CHAPTER X.

MISCELLANEOUS EQUATIONS.

129. In this chapter we propose to consider some miscellaneous equations; it will be seen that many of these can be solved by the ordinary rules for quadratic equations, but others require some special artifice for their solution.

Example 1. Solve $$8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63.$$

Multiply by $x^{\frac{3}{2n}}$ and transpose; thus

$$8x^{\frac{3}{n}} - 63x^{\frac{3}{2n}} - 8 = 0;$$

$$(x^{\frac{3}{2n}} - 8)(8x^{\frac{3}{2n}} + 1) = 0;$$

$$x^{\frac{3}{2n}} = 8, \text{ or } -\frac{1}{8};$$

$$x = (2^3)^{\frac{2n}{3}}, \text{ or } \left(-\frac{1}{2^3}\right)^{\frac{2n}{3}};$$

$$\therefore\ x = 2^{2n}, \text{ or } \frac{1}{2^{2n}}.$$

Example 2. Solve $$2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}.$$

Let $$\sqrt{\frac{x}{a}} = y; \text{ then } \sqrt{\frac{a}{x}} = \frac{1}{y};$$

$$\therefore\ 2y + \frac{3}{y} = \frac{b}{a} + \frac{6a}{b};$$

$$2aby^2 - 6a^2y - b^2y + 3ab = 0;$$

$$(2ay - b)(by - 3a) = 0;$$

$$y = \frac{b}{2a}, \text{ or } \frac{3a}{b};$$

$$\therefore\ \frac{x}{a} = \frac{b^2}{4a^2}, \text{ or } \frac{9a^2}{b^2};$$

that is, $$x = \frac{b^2}{4a}, \text{ or } \frac{9a^3}{b^2}.$$

Example 3. Solve $(x-5)(x-7)(x+6)(x+4)=504$.

We have $$(x^2-x-20)(x^2-x-42)=504;$$
which, being arranged as a quadratic in x^2-x, gives
$$(x^2-x)^2-62(x^2-x)+336=0;$$
$$\therefore\ (x^2-x-6)(x^2-x-56)=0;$$
$$\therefore\ x^2-x-6=0, \text{ or } x^2-x-56=0;$$
whence $$x=3,\ -2,\ 8,\ -7.$$

130. Any equation which can be thrown into the form
$$ax^2+bx+c+p\sqrt{ax^2+bx+c}=q$$
may be solved as follows. Putting $y=\sqrt{ax^2+bx+c}$, we obtain
$$y^2+py-q=0.$$

Let α and β be the roots of this equation, so that
$$\sqrt{ax^2+bx+c}=\alpha, \quad \sqrt{ax^2+bx+c}=\beta;$$
from these equations we shall obtain *four* values of x.

When no sign is prefixed to a radical it is usually understood that it is to be taken as positive; hence, if α and β are both positive, all the four values of x satisfy the *original* equation. If however α or β is negative, the roots found from the resulting quadratic will satisfy the equation
$$ax^2+bx+c-p\sqrt{ax^2+bx+c}=q,$$
but not the original equation.

Example. Solve $x^2-5x+2\sqrt{x^2-5x+3}=12$.

Add 3 to each side; then
$$x^2-5x+3+2\sqrt{x^2-5x+3}=15.$$

Putting $\sqrt{x^2-5x+3}=y$, we obtain $y^2+2y-15=0$; whence $y=3$ or -5.

Thus $\sqrt{x^2-5x+3}=+3$, or $\sqrt{x^2-5x+3}=-5$.

Squaring, and solving the resulting quadratics, we obtain from the first $x=6$ or -1; and from the second $x=\frac{5\pm\sqrt{113}}{2}$. The first pair of values satisfies the given equation, but the second pair satisfies the equation
$$x^2-5x-2\sqrt{x^2-5x+3}=12.$$

131. Before clearing an equation of radicals it is advisable to examine whether any common factor can be removed by division.

Example. Solve $\sqrt{x^2 - 7ax + 10a^2} - \sqrt{x^2 + ax - 6a^2} = x - 2a$.

We have

$$\sqrt{(x-2a)(x-5a)} - \sqrt{(x-2a)(x+3a)} = x - 2a.$$

The factor $\sqrt{x-2a}$ can now be removed from every term;

$$\therefore \sqrt{x-5a} - \sqrt{x+3a} = \sqrt{x-2a};$$

$$x - 5a + x + 3a - 2\sqrt{(x-5a)(x+3a)} = x - 2a;$$

$$x = 2\sqrt{x^2 - 2ax - 15a^2};$$

$$3x^2 - 8ax - 60a^2 = 0;$$

$$(x-6a)(3x+10a) = 0;$$

$$x = 6a, \text{ or } -\frac{10a}{3}.$$

Also by equating to zero the factor $\sqrt{x-2a}$, we obtain $x = 2a$.

