $x^m$ in the expansion of $[(1 + x)^n + (1 + x)^{n+1} + (1 + x)^{n-2} + \cdots + (1 + x)^m]$

$= $ Coefficient of $x^m$ in $[(1 + x)^m + (1 + x)^{m-1} + (1 + x)^{m-2} + \cdots + (1 + x)^n]$ (writing in reserve order)

$= $ Coefficient of $x^m$ $\left[(1+x)^m \dfrac{\{(1+x)^{n-m+1} - 1\}}{1+x-1}\right]$ [sum of G.P.]

$= $ Coefficient of $x^m$ in $\dfrac{\left[(1+x)^{n+1} - (1+x)^m\right]}{x}$

$= $ Coefficient of $x^{m+1}$ in $[(1 + x)^{n+1} - (1 + x)^m]$

$= {}^{n+1}C_{m+1} - 0$

$= {}^{n+1}C_{m+1}$

Now, we have to prove

$${}^nC_m + 2\,{}^{n-1}C_m + 3\,{}^{n-2}C_m + \dots + (n - m + 1)\,{}^mC_m = {}^{n+2}C_{m+2}$$

Let us consider

$$S = (1 + x)^n + 2(1 + x)^{n-1} + 3(1 + x)^{n-2} + \cdots + (n - m + 1) \times (1 + x)^m \quad (1)$$

$$(1 + x)\,S = (1 + x)^{n+1} + 2(1 + x)^n + 3(1 + x)^{n-1} + \cdots + (n - m + 1)(1 + x)^{m+1} \quad (2)$$

Subtracting (1) from (2), we get

$$x\,S = (1 + x)^{n+1} + (1 + x)^n + (1 + x)^{n-1} + (1 + x)^{n-2} + \cdots + (1 + x)^m + (n - m + 1)(1 + x)^m$$

$$= \frac{(1+x)^{n+1}\left[1 - \left(\frac{1}{1+x}\right)^{n+1-m}\right]}{1 - \frac{1}{1+x}} + (n - m + 1)(1 + x)^m$$

$$= \frac{(1+x)^{n+1}\left[(1+x)^{n+1-m} - 1\right]}{x(1+x)^{n-m}} + (n - m + 1)(1 + x)^m$$

$$= \frac{(1+x)^{m+1}\left[(1+x)^{n+1-m} - 1\right]}{x} + (n - m + 1)(1 + x)^m$$

$$\Rightarrow \quad S = \frac{(1+x)^{n+2} - (1+x)^{m+1}}{x^2} + \frac{(n-m+1)(1+x)^m}{x}$$

Now,

${}^nC_m + 2 \times {}^{n-1}C_m + 3 \times {}^{n-2}C_m + \dots + (n - m + 1)\,{}^mC_m$

$= $ Coefficient of $x^m$ is $S$

$= $ Coefficient of $x^m$ in

$$\left[\frac{(1+x)^{n+2} - (1+x)^{m+1}}{x^2} + \frac{(n-m+1)(1+x)^m}{x}\right]$$

$= $ Coefficient of $x^{m+2}$ in $[(1 + x)^{n+2} - (1 + x)^{m+1}]$

$= {}^{n+2}C_{m+2}$

**12.** $(25)^{n+1} - 24n + 5735$

$= (1 + 24)^{n+1} - 24n + 5735$

$= {}^{n+1}C_0 + {}^{n+1}C_1\,24 + {}^{n+1}C_2 24^2 + \cdots - 24n + 5735$

$= 1 + 24(n + 1) + {}^{n+1}C_2 24^2 + \cdots + {}^{n+1}C_{n+1} 24^{n+1} - 24n + 5735$

$= 5760 + 24^2({}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} 24^{n-1})$

$= 24^2[10 + ({}^{n+1}C_2 + \cdots + {}^{n+1}C_{n+1} 24^{n-1})]$

which is divisible by $24^2$.

13. $S = 2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0}$

$= 2^k\, {}^nC_0\, {}^nC_k - 2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1} + \cdots + (-1)^k\, {}^nC_k\, {}^{n-k}C_0$

$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\, {}^nC_r\, {}^{n-r}C_{k-r}$

$= \sum_{r=0}^{k}(-1)^r 2^{k-r}\dfrac{n!}{(n-r)!r!}\dfrac{(n-r)!}{(k-r)!(n-k)!}$

$= \dfrac{n!}{k!(n-k)!}\sum_{r=0}^{k}(-1)^r 2^{k-r}\dfrac{k!}{r!(k-r)!}$

$= {}^nC_k\sum_{r=0}^{k}(-1)^r 2^{k-r}\, {}^kC_r$

$= {}^nC_k\,[{}^kC_0 2^k - {}^kC_1 2^{k-1} + {}^kC_2 2^{k-2} - \cdots + (-1)^k\, {}^kC_k]$

