22. Brief Review of Cartesian System of Rectangular Co-ordinates

Exercise 22.1

1. Question

If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtends an angle α at the origin O, prove that : OP. OQ cos α = $x_1 x_2 + y_1 y_2$.

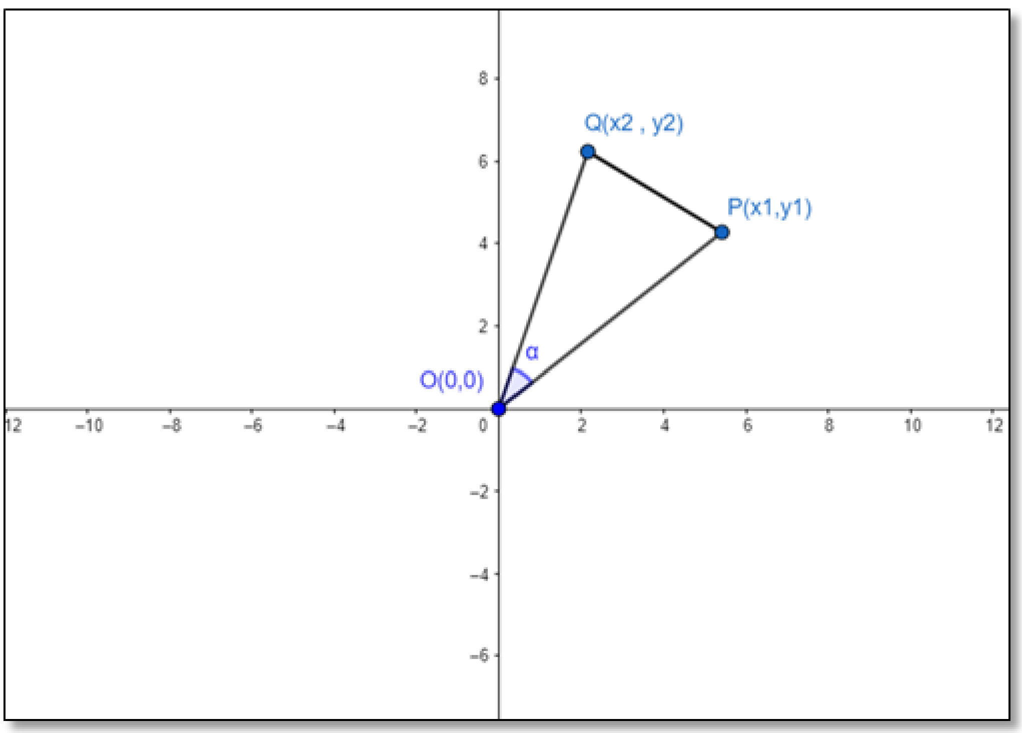
Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is given by- PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Given,

Two points P and Q subtends an angle α at the origin as shown in figure:



From figure we can see that points O,P and Q forms a triangle.

Clearly in ΔOPQ we have:

$\cos \alpha = \frac{OP^2 + OQ^2 - PQ^2}{2OP.OQ}$ {from cosine formula in a triangle}

$\Rightarrow 2\ OP.\ OQ \cos\alpha = OP^2 + OQ^2 - PQ^2$ …..equation 1

From distance formula we have-

OP = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

As, coordinates of O are (0, 0) ⇒ $x_2 = 0$ and $y_2 = 0$

Coordinates of P are (x_1, y_1) ⇒ $x_1 = x_1$ and $y_1 = y_1$

= $\sqrt{(x_1 - 0)^2 + (y_1 - 0)^2}$

= $\sqrt{x_1^2 + y_1^2}$

Similarly, OQ = $\sqrt{(x_2 - 0)^2 + (y_2 - 0)^2}$

= $\sqrt{x_2^2 + y_2^2}$

And, PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

∴ $OP^2 + OQ^2 - PQ^2 = \left(\sqrt{x_1^2 + y_1^2}\right)^2 + \left(\sqrt{x_2^2 + y_2^2}\right)^2 - \left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)^2$

⇒ $OP^2 + OQ^2 - PQ^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}$

Using $(a-b)^2 = a^2 + b^2 - 2ab$

∴ $OP^2 + OQ^2 - PQ^2 = 2x_1 x_2 + 2y_1 y_2$ ….equation 2

From equation 1 and 2 we have:

$2OP.\ OQ \cos\alpha = 2x_1x_2 + 2y_1y_2$

⇒ $OP.\ OQ \cos\alpha = x_1x_2 + y_1y_2$ …Proved.

2. Question

The vertices of a triangle ABC are A(0, 0), B (2, -1) and C (9, 0). Find cos B.

Answer

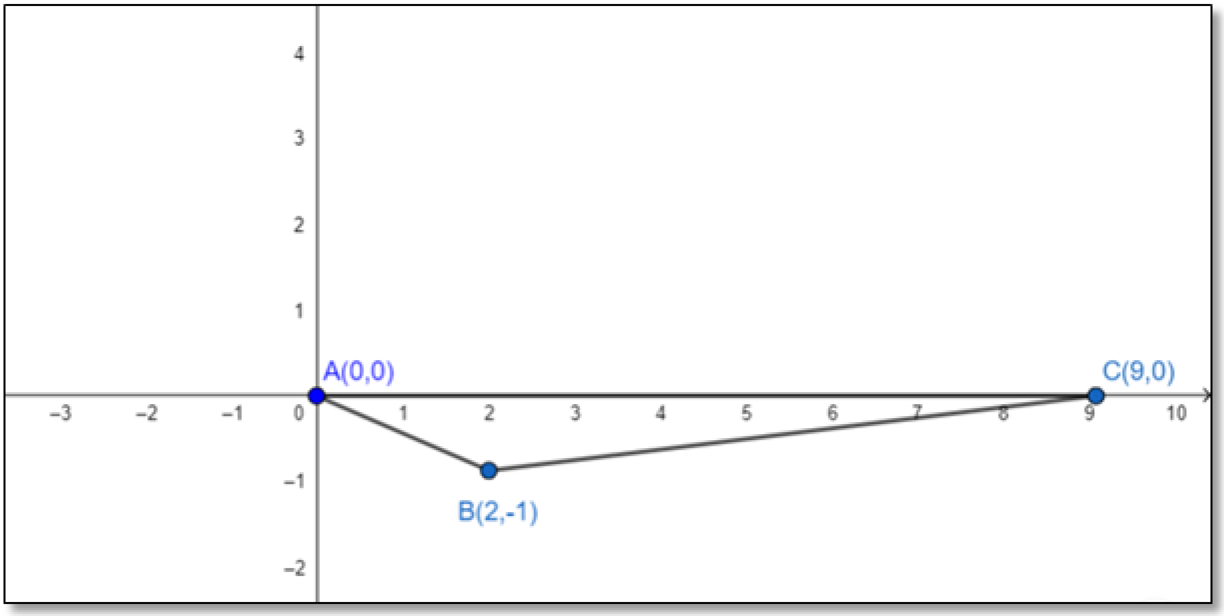
Key points to solve the problem:

• Idea of distance formula- Distance between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is given by- PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Given,

Coordinates of triangle and we need to find cos B which can be easily found using cosine formula.

See the figure:



From cosine formula in ΔABC , We have:

cos B = $\frac{AB^2+BC^2-AC^2}{2AB.BC}$

using distance formula we have:

AB = $\sqrt{(2-0)^2+(-1-0)^2} = \sqrt{5}$

BC = $\sqrt{(9-2)^2+(0-(-1))^2} = \sqrt{7^2+1^2} = \sqrt{50}$

And, AC = $\sqrt{(9-0)^2+(0-0)^2} = 9$

∴ cos B = $\frac{(\sqrt{5})^2+(\sqrt{50})^2-9^2}{2\sqrt{5}\sqrt{50}} = \frac{55-81}{2\sqrt{5}\sqrt{2\times 25}} = \frac{-26}{10\sqrt{10}} = \frac{-13}{5\sqrt{10}}$

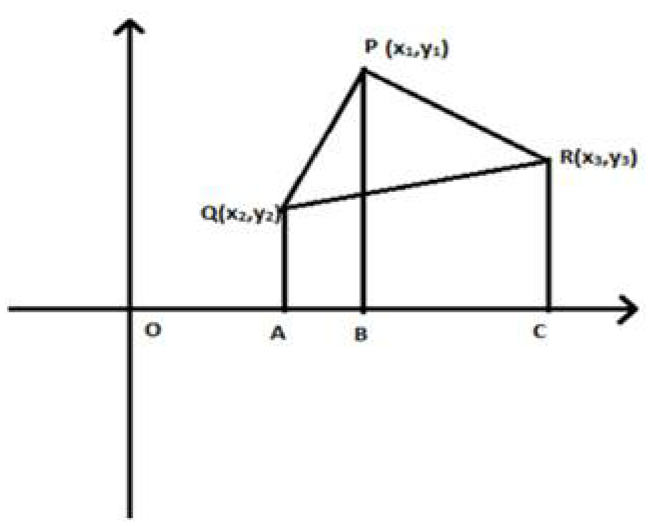
3. Question

Four points A (6, 3), B(-3, 5), C(4, -2) and D(x, 3x) are given in such a way that $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.