On trial it will be found that $x = 6a$ does not satisfy the equation: thus the roots are $-\frac{10a}{3}$ and $2a$.

The student may compare a similar question discussed in the *Elementary Algebra*, Art. 281.

132. The following artifice is sometimes useful.

Example. Solve $\sqrt{3x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 9$ ....(1).

We have *identically*

$$(3x^2 - 4x + 34) - (3x^2 - 4x - 11) = 45 \text{ ....(2).}$$

Divide each member of (2) by the corresponding member of (1); thus

$$\sqrt{3x^2 - 4x + 34} - \sqrt{3x^2 - 4x - 11} = 5 \text{ ....(3).}$$

Now (2) is an *identical equation* true for *all* values of x, whereas (1) is an equation which is true only for certain values of x; hence also equation (3) is only true for these values of x.

From (1) and (3) by addition

$$\sqrt{3x^2 - 4x + 34} = 7;$$

$$\text{whence } x = 3, \text{ or } -\frac{5}{3}.$$

133. The solution of an equation of the form

$$ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0,$$

in which the coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of a quadratic. Equations of this type are known as *reciprocal equations*, and are so named because they are not altered when x is changed into its reciprocal $\frac{1}{x}$.

For a more complete discussion of reciprocal equations the student is referred to Arts. 568—570.

Example. Solve $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0.$

Dividing by x^2 and rearranging,

$$12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0.$$

Put $x + \frac{1}{x} = z$; then $x^2 + \frac{1}{x^2} = z^2 - 2$;

$$\therefore\ 12(z^2 - 2) - 56z + 89 = 0;$$

whence we obtain $z = \frac{5}{2}$, or $\frac{13}{6}$.

$$\therefore\ x + \frac{1}{x} = \frac{5}{2}, \text{ or } \frac{13}{6}.$$

By solving these equations we find that $x = 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}$.

134. The following equation though not *reciprocal* may be solved in a similar manner.

Example. Solve $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0.$

We have $6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0;$

whence $6\left(x - \frac{1}{x}\right)^2 - 25\left(x - \frac{1}{x}\right) + 24 = 0;$

$$\therefore\ 2\left(x - \frac{1}{x}\right) - 3 = 0, \text{ or } 3\left(x - \frac{1}{x}\right) - 8 = 0;$$

whence we obtain $x = 2, -\frac{1}{2}, 3, -\frac{1}{3}$.

135. When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of quadratic equations proved in Art. 114.

Example. Solve $(1-a^2)(x+a)-2a(1-x^2)=0$.

This is a quadratic, one of whose roots is clearly a.

Also, since the equation may be written

$$2ax^2+(1-a^2)x-a(1+a^2)=0,$$

the product of the roots is $-\dfrac{1+a^2}{2}$; and therefore the other root is $-\dfrac{1+a^2}{2a}$.

EXAMPLES. X. a.

Solve the following equations:

1. $x^{-2}-2x^{-1}=8$.
2. $9+x^{-4}=10x^{-2}$.
3. $2\sqrt{x}+2x^{-\frac{1}{2}}=5$.
4. $6x^{\frac{3}{4}}=7x^{\frac{1}{4}}-2x^{-\frac{1}{4}}$.
5. $x^{\frac{2}{n}}+6=5x^{\frac{1}{n}}$.
6. $3x^{\frac{1}{2n}}-x^{\frac{1}{n}}-2=0$.
7. $5\sqrt{\dfrac{3}{x}}+7\sqrt{\dfrac{x}{3}}=22\tfrac{2}{3}$.
8. $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}}=2\tfrac{1}{6}$.
9. $6\sqrt{x}=5x^{-\frac{1}{2}}-13$.
10. $1+8x^{\frac{6}{5}}+9\sqrt[5]{x^3}=0$.
11. $3^{2x}+9=10 \cdot 3^x$.
12. $5(5^x+5^{-x})=26$.
13. $2^{2x+8}+1=32 \cdot 2^x$.
14. $2^{2x+3}-57=65(2^x-1)$.
15. $\sqrt{2^x}+\dfrac{1}{\sqrt{2^x}}=2$.
16. $\dfrac{3}{\sqrt{2x}}-\dfrac{\sqrt{2x}}{5}=5\tfrac{9}{10}$.
17. $(x-7)(x-3)(x+5)(x+1)=1680$.
18. $(x+9)(x-3)(x-7)(x+5)=385$.
19. $x(2x+1)(x-2)(2x-3)=63$.
20. $(2x-7)(x^2-9)(2x+5)=91$.
21. $x^2+2\sqrt{x^2+6x}=24-6x$.
22. $3x^2-4x+\sqrt{3x^2-4x-6}=18$.
23. $3x^2-7+3\sqrt{3x^2-16x+21}=16x$.
24. $8+9\sqrt{(3x-1)(x-2)}=3x^2-7x$.
25. $\dfrac{3x-2}{2}+\sqrt{2x^2-5x+3}=\dfrac{(x+1)^2}{3}$.