$= {}^nC_k\,(2-1)^k = {}^nC_k$

## Objective Type

*Fill in the blanks*

1. We have,

$$101^{50} = (100+1)^{50} = 100^{50} + 50\times 100^{49} + \frac{50\times 49}{2\times 1}100^{48} + \cdots \quad (1)$$

$$99^{50} = (100-1)^{50} = 100^{50} - 50\times 100^{49} + \frac{50\times 49}{2\times 1}100^{48} - \cdots \quad (2)$$

Subtracting (2) from (1), we get

$$101^{50} - 99^{50} = 100^{50} + 2\,\frac{50\times 49\times 48}{1\times 2\times 3}100^{47} + \cdots > 100^{50}$$

Hence, $101^{50} > 100^{50} + 99^{50}$.

2. If we put $x = 1$ in the expansion of $(1 + x - 3x^2)^{2163} = A_0 + A_1x + A_2x^2 + \cdots$ we will get the sum of coefficients of the given polynomial, which is equal to $-1$.

3. $(1 + ax)^n = 1 + 8x + 24x^2 + \cdots$

$\Rightarrow\ 1 + nxa + \dfrac{n(n-1)}{2!}a^2x^2 + \cdots = 1 + 8x + 24x^2 + \cdots$

Comparing like powers of $x$, we get

$nax = 8x \Rightarrow na = 8$ (1)

$\dfrac{n(n-1)a^2}{2} = 24 \Rightarrow n(n-1)a^2 = 48$ (2)

Solving (1) and (2), $n = 4$, $a = 2$.

4. Let $T_{r+1}$ be the general term in the expansion of $\left(\sqrt{2} + 3^{1/5}\right)^{10}$

$\therefore\ T_{r+1} = {}^{10}C_r(\sqrt{2})^{10-r}(3^{1/5})^r \quad (0 \le r \le 10)$

$= \dfrac{10!}{r!(10-r)!}2^{5-r/2}3^{r/5}$

$T_{r+1}$ will be rational if $2^{5-r/2}$ and $3^{r/5}$ are rational numbers. Hence, $5 - r/2$ and $r/5$ are integers. So, $r = 0$ and $r = 10$. Therefore, $T_1$ and $T_{11}$ are rational terms. Now, sum of $T_1$ and $T_{11}$ is ${}^{10}C_0 2^{5-0}\times 3^0 + {}^{10}C_{10}2^{5-5}\times 3^2 = 32 + 9 = 41$.

*Multiple choice questions with one correct answer*

1. **a.** Given that $r$ and $n$ are +ve integers such that $r > 1$, $n > 2$. Also, in the expansion of $(1 + x)^{2n}$,

Coefficient of $3r^{th}$ term = coefficient of $(r + 2)^{th}$ term

$\Rightarrow\ {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow\ 3r - 1 = r + 1$ or $3r - 1 + r + 1 = 2n$

[using ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$]

$\Rightarrow\ r = 1$ or $2r = n$.

But $r > 1$

$\therefore\ n = 2r$

2. **a.** $\left(\dfrac{x}{2} - \dfrac{3}{x^2}\right)^{10}$

General term in this expansion is

$$T_{r+1} = {}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(\frac{-3}{x^2}\right)^r = {}^{10}C_r\,x^{10-3r}\frac{(-1)^r3^r}{2^{10-r}}$$

For coefficient of $x^4$, we should have $r = 2$.

Therefore, coefficient of $x^4$ is ${}^{10}C_2\,\dfrac{(-1)^2 3^2}{2^8} = \dfrac{405}{256}$

3. **c.** Since $n$ is even, let $n = 2m$. Then,

$$\text{L.H.S.} = S = \frac{2m!m!}{(2m)!}\left[C_0^2 - 2C_1^2 + 3C_2^2 + \cdots + (-1)^{2m}\times(2m+1)C_{2m}^2\right] \quad (1)$$

$$\Rightarrow\ S = \frac{2m!m!}{(2m)!}\left[(2m+1)C_0^2 - 2mC_1^2 + (2m-1)\times C_2^2 + \cdots + C_0^2\right] \quad (2)\ (\text{Using } C_r = C_{n-r})$$

Adding (1) and (2), we get

$$2S = 2\frac{m!m!}{(2m)!}(2m+2)\left[C_0^2 - C_1^2 + C_2^2 + \cdots + C_{2m}^2\right]$$