Answer

Key points to solve the problem:

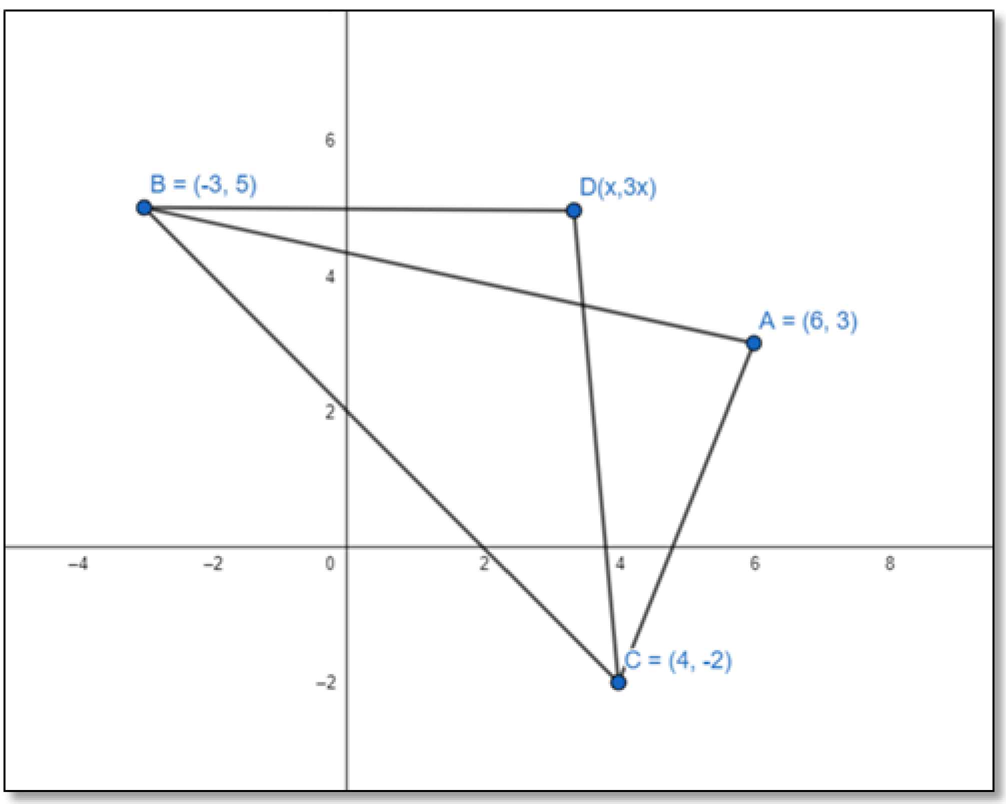
• Idea of distance formula- Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



• Area of a ΔPQR – Let $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ be the 3 vertices of ΔPQR.

Ar(ΔPQR) = $\frac{1}{2}[x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)]$

Given, coordinates of triangle as shown in figure.



Also, $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$

ar(ΔDBC) = $\frac{1}{2}[x(5 - (-2)) + (-3)(-2 - 3x) + 4(3x - 5)]$

= $\frac{1}{2}[7x + 6 + 9x + 12x - 20] = 14x - 7$

Similarly, ar(ΔABC) = $\frac{1}{2}[6(5-(-2))+(-3)(-2-3)+4(3-5)]$

= $\frac{1}{2}[42+15-8]=\frac{49}{2}=24.5$

$\therefore \frac{\Delta DBC}{\Delta ABC}=\frac{1}{2}=\frac{14x-7}{24.5}$

⇒ 24.5 = 28x − 14

⇒ 28x = 38.5

⇒ x = 38.5/28 = 1.375

4. Question

The points A (2, 0), B(9, 1), C (11, 6) and D (4, 4) are the vertices of a quadrilateral ABCD. Determine whether ABCD is a rhombus or not.

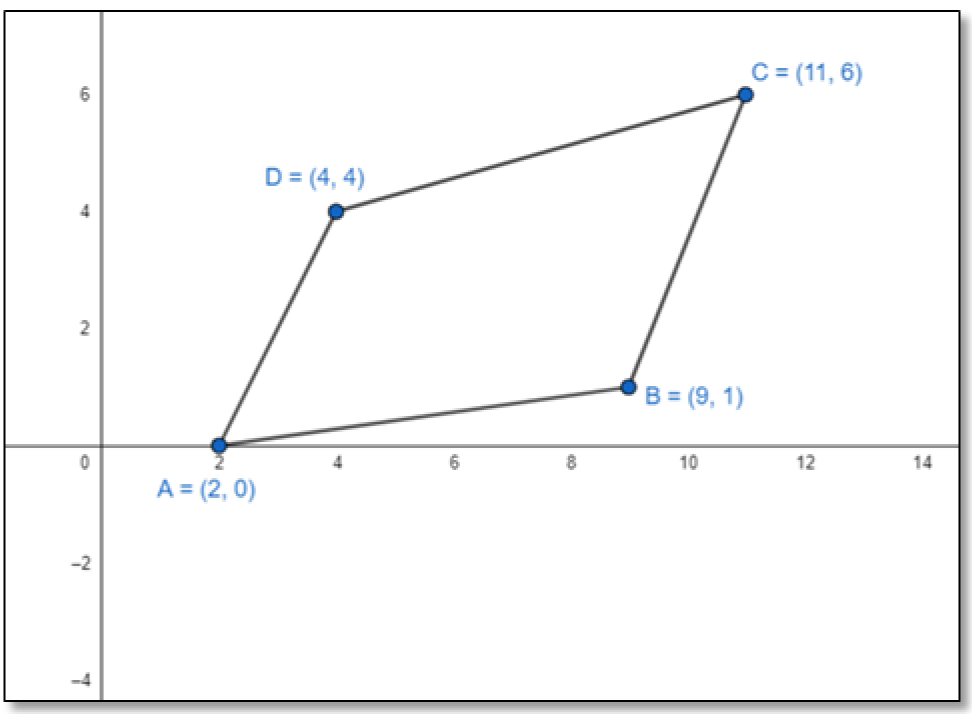
Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is given by- PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• Idea of Rhombus – It is a quadrilateral with all four sides equal.

Given, coordinates of 4 points that form a quadrilateral as shown in fig:



Using distance formula, we have:

AB = $\sqrt{(9-2)^2+(1-0)^2}=\sqrt{7^2+1}=\sqrt{50}$

BC = $\sqrt{(11-9)^2+(6-1)^2} = \sqrt{2^2+5^2} = \sqrt{29}$

Clearly, AB ≠ BC ⇒ quad ABCD does not have all 4 sides equal.