26. $7x - \dfrac{\sqrt{3x^2-8x+1}}{x} = \left(\dfrac{8}{\sqrt{x}} + \sqrt{x}\right)^2.$

27. $\sqrt{4x^2-7x-15} - \sqrt{x^2-3x} = \sqrt{x^2-9}.$

28. $\sqrt{2x^2-9x+4} + 3\sqrt{2x-1} = \sqrt{2x^2+21x-11}.$

29. $\sqrt{2x^2+5x-7} + \sqrt{3(x^2-7x+6)} - \sqrt{7x^2-6x-1} = 0.$

30. $\sqrt{a^2+2ax-3x^2} - \sqrt{a^2+ax-6x^2} = \sqrt{2a^2+3ax-9x^2}.$

31. $\sqrt{2x^2+5x-2} - \sqrt{2x^2+5x-9} = 1.$

32. $\sqrt{3x^2-2x+9} + \sqrt{3x^2-2x-4} = 13.$

33. $\sqrt{2x^2-7x+1} - \sqrt{2x^2-9x+4} = 1.$

34. $\sqrt{3x^2-7x-30} - \sqrt{2x^2-7x-5} = x-5.$

35. $x^4+x^3-4x^2+x+1=0.$

36. $x^4 + \dfrac{8}{9}x^2 + 1 = 3x^3 + 3x.$

37. $x^4+1-3(x^3+x) = 2x^2.$

38. $10(x^4+1) - 63x(x^2-1) + 52x^2 = 0.$

39. $\dfrac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}} = \dfrac{\sqrt{a}+1}{\sqrt{a}-1}.$

40. $\dfrac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}} = \dfrac{5x}{a}.$

41. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} - \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 8x\sqrt{x^2-3x+2}.$

42. $\sqrt{x^2+x} + \dfrac{\sqrt{x-1}}{\sqrt{x^3-x}} = \dfrac{5}{2}.$

43. $\dfrac{x^3+1}{x^2-1} = x + \sqrt{\dfrac{6}{x}}.$

44. $2^{x^2} : 2^{2x} = 8 : 1.$

45. $a^{2x}(a^2+1) = (a^{3x}+a^x)a.$

46. $\dfrac{8\sqrt{x-5}}{3x-7} = \dfrac{\sqrt{3x-7}}{x-5}.$

47. $\dfrac{18(7x-3)}{2x+1} = \dfrac{250\sqrt{2x+1}}{3\sqrt{7x-3}}.$

48. $(a+x)^{\frac{2}{3}} + 4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}.$

49. $\sqrt{x^2+ax-1} - \sqrt{x^2+bx-1} = \sqrt{a} - \sqrt{b}.$

50. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 98.$

51. $x^4-2x^3+x=380.$

52. $27x^3+21x+8=0.$

136. We shall now discuss some simultaneous equations of two unknown quantities.

Example 1. Solve $x+2+y+3+\sqrt{(x+2)(y+3)}=39$.

$$(x+2)^2+(y+3)^2+(x+2)(y+3)=741.$$

Put $x+2=u$, and $y+3=v$; then

$$u+v+\sqrt{uv}=39 \quad \text{.......(1)},$$

$$u^2+v^2+uv=741 \quad \text{.......(2)},$$

hence, from (1) and (2), we obtain by division,

$$u+v-\sqrt{uv}=19 \quad \text{.......(3)}.$$

From (1) and (3), $u+v=29$;

and $\sqrt{uv}=10$,

or $uv=100$;

whence $u=25$, or 4; $v=4$, or 25;

thus $x=23$, or 2; $y=1$, or 22.

Example 2. Solve

$$x^4+y^4=82 \quad \text{.......(1)},$$

$$x-y=2 \quad \text{.......(2)}.$$

Put $x=u+v$, and $y=u-v$;

then from (2) we obtain $v=1$.

Substituting in (1), $(u+1)^4+(u-1)^4=82$;

$$\therefore\ 2(u^4+6u^2+1)=82;$$

$$u^4+6u^2-40=0;$$

whence $u^2=4$, or -10;

and $u=\pm 2$, or $\pm\sqrt{-10}$.