Now keeping in mind that $C_0^2 - C_1^2 + C_2^2 - \cdots + C_n^2 = (-1)^{n/2}\,{}^nC_{n/2}$ if $n$ is even, we get

$$S = 2\frac{m!\,m!}{(2m)!}(m+1)\left[(-1)^m\,{}^{2m}C_m\right]$$

$$= 2\left(\frac{n}{2}+1\right)(-1)^{n/2}$$

$$= (-1)^{n/2}(n+2)$$

4. **c.** Let,

$$b = \sum_{r=0}^{n}\frac{r}{{}^nC_r} \quad (1)$$

$$= \sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}} \quad \text{(we can replace } r \text{ by } n-r)$$

$$= \sum_{r=0}^{n}\frac{n-r}{{}^nC_r} \quad (2)$$

Adding (1) and (2), we have

$$2b = \sum_{r=0}^{n}\frac{r}{{}^nC_r} + \sum_{r=0}^{n}\frac{n-r}{{}^nC_r}$$

$= n\sum_{r=0}^{n} \frac{1}{^nC_r}$

$= na_n$

$\Rightarrow \quad b = \frac{n}{2} a_n$

**5.** **c.** The given expression is $\left(x+\sqrt{x^3-1}\right)^5 + \left(x-\sqrt{x^3-1}\right)^5$.
We know that

$(x+a)^n + (x-a)^n = 2\,[{}^nC_0 x^n + {}^nC_2 x^{n-2}a^2 + {}^nC_4 x^{n-4}a^4 + \cdots]$

Therefore the given expression is equal to $2[{}^5C_0 x^5 + {}^5C_2 x^3 (x^3-1) + {}^5C_4 x(x^3-1)^2]$.
Maximum power of $x$ involved here is 7, also only +ve integral powers of $x$ are involved, therefore the given expression is a polynomial of degree 7.

**6.** **d.** $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

$= \left[\binom{n}{r} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r-2}\right]$

$= \binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r} \quad [\because\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

**7.** **b.** $(a-b)^n, n \ge 5$

In the binomial expansion,

$T_5 + T_6 = 0$

$\Rightarrow\ {}^nC_4 a^{n-4} b^4 - {}^nC_5 a^{n-5} b^5 = 0$

$\Rightarrow\ \frac{{}^nC_4}{{}^nC_5}\frac{a}{b} = 1 \Rightarrow \frac{4+1}{n-4}\frac{a}{b} = 1 \left[\text{Using } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}\right]$

$\Rightarrow\ \frac{a}{b} = \frac{n-4}{5}$

**8.** **c.** $\sum_{i=0}^{m} {}^{10}C_i\, {}^{20}C_{m-i} = {}^{10}C_0\, {}^{20}C_m + {}^{10}C_1\, {}^{20}C_{m-1} + {}^{10}C_2\, {}^{20}C_{m-2} + \cdots + {}^{10}C_m\, {}^{20}C_0$

= Coefficient of $x^m$ in the expansion of product $(1+x)^{10}(x+1)^{20}$

= Coefficient of $x^m$ in the expansion of $(1+x)^{30}$

$= {}^{30}C_m$

Hence, the maximum value ${}^{30}C_m$ is ${}^{30}C_{15}$.

**9.** **d.** $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$= (1+t^{12}+t^{24}+t^{36})(1+t^2)^{12}$

$\therefore$ Coefficient of $t^{24}$

$= 1 \times$ coefficient of $t^{24}$ in $(1+t^2)^{12} + 1 \times$ coefficient of $t^{12}$ in $(1+t^2)^{12} + 1 \times$ constant term in $(1+t^2)^{12}$

$= {}^{12}C_{12} + {}^{12}C_6 + {}^{12}C_0 = 1 + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

**10.** **d.** ${}^{n-1}C_r = {}^nC_{r+1}(k^2-3)$

$\Rightarrow\ k^2 - 3 = \frac{{}^{n-1}C_r}{{}^nC_{r+1}} = \frac{r+1}{n}$

Now,

$0 \le r \le n-1$

$\Rightarrow\ 1 \le r+1 \le n$

$\Rightarrow\ \frac{1}{n} \le \frac{r+1}{n} \le 1$

$\Rightarrow \frac{1}{n} \le k^2 - 3 \le 1$

$\Rightarrow\ 3 + \frac{1}{n} \le k^2 \le 4 \Rightarrow \sqrt{3+\frac{1}{n}} \le k \le 2$

When $n \to \infty$, we have

$\sqrt{3} < k \le 2$

$\Rightarrow\ k \in (\sqrt{3}, 2]$

**11.** **a.** Given series is

${}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} - \cdots + {}^{30}C_{20}\,{}^{30}C_{30}$

which is

${}^{30}C_0\,{}^{30}C_{20} - {}^{30}C_1\,{}^{30}C_{19} + {}^{30}C_2\,{}^{30}C_{18} - \cdots + {}^{30}C_{20}\,{}^{30}C_0$

= Coefficient of $x^{20}$ in the expansion of $(x+1)^n(1-x)^n$

= Coefficient of $x^{20}$ in the expansion of $(1-x^2)^n$

$= {}^{30}C_{10}$