CHAPTER

7

Determinants

- Introduction
- Properties of Determinants
- Some Important Determinants
- Use of Determinant in Coordiante Geometry
- Product of Two Determinants
- Differentiation of a Determinant
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INTRODUCTION

A system of equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

can be represented as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Now whether this system of equations has a unique solution or not, is determined by the number $a_1b_2 - a_2b_1$. The number $a_1b_2 - a_2b_1$ which determines the uniqueness of solution is associated with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called determinant of A or det(A). Determinants have wide applications in engineering, science, economics, social science, etc.

Definition

Let a, b, c, d be any four numbers, real or complex. Then the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes $ad - bc$ and is called a determinant of second order; a, b, c, d are called elements of the determinant and $ad - bc$ is called its value. As shown above, the elements of a determinant are arranged in the form of a square in its designation. The diagonal on which the elements a and d are situated is called the principal diagonal and the diagonal on which the elements c and b are situated is called the secondary diagonal. The elements which lie in the same horizontal line constitute one row and the elements which lie in the same vertical line constitute one column.

Let $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ be any nine numbers. Then the symbol $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is another way of denoting

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

i.e., $a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$.

Here we see that +, – and + signs occur before a_1, a_2 and a_3, respectively.

Minors and Cofactors

Let us consider a determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1)$$

In the above determinant, if we leave the row and the column passing through the element a_{ij}, then the second order determinant thus obtained is called the minor of a_{ij}, and is denoted by M_{ij}. Thus we can get 9 minors corresponding to the 9 elements.

For example, in determinant (1) the minor of the element a_{21} is

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

The minor of the element a_{32} is $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

In terms of the notation of minors if we expand the determinant along the first row, then

$$\Delta = (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13}$$
$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly expanding Δ along the second column, we have

$$\Delta = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij}.

If we denote the cofactor of the element a_{ij}, by C_{ij}, then cofactor of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor of the element a_{21} is

$$C_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

In terms of the notation of the cofactors,

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

Also, $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$, etc. Therefore, in a determinant the sum of the products of the elements of any row or column with the corresponding cofactors is equal to value of the determinant. Also the sum of the products of the elements of any row or column with the cofactors of the corresponding elements of any other row or column is zero.

Value of n-order determinant,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

(when expanded along first row)

Note: *For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.*

Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.

Rule: The three diagonals sloping down to the right give the three positive terms and the three diagonals sloping down to the left the three negative terms.

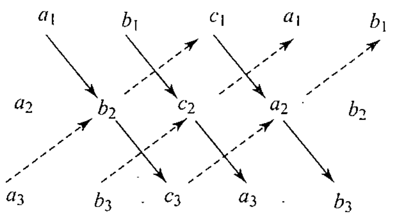

Fig. 7.1

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Example 7.1 **A determinant of second order is made with the elements 0 and 1. Find the number of determinants with non-negative values.**

Sol. There are only three determinants of second order with negative values, viz.

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

The Number of possible determinants with elements 0 and 1 is $2^4 = 16$. Therefore number of determinants with non-negative values is 13.

Example 7.2 **Find the value of** $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ \mathbf{4} & \mathbf{1} & \mathbf{0} \end{vmatrix}$

Sol. Here in the third column, two entries are zero. So expanding along third column (C_3), we get

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 4(-1 - 12) - 0 + 0$$

$$= -52$$

Example 7.3 **Find the largest value of a third-order determinant whose elements are 0 or 1.**

Sol. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3. Then,

$$\Delta = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$$= (a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$$

Since each element of Δ is either 1 or 0, therefore the value of the determinant cannot exceed 3.

Clearly, the value of Δ is maximum when the value of each term in first bracket is 1 and the value of each term in the second bracket is zero. But $a_1b_2c_3 = a_3b_1c_1 = 1$ implies that every element of the determinant Δ is 1 and in that case $\Delta = 0$. Thus, we may have

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

Example 7.4 **If $a, b, c \in R$, then find the number of real roots of the equation** $\Delta = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0$.

Sol. From the symmetry of the determinant, it is simple to expand by Sarrus rule.

$$\Delta = x^3 + abc - abc + (b^2x + a^2x + c^2x) = 0$$

$$\Rightarrow x^3 + x(a^2 + b^2 + c^2) = 0$$

$$\Rightarrow x = 0 \text{ or } x^2 = -(a^2 + b^2 + c^2)$$

$$\Rightarrow x = 0 \text{ or } x = \pm i\sqrt{a^2 + b^2 + c^2}$$

Example 7.5 **If $x + y + z = 0$, prove that**

$$\begin{vmatrix} ax & by & cz \\ cy & az & bx \\ bz & cx & ay \end{vmatrix} = xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Sol. Since determinant is symmetrical it is simple to expand by Sarrus rule.

$$\begin{vmatrix} ax & by & cz \\ cy & az & bx \\ bz & cx & ay \end{vmatrix} = xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3)$$

$$= xyz(a^3 + b^3 + c^3 - 3abc) - abc(x^3 + y^3 + z^3 - 3xyz)$$

$$= xyz(a^3 + b^3 + c^3 - 3abc) - abc(x + y + z) \times (x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= xyz(a^3 + b^3 + c^3 - 3abc)$$

$$= xyz\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Concept Application Exercise 7.1

1. If A, B and C are the angles of non-right angled triangle ABC, then find the value of
$$\begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$$

2. If $e^{i\theta} = \cos\theta + i\sin\theta$, find the value of
$$\begin{vmatrix} 1 & e^{i\pi/3} & e^{i\pi/4} \\ e^{-i\pi/3} & 1 & e^{i2\pi/3} \\ e^{-i\pi/4} & e^{-i2\pi/3} & 1 \end{vmatrix}$$

3. Find the number of real roots of the equation
$$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0,\ a \neq b \neq c \text{ and } b(a+c) > ac$$

4. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$ and $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$, $\alpha \neq \beta \neq \gamma$, then find the equation whose roots are $\alpha + \beta - \gamma$, $\beta + \gamma - \alpha$ and $\gamma + \alpha - \beta$.

Some Operations

First, second and third rows of a determinant are denoted by R_1, R_2 and R_3, respectively and the first, second and third columns by C_1, C_2 and C_3, respectively.

(i) The interchange of its i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$.

(ii) The interchange of i^{th} column and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

(iii) The addition of m-times the elements of j^{th} row of the corresponding elements of i^{th} row is denoted by $R_i \to R_i + mR_j$.

(iv) The addition of m-times the elements of j^{th} column to the corresponding elements of i^{th} column is denoted by $C_i \to C_i + mC_j$.

(v) The addition of m-times the elements of j^{th} row to n-times the elements of i^{th} row is denoted by $R_i \to nR_i + mR_j$.

PROPERTIES OF DETERMINANTS

Property I. The value of the determinant is not changed when rows are changed into corresponding columns.

Naturally when rows are changed into corresponding columns, then columns will change into corresponding rows.

Proof: Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row,

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1)$$

If Δ' be the value of the determinant when rows of determinant Δ are changed into corresponding columns, then

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (2)$$

From Eqs. (1) and (2), $\Delta' = \Delta$

Property II. If any two rows or columns of a determinant are interchanged, the sign of the value of the determinant is changed.

Proof: Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row,

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1)$$

Now,

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \quad [R_1 \leftrightarrow R_3]$$

$$= a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2)$$

$$= a_3b_2c_1 - a_3b_1c_2 - b_3a_2c_1 + a_1b_3c_2 + c_3a_2b_1 - a_1b_2c_3$$

$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2) \quad (2)$$

From Eqs. (1) and (2), $\Delta' = -\Delta$

Property III. The value of a determinant is zero if any two rows of columns are identical.