Thus $x=3,\ -1,\ 1\pm\sqrt{-10}$;

$$y=1,\ -3,\ -1\pm\sqrt{-10}.$$

Example 3. Solve

$$\frac{2x+y}{3x-y}-\frac{x-y}{x+y}=2\tfrac{8}{15} \quad \text{.......(1)},$$

$$7x+5y=29 \quad \text{.......(2)}.$$

From (1), $15(2x^2+3xy+y^2-3x^2+4xy-y^2)=38(3x^2+2xy-y^2)$;

$$\therefore\ 129x^2-29xy-38y^2=0;$$

$$\therefore\ (3x-2y)(43x+19y)=0.$$

Hence $3x=2y \quad \text{.......(3)},$

or $43x=-19y \quad \text{.......(4)}.$

From (3), $$\frac{x}{2}=\frac{y}{3}=\frac{7x+5y}{29}$$

$$=1, \text{ by equation (2).}$$

$$\therefore\ x=2,\ y=3.$$

Again, from (4), $$\frac{x}{19}=\frac{y}{-43}=\frac{7x+5y}{-82}$$

$$=-\frac{29}{82}, \text{ by equation (2),}$$

$$\therefore\ x=-\frac{551}{82},\ y=\frac{1247}{82}.$$

Hence $$x=2,\ y=3;\ \text{or } x=-\frac{551}{82},\ y=\frac{1247}{82}.$$

Example 4. Solve $$4x^3+3x^2y+y^3=8,$$
$$2x^3-2x^2y+xy^2=1.$$

Put $y=mx$, and substitute in both equations. Thus

$$x^3(4+3m+m^3)=8 \quad \text{.............................(1).}$$

$$x^3(2-2m+m^2)=1 \quad \text{.............................(2).}$$

$$\therefore\ \frac{4+3m+m^3}{2-2m+m^2}=8;$$

$$m^3-8m^2+19m-12=0;$$

that is, $$(m-1)(m-3)(m-4)=0;$$

$$\therefore\ m=1, \text{ or } 3, \text{ or } 4.$$

(i) Take $m=1$, and substitute in either (1) or (2).

From (2), $$x^3=1;\ \therefore\ x=1;$$

and $$y=mx=x=1.$$

(ii) Take $m=3$, and substitute in (2);

thus $$5x^3=1;\ \therefore\ x=\sqrt[3]{\frac{1}{5}};$$

and $$y=mx=3x=3\sqrt[3]{\frac{1}{5}}.$$

(iii) Take $m=4$; we obtain

$$10x^3=1;\ \therefore\ x=\sqrt[3]{\frac{1}{10}};$$

and $$y=mx=4x=4\sqrt[3]{\frac{1}{10}}.$$

Hence the complete solution is

$$x=1, \quad \sqrt[3]{\frac{1}{5}}, \quad \sqrt[3]{\frac{1}{10}}.$$

$$y=1, \quad 3\sqrt[3]{\frac{1}{5}}, \quad 4\sqrt[3]{\frac{1}{10}}.$$

NOTE. The above method of solution may always be used when the equations are *of the same degree and homogeneous.*

Example 5. Solve $31x^2y^2-7y^4-112xy+64=0$ ....(1),

$$x^2-7xy+4y^2+8=0 \text{ ....(2).}$$

From (2) we have $-8=x^2-7xy+4y^2$; and, substituting in (1),

$$31x^2y^2-7y^4+14xy\,(x^2-7xy+4y^2)+(x^2-7xy+4y^2)^2=0;$$

$$\therefore\ 31x^2y^2-7y^4+(x^2-7xy+4y^2)\,(14xy+x^2-7xy+4y^2)=0;$$

$$\therefore\ 31x^2y^2-7y^4+(x^2+4y^2)^2-(7xy)^2=0;$$

that is, $$x^4-10x^2y^2+9y^4=0 \text{ ....(3).}$$

$$\therefore\ (x^2-y^2)\,(x^2-9y^2)=0;$$

hence $$x=\pm y, \text{ or } x=\pm 3y.$$

Taking these cases in succession and substituting in (2), we obtain

$$x=y=\pm 2;$$

$$x=-y=\pm\sqrt{-\frac{2}{3}};$$

$$x=\pm 3, \quad y=\pm 1;$$

$$x=\pm 3\sqrt{-\frac{4}{17}}, \quad y=\mp\sqrt{-\frac{4}{17}}.$$

NOTE. It should be observed that equation (3) is *homogeneous.* The method here employed by which one equation is made homogeneous by a suitable combination with the other is a valuable artifice. It is especially useful in Analytical Geometry.