∴ ABCD is not a Rhombus

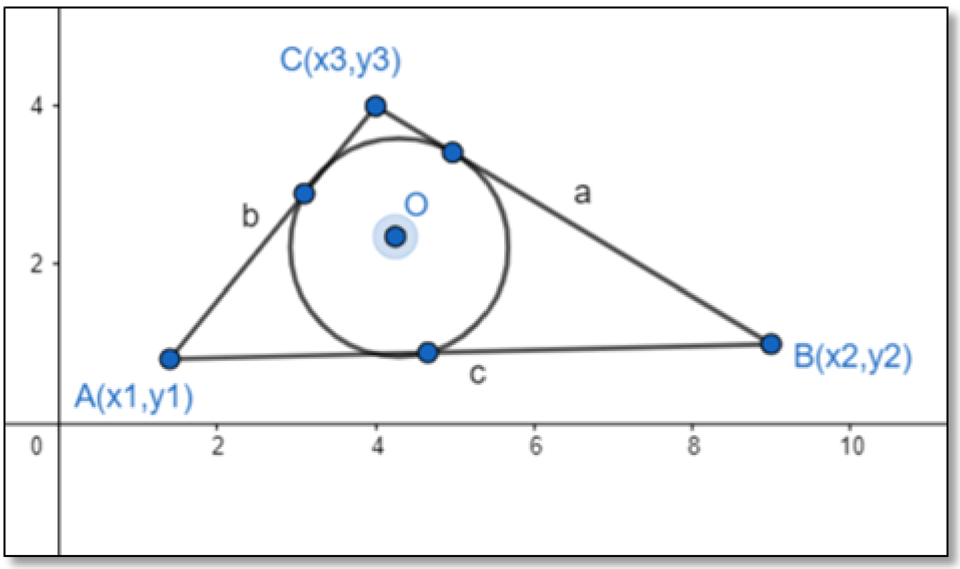
5. Question

Find the coordinates of the centre of the circle inscribed in a triangle whose vertices are (-36, 7), (20, 7) and (0, -8).

Answer

Key points to solve the problem:

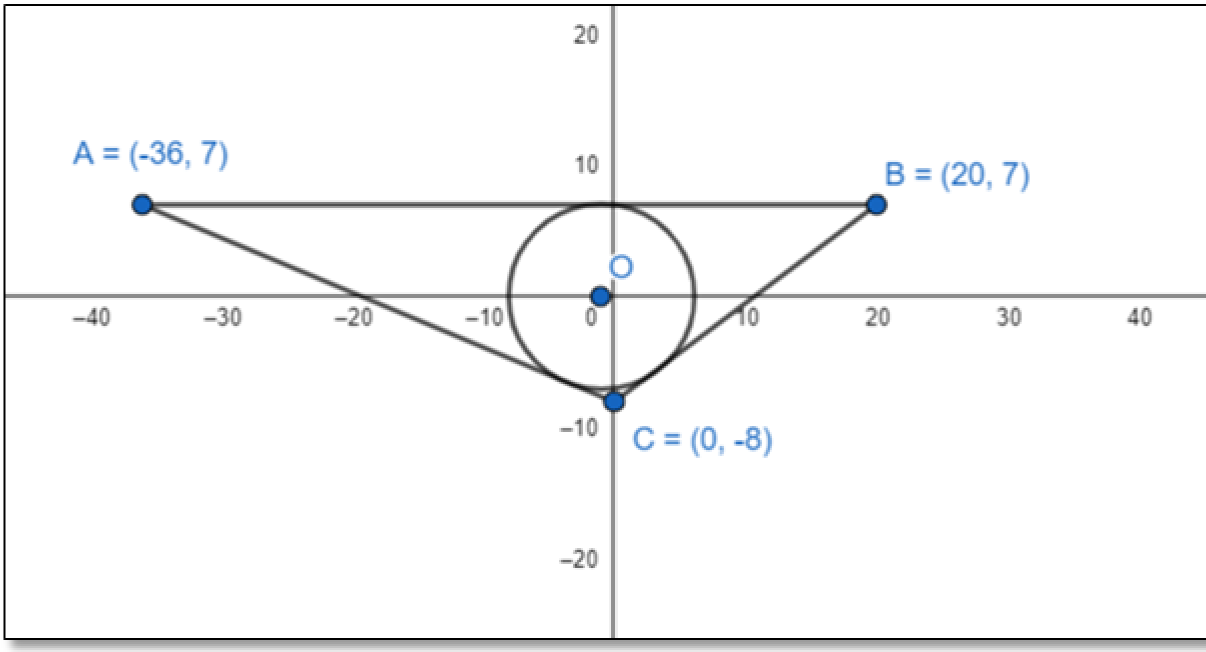
• Idea of distance formula- Distance between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is given by- PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



• Incentre of a triangle - Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the 3 vertices of ΔABC and O be the centre of circle inscribed in ΔABC

O = $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$ where a, b and c are length of sides opposite to ∠ A , ∠ B and ∠ C respectively.

Given, coordinates of vertices of triangle as shown in figure:



We need to find the coordinates of O:

Before that we have to find a ,b and c. We will use distance formula to find the same.

As, a = BC = $\sqrt{(20-0)^2+(7-(-8))^2} = \sqrt{20^2+15^2} = 25$

b = AC = $\sqrt{(-36-0)^2+(7-(-8))^2} = \sqrt{36^2+15^2} = \sqrt{1521} = 39$

and c = AB = $\sqrt{(-36-20)^2+(7-7)^2} = 56$

∴ coordinates of O = $\left(\frac{25(-36)+39(20)+56(0)}{25+39+56}, \frac{25(7)+39(7)+56(-8)}{25+39+56}\right)$

$= \left(\frac{-1}{120}, \frac{0}{120}\right) = (-1\,,0)$

6. Question

The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points P(x_1,y_1) and Q(x_2,y_2) is given by- PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• Equilateral triangle- triangle with all 3 sides equal.

• Coordinates of midpoint of a line segment – Let P(x_1,y_1) and Q(x_2,y_2) be the end points of line segment PQ. Then coordinated of midpoint of PQ is given by – $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$

Given, an equilateral triangle with base along y axis and midpoint at (0,0)

∴ coordinates of triangle will be A(0,y_1) B(0,y_2) and C(x,0)

As midpoint is at origin ⇒ $y_1+y_2 = 0$ ⇒ $y_1 = -y_2$ …..eqn 1

Also length of each side = 2a (given)

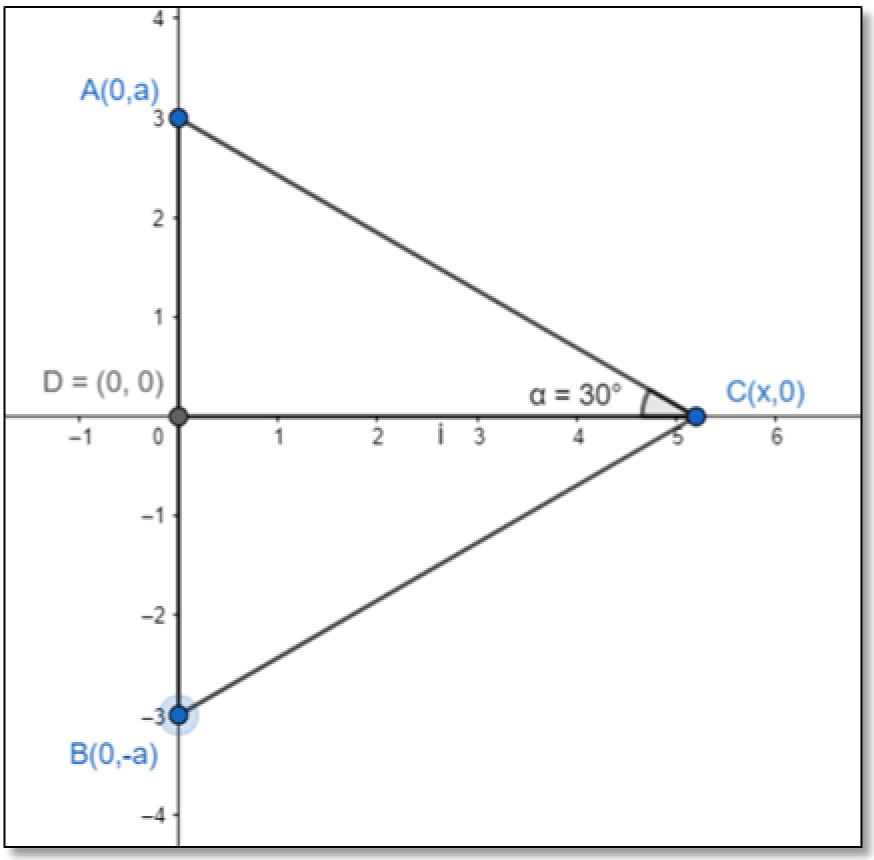
∴ AB = $\sqrt{(0-0)^2+(y_2-y_1)^2} = y_2 - y_1 = 2a$ ….eqn 2