Proof: Let,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Then,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = -\Delta \qquad [\text{by } R_1 \leftrightarrow R_3]$$

Thus,

$$\Delta = -\Delta$$

$$\Rightarrow 2\Delta = 0$$

$$\Rightarrow \Delta = 0$$

Property IV. A common factor of all elements of any row (or of any column) may be taken outside the sign of the determinant. In other words, if all the elements of the same row (or the same column) are multiplied by a certain number, then the determinant becomes multiplied by that number.

Proof: Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding the determinant along the first row, we get

$$\Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad (1)$$

and

$$\Delta' = \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= ma_1(b_2c_3 - b_3c_2) - mb_1(a_2c_3 - a_3c_2) + mc_1(a_2b_3 - a_3b_2)$$

$$= m\Delta \qquad [\text{from (1)}]$$

Thus,

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Example: $\begin{vmatrix} 32 & 24 & 16 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix} = 8 \times \begin{vmatrix} 4 & 3 & 2 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix}$

[taking 8 common from first row]

$$= 8 \times 4 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 3 \end{vmatrix}$$

[taking 4 common from the first column]

Property V. If every element of some column or (row) is the sum of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant.

Proof: We have to prove that

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Let,

$$\Delta = \begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix}$$

Then,

$$\Delta = (a_1+\alpha_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (a_2+\alpha_2)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (a_3+\alpha_3)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + \alpha_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$- \alpha_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + \alpha_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Note:

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 & e_1 \\ a_2+b_2 & c_2+d_2 & e_2 \\ a_3+b_3 & c_3+d_3 & e_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} a_1 & d_1 & e_1 \\ a_2 & d_2 & e_2 \\ a_3 & d_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \\ b_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & e_1 \\ b_2 & d_2 & e_2 \\ b_3 & d_3 & e_3 \end{vmatrix}$$

Property VI. The value of a determinant does not change when any row or column is multiplied by a number or an expression and is then added to or subtracted from any other row or column.

Here it should be noted that if the row or column which is changed is multiplied by a number, then the determinant will have to be divided by that number.

Proof: To prove $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$

Let, $\Delta = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$

Then,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \left[\because \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0\right]$$

Example: Let, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix} = -7$

$$\Delta' = \begin{vmatrix} 5 & 2 & 13 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix} \quad [R_1 \to R_1 + 2R_3]$$

$$= 5(15-0) - 2(10-8) + 13(0-6)$$
$$= 75 - 4 - 78 = -7$$

$$\Delta'' = \frac{1}{3}\begin{vmatrix} 7 & 6 & 19 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{vmatrix}$$

[Here Δ'' has been obtained from Δ by applying $R_1 \to 3R_1 + 2R_3$]

$$= \frac{1}{3}[7(15-0) - 6(10-8) + 19(0-6)] = \frac{1}{3}(-21) = -7$$

In obtaining Δ'' from Δ, R_1 has been changed and it has been multiplied by 3, therefore, the determinant has been divided by 3.

Note:

- *If more than one operation like $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.*
- *Many times we use this operation to get as many zeros as we can.*

Property VII. If $\Delta_r = \begin{vmatrix} f_1(r) & f_2(r) & f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$ where $f_1(r), f_2(r), f_3(r)$ are functions of r and a, b, c, d, e, f are constants. Then,

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) & \sum_{r=1}^{n} f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$$

Also for $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a & b & c \\ d & e & f \end{vmatrix}$ where $f_1(x), f_2(x), f_3(x)$ are functions of x and a, b, c, d, e, f are constants, we have

$$\int_p^q \Delta(x)dx = \begin{vmatrix} \int_p^q f_1(x)dx & \int_p^q f_2(x)dx & \int_p^q f_3(x)dx \\ a & b & c \\ d & e & f \end{vmatrix}$$

Note: *All the above properties are applicable for n-order determinants also.*

SOME IMPORTANT DETERMINANTS

1. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Proof:

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x & (y-x) & (z-y) \\ x^3 & (y-x)(y+x) & (z-y)(z+y) \end{vmatrix}$$

$$= (y-x)(z-y) \times \begin{vmatrix} 1 & 1 \\ y+x & z+y \end{vmatrix}$$

(Expanding along R_1)

$$= (x-y)(y-z)(z-x)$$

2. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

3. $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

4. **Circulant:** Let a, b, c be positive and not all equal. Then the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

Proof: $\Delta = a[bc - a^2] - b[b^2 - ac] + c[ab - c^2]$

$$= -[a^3 + b^3 + c^3 - 3abc]$$

$$= -(a+b+c)[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] < 0$$

As $a + b + c > 0$, a, b, c are all positive and not all equal.

Example 7.6 **Without expanding at any stage, prove that the value of each of the following determinants is zero.**

a. $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$ **b.** $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$ **c.** $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

where w is cube root of unity

Sol.

a. Let, $\Delta = \begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & q-p & r-p \\ p-q & 0 & r-q \\ p-r & q-r & 0 \end{vmatrix}$$ [Taking transpose]

$$= (-1)^3 \begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$$

[Taking (–1) common from each row]

$\therefore \Delta = -\Delta$ or $2\Delta = 0$ or $\Delta = 0$

b. $\Delta = \begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

Applying $C_1 \to C_1 + (-8)C_3$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix} = 0$$ [$\because C_1$ and C_2 are identical]

c. Let, $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix}$$

$$= \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix}$$ [$\because 1 + w + w^2 = 0$]

$= 0$ [$\because C_1$ consists of all zeros]

Example 7.7 **If $\Delta = \begin{vmatrix} abc & b^2c & c^2b \\ abc & c^2a & ca^2 \\ abc & a^2b & b^2a \end{vmatrix} = 0$, ($a, b, c \in R$ and are all different and non-zero) then prove that $a + b + c = 0$.**

Sol.

$$\Delta = 0 \Rightarrow bc.ca.ab\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow \quad -a^2b^2c^2(a^3 + b^3 + c^3 - 3abc) = 0$$

(expanding by Sarrus Rule)

$$\Rightarrow \quad a^2b^2c^2(a+b+c)\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a^2)] = 0$$

$$\Rightarrow \quad a + b + c = 0$$

Example 7.8 **Prove that $a \neq 0$, $\begin{vmatrix} x+1 & x & x \\ x & x+a & x \\ x & x & x+a^2 \end{vmatrix} = 0$ represents a straight line parallel to y-axis.**

Sol.

$$\Delta = 0 \Rightarrow \begin{vmatrix} x+1 & -1 & -1 \\ x & a & 0 \\ x & 0 & a^2 \end{vmatrix} = 0 \quad \begin{bmatrix} C_3 \to C_3 - C_1 \\ C_2 \to C_2 - C_1 \end{bmatrix}$$

$$\Rightarrow (a^3 + a^2 + a)x = -a^3$$

$$\Rightarrow x = \frac{-a^3}{a^3 + a^2 + a}$$ which is a straight line parallel to y-axis

Example 7.9 **Prove that the value of the determinant $\begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix}$ is real.**

Sol. Let $$z = \begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix} \quad (1)$$

To prove that this number (z) is real we have to prove that $\bar{z} = z$

Now we know that conjugate of complex number is distributive over all algebraic operations.

Hence to take conjugate of z in (1) we need not to expand determinant.