Example 6. Solve $(x+y)^{\frac{2}{3}}+2\,(x-y)^{\frac{2}{3}}=3\,(x^2-y^2)^{\frac{1}{3}}$ ....(1).

$$3x-2y=13 \text{ ....(2).}$$

Divide each term of (1) by $(x^2-y^2)^{\frac{1}{3}}$, or $(x+y)^{\frac{1}{3}}\,(x-y)^{\frac{1}{3}}$;

$$\therefore\ \left(\frac{x+y}{x-y}\right)^{\frac{1}{3}}+2\left(\frac{x-y}{x+y}\right)^{\frac{1}{3}}=3.$$

This equation is a quadratic in $\left(\frac{x+y}{x-y}\right)^{\frac{1}{3}}$, from which we easily find,

$$\left(\frac{x+y}{x-y}\right)^{\frac{1}{3}}=2 \text{ or } 1; \quad \text{whence } \frac{x+y}{x-y}=8 \text{ or } 1;$$

$$\therefore\ 7x=9y, \text{ or } y=0.$$

Combining these equations with (2), we obtain

$$x=9,\ y=7; \text{ or } x=\frac{13}{3},\ y=0.$$

EXAMPLES. X. b.

Solve the following equations:

1. $3x-2y=7$, $xy=20$.
2. $5x-y=3$, $y^2-6x^2=25$.
3. $4x-3y=1$, $12xy+13y^2=25$.
4. $x^4+x^2y^2+y^4=931$, $x^2-xy+y^2=19$.
5. $x^2+xy+y^2=84$, $x-\sqrt{xy}+y=6$.
6. $x+\sqrt{xy}+y=65$, $x^2+xy+y^2=2275$.
7. $x+y=7+\sqrt{xy}$, $x^2+y^2=133-xy$.
8. $3x^2-5y^2=7$, $3xy-4y^2=2$.
9. $5y^2-7x^2=17$, $5xy-6x^2=6$.
10. $3x^2+165=16xy$, $7xy+3y^2=132$.
11. $3x^2+xy+y^2=15$, $31xy-3x^2-5y^2=45$.
12. $x^2+y^2-3=3xy$, $2x^2-6+y^2=0$.
13. $x^4+y^4=706$, $x+y=8$.
14. $x^4+y^4=272$, $x-y=2$.
15. $x^5-y^5=992$, $x-y=2$.
16. $x+\frac{4}{y}=1$, $y+\frac{4}{x}=25$.
17. $\frac{x^2}{y}+\frac{y^2}{x}=\frac{9}{2}$, $\frac{3}{x+y}=1$.
18. $\frac{x}{2}+\frac{y}{5}=5$. $\frac{2}{x}+\frac{5}{y}=\frac{5}{6}$.
19. $x+y=1072$, $x^{\frac{1}{3}}+y^{\frac{1}{3}}=16$.
20. $xy^{\frac{1}{2}}+yx^{\frac{1}{2}}=20$, $x^{\frac{3}{2}}+y^{\frac{3}{2}}=65$.
21. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=5$, $6\left(x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\right)=5$.

22. $\sqrt{x+y}+\sqrt{x-y}=4,$
$x^2-y^2=9.$

23. $y+\sqrt{x^2-1}=2,$
$\sqrt{x+1}-\sqrt{x-1}=\sqrt{y}.$

24. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3},$
$x+y=10.$

25. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}+\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\frac{17}{4},$
$x^2+y^2=706.$

26. $x^2+4y^2-15x=10(3y-8), \quad xy=6.$

27. $x^2y^2+400=41xy, \quad y^2=5xy-4x^2.$

28. $4x^2+5y=6+20xy-25y^2+2x, \quad 7x-11y=17.$

29. $9x^2+33x-12=12xy-4y^2+22y, \quad x^2-xy=18.$

30. $(x^2-y^2)(x-y)=16xy, \quad (x^4-y^4)(x^2-y^2)=640x^2y^2.$

31. $2x^2-xy+y^2=2y, \quad 2x^2+4xy=5y.$

32. $\frac{x^3+y^3}{(x+y)^2}+\frac{x^3-y^3}{(x-y)^2}=\frac{43x}{8}, \quad 5x-7y=4.$

33. $y(y^2-3xy-x^2)+24=0, \quad x(y^2-4xy+2x^2)+8=0.$

34. $3x^3-8xy^2+y^3+21=0, \quad x^2(y-x)=1.$

35. $y^2(4x^2-108)=x(x^3-9y^3), \quad 2x^2+9xy+y^2=108.$

36. $6x^4+x^2y^2+16=2x(12x+y^3), \quad x^2+xy-y^2=4.$

37. $x(a+x)=y(b+y), \quad ax+by=(x+y)^2.$

38. $xy+ab=2ax, \quad x^2y^2+a^2b^2=2b^2y^2.$

39. $\frac{x-a}{a^2}+\frac{y-b}{b^2}=\frac{1}{x-b}-\frac{1}{y-a}-\frac{1}{a-b}=0.$

40. $bx^3=10a^2bx+3a^3y, \quad ay^3=10ab^2y+3b^3x.$

41. $2a\left(\frac{x}{y}-\frac{y}{x}\right)+4a^2=4x^2+\frac{xy}{2a}-\frac{y^2}{a^2}=1.$

137. Equations involving three or more unknown quantities can only be solved in special cases. We shall here consider some of the most useful methods of solution.