∴ from eqn 1 and 2:

y_1 = a and y_2 = -a

∴ 2 coordinates are – A(0,a) and B(0,-a)

See the figure:



Clearly from figure:

DC = x

Also in ΔADC: cos 30° = $\frac{DC}{AC} = \frac{x}{\sqrt{(0-x)^2+(a-0)^2}}$

∴ $\frac{\sqrt{3}}{2} = \frac{x}{\sqrt{x^2+a^2}}$

Squaring both sides:

$$3(x^2+a^2) = 4x^2 \ \Rightarrow \ x^2 = 3a^2$$

$\therefore x = \pm\sqrt{3a}$

∴ Coordinates of C are (√3a,0) or (-√3a,0)

7. Question

Find the distance between P(x_1, y_1) and Q(x_2, y_2) when (i) PQ is parallel to the y-axis (ii) PQ is parallel to the x-axis.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Given, P(x_1, y_1) and Q(x_2, y_2) are two points.

i) When PQ is parallel to y-axis

This implies that x – coordinate is constant ⇒ $x_2 = x_1$

∴ from distance formula:

PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{0+(y_2-y_1)^2}$ = $|y_2 - y_1|$

ii) When PQ is parallel to x-axis

This implies that y – coordinate is constant ⇒ $y_2 = y_1$

∴ from distance formula:

PQ = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{0+(x_2-x_1)^2}$ = $|x_2 - x_1|$

Note: we take modulus because square root gives both positive and negative values but distance is always positive so we make it positive using modulus function.

8. Question

Find a point on the x-axis, which is equidistant from the point (7, 6) and (3, 4).

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

As, the point is on x-axis so y-coordinate is 0.

Let the coordinate be (x,0)

Given distance of (x,0) from (7,6) and (3,4) is same.

∴ using distance formula we have:

$$\sqrt{(x-7)^2+(0-6)^2} = \sqrt{(x-3)^2+(0-4)^2}$$

squaring both sides, we have:

$$(x-7)^2+(0-6)^2 = (x-3)^2+(0-4)^2$$

$$\Rightarrow x^2+49-14x+36 = x^2+9-6x+16$$

$$\Rightarrow 8x = 60 \Rightarrow x = \frac{60}{8} = \frac{15}{2} = 7.5$$

∴ point on x-axis is (7.5,0)

Exercise 22.2

1. Question

Find the locus of a point equidistant from the point (2, 4) and the y-axis.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k)

As we need to maintain a same distance of (h,k) from (2,4) and y-axis.

So we select point (0,k) on y-axis.

From distance formula:

Distance of (h,k) from (2,4) = $\sqrt{(h-2)^2+(k-4)^2}$

Distance of (h,k) from (0,k) = $\sqrt{(h-0)^2+(k-k)^2}$

According to question both distance are same.

$$\therefore \sqrt{(h-2)^2+(k-4)^2} = \sqrt{(h-0)^2+(k-k)^2}$$

Squaring both sides:

$$(h-2)^2+(k-4)^2 = (h-0)^2+(k-k)^2$$

$$\Rightarrow h^2+4-4h+k^2-8k+16 = h^2+0$$

$$\Rightarrow k^2-4h-8k+20 = 0$$

Replace (h,k) with (x,y)

Thus, locus of point equidistant from (2,4) and y-axis is-

$$y^2 - 4x - 8y + 20 = 0$$

2. Question

Find the equation of the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5 : 4.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the point whose locus is to be determined be (h,k)

Distance of (h,k) from (2,0) = $\sqrt{(h-2)^2 + (k-0)^2}$

Distance of (h,k) from (1,3) = $\sqrt{(h-1)^2 + (k-3)^2}$

According to question:

$$\frac{\sqrt{(h-2)^2 + (k-0)^2}}{\sqrt{(h-1)^2 + (k-3)^2}} = \frac{5}{4}$$

Squaring both sides:

$$16\{(h-2)^2 + k^2\} = 25\{(h-1)^2 + (k-3)^2\}$$

$$\Rightarrow 16\{h^2 + 4 - 4h + k^2\} = 25\{h^2 - 2h + 1 + k^2 - 6k + 9\}$$

$$\Rightarrow 9h^2 + 9k^2 + 14h - 150k + 186 = 0$$

Replace (h,k) with (x,y)

Thus, the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5 : 4 is –

$$9x^2 + 9y^2 + 14x - 150y + 186 = 0$$

3. Question

A point moves as so that the difference of its distances from (ae, 0) and (-ae, 0) is 2a, prove that the equation to its locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the point whose locus is to be determined be (h,k)

Distance of (h,k) from (ae,0) = $\sqrt{(h-ae)^2+(k-0)^2}$

Distance of (h,k) from (-ae,0) = $\sqrt{(h-(-ae))^2+(k-0)^2}$

According to question:

$$\sqrt{(h-ae)^2+(k-0)^2} - \sqrt{\left(h-(-ae)\right)^2+(k-0)^2} = 2a$$

$$\Rightarrow \sqrt{(h-ae)^2+(k-0)^2} = 2a + \sqrt{(h+ae)^2+(k-0)^2}$$

Squaring both sides:

$$(h-ae)^2+(k-0)^2 = \left\{2a + \sqrt{(h+ae)^2+(k-0)^2}\right\}^2$$

$$\Rightarrow h^2+a^2e^2-2aeh+k^2 = 4a^2+\{(h+ae)^2+k^2\}+4a\sqrt{(h+ae)^2+(k-0)^2}$$

$$\Rightarrow h^2+a^2e^2-2aeh+k^2$$
$$= 4a^2+h^2+2aeh+a^2e^2+k^2+4a\sqrt{(h+ae)^2+(k-0)^2}$$

$$\Rightarrow -4aeh-4a^2 = 4a\sqrt{(h+ae)^2+(k-0)^2}$$

$$\Rightarrow -4a(eh+a) = 4a\sqrt{(h+ae)^2+(k-0)^2}$$

Again squaring both sides:

$$(eh+a)^2 = (h+ae)^2+(k-0)^2$$

$$\Rightarrow e^2h^2+a^2+2aeh = h^2+a^2e^2+2aeh+k^2$$

$$\Rightarrow h^2(e^2-1)-k^2 = a^2(e^2-1)$$

$$\therefore \frac{h^2}{a^2}-\frac{k^2}{a^2(e^2-1)} = 1$$

$\Rightarrow \frac{h^2}{a^2}-\frac{k^2}{b^2} = 1$ where $b^2 = a^2(e^2-1)$

Replace (h,k) with (x,y)

Thus, locus of a point such that difference of its distances from (ae, 0) and (-ae, 0) is 2a:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$ ….proved

4. Question

Find the locus of a point such that the sum of its distances from (0, 2) and (0, -2) is 6.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the point whose locus is to be determined be (h,k)

Distance of (h,k) from (0,2) = $\sqrt{(h-0)^2 + (k-2)^2}$

Distance of (h,k) from (0,-2) = $\sqrt{(h-0)^2 + (k-(-2))^2}$

According to question:

$\sqrt{(h)^2 + (k-2)^2} + \sqrt{(h)^2 + (k+2)^2} = 6$

⇒ $\sqrt{(h)^2 + (k-2)^2} = 6 - \sqrt{(h)^2 + (k+2)^2}$

Squaring both sides:

$h^2 + (k-2)^2 = \left\{6 - \sqrt{h^2 + (k+2)^2}\right\}^2$

⇒ $h^2 + 4 - 4k + k^2 = 36 + \{h^2 + k^2 + 4k + 4\} - 12\sqrt{h^2 + (k+2)^2}$

⇒ $-8k - 36 = -12\sqrt{h^2 + (k+2)^2}$

⇒ $-4(2k+9) = -12\sqrt{h^2 + (k+2)^2}$

Again squaring both sides:

$(2k+9)^2 = \left\{3\sqrt{h^2 + (k+2)^2}\right\}^2$

⇒ $4k^2 + 81 + 36k = 9(h^2 + k^2 + 4k + 4)$

⇒ $9h^2 + 5k^2 = 45$

Replace (h,k) with (x,y)

Thus, locus of a point such that sum of its distances from (0,2) and (0,-2) is 6:

$9x^2 + 5y^2 = 45$ ....proved

5. Question

Find the locus of a point which is equidistant from (1, 3) and x-axis.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k)

As we need to maintain a same distance of (h,k) from (2,4) and x-axis.