To get the conjugate of z we can take conjugate of each element of determinant.

$$\Rightarrow \quad \bar{z} = \begin{vmatrix} -7 & 5-3i & \frac{2}{3}+4i \\ 5+3i & 8 & 4-5i \\ \frac{2}{3}-4i & 4+5i & 9 \end{vmatrix} \quad (2)$$

Now interchanging rows into columns (taking transpose) in (2)

we have $$\bar{z} = \begin{vmatrix} -7 & 5+3i & \frac{2}{3}-4i \\ 5-3i & 8 & 4+5i \\ \frac{2}{3}+4i & 4-5i & 9 \end{vmatrix} \quad (3)$$

or $\bar{z} = z$ (4) (from (1) & (3))

$\Rightarrow$ z is purely real

Example 7.10 **If $a_r = (\cos 2r\pi + i \sin 2r\pi)^{\frac{1}{9}}$, then prove that $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = 0$.**

Sol. $a_r = (\cos 2r\pi + i\sin 2r\pi)^{\frac{1}{9}} = e^{i\frac{2r\pi}{9}}$

$$\Rightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

$$= \begin{vmatrix} e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{8\pi}{9}} & e^{i\frac{10\pi}{9}} & e^{i\frac{12\pi}{9}} \\ e^{i\frac{14\pi}{9}} & e^{i\frac{16\pi}{9}} & e^{i\frac{18\pi}{9}} \end{vmatrix}$$

$$= e^{i\frac{6\pi}{9}} \begin{vmatrix} e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{14\pi}{9}} & e^{i\frac{16\pi}{9}} & e^{i\frac{18\pi}{9}} \end{vmatrix}$$ [taking $e^{i\frac{6\pi}{9}}$ common from R_2]

$= 0$ [R_1 and R_2 are identical]

Example 7.11 **Without expanding the determinants, prove that**

$$\begin{vmatrix} \mathbf{103} & \mathbf{115} & \mathbf{114} \\ \mathbf{111} & \mathbf{108} & \mathbf{106} \\ \mathbf{104} & \mathbf{113} & \mathbf{116} \end{vmatrix} + \begin{vmatrix} \mathbf{113} & \mathbf{116} & \mathbf{104} \\ \mathbf{108} & \mathbf{106} & \mathbf{111} \\ \mathbf{115} & \mathbf{114} & \mathbf{103} \end{vmatrix} = \mathbf{0}$$

Sol. $$D = \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_1} + \underbrace{\begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}}_{D_2}$$

In D_2, interchanging C_1 and C_3,

$$D = \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_1} - \underbrace{\begin{vmatrix} 104 & 116 & 113 \\ 111 & 106 & 108 \\ 103 & 114 & 115 \end{vmatrix}}_{D_2}$$

In D_2, interchanging C_2 and C_3,

$$D = \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_1} + \underbrace{\begin{vmatrix} 104 & 113 & 116 \\ 111 & 108 & 106 \\ 103 & 115 & 114 \end{vmatrix}}_{D_2}$$

In D_2, interchanging R_1 and R_3,

$$D = \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_1} - \underbrace{\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix}}_{D_2} = 0$$

Example 7.12 **Find the value of the determinant**

$$\begin{vmatrix} \sqrt{(13)}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{(15)}+\sqrt{(26)} & 5 & \sqrt{(10)} \\ 3+\sqrt{(65)} & \sqrt{(15)} & 5 \end{vmatrix}$$

Sol. $\Delta = (\sqrt{5})^2 \begin{vmatrix} \sqrt{(13)}+\sqrt{3} & 2 & 1 \\ \sqrt{(15)}+\sqrt{(26)} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix} \ (C_2 \to \frac{1}{\sqrt{5}}C_2, C_3 \to \frac{1}{\sqrt{5}}C_3)$

Now applying $C_1 \to C_1 - \sqrt{3}C_2 - \sqrt{(13)}C_3$, we get

$$\Delta = 5 \begin{vmatrix} -\sqrt{3} & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$= -5\sqrt{3}\,(5-\sqrt{6})$ (expanding along C_1)

Example 7.13 **Using properties of determinants, evaluate**

$$\begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$$

Sol. Let $D = \begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$

Let first reduce the value of elements by performing some operation.

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 5R_1$

$$D = \begin{vmatrix} 18 & 40 & 89 \\ 4 & 9 & 20 \\ -1 & -2 & -5 \end{vmatrix}$$

Applying $C_3 \to C_3 - 2C_2$

$$D = \begin{vmatrix} 18 & 40 & 9 \\ 4 & 9 & 2 \\ -1 & -2 & -1 \end{vmatrix}$$

Now applying $C_1 \to C_1 - 2C_3$ to get zeros in C_1

$$D = \begin{vmatrix} 0 & 40 & 9 \\ 0 & 9 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 40 & 9 \\ 9 & 2 \end{vmatrix} = 80 - 81 = -1$$

Example 7.14 **Solve for x:** $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$

Sol. $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$ then we get

or $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0$

or $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$

expanding along 1st row

or $(x-2).6 - (2x-3).4 + (3x-4).1 = 0$

or $x = 4$

Example 7.15 **Prove that**

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix} = 0$$

Sol. $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

$= \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta + \sin\delta\cos\alpha \\ \sin\beta & \cos\beta & \sin\beta\cos\delta + \sin\delta\cos\beta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta + \sin\delta\cos\gamma \end{vmatrix}$

$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \ [R_3 \to R_3 - \cos\delta R_1 - \sin\delta R_2]$

$= 0$

Example 7.16 **Find the value of the determinant**

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$$

Sol. We have,

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$]

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 9 & 19 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{vmatrix}$$

[Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$]
$= (10 - 9) = 1$

Example 7.17 **Prove that** $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ $= (a + b + c)^3$.

Sol. Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ and taking $(a + b + c)$ common from each of C_1 and C_2, we get

$$D = (a+b+c)^2 \times \begin{vmatrix} -1 & 0 & 2a \\ 1 & -1 & 2a \\ 0 & 1 & c-a-b \end{vmatrix}$$

Now, $R_3 \to R_3 + R_1 + R_2$ gives

$$D = (a+b+c)^2 \times \begin{vmatrix} -1 & 0 & 2a \\ 1 & -1 & 2b \\ 0 & 0 & (a+b+c) \end{vmatrix}$$

Expanding along R_3, we get

$$D = (a+b+c)^2(a+b+c) = (a+b+c)^3$$

Example 7.18 **Prove that**

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix} = -(a+b+c) \times (a-b)(b-c)(c-a).$$

Sol. $\Delta = \begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$

$$= (a+b+c) \times \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

[Applying $C_2 \to C_2 + C_1$ and taking $(a + b + c)$ common from C_2]

$$= -(a+b+c) \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= -(a+b+c)(a-b)(b-c)(c-a)$$

Example 7.19 **Prove that** $\begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix}$ $= (x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$.

Sol. $D = \begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix}$

$$= \begin{vmatrix} x^2 & -(x^2+y^2+z^2) & yz \\ y^2 & -(x^2+y^2+z^2) & zx \\ z^2 & -(x^2+y^2+z^2) & xy \end{vmatrix}$$

[Operating $C_2 \to C_2 - 2C_1 - 2C_3$]

$$= -(x^2+y^2+z^2)\begin{vmatrix} x^2 & 1 & yz \\ y^2 & 1 & zx \\ z^2 & 1 & xy \end{vmatrix}$$

$$= -\frac{(x^2+y^2+z^2)}{xyz}\begin{vmatrix} x^3 & x & xyz \\ y^3 & y & xyz \\ z^3 & x & xyz \end{vmatrix}$$

(Multiplying R_1, R_2, R_3 by x, y, z, respectively)

$$= -(x^2+y^2+z^2)\begin{vmatrix} x^3 & x & 1 \\ y^3 & y & 1 \\ z^3 & x & 1 \end{vmatrix}$$

$$= (x^2+y^2+z^2)\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & x & z^3 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$$

Example 7.20 **If a, b, c are all different and**

$\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, **show that** $abc(ab + bc + ca) = a + b + c$.

Sol. Expressing the given determinant as sum of two determinants, we get

$$\begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

$$\text{or } abc\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b-a & b^3-a^3 & 0 \\ c-a & c^3-a^3 & 0 \end{vmatrix}$$

[We take a, b, c common from the first determinant and apply $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ in both determinants.]