Example 1. Solve

$$x+y+z=13 \quad\ldots\ldots(1),$$
$$x^2+y^2+z^2=65 \quad\ldots\ldots(2),$$
$$xy=10 \quad\ldots\ldots(3).$$

From (2) and (3),
$$(x+y)^2+z^2=85.$$

Put u for $x+y$; then this equation becomes

$$u^2+z^2=85.$$

Also from (1), $u+z=13$;

whence we obtain $u=7$ or 6; $z=6$ or 7.

Thus we have $\left.\begin{array}{r}x+y=7,\\ xy=10\end{array}\right\}$ and $\left.\begin{array}{r}x+y=6,\\ xy=10\end{array}\right\}$

Hence the solutions are

$$\left.\begin{array}{l}x=5, \text{ or } 2,\\ y=2, \text{ or } 5,\\ z=6;\end{array}\right\} \text{ or } \left.\begin{array}{l}x=3\pm\sqrt{-1},\\ y=3\mp\sqrt{-1},\\ z=7.\end{array}\right\}$$

Example 2. Solve

$$(x+y)(x+z)=30,$$
$$(y+z)(y+x)=15,$$
$$(z+x)(z+y)=18.$$

Write u, v, w for $y+z$, $z+x$, $x+y$ respectively; thus

$$vw=30, \quad wu=15, \quad uv=18 \quad \text{..............(1)}.$$

Multiplying these equations together, we have

$$u^2v^2w^2=30\times15\times18=15^2\times6^2;$$
$$\therefore\ uvw=\pm 90.$$

Combining this result with each of the equations in (1), we have

$$u=3,\ v=6,\ w=5; \text{ or } u=-3,\ v=-6,\ w=-5;$$

$$\therefore\ \left.\begin{array}{r}y+z=3,\\ z+x=6,\\ x+y=5;\end{array}\right\} \text{ or } \left.\begin{array}{r}y+z=-3,\\ z+x=-6,\\ x+y=-5,\end{array}\right\}$$

whence $x=4,\ y=1,\ z=2$; or $x=-4,\ y=-1,\ z=-2$.

Example 3. Solve

$$y^2+yz+z^2=49 \quad \text{..............(1)},$$
$$z^2+zx+x^2=19 \quad \text{..............(2)},$$
$$x^2+xy+y^2=39 \quad \text{..............(3)}.$$

Subtracting (2) from (1)

$$y^2-x^2+z(y-x)=30;$$

that is, $$(y-x)(x+y+z)=30 \quad \text{..............(4)}.$$

Similarly from (1) and (3)

$$(z-x)(x+y+z)=10 \quad \text{..............(5)}.$$

Hence from (4) and (5), by division

$$\frac{y-x}{z-x}=3;$$

whence $$y=3z-2x.$$

Substituting in equation (3), we obtain

$$x^2 - 3xz + 3z^2 = 13.$$

From (2), $$x^2 + xz + z^2 = 19.$$

Solving these homogeneous equations as in Example 4, Art. 136, we obtain

$$x = \pm 2, \quad z = \pm 3; \text{ and therefore } y = \pm 5;$$

or $$x = \pm\frac{11}{\sqrt{7}}, \quad z = \pm\frac{1}{\sqrt{7}}; \text{ and therefore } y = \mp\frac{19}{\sqrt{7}}.$$

Example 4. Solve $x^2 - yz = a^2$, $y^2 - zx = b^2$, $z^2 - xy = c^2$.

Multiply the equations by y, z, x respectively and add; then

$$c^2x + a^2y + b^2z = 0 \quad \text{..................................(1).}$$

Multiply the equations by z, x, y respectively and add; then

$$b^2x + c^2y + a^2z = 0 \quad \text{.............................. (2).}$$

From (1) and (2), by cross multiplication,

$$\frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2} = k \text{ suppose.}$$

Substitute in any one of the given equations; then

$$k^2(a^6 + b^6 + c^6 - 3a^2b^2c^2) = 1;$$

$$\therefore \frac{x}{a^4 - b^2c^2} = \frac{y}{b^4 - c^2a^2} = \frac{z}{c^4 - a^2b^2} = \pm\frac{1}{\sqrt{a^6 + b^6 + c^6 - 3a^2b^2c^2}}.$$

EXAMPLES. X. c.