So we select point (h,0) on x-axis.

From distance formula:

Distance of (h,k) from (1,3) = $\sqrt{(h-1)^2+(k-3)^2}$

Distance of (h,k) from (h,0) = $\sqrt{(h-h)^2+(k-0)^2}$

According to question both distance are same.

$\therefore \sqrt{(h-1)^2+(k-3)^2} = \sqrt{(h-h)^2+(k-0)^2}$

Squaring both sides:

$(h-1)^2+(k-3)^2 = (h-h)^2+(k-0)^2$

$\Rightarrow h^2+1-2h+k^2-6k+9 = k^2+0$

$\Rightarrow h^2-2h-6k+10 = 0$

Replace (h,k) with (x,y)

Thus, locus of point equidistant from (1,3) and x-axis is-

$x^2-2x-6y+10 = 0$

6. Question

Find the locus of a point which moves such that its distance from the origin is three times is distance from x-axis.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k)

As we need to maintain a distance of (h,k) from origin such that it is 3 times the distance from x-axis.

So we select point (h,0) on x-axis.

From distance formula:

Distance of (h,k) from (0,0) = $\sqrt{(h-0)^2+(k-0)^2}$

Distance of (h,k) from (h,0) = $\sqrt{(h-h)^2+(k-0)^2}$

According to question both distance are same.

$\therefore \sqrt{(h-0)^2+(k-0)^2} = 3\sqrt{(h-h)^2+(k-0)^2}$

Squaring both sides:

$h^2+k^2 = 9k^2$

$\Rightarrow h^2 = 8k^2$

Replace (h,k) with (x,y)

Thus, locus of point is $x^2 = 8y^2$ …

7. Question

A(5, 3), B(3, -2) are two fixed points, find the equation to the locus of a point P which moves so that the area of the triangle PAB is 9 units.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

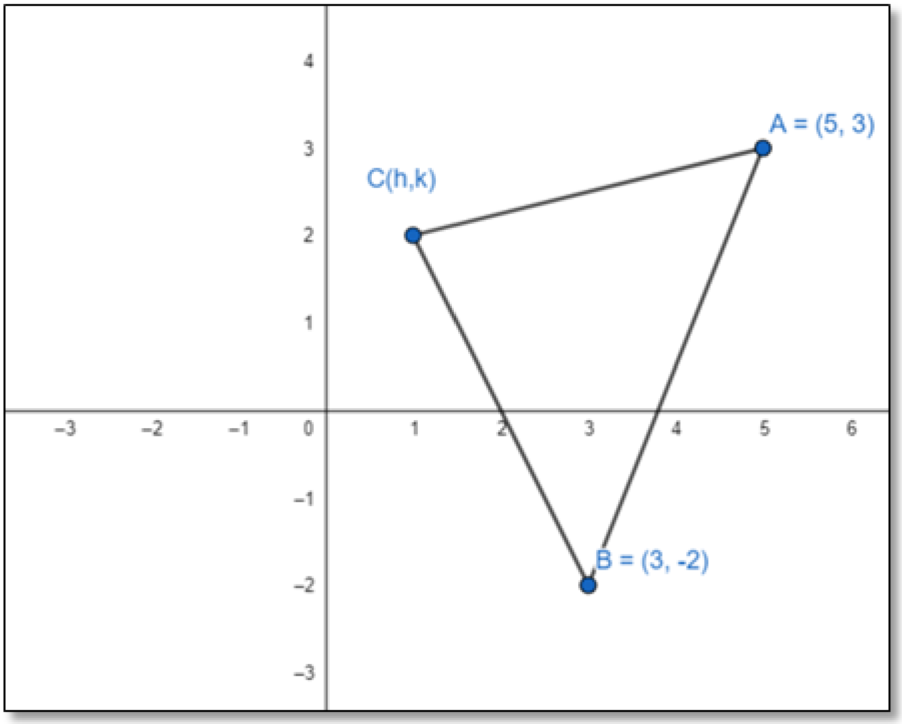
Area of a ΔPQR – Let P(x_1,y_1) , Q(x_2,y_2) and R(x_3,y_3) be the 3 vertices of ΔPQR.

Ar(ΔPQR) = $\frac{1}{2}|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k). Name the moving point be C

Given area of ΔABC = 9



According to question:

9 = $\frac{1}{2}|5(-2-k)+3(k-3)+h\left((3-(-2))\right|$

⇒ $18 = |-10-5k+3k-9+3h+2h|$

⇒ $|5h-2k-19| = 18$

∴ $5h-2k-19 = 18 \text{ or } 5h-2k-19 = -18$

⇒ $5h-2k-37 = 0 \text{ or } 5h-2k-1 = 0$

Replace (h,k) with (x,y)

Thus, locus of point is $5x-2y-37 = 0 \text{ or } 5x-2y-1 = 0$ ...

8. Question

Find the locus of a point such that the line segments having end points (2, 0) and (-2, 0) subtend a right angle at that point.

Answer

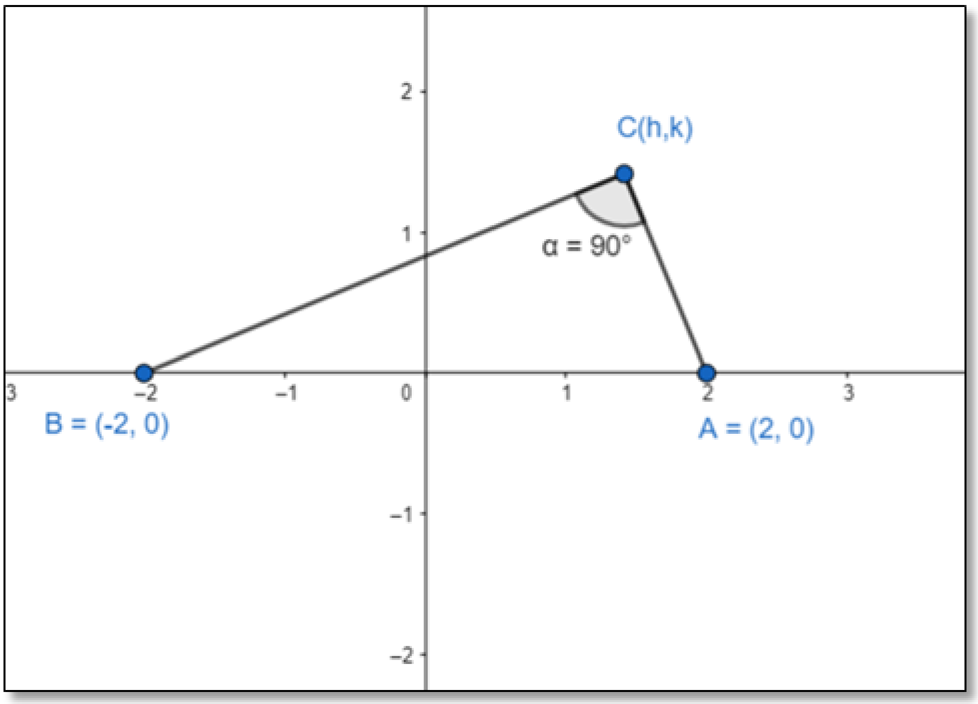
Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• Pythagoras theorem: In right triangle ΔABC : sum of the square of two sides is equal to square of its hypotenuse.

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k) and name the moving point be C.



According to question on drawing the figure we get a right triangle Δ ABC.