As a, b, c are all distinct, canceling out $b - a$ and $c - a$, we get

$$abc\begin{vmatrix} b+a & b^2+a^2+ab \\ c+a & c^2+a^2+ac \end{vmatrix} = \begin{vmatrix} 1 & b^2+a^2+ab \\ 1 & c^2+a^2+ac \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and then cancelling $c - b$ on both sides, we get

$$abc\begin{vmatrix} b+a & b^2+a^2+ab \\ 1 & a+b+c \end{vmatrix} = \begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & a+b+c \end{vmatrix}$$

$\therefore \quad abc(ab + b^2 + bc + a^2 + ab + ac - b^2 - a^2 - ab) = a + b + c$

or $\quad abc(ab + bc + ca) = a + b + c$

Hence the result.

Example 7.21 **If $x_i = a_i b_i c_i$, $i = 1, 2, 3$ are three-digit positive integers such that each x_i is a multiple of 19, then for some integer n, prove that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is divisible by 19.**

Sol. $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ (100a_1 + 10b_1 + c_1) & (100a_2 + 10b_2 + c_2) & (100a_3 + 10b_3 + c_3) \end{vmatrix}$

$[R_3 \to R_3 + 100R_1 + 10R_2]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 19m_1 & 19m_2 & 19m_3 \end{vmatrix} \quad \text{[where each } m \in N]$$

$$= 19\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 19n$$

where $n = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ m_1 & m_2 & m_3 \end{vmatrix}$ is certainly an integer.

Example 7.22 **Prove that** $\begin{vmatrix} (b+c)^2 & ba & ac \\ ba & (c+a)^2 & cb \\ ca & cb & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Sol. Multiplying R_1, R_2, R_3 by a, b, c, respectively, and dividing by abc, we get

$$\Delta = \frac{1}{abc} \times \begin{bmatrix} a(b+c)^2 & ba^2 & ca^2 \\ ab^2 & b(c+a)^2 & cb^2 \\ ac^2 & bc^2 & c(a+b)^2 \end{bmatrix}$$

Taking a, b, c common from C_1, C_2 and C_3, respectively, we get

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Now, applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ gives

$$\Delta = \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a-b-c)(a+b+c) \\ b^2 & (c+a+b)(c+a-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2 \times \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$ and then taking 2 common from R_1, we get

$$\Delta = 2(a+b+c)^2 \times \begin{vmatrix} bc & -c & -b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Now, applying $C_2 \to bC_2 + C_1$, $C_3 \to C_3 + C_1$ gives

$$\Delta = \frac{2(a+b+c)^2}{bc} \times \begin{vmatrix} bc & 0 & 0 \\ b^2 & b(c+a) & b^2 \\ c^2 & c^2 & c(a+b) \end{vmatrix}$$

$$= \frac{2(a+b+c)^2}{bc} bc[(bc+ba)(ca+cb) - b^2c^2]$$

$= 2(a + b + c)^2[bc(ac + bc + ab + a^2 - bc)]$

$= 2abc(a + b + c)^3$

Example 7.23 **Show that** $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$

$= (a + b + c)(a^2 + b^2 + c^2)$.

Sol. Let, $\Delta = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$

$$\Rightarrow \Delta = \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix} \quad \text{[Multiplying first column by } a]$$

$$= \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$$

[Applying $C_1 \to C_1 + bC_2 + cC_3$]

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

[Taking $a^2 + b^2 + c^2$ common from C_1]

$$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= \frac{1}{a}(a^2+b^2+c^2) \times 1 \times \begin{vmatrix} c & -a-b \\ a+c & -b \end{vmatrix}$$

[Expanding along C_1]

$= \frac{1}{a}(a^2 + b^2 + c^2)(-bc + a^2 + ac + ba + bc)$

$= (a^2 + b^2 + c^2)(a + b + c)$

Example 7.24 **Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation**

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$

represents a straight line.

Sol. Given,

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (a^2+b^2+c^2)x & bx+ay & cx+a \\ (a^2+b^2+c^2)y & -ax+by-c & cy+b \\ (a^2+b^2+c^2) & cy+b & -ax-by+c \end{vmatrix} = 0$$

[Applying $C_1 \to aC_1 + bC_2 + cC_3$]

$$\Rightarrow \begin{vmatrix} x & bx+ay & cx+a \\ y & -ax+by-c & cy+b \\ 1 & cy+b & -ax-by+c \end{vmatrix} = 0 \quad [\because a^2+b^2+c^2=1]$$

$$\Rightarrow \begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

[Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$]

$$\Rightarrow \begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ x^2+y^2+1 & 0 & 0 \end{vmatrix} = 0$$

[Applying $R_3 \to R_3 + xR_1 + yR_2$, we get]

$\Rightarrow (x^2 + y^2 + 1)(aby + a^2x + ac) = 0 \Rightarrow ax + by + c = 0$

Example 7.25 **Prove that** $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$ **$= 1 + a^2 + b^2 + c^2$.**

Sol. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & c^2 \\ a^2 & b^2 & c(c^2+1) \end{vmatrix}$$

[Multiplying C_1, C_2, C_3 by a, b, c, respectively]

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

[Taking common a, b, c from R_1 R_2, R_3, respectively]

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (1 + a^2 + b^2 + c^2)$

Example 7.26 **If $a^2 + b^2 + c^2 = 1$, then prove that**

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

is independent of a, b, c.

Sol. Multiplying C_1 by a, C_2 by b and C_3 by c, we have

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^3+a(b^2+c^2)\cos\phi & ab^2(1-\cos\phi) & ac^2(1-\cos\phi) \\ ba^2(1-\cos\phi) & b^3+b(c^2+a^2)\cos\phi & bc^2(1-\cos\phi) \\ ca^2(1-\cos\phi) & cb^2(1-\cos\phi) & c^3+c(a^2+b^2)\cos\phi \end{vmatrix}$$

Now take a, b and c common from R_1, R_2 and R_3, respectively, to give

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

[$\because a^2 + b^2 + c^2 = 1$]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ 0 & (a^2+b^2+c^2)\cos\phi & 0 \\ 0 & 0 & (a^2+b^2+c^2)\cos\phi \end{vmatrix}$$

Expanding along C_1, we get $\begin{vmatrix} \cos\phi & 0 \\ 0 & \cos\phi \end{vmatrix} = \cos^2\phi.$

Example 7.27 **Let** $\Delta_r = \begin{vmatrix} r-1 & n & 6 \\ (r-1)^2 & 2n^2 & 4n-2 \\ (r-1)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$. **Show that** $\sum_{r=1}^{n} \Delta_r$ **is constant.**

Sol. Since C_1 has variable terms and C_2 and C_3 are constant, summation runs on C_1.

$$\therefore \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_1^n (r-1) & n & 6 \\ \sum_1^n (r-1)^2 & 2n^2 & 4n-2 \\ \sum_1^n (r-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2}(n-1)n & n & 6 \\ \frac{1}{6}(n-1)n(2n-1) & 2n^2 & 4n-2 \\ \frac{1}{4}(n-1)^2 n^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

Taking $\frac{1}{12}n(n-1)$ common from C_1 and n common from C_2, we get

$$\Sigma\Delta_r = \frac{1}{12}n^2(n-1) \times \begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix}$$

= 0, which is constant [$\because$ C_1 and C_3 are identical]

Example 7.28 **Prove that** $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$

$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$. **Hence find the value of the determinant if a, b, c, d are the roots of the equation $px^4 + qx^3 + rx^2 + sx + t = 0$.**

Sol. Applying $R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2, R_3 \to \frac{1}{c}R_3, R_4 \to \frac{1}{d}R_4$, we get

$$\Delta = abcd \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3 + R_4$ and taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$ common, we get

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$,

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding again along R_1, we get

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

2nd part: $\Delta = abcd + (bcd + acd + abd + abc)$

$$= \frac{t}{p} - \frac{s}{p} = \frac{t-s}{p}$$

USE OF DETERMINANT IN COORDINATE GEOMETRY

Area of Triangle

The area of a triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

Condition of Concurrency of Three Lines

Three lines are said to be concurrent if they pass through a common point i.e., they meet at a point.