Solve the following equations:

1. $9x + y - 8z = 0$,
$4x - 8y + 7z = 0$,
$yz + zx + xy = 47$.

2. $3x + y - 2z = 0$,
$4x - y - 3z = 0$,
$x^3 + y^3 + z^3 = 467$.

3. $x - y - z = 2$,
$x^2 + y^2 - z^2 = 22$,
$xy = 5$.

4. $x + 2y - z = 11$,
$x^2 - 4y^2 + z^2 = 37$,
$xz = 24$.

5. $x^2 + y^2 - z^2 = 21$,
$3xz + 3yz - 2xy = 18$,
$x + y - z = 5$.

6. $x^2 + xy + xz = 18$,
$y^2 + yz + yx + 12 = 0$,
$z^2 + zx + zy = 30$.

7. $x^2 + 2xy + 3xz = 50$,
$2y^2 + 3yz + yx = 10$,
$3z^2 + zx + 2zy = 10$.

8. $(y - z)(z + x) = 22$,
$(z + x)(x - y) = 33$,
$(x - y)(y - z) = 6$.

9. $x^2y^2z^2u=12,\ x^2y^2zu^2=8,\ x^2yz^2u^2=1,\ 3xy^2z^2u^2=4.$

10. $x^3y^2z=12,\ x^3yz^3=54,\ x^7y^3z^2=72.$

11. $xy+x+y=23,$
$xz+x+z=41,$
$yz+y+z=27.$

12. $2xy-4x+y=17,$
$3yz+y-6z=52,$
$6xz+3z+2x=29.$

13. $xz+y=7z,\ yz+x=8z,\ x+y+z=12.$

14. $x^3+y^3+z^3=a^3,\ x^2+y^2+z^2=a^2,\ x+y+z=a.$

15. $x^2+y^2+z^2=yz+zx+xy=a^2,\ 3x-y+z=a\sqrt{3}.$

16. $x^2+y^2+z^2=21a^2,\ yz+zx-xy=6a^2,\ 3x+y-2z=3a.$

Indeterminate Equations.

138. Suppose the following problem were proposed for solution:

A person spends £461 in buying horses and cows; if each horse costs £23 and each cow £16, how many of each does he buy?

Let x, y be the number of horses and cows respectively; then

$$23x+16y=461.$$

Here we have *one* equation involving *two* unknown quantities, and it is clear that by ascribing any value we please to x, we can obtain a corresponding value for y; thus it would appear at first sight that the problem admits of an infinite number of solutions. But it is clear from the nature of the question that x and y must be positive integers; and with this restriction, as we shall see later, the number of solutions is limited.

If the number of unknown quantities is greater than the number of independent equations, there will be an unlimited number of solutions, and the equations are said to be **indeterminate.** In the present section we shall only discuss the simplest kinds of indeterminate equations, confining our attention to *positive integral values* of the unknown quantities; it will be seen that this restriction enables us to express the solutions in a very simple form.

The general theory of indeterminate equations will be found in Chap. XXVI.

Example 1. Solve $7x+12y=220$ in positive integers.

Divide throughout by 7, the smaller coefficient; thus

$$x+y+\frac{5y}{7}=31+\frac{3}{7};$$

$$\therefore\ x+y+\frac{5y-3}{7}=31 \quad\text{.........................}\quad (1)$$

Since x and y are to be integers, we must have

$$\frac{5y-3}{7}=\text{integer};$$

and therefore $$\frac{15y-9}{7}=\text{integer};$$

that is, $$2y-1+\frac{y-2}{7}=\text{integer};$$

and therefore $$\frac{y-2}{7}=\text{integer}=p \text{ suppose.}$$

$$\therefore\ y-2=7p,$$

or $$y=7p+2 \quad\text{................................}\quad (2).$$

Substituting this value of y in (1),

$$x+7p+2+5p+1=31;$$

that is, $$x=28-12p \quad\text{...}\quad (3).$$

If in these results we give to p any integral value, we obtain corresponding integral values of x and y; but if $p>2$, we see from (3) that x is negative; and if p is a negative integer, y is negative. Thus the only *positive integral* values of x and y are obtained by putting $p=0, 1, 2$.

The complete solution may be exhibited as follows:

$$\left.\begin{array}{llll} p= & 0, & 1, & 2, \\ x= & 28, & 16, & 4, \\ y= & 2, & 9, & 16. \end{array}\right\}$$

NOTE. When we obtained $\frac{5y-3}{7}=\text{integer}$, we multiplied by 3 *in order to make the coefficient of y differ by unity from a multiple of* 7. A similar artifice should always be employed before introducing a symbol to denote the integer.