From Pythagoras theorem we have:

$BC^2 + AC^2 = AB^2$

From distance formula:

BC = $\sqrt{(h-(-2))^2+(k-0)^2}$

AC = $\sqrt{(h-2)^2+(k-0)^2}$

And AB = 4

$\therefore \left\{\sqrt{(h-(-2))^2+(k-0)^2}\right\}^2+\left\{\sqrt{(h-2)^2+(k-0)^2}\right\}^2=16$

$\Rightarrow (h+2)^2+k^2+(h-2)^2+k^2=16$

$\Rightarrow h^2+4+4h+k^2+h^2-4h+4+k^2=16$

$\Rightarrow 2h^2+2k^2-8=0$

$\Rightarrow h^2 + k^2 = 4$

Replace (h,k) with (x,y)

Thus, locus of point is $x^2 + y^2 = 4$

9. Question

If A (-1, 1) and B (2, 3) are two fixed points, find the locus of a point P so that the area d ΔPAB = 8 sq. units.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

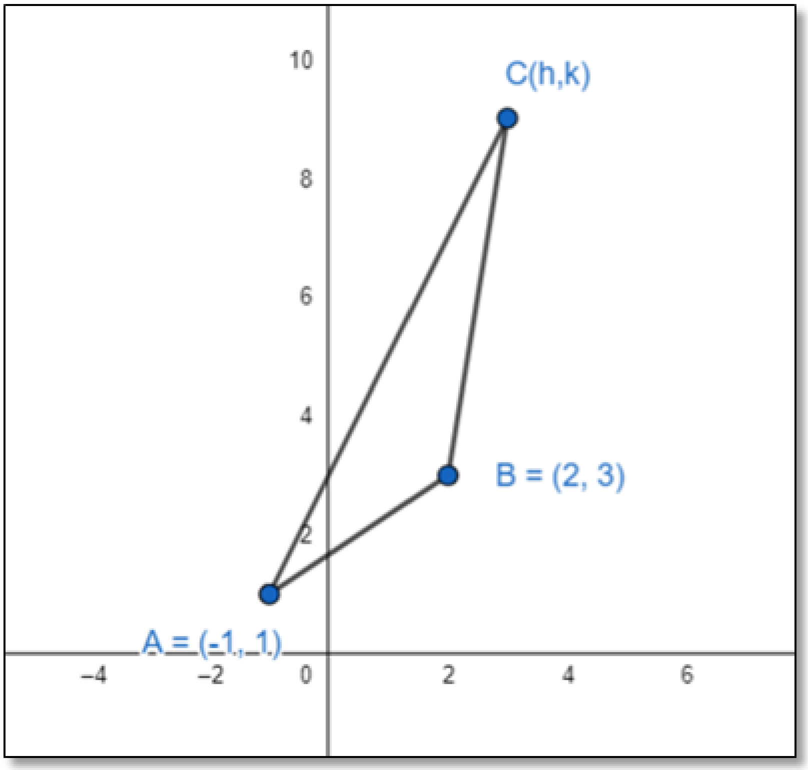
Area of a ΔPQR – Let $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ be the 3 vertices of ΔPQR.

Ar(ΔPQR) = $\frac{1}{2}|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k). Name the moving point be C

Given area of ΔABC = 8



According to question:

$8 = \frac{1}{2}\left| -1(3-k) + 2(k-1) + h\left((1-3)\right|\right.$

$\Rightarrow 16 = |-3 + k + 2k - 2 + h - 3h|$

$\Rightarrow |3k - 2h - 5| = 16$

$\therefore 3k - 2h - 5 = 16 \text{ or } 3k - 2h - 5 = -16$

$\Rightarrow 3k - 2h - 21 = 0 \text{ or } 3k - 2h + 11 = 0$

Replace (h,k) with (x,y)

Thus, locus of point is $3y - 2x - 21 = 0 \text{ or } 3y - 2x + 11 = 0 \ldots$

10. Question

A rod of length l slides between the two perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points A(x_1,y_1) and B(x_2,y_2) is given by- AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

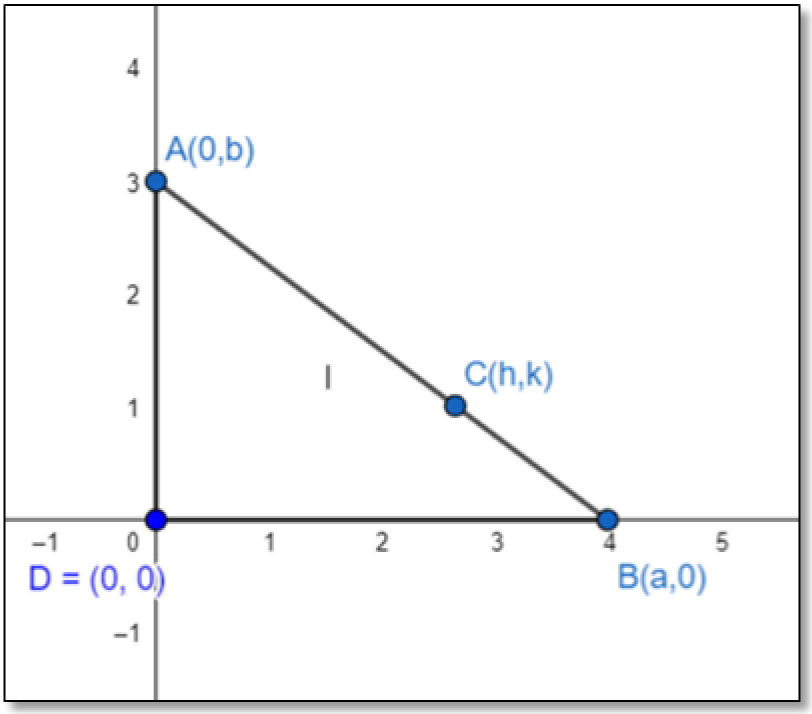
• Idea of section formula- Let two points A(x_1,y_1) and B(x_2,y_2) forms a line segment. If a point C(x,y) divides line segment AB in ratio of m:n internally, then coordinates of C is given as:

C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k). Name the moving point be C

Assume the two perpendicular lines on which rod slides are x and y axis respectively.



Here line segment AB represents the rod of length l also ΔADB formed is a right triangle. Coordinates of A and B are assumed to be (0,b) and (a,0) respectively.

∴ $a^2 + b^2 = l^2$ ...eqn 1

As, (h,k) divides AB in ratio of 1:2

∴ from section formula we have coordinate of point C as-

C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right) = \left(\frac{1\times0+2\times a}{2+1}, \frac{1\times b+2\times0}{2+1}\right) = \left(\frac{2a}{3}, \frac{b}{3}\right)$

As, a and b are assumed parameters so we have to remove it.

∵ h = 2a/3 ⇒ a = 3h/2

And k = b/3 ⇒ b = 3k

From eqn 1:

$a^2 + b^2 = l^2$

∴ $\left(\frac{3h}{2}\right)^2 + (3k)^2 = l^2$

⇒ $\frac{9h^2}{4} + 9k^2 = l^2 \quad \Rightarrow \frac{h^2}{4} + k^2 = \frac{l^2}{9}$

Replace (h,k) with (x,y)

Thus, locus of point on rod is: $\frac{x^2}{4} + y^2 = \frac{l^2}{9}$

11. Question

Find the locus of the mid-point of the portion of the x cos α + y sin α = p which is intercepted between the axes.

Answer

Key points to solve the problem:

• Idea of distance formula- Distance between two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by- AB = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

• Idea of section formula- Let two points $A(x_1,y_1)$ and $B(x_2,y_2)$ forms a line segment. If a point C(x,y) divides line segment AB in ratio of m:n internally, then coordinates of C is given as:

C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ when m = n =1 , C becomes the midpoint of AB and C is given as C = $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k). Name the moving point be C

Given that (h,k) is midpoint of line x cos α + y sin α = p intercepted between axes.

So we need to first find the points at which x cos α + y sin α = p cuts the axes after which we will apply the section formula to get the locus.

Put y = 0

∴ x = p/cos α ⇒ coordinates on x-axis is (p/cos α , 0). Name the point A

Similarly, Put x = 0

∴ y = p/sin α ⇒ coordinates on y-axis is (0, p/sin α). Name this point B

As C(h,k) is midpoint of AB

∴ coordinate of C is given by:

C = $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right) = \left(\frac{\frac{p}{\cos\alpha}+0}{2}, \frac{0+\frac{p}{\sin\alpha}}{2}\right) = \left(\frac{p}{2\cos\alpha}, \frac{p}{2\sin\alpha}\right)$

Thus,

$h = \frac{p}{2\cos\alpha} \Rightarrow \frac{p}{2h} = \cos\alpha$ ...equation 1

and $k = \frac{p}{2\sin\alpha} \Rightarrow \frac{p}{2k} = \sin\alpha$ ...equation 2

Squaring and adding equation 1 and 2:

$$\frac{p^2}{4h^2} + \frac{p^2}{4k^2} = \cos^2\alpha + \sin^2\alpha$$

$\Rightarrow \frac{p^2}{4h^2} + \frac{p^2}{4k^2} = 1$

Replace (h,k) with (x,y)

Thus, locus of point on rod is: $\frac{p^2}{4x^2} + \frac{p^2}{4y^2} = 1$

12. Question

If O is the origin and Q is a variable point on $y^2 = x$, Find the locus of the mid-point of OQ.