Let $a_1 x + b_1 y + c_1 = 0$ (1)

$a_2 x + b_2 y + c_2 = 0$ (2)

$a_3 x + b_3 y + c_3 = 0$ (3)

be three concurrent lines then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Condition for General 2nd Degree Equation in x and y Represent Pair of Straight Lines

The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example 7.29 **Find the area of a triangle whose vertices are $A(3, 2)$, $B(11, 8)$ and $C(8, 12)$.**

Sol. The area of triangle is $A = \frac{1}{2}\left\|\begin{matrix} 3 & 2 & 1 \\ 11 & 8 & 1 \\ 8 & 12 & 1 \end{matrix}\right\|$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow A = \frac{1}{2}\left\|\begin{matrix} -8 & -6 & 0 \\ 3 & -4 & 0 \\ 8 & 12 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\begin{vmatrix} -8 & -6 \\ 3 & -4 \end{vmatrix}$$

$= 25$ sq. units

Example 7.30 **If x_1, x_2, x_3, as well as y_1, y_2, y_3 are in G.P. with same common ratio, then prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.**

Sol. Since points are collinear, we have to prove that area is zero. $x_2 = x_1r$, $x_3 = x_1r^2$ and so is $y_2 = y_1r$, $y_3 = y_1r^2$

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$$

$$= r.r^2\begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}$$

$= 0$

Hence the points are collinear.

Example 7.31 **If the lines $a_1x + b_1y + 1 = 0$, $a_2x + b_2y + 1 = 0$ and $a_3x + b_3y + 1 = 0$ are concurrent, show that the points (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear.**

Sol. The given lines are

$a_1x + b_1y + 1 = 0$ (1)

$a_2x + b_2y + 1 = 0$ (2)

and $a_3x + b_3y + 1 = 0$ (3)

If these lines are concurrent, we must have $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$,

which is the condition of collinearity of three points (a_1, b_1), (a_2, b_2) and (a_3, b_3).

Hence, if the given lies are concurrent, the given points are collinear.

Example 7.32 **Find the values of 'a' for which the lines**

$\mathbf{2x + y - 1 = 0}$

$\mathbf{ax + 3y - 3 = 0}$

$\mathbf{3x + 2y - 2 = 0}$

are concurrent.

Sol. Lines are concurrent if $\begin{vmatrix} 2 & 1 & -1 \\ a & 3 & -3 \\ 3 & 2 & -2 \end{vmatrix} = 0$

Since C_1 and C_3 are proportional, lines are concurrent for infinite values of a.

Example 7.33 **If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b, c being distinct and different from 1) are concurrent, then prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$.**

Sol. If the given lines are concurrent, then $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

(Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$\Rightarrow a(b-1)(c-1) - (c-1)(1-a) - (b-1)(1-a) = 0$$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

(Dividing by $(1-a)(1-b)(1-c)$)

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Example 7.34 **If lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent then prove that $p + q + r = 0$ (where p, q, r are distinct).**

Sol. For concurrency of three lines

$px + qy + r = 0$; $qx + ry + p = 0$; $rx + py + q = 0$

We must have, $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$$\Rightarrow 3pqr - p^3 - q^3 - r^3 = 0$$

$$\Rightarrow (p + q + r)(p^2 + q^2 + r^2 - pq - pr - rq) = 0$$

$$\Rightarrow p + q + r = 0$$

Example 7.35 **Find the value of λ if $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ represent a pair of straight lines.**

Sol. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 7/2 & 4 \\ 7/2 & 3 & 7 \\ 4 & 7 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 6\lambda + 2(7)(4)\left(\frac{7}{2}\right) - 2(7)^2 - 3(4)^2 - \lambda\left(\frac{7}{2}\right)^2 = 0$$

$$\Rightarrow 6\lambda + 196 - 98 - 48 - \frac{49\lambda}{4} = 0$$

$$\Rightarrow \frac{49\lambda}{4} - 6\lambda = 196 - 146 = 50$$

$$\Rightarrow \frac{25\lambda}{4} = 50 \therefore \lambda = \frac{200}{25} = 8$$

Concept Application Exercise 7.2

1. Prove that the value of each the following determinants is zero.

a. $\begin{vmatrix} a_1 & la_1 + mb_1 & b_1 \\ a_2 & la_2 + mb_2 & b_2 \\ a_3 & la_3 + mb_3 & b_3 \end{vmatrix}$

b. $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$

c. $\begin{vmatrix} \log x & \log y & \log z \\ \log 2x & \log 2y & \log 2z \\ \log 3x & \log 3y & \log 3z \end{vmatrix}$

d. $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^y + b^{-y})^2 & (b^y - b^{-y})^2 & 1 \\ (c^z + c^{-z})^2 & (c^z - c^{-z})^2 & 1 \end{vmatrix}$

e. $\begin{vmatrix} \sin^2\left(x + \frac{3\pi}{2}\right) & \sin^2\left(x + \frac{5\pi}{2}\right) & \sin^2\left(x + \frac{7\pi}{2}\right) \\ \sin\left(x + \frac{3\pi}{2}\right) & \sin\left(x + \frac{5\pi}{2}\right) & \sin\left(x + \frac{7\pi}{2}\right) \\ \sin\left(x - \frac{3\pi}{2}\right) & \sin\left(x - \frac{5\pi}{2}\right) & \sin\left(x - \frac{7\pi}{2}\right) \end{vmatrix}$

2. Prove that $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$.

3. If $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = k(a-b)(b-c)(c-a)$, then find the value of k.

4. Prove that $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc + a^2 & ac + b^2 & ab + c^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)$

5. Show that $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$.

6. Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

7. Show that $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

8. Find the value of $\begin{vmatrix} yz & zx & xy \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$, where x, y, z are respectively p^{th}, $(2q)^{th}$ and $(3r)^{th}$ terms of an H.P.

9. Show that if $x_1, x_2, x_3 \neq 0$

$$\begin{vmatrix} x_1 + a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2 + a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3 + a_3b_3 \end{vmatrix} = x_1x_2x_3\left(1 + \frac{a_1b_1}{x_1} + \frac{a_2b_2}{x_2} + \frac{a_3b_3}{x_3}\right).$$

10. If $\Delta_r = \begin{vmatrix} 2^r - 1 & 2.3^r - 1 & 4.5^r - 1 \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, then find the value of Δ.

11. Solve the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ where $a + b + c \neq 0$.

12. Solve for x, $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$.

13. If $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ are three three-digit numbers, each of which is divisible by k, then prove that

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$ is divisible by k.

PRODUCT OF TWO DETERMINANTS

Let, $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then row by row multiplication of Δ_1 and Δ_2 is given by

$$\Delta_1 \times \Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column; column by row or column by column as required in the problem.

To express a determinant as product of two determinants, one requires a lots of practice and this can be done only by inspection and trial.