Example 2. Solve in positive integers, $14x-11y=29$..................(1).

Divide by 11, the smaller coefficient; thus

$$x+\frac{3x}{11}-y=2+\frac{7}{11};$$

$$\therefore\ \frac{3x-7}{11}=2-x+y=\text{integer};$$

hence $$\frac{12x-28}{11}=\text{integer};$$

that is, $$x-2+\frac{x-6}{11}=\text{integer};$$

$$\therefore \frac{x-6}{11}=\text{integer}=p \text{ suppose};$$

and, from (1), $$\therefore \left.\begin{aligned} x&=11p+6 \\ y&=14p+5 \end{aligned}\right\}.$$

This is called the *general solution* of the equation, and by giving to p any positive integral value or zero, we obtain positive integral values of x and y; thus we have

$$\left.\begin{aligned} p&=0,\ \ 1,\ \ 2,\ \ 3,\ \ldots\ldots\ldots\ldots \\ x&=6,\ 17,\ 28,\ 39,\ \ldots\ldots\ldots\ldots \\ y&=5,\ 19,\ 33,\ 47,\ \ldots\ldots\ldots\ldots \end{aligned}\right\},$$

the number of solutions being infinite.

Example 3. In how many ways can £5 be paid in half-crowns and florins?

Let x be the number of half-crowns, y the number of florins; then

$$5x+4y=200;$$

$$\therefore x+y+\frac{x}{4}=50;$$

$$\therefore \frac{x}{4}=\text{integer}=p \text{ suppose};$$

$$\therefore x=4p,$$

and $$y=50-5p.$$

Solutions are obtained by ascribing to p the values 1, 2, 3, ...9; and therefore the number of ways is 9. If, however, the sum may be paid *either* in half-crowns *or* florins, p may also have the values 0 and 10. If $p=0$, then $x=0$, and the sum is paid entirely in florins; if $p=10$, then $y=0$, and the sum is paid entirely in half-crowns. Thus if zero values of x and y are admissible the number of ways is 11.

Example 4. The expenses of a party numbering 43 were £5. 14*s*. 6*d*.; if each man paid 5*s*., each woman 2*s*. 6*d*., and each child 1*s*., how many were there of each?

Let x, y, z denote the number of men, women, and children, respectively; then we have

$$x+y+z=43 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$10x+5y+2z=229.$$

Eliminating z, we obtain $$8x+3y=143.$$

The general solution of this equation is

$$x=3p+1,$$
$$y=45-8p;$$

Hence by substituting in (1), we obtain

$$z = 5p - 3.$$

Here p cannot be negative or zero, but may have positive integral values from 1 to 5. Thus

$$\begin{aligned} p &= 1,\ 2,\ 3,\ 4,\ 5; \\ x &= 4,\ 7,\ 10,\ 13,\ 16; \\ y &= 37,\ 29,\ 21,\ 13,\ 5; \\ z &= 2,\ 7,\ 12,\ 17,\ 22. \end{aligned}$$

EXAMPLES. X. d.

Solve in positive integers:

1. $3x + 8y = 103$. 2. $5x + 2y = 53$. 3. $7x + 12y = 152$.

4. $13x + 11y = 414$. 5. $23x + 25y = 915$. 6. $41x + 47y = 2191$.

Find the general solution in positive integers, and the least values of x and y which satisfy the equations:

7. $5x - 7y = 3$. 8. $6x - 13y = 1$. 9. $8x - 21y = 33$.

10. $17y - 13x = 0$. 11. $19y - 23x = 7$. 12. $77y - 30x = 295$.

13. A farmer spends £752 in buying horses and cows; if each horse costs £37 and each cow £23, how many of each does he buy?

14. In how many ways can £5 be paid in shillings and sixpences, including zero solutions?

15. Divide 81 into two parts so that one may be a multiple of 8 and the other of 5.

16. What is the simplest way for a person who has only guineas to pay 10*s.* 6*d.* to another who has only half-crowns?

17. Find a number which being divided by 39 gives a remainder 16, and by 56 a remainder 27. How many such numbers are there?

18. What is the smallest number of florins that must be given to discharge a debt of £1. 6*s.* 6*d.*, if the change is to be paid in half-crowns only?

19. Divide 136 into two parts one of which when divided by 5 leaves remainder 2, and the other divided by 8 leaves remainder 3.

20. I buy 40 animals consisting of rams at £4, pigs at £2, and oxen at £17: if I spend £301, how many of each do I buy?

21. In my pocket I have 27 coins, which are sovereigns, half-crowns or shillings, and the amount I have is £5. 0*s.* 6*d.*; how many coins of each sort have I?