Answer

Key points to solve the problem:

• Idea of section formula- Let two points $A(x_1,y_1)$ and $B(x_2,y_2)$ forms a line segment. If a point C(x,y) divides line segment AB in ratio of m:n internally, then coordinates of C is given as:

C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ when m = n =1 , C becomes the midpoint of AB and C is given as C = $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the coordinates of point whose locus is to be determined be (h, k). Name the moving point be C

As, coordinate of mid point is (h,k) {by our assumption},

Let Q(a,b) be the point such that Q lies on curve $y^2 = x$

$b^2 = a$ ……equation 1

According to question C is midpoint of OQ

$\because$ C = $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right) \Rightarrow$ C = $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

$\therefore h = \frac{a}{2}$ or $a = 2h$

Similarly, $k = \frac{b}{2}$ or $b = 2k$

Putting values of a and b in equation 1,we have:

$(2k)^2 = 2h \Rightarrow 4k^2 = 2h \Rightarrow 2k^2 = h$

Replace (h,k) with (x,y)

Thus, locus of point is: $2y^2 = x$

Exercise 22.3

1. Question

What does the equation $(x - a)^2 + (y - b)^2 = r^2$ become when the axes are transferred to parallel axes through the point (a-c, b)?

Answer

Given, equation $(x - a^2) + (y - b)^2 = r^2$. For curious readers- this equation represents a circle in the space centered at point (a, b) having a radius of r units.

To find: Transformed equation of given equation when the coordinate axes are transformed parallelly at point (a - c, b).

We know that, when we transform origin from (0, 0) to an arbitrary point (p, q), the new coordinates for the point (x, y) becomes (x + p, y + q), and hence an equation with two variables x and y must be transformed accordingly replacing x with x + p, and y with y + q in original equation.

Since, origin has been shifted from (0, 0) to (a − c, b); therefore any arbitrary point (x, y) will also be converted as (x + (a − c), y + b) or (x + a - c, y + b).

The given equation $(x - a)^2 + (y - b)^2 = r^2$ will hence be transformed into the new equation by changing x by x − a + c and y by y − b, i.e. substitution of x by x + a and y by y + b.

$= ((x + a - c) - a)^2 + ((y - b) - b)^2 = r^2$

$= (x - c)^2 + y^2 = r^2$

$= x^2 + c^2 - 2cx + y^2 = r^2$

$= x^2 + y^2 = r^2 - c^2 + 2cx$

Hence, the transformed equation is $x^2 + y^2 = r^2 - c^2 + 2cx$.

2. Question

What does the equation $(a - b) (x^2 + y^2) - 2abx = 0$ become if the origin is shifted to the point (ab/(a-b), 0) without rotation?

Answer

Given, equation $(a - b)(x^2 + y^2) - 2abx = 0$

To find: Transformed equation of given equation when the origin (0, 0) is shifted at point (ab/(a − b), 0).

We know that, when we transform origin from (0, 0) to an arbitrary point (p, q), the new coordinates for the point (x, y) becomes (x + p, y + q), and hence an equation with two variables x and y must be transformed accordingly replacing x with x + p, and y with y + q in original equation.

Since, origin has been shifted from (0, 0) to (ab/(a − b), 0); therefore any arbitrary point (x, y) will also be converted as (x + (ab / (a - b)), y + 0) or (x + ab / (a - b), y).

The given equation $(a - b)(x^2 + y^2) - 2abx = 0$will hence be transformed into new equation by changing x by x + ab/(a-b) and y by y as

$$\Rightarrow (a-b)\left(\left(x+\frac{ab}{a-b}\right)^2+y^2\right)-2ab\left(x+\frac{ab}{(a-b)}\right)=0$$

$$\Rightarrow (a-b)\left(\left(x^2+\left(\frac{ab}{a-b}\right)^2+2x\frac{ab}{a-b}\right)^2+y^2\right)-2ab\left(\frac{(a-b)x+ab}{(a-b)}\right)=0$$

$$\Rightarrow (a-b)^2(x^2+y^2)=a^2b^2$$

Hence, the transformed equation is $(a - b)^2 (x^2 + y^2) = a^2 b^2$.

3. Question

Find what the following equations become when the origin is shifted to the point (1, 1)?

(i) $x^2 + xy - 3x - y + 2 = 0$

(ii) $x^2 - y^2 - 2x + 2y = 0$

(iii) $xy - x - y + 1 = 0$

(iv) $xy - y^2 - x + y = 0$

Answer

To find: Transformed equation of given equation when the origin (0, 0) is shifted at point (ab/(a − b), 0).

We know that, when we transform origin from (0, 0) to an arbitrary point (p, q), the new coordinates for the point (x, y) becomes (x + p, y + q), and hence an equation with two variables x and y must be transformed accordingly replacing x with x + p, and y with y + q in original equation.

Since, origin has been shifted from (0, 0) to (1, 1); therefore any arbitrary point (x, y) will also be converted as (x + 1, y + 1) or (x + 1, y + 1).

(i) $x^2 + xy - 3x - y + 2 = 0$

Substituting the value of x by x + 1 and y by y + 1, we have

$= (x + 1)^2 + (x + 1)(y + 1) - 3(x + 1) - (y + 1) + 2 = 0$

$= x^2 + 1 + 2x + xy + x + y + 1 - 3x - 3 - y - 1 + 2 = 0$

$= x^2 + xy = 0$

Hence, the transformed equation is $x^2 + xy = 0$.

(ii) $x^2 - y^2 - 2x + 2y = 0$

Substituting the value of x and y by x + 1 and y + 1 respectively, we have

$= (x + 1)^2 - (y + 1)^2 - 2(x + 1) + 2(y + 1) = 0$

$= x^2 + 1 + 2x - y^2 - 1 - 2y - 2x - 2 + 2y + 2 = 0$

$= x^2 - y^2 = 0$

Hence, the transformed equation is $x^2 - y^2 = 0$.

(iii) $xy - x - y + 1 = 0$

Substituting the value of x and y by x + 1 and y + 1 respectively, we have

$= (x + 1)(y + 1) - (x + 1) - (y + 1) + 1 = 0$

$= xy + x + y + 1 - x - 1 - y - 1 + 1 = 0$

$= xy = 0$

Hence, the transformed equation is $xy = 0$.

(iv) $xy - y^2 - x + y = 0$

Substituting the value of x and y by x + 1 and y + 1 respectively, we have

$= (x + 1)(y + 1) - (y + 1)^2 - (x + 1) + (y + 1) = 0$

$= xy + x + y + 1 - y^2 - 1 - 2y - x - 1 + y + 1 = 0$

$= xy - y^2 = 0$

Hence, the transformed equation is $xy - y^2 = 0$.

4. Question

At what point the origin be shifted so that the equation $x^2 + xy - 3x + 2 = 0$ does not contain any first-degree term and constant term?

Answer

Given, equation $x^2 + xy - 3x + 2 = 0$

Let's assume that the origin is shifted at point (p, q).

To find: The shifted point (p, q) satisfying the question's conditions.

We know that, when we transform origin from (0, 0) to an arbitrary point (p, q), the new coordinates for the point (x, y) becomes (x + p, y + q), and hence an equation with two variables x and y must be transformed accordingly replacing x with x + p, and y with y + q in original equation.

Since, origin has been shifted from (0, 0) to (p, q); therefore any arbitrary point (x, y) will also be converted as (x + p, y + q).