Property: If $A_1, B_1, C_1, \ldots$ are respectively the cofactors of the elements $a_1, b_1, c_1, \ldots$ of the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \ \Delta \neq 0, \text{ then } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Proof: Given,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $A_1, B_1, C_1, \ldots$ are cofactors of $a_1, b_1, c_2, \ldots$ Hence,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_3 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}$$ (as $a_iA_i + b_iB_i + c_iC_i = \Delta$, $i = 1, 2, 3$ and $a_iA_j + b_iB_j + c_iC_j = 0$)

$= \Delta^3$

$$\Rightarrow \Delta \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^3 \quad \text{or} \quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Note: *For n-order determinant $\Delta_c = \Delta^{n-1}$, where Δ_c is the determinant formed by the cofactors of Δ and n is order of determinant. This property is very useful in studying adjoint of matrix.*

Example 7.36 **Prove that**

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_2\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0$$

Sol. The given determinant is the product of the determinants

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} = 0$$

Example 7.37 **If α, β, γ are real numbers, then without expanding at any stage, show that**

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$$

Sol.

$$\Delta = \begin{vmatrix} 1 & \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta & 1 & \cos\gamma\cos\beta + \sin\gamma\sin\beta \\ \cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \cos\beta\cos\gamma + \sin\beta\sin\gamma & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= 0 \times 0 = 0$

Example 7.38 **If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 3$, then find the value of**

$$\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$$

Sol. $D_c = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$ is the determinant formed by the cofactors of determinant

$$D = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}.$$

Hence, $D_c = D^2 = 3^2 = 9$.

Example 7.39 **Prove that** $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = 2(b-c)(c-a)(a-b)$$

$\times (y-z)(z-x)(x-y)$.

Sol. $\Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & 2a & 1 \\ b^2 & 2b & 1 \\ c^2 & 2c & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \text{ (in second determinant } C_1 \leftrightarrow C_2)$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix}$$

$$= 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$$

Multiplying row by row, we get

$$\Delta = \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

Example 7.40 **Express** $\Delta = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

as square of a determinant and hence evaluate it.

Sol. Keeping in mind the term $2bc-a^2$, we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = \Delta \quad \text{[row by row multiplication]}$$

Therefore,

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= [a(bc-a^2) + b(ac-b^2) + c(ab-c^2)]^2$$

$$= [3abc - a^3 - b^3 - c^3]^2$$

Example 7.41 **Prove without expansion that**

$$\begin{vmatrix} ah+bg & g & ab+ch \\ bf+ba & f & hb+bc \\ af+bc & c & bg+fc \end{vmatrix} = a \begin{vmatrix} ah+bg & a & h \\ bf+ba & h & b \\ af+bc & g & f \end{vmatrix}$$

Sol. Rewriting the given determinant, we have

$$\Delta = \frac{1}{c}\begin{vmatrix} ah+bg & gc & ab+ch \\ bf+ba & fc & hb+bc \\ af+bc & c^2 & bg+fc \end{vmatrix} = -\frac{a}{c}\begin{vmatrix} ah+bg & a & ch \\ bf+ba & h & bc \\ af+bc & g & fc \end{vmatrix}$$

By operating in second determinant $C_3 \rightarrow C_3 + bC_2$, we get

$$\Delta = \frac{1}{c}\Delta_1 - \frac{a}{c}\begin{vmatrix} ah+bg & a & ab+ch \\ bf+ba & h & hb+bc \\ af+bg & g & bg+fc \end{vmatrix}$$

$$= \frac{1}{c}\Delta_1 + \frac{1}{c}\begin{vmatrix} ah+bg & -a^2 & ab+ch \\ bf+ba & -ah & hb+bc \\ af+bc & -ag & bg+fc \end{vmatrix}$$

$$= \frac{1}{c}\begin{vmatrix} ah+bg & gc-a^2 & ab+ch \\ bf+ba & fc-ah & hb+bc \\ af+bc & c^2-ag & bg+fc \end{vmatrix}$$

$$= -\frac{1}{c}\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

Concept Application Exercise 7.3

1. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$$

2. Show that the determinant

$$\begin{vmatrix} a^2+b^2+c^2 & bc+ca+ab & bc+ca+ab \\ bc+ca+ab & a^2+b^2+c^2 & bc+ca+ab \\ bc+ca+ab & bc+ca+ab & a^2+b^2+c^2 \end{vmatrix}$$

is always non-negative. When is the determinant zero?

3. Prove that $\begin{vmatrix} (b+x)(c+x) & (c+x)(a+x) & (a+x)(b+x) \\ (b+y)(c+y) & (c+y)(a+y) & (a+y)(b+y) \\ (b+z)(c+z) & (c+z)(a+z) & (a+z)(b+z) \end{vmatrix}$

$= (b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$.

4 Factorize the following:

$$\begin{vmatrix} 3 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$$

DIFFERENTIATION OF A DETERMINANT

I. Let $\Delta(x)$ be a determinant of order two. If we write

$\Delta(x) = [C_1\ C_2]$, where C_1 and C_2 denote the first and second columns then

$$\Delta'(x) = [C_1'\ C_2] + [C_1\ C_2'],$$

where C_i' denotes the column which contains the derivative of all the functions in the i^{th} column C_i. In a similar fashion, if we write

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$$

Example: Let, $\Delta(x) = \begin{vmatrix} \sin x & \log x \\ e & 1/x \end{vmatrix}, x > 0$

Then, $\Delta'(x) = \begin{vmatrix} \cos x & \log x \\ 0 & 1/x \end{vmatrix} + \begin{vmatrix} \sin x & 1/x \\ e & -1/x^2 \end{vmatrix}$

II. Let $\Delta(x)$ be a determinant of order three. If we write $\Delta(x) = [C_1\ C_2\ C_3]$, then

$\Delta'(x) = [C_1'\ C_2\ C_3] + [C_1\ C_2'\ C_3] + [C_1\ C_2\ C_3']$ and similarly if we consider

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$$

III. If only one row (column) consists functions of x and other rows are constant, viz., let

$$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ (say)}$$

Then

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and in general

$$\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_2 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of $f(x)$.

Example 7.42 **If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, find $\dfrac{dy}{dx}$.**

Sol. $\dfrac{dy}{dx} = \begin{vmatrix} \cos x & -\sin x & \cos x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & \cos x & \sin x \\ -\sin x & -\cos x & -\sin x \\ x & 1 & 1 \end{vmatrix}$

$$+ \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0 - \begin{vmatrix} \sin x & \cos x & \sin x \\ \sin x & \cos x & \sin x \\ x & 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= 0 + (\cos^2 x + \sin^2 x)$$

Example 7.43 **If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\dfrac{n\pi}{2} & 4 \\ \sin x & \sin\dfrac{n\pi}{2} & 8 \end{vmatrix}$, then find the value of $\dfrac{d^n}{dx^n}[f(x)]_{x=0} (n \in Z)$.**

Sol. $\dfrac{d^n}{dx^n}[f(x)] = \begin{vmatrix} \dfrac{d^n}{dx^n}(x^n) & n! & 2 \\ \dfrac{d^n}{dx^n}(\cos x) & \cos\dfrac{n\pi}{2} & 4 \\ \dfrac{d^n}{dx^n}(\sin x) & \sin\dfrac{n\pi}{2} & 8 \end{vmatrix}$

$$= \begin{vmatrix} n! & n! & 2 \\ \cos\left(x + \dfrac{n\pi}{2}\right) & \cos\dfrac{n\pi}{2} & 4 \\ \sin\left(x + \dfrac{n\pi}{2}\right) & \sin\dfrac{n\pi}{2} & 8 \end{vmatrix} \quad (n \in Z)$$

$$\Rightarrow \frac{d^n}{dx^n}[f(x)]_{x=0} = 0$$

Example 7.44 **If f, g and h are differentiable functions of x and $\Delta(x) = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$, prove that**

$$\Delta'(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Sol. $(xf)' = xf' + f$ and $(x^2f)'' = [2xf + x^2f']' = 2f + 4xf' + x^2f''$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf'+f & xg'+g & xh'+h \\ 2f+4xf'+x^2f'' & 2g+4xg'+x^2g'' & 2h+4xh'+x^2h'' \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and then $R_3 \to R_3 - 4R_2 - 2R_1$

$$\Rightarrow \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}$$

Taking x common from R_2 and multiplying with R_3, we have

$$\Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$\Rightarrow \frac{d\Delta}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Example 7.45 **Let α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degrees 3, 4 and 5, respectively, then show that**

$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ **is divisible by $f(x)$ where prime (′) denotes the derivatives.**

Sol. Since α is a repeated root of the quadratic equation $f(x) = 0$, $f(x)$ can be written as

$f(x) = k(x - \alpha)^2$, where k is some non-zero constant.