The New equation hence becomes:

$= (x + p)^2 + (x + p)(y + q) - 3(x + p) + 2 = 0$

$= x^2 + p^2 + 2px + xy + py + qx + pq - 3x - 3p + 2 = 0$

$= x^2 + xy + x(2p + q - 3) + y(q - 1) + p^2 + pq - 3p - q + 2 = 0$

For no first degree term, we have $2p + q - 3 = 0$ and $p - 1 = 0$, and for no constant term we have $p^2 + pq - 3p - q + 2 = 0$.

Solving these simultaneous equations we have p = 1 and q = 1 from first equation. And, p = 1 and q = 1 satisfies $p^2 + pq - 3p - q + 2 = 0$.

Hence, the point to which origin must be shifted is (p, q) = (1, 1).

5. Question

Verify that the area of the triangle with vertices (2, 3), (5, 7) and (-3 -1) remains invariant under the translation of axes when the origin is shifted to the point (-1, 3).

Answer

Given points (2, 3), (5, 7), and (-3, -1).

To show: The area of a triangle is invariant to shifting of origin.

The area of triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Hence, the area of given triangle $= \frac{1}{2}[2(7+1) + 5(-1-3) - 3(3-7)]$

$= \frac{1}{2}[16 - 20 + 12]$

$= \frac{1}{2}[8]$

$= 4$

Origin shifted to point (-1, 3), the new coordinates of the triangle are (3, 0), (6, 4), and (-2, -4) obtained from subtracting a point (-1, 3).

Hence, the new area of triangle $= \frac{1}{2}[3(4-(-4)) + 6(-4-0) - 2(0-4)]$

$= \frac{1}{2}[24-24+8]$

$= \frac{1}{2}[8]$

$= 4$

Since the area of the triangle before and after the translation after shifting of origin remains same, i.e. 4. Therefore we can say that the area of a triangle is invariant to shifting of origin.

6. Question

Find, what the following equations become when the origin is shifted to the point (1, 1).

(i) $x^2 + xy - 3y^2 - y + 2 = 0$

(ii) $xy - y^2 - x + y = 0$

(iii) $xy - x - y + 1 = 0$

(iv) $x^2 - y^2 - 2x + 2y = 0$

Answer

To find: Transformed equations of given equations when the origin (0, 0) is shifted at point (1, 1).

We know that, when we transform origin from (0, 0) to an arbitrary point (p, q), the new coordinates for the point (x, y) becomes (x + p, y + q), and hence an equation with two variables x and y must be transformed accordingly replacing x with x + p, and y with y + q in original equation.

Since, origin has been shifted from (0, 0) to (1, 1); therefore any arbitrary point (x, y) will also be converted as (x + 1, y + 1).

(i) $x^2 + xy - 3y^2 - y + 2 = 0$

Substituting x and y with (x+1) and (y+1) respectively, we have

$= (x+1)^2 + (x+1)(y+1) - 3y^2 - (y + 1) + 2 = 0$

$= x^2 + 1 + 2x + xy + x + y + 1 - 3y^2 - y - 1 + 2 = 0$

$= x^2 - 3y^2 + xy + 3x - 6y = 0$

Hence, the transformed equation is $x^2 - 3y^2 + xy + 3x - 6y = 0$

(ii) $xy - y^2 - x + y = 0$

Substituting x and y with (x+1) and (y+1) respectively, we have

$= (x+1)(y+1) - y^2 - (x + 1) + (y + 1) = 0$

$= xy + x + y + 1 - y^2 - x - 1 - y - 1 = 0$

$= xy - y^2 = 0$

Hence, the transformed equation is $xy - y^2 = 0$

(iii) $xy - x - y + 1 = 0$

Substituting x and y with (x+1) and (y+1) respectively, we have

$= (x+1)(y+1) - (x + 1) - (y + 1) + 1 = 0$

$= xy + x + y + 1 - y - 1 - x - 1 + 1 = 0$

$= xy = 0$

Hence, the transformed equation is $xy = 0$.

(iv) $x^2 - y^2 - 2x + 2y = 0$

Substituting x and y with (x+1) and (y+1) respectively, we have

$= (x+1)^2 - (y + 1)^2 - 2(x + 1) + 2(y + 1) = 0$

$= x^2 + 1 + 2x - y^2 - 1 - 2y - 2x - 2 + 2y + 2 = 0$

$= x^2 - y^2 = 0$

Hence, the transformed equation is $x^2 - y^2 = 0$.

7. Question

Find the point to which the origin should be shifted after a translation of axes so that the following equations will have no first degree terms:

(i) $y^2 + x^2 - 4x - 8y + 3 = 0$

(ii) $x^2 + y^2 - 5x + 2y - 5 = 0$

(iii) $x^2 - 12x + 4 = 0$

Answer

To find: The point to which origin has to be shifted such that there are no first-degree terms, i.e. there are no terms with $(\text{variable})^1$

We know that, when we transform origin from (0, 0) to an arbitrary point (p, q), the new coordinates for the point (x, y) becomes (x + p, y + q), and hence an equation with two variables x and y must be transformed accordingly replacing x with x + p, and y with y + q in original equation.

In following subproblems, we assume that origin has been shifted from (0, 0) to (p, q); therefore any arbitrary point (x, y) will also be converted as (x + p, y + q).

(i) $y^2 + x^2 - 4x - 8y + 3 = 0$

Substituting x and y with (x+p) and (y+q) respectively, we have

$= (x+p)^2 + (y + q)^2 - 4(x + p) - 8(y + q) + 3 = 0$

$= x^2 + p^2 + 2px - y^2 - q^2 - 2qy - 4x - 4p - 8y - 8q + 3 = 0$

$= x^2 + y^2 + x(2p - 4) + y(2q - 8) + p^2 + q^2 - 4p - 8q + 3 = 0$

For first degree term to be zero we have,

$2p - 4 = 0$ and $2q - 8 = 0$

Giving us, p = 2 and q = 4.

Hence, the shifted point is (p, q) = (2, 4).

(ii) $x^2 + y^2 - 5x + 2y - 5 = 0$

Substituting x and y with (x+p) and (y+q) respectively, we have

$= (x+p)^2 + (y + q)^2 - 5(x + p) + 2(y + q) - 5 = 0$

$= x^2 + p^2 + 2px - y^2 - q^2 - 2qy - 5x - 5p + 2y + 2q - 5 = 0$

$= x^2 + y^2 + x(2p - 5) + y(2q + 2) + p^2 + q^2 - 5p + 2q - 5 = 0$

For first degree term to be zero we have,

$2p - 5 = 0$ and $2q + 2 = 0$

Giving us, p = 5/2 and q = 1.

Hence, the shifted point is (p, q) = (5/2, 1).

(iii) $x^2 - 12x + 4 = 0$

Substituting x and y with (x+p) and (y+q) respectively, we have

$= (x+p)^2 - 12(x + p) + 4 = 0$

$= x^2 + p^2 + 2px - 12x - 12p + 4 = 0$

$= x^2 + x(2p - 12) + p^2 - 12p + 4 = 0$

For first degree term to be zero we have,

$2p - 12 = 0.$

Giving us, p = 2.

Hence, the shifted point is (p, q) = (2, q), where q can be any real number.

8. Question

Verify that the area of the triangle with vertices (4, 6), (7, 10) and (1, -2) remains invariant under the translation of axes when the origin is shifted to the point (-2, 1).

Answer

Given points (4, 6), (7, 10), and (1, -2).

To show: The area of a triangle is invariant to shifting of origin.

The area of triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Hence, the area of given triangle $= \frac{1}{2}[4(10-(-2)) + 7(-2-6) + 1(6-10)]$

$= \frac{1}{2}[48 - 56 - 4]$

$= \frac{1}{2}[-12]$

$= -6$

we takes modulus value of -6 i.e. 6 since the area cannot be negative.

Origin shifted to point (-2, 1), the new coordinates of the triangle are (6, 5), (9, 9), and (3, -3) obtained from subtracting a point (-2, 1).

Hence, the new area of triangle $= \frac{1}{2}[6(9-(-3)) + 9(-3-5) + 3(5-9)]$

$= \frac{1}{2}[72 - 72 + (-12)]$

$= \frac{1}{2}[-12]$

$= -6$

we takes modulus value of -6 i.e. 6 sq. sq. units since the area cannot be negative.

Since the area of the triangle before and after the translation after shifting of origin remains same, i.e. 6 sq. units, therefore we can say that the area of a triangle is invariant to shifting of origin.