Let, $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$g(x)$ is divisible by $f(x)$ if it is divisible by $(x - \alpha)^2$, i.e., $g(\alpha) = 0$ and $g'(\alpha) = 0$.

As $A(x)$, $B(x)$ and $C(x)$ are polynomials of degrees 3, 4 and 5, respectively, deg. $g(x) \geq 2$.

Now,

$$g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

(R_1 and R_2 are identical)

Also

$$g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$$\therefore \; g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

(R_1 and R_3 are identical)

This implies that $f(x)$ divides $g(x)$.

Concept Application Exercise 7.4

1. Let $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$. Find the value of $f'(0)$.

2. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that

   $\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$ is a constant polynomial.

3. If $g(x) = \dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree <3, then prove that

   $$\frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

SYSTEM OF LINEAR EQUATIONS

System of consistent linear equations: A system of (linear) equations is said to be consistent if it has at least one solution.

Example: (i) System of equations $\left.\begin{array}{l}x+y=2\\2x+2y=5\end{array}\right\}$ is inconsistent because it has no solutions, i.e., there is no value of x and y which satisfy both the equations.

Here the two straight lines are parallel.

(ii) System of equations $\left.\begin{array}{l}x+y=2\\x-y=0\end{array}\right\}$ is consistent because it has a solution $x = 1, y = 1$.

Here the two lines intersect at one point.

Cramer's Rule

I. System of linear equations in two variables:

Let the given system of equations be

$$\left.\begin{array}{l}a_1x+b_1y+c_1=0\\a_2x+b_2y+c_2=0\end{array}\right\} \quad (1)$$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

Solving by cross-multiplication, we have

$$\frac{x}{b_1c_2-b_2c_1}=\frac{-y}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

or

$$\frac{x}{\begin{vmatrix}b_1 & c_1\\b_2 & c_2\end{vmatrix}}=\frac{-y}{\begin{vmatrix}a_1 & c_1\\a_2 & c_2\end{vmatrix}}=\frac{1}{\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}}$$

II. System of linear equations in three variables:

Let the given system of linear equations in three variables x, y and z be

$$a_1x + b_1y + c_1z = d_1 \quad (1)$$
$$a_2x + b_2y + c_2z = d_2 \quad (2)$$
$$a_3x + b_3y + c_3z = d_3 \quad (3)$$

Let, $\Delta=\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}, \Delta_1=\begin{vmatrix}d_1 & b_1 & c_1\\d_2 & b_2 & c_2\\d_3 & b_3 & c_3\end{vmatrix}, \Delta_2=\begin{vmatrix}a_1 & d_1 & c_1\\a_2 & d_2 & c_2\\a_3 & d_3 & c_3\end{vmatrix},$

$$\Delta_3=\begin{vmatrix}a_1 & b_1 & d_1\\a_2 & b_2 & d_2\\a_3 & b_3 & d_3\end{vmatrix}$$

Let, $\Delta \neq 0$. Now,

$$\Delta_1=\begin{vmatrix}d_1 & b_1 & c_1\\d_2 & b_2 & c_2\\d_3 & b_3 & c_3\end{vmatrix}=\begin{vmatrix}a_1x+b_1y+c_1z & b_1 & c_1\\a_2x+b_2y+c_2z & b_2 & c_2\\a_3x+b_3y+c_3z & b_3 & c_3\end{vmatrix}$$

$$=\begin{vmatrix}a_1x & b_1 & c_1\\a_2x & b_2 & c_2\\a_3x & b_3 & c_3\end{vmatrix}=x\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}=x\Delta$$

$$[C_1 \to C_1 - yC_2 - zC_3]$$

$\therefore \quad x = \frac{\Delta_1}{\Delta}$, where $\Delta \neq 0$

Similarly, $\Delta_2 = y\Delta \quad \therefore y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$

Thus $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$, where $\Delta \neq 0$ (4)

The rule given in Eq. (4) to find the values of x, y, z is called the Cramer's rule.

Note:

(i) *Δ_i is obtained by replacing elements of i^{th} column by d_1, d_2, d_3 where $i = 1, 2, 3$.*

(ii) *Cramer's rule can be used only when $\Delta \neq 0$.*

Nature of Solution of System of Linear Equations

Let the given system of linear equations be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Now there are two possibilities.

Case I: $\Delta \neq 0$

In this case, from (i), (ii) and (iii), we have

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}$$

Hence, unique value of x, y, z will be obtained and the system of equations will have unique solution.

Case II: $\Delta = 0$

(a) When at least one of Δ_1, Δ_2, Δ_3 is non-zero

Let $\Delta_1 \neq 0$, then from (i), $\Delta_1 = x\Delta$ will not be satisfied for any value of x because $\Delta = 0$ and $\Delta_1 \neq 0$ and hence no value of x is possible in this case.

Similarly when $\Delta_2 \neq 0$, $\Delta_2 = y\Delta$ will not be possible for any value of y and hence no value of y will be possible when $\Delta_3 \neq 0$, $\Delta_3 = z\Delta$ will not be possible for any value of z and hence no value of z will be possible.

Thus if $\Delta = 0$ and any of Δ_1, Δ_2 and Δ_3 is non-zero, then no solution is possible and hence system of equations will be inconsistent.

(b) when $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$

In this case $\left.\begin{array}{l}\Delta_1 = x\Delta\\\Delta_2 = y\Delta\\\Delta_3 = z\Delta\end{array}\right\}$ will be true for all values of x, y and z.

But since $a_1x + b_2y + c_1z = d_1$, therefore, only two of x, y, z will be independent and the third will be dependent on other two.

Thus infinitely many values of x, y, z are possible and out of x, y, z only two can be given independent values.

Hence if $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system of equations will be consistent and it will have infinitely many solutions.

Summary:

(i) If $\Delta \neq 0$, then given system of equations is consistent and it has unique (one) solution.

(ii) If $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, then given system of equations is inconsistent and it will have no solution.

(iii) If all of Δ_1, Δ_2 and Δ_3 are zero, then given system of equations is consistent and has infinitely many solutions.

Conditions for Consistency of Three Linear Equations in Two Unknowns

System of three linear equations in x and y

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$

will be consistent if the values of x and y obtained from any two equations satisfy the third equation.

Solving first two equations by Cramer's rule, we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

These values of x and y will satisfy the third equation if

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition for consistency of three linear equations in two unknown. If such system of equations is consistent, then number of solutions is one.

System of Homogeneous Linear Equations

A system of linear equations is said to be homogeneous if the sum of powers of variable in each term is 1.

Let the three homogeneous linear equations in three unknown x, y, z be

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0 \quad (i) \\ a_2x + b_2y + c_2z &= 0 \quad (ii) \\ a_3x + b_3y + c_3z &= 0 \quad (iii) \end{aligned}\right\} \quad (A)$$

Clearly $x = 0$, $y = 0$, $z = 0$ is a solution of system of Eq. (A). This solution is called a trivial solution. Any other solution is called a non-trivial solution. Let, system of Eq. (A) has non-trivial solution.

Let, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

From (i) and (ii), we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} = k \text{ (say)}$$

$$\therefore \quad x = k(b_1c_2 - b_2c_1)$$
$$y = -k(a_1c_2 - a_2c_1)$$
$$z = k(a_1b_2 - a_2b_1)$$

Putting these values of x, y, z in (iii), we get

$$k[a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1)] = 0$$

or $a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1) = 0$ $\quad [\because \ k \neq 0]$

or $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

or $\Delta = 0$

This is the condition for system of Eq. (A) to have nontrivial solution.

Summary:

(i) If $\Delta \neq 0$, then given system of equations has only trivial solution and the number of solutions in this case is one.

(ii) If $\Delta = 0$, then given system of equations has non-trivial solution as well as trivial solution and number of solutions in this case is infinite.

Example 7.46 **Solve by Cramer's rule**

$$x + y + z = 6$$
$$x - y + z = 2$$
$$3x + 2y - 4z = -5$$

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 3 & -1 & -7 \end{vmatrix} = 14$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ -5 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 1 \\ -4 & -2 & 0 \\ 19 & 6 & 0 \end{vmatrix} = 14$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 0 & -4 & 0 \\ 0 & -23 & -7 \end{vmatrix} = 28$$

and $\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 3 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 0 & -2 & -4 \\ 0 & -1 & -23 \end{vmatrix} = 42$

Hence by Cramer's rule,

$$x = \frac{\Delta_x}{\Delta} = 1, \ y = \frac{\Delta_y}{\Delta} = 2, \ z = \frac{\Delta_z}{\Delta} = 3$$

Example 7.47 **For what values of p and q, the system of equations $2x + py + 6z = 8, x + 2y + qz = 5, x + y + 3z = 4$ has**
(i) no solution,
(ii) a unique solution,
(iii) infinitely many solutions.

Sol. The given system of equation is

$$2x + py + 6z = 8$$
$$x + 2y + qz = 5$$
$$x + y + 3z = 4$$

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = (2-p)(3-q)$$

By Cramer's rule, if $\Delta \neq 0$, i.e., $p \neq 2$, $q \neq 3$, the system has unique solution.
If $p = 2$ or $q = 3$, $\Delta = 0$, then if $\Delta_x = \Delta_y = \Delta_z = 0$, the system has infinite solutions and if any one of $\Delta_x, \Delta_y, \Delta_z \neq 0$, system has no solution. Now,

$$\Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$$
$$= 30 - 8q - 15p + 4pq = (p-2)(4q-15)$$
$$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$$
$$= -8q + 8q = 0$$
$$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$$
$$= p - 2$$

Thus if $p = 2$, $\Delta_x = \Delta_y = \Delta_z = 0$ for all $q \in R$, so the system has infinite solutions.

And if $p \neq 2$, $q = 3$, $\Delta_x, \Delta_z \neq 0$, the system has no solution.

Hence the system has
(i) no solution, if $p \neq 2, q = 3$,
(ii) a unique solution, if $p \neq 2, q \neq 3$,
(iii) infinitely many solutions, $p = 2, q \in R$.

Example 7.48 **Find λ for which the system of equations $x + y - 2z = 0, 2x - 3y + z = 0, x - 5y + 4z = \lambda$ is consistent and find the solutions for all such values of λ.**

Sol. The given system is

$$x - 5y + 4z = \lambda \quad (1)$$
$$x + y - 2z = 0 \quad (2)$$
$$2x - 3y + z = 0 \quad (3)$$

$$\Delta = \begin{vmatrix} 1 & -5 & 4 \\ 1 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -5 & 4 \\ 0 & 6 & -6 \\ 0 & 7 & -7 \end{vmatrix} = 0$$

Hence, system is consistent only when $\Delta_x = \Delta_y = \Delta_z = 0$. Now,

$$\Delta_x = \begin{vmatrix} \lambda & -5 & 4 \\ 0 & 1 & -2 \\ 0 & -3 & 1 \end{vmatrix} = -5\lambda = 0$$

$\Rightarrow \lambda = 0$

For $\lambda = 0$, clearly $\Delta_y = \Delta_z = 0$.

Therefore, system is consistent if $\lambda = 0$. Then on eliminating x from (1), (2) and (3), we have $y - z = 0$.

Let, $y = z = k \in R$. Then from (1), we have
$x = 5k - 4k = k$

Hence, solution is $x = y = z = k \in R$.

Example 7.49 **For what values of k, the following system of equations possesses a non-trivial solution over the set of rationals: $x + ky + 3z = 0, 3x + ky - 2z = 0, 2x + 3y - 4z = 0$. Also find the solution for this value of k.**

Sol. The system

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 3y - 4z = 0$$

has non-trivial solution (i.e., non-zero solution) if the determinant of coefficients of x, y and z is zero. Here,

$$\Delta = \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$\Rightarrow 2k - 33 = 0$ or $k = 33/2$ (1)

Then the equations become

$$2x + 33y + 6z = 0 \quad (2)$$
$$6x + 33y - 4z = 0 \quad (3)$$
$$2x + 3y - 4z = 0 \quad (4)$$

Eliminating x, we get from (2) and (4),

$30y + 10z = 0$, i.e., $3y + z = 0$ (5)

Let, $y = \lambda \in R$. Then $z = -3\lambda$ and so

$$2x = -33\lambda + 18\lambda = -15\lambda$$
$$\therefore \quad x = -\frac{15}{2}\lambda$$

Hence, $x = -\frac{15}{2}\lambda, y = \lambda, z = -3\lambda, \lambda \in R$.

Example 7.50 **If $2ax - 2y + 3z = 0, x + ay + 2z = 0$ and $2x + az = 0$ have a non-trivial solution, find the value of a.**

Sol. For non-trivial solution, we must have

$$\begin{vmatrix} 2a & -2 & 3 \\ 1 & a & 2 \\ 2 & 0 & a \end{vmatrix} = 0$$

$\Rightarrow 2a(a^2 - 0) + 2(a - 4) + 3(0 - 2a) = 0$
$\Rightarrow 2a^3 + 2a - 8 + 0 - 6a = 0$
$\Rightarrow 2a^3 - 4a - 8 = 0$

$\Rightarrow \quad a^3 - 2a - 4 = 0$

$\Rightarrow \quad a^3 - 2a^2 + 2a^2 - 4a + 2a - 4 = 0$

$\Rightarrow \quad a^2(a-2) + 2a(a-2) + 2(a-2) = 0$

$\Rightarrow \quad (a-2)(a^2 + 2a + 2) = 0$

$\Rightarrow \quad a = 2$

Example 7.51 **If x, y and z are not all zero and connected by the equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ and $(p_1 + \lambda q_1)x + (p_2 + \lambda q_2)y + (p_3 + \lambda q_3)z = 0$, show that**

$$\lambda = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 & p_2 & p_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

Sol. Since x, y and z are not all zero, the determinant of the coefficient of the given set of equations must satisfy

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 + \lambda q_1 & p_2 + \lambda q_2 & p_3 + \lambda q_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 & p_2 & p_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

Concept Application Exercise 7.5

1. If the equations $2x + 3y + 1 = 0$, $3x + y - 2 = 0$ and $ax + 2y - b = 0$ are consistent, then prove that $a - b = 2$.
2. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$ where x, y, z are not all zeros, then find the value of $a^2 + b^2 + c^2 + 2abc$.
3. If the following system of equations is consistent,
   $(a+1)^3x + (a+2)^3y = (a+3)^3$
   $(a+1)x + (a+2)y = a+3$
   $x + y = 1,$
   then find the value of a.
4. Solve the system of the equations:
   $ax + by + cz = d$
   $a^2x + b^2y + c^2z = d^2$
   $a^3x + b^3y + c^3z = d^3$
   Will the solution always exist and be unique?

EXERCISES

Subjective Type

Solutions on page 7.38

1. Solve for x, $\begin{vmatrix} x^2 - a^2 & a^2 - b^2 & x^2 - c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x+a)^3 & (x+b)^3 & (x+c)^2 \end{vmatrix} = 0, a \neq b \neq c.$

2. Prove that $\Delta = \begin{vmatrix} a & c & c-a & a+c \\ c & b & b-c & b+c \\ a-b & b-c & 0 & a-c \\ x & y & z & 1+x+y \end{vmatrix} = 0$ implies that a, b, c are in A.P. or a, c, b are in G.P.

3. If $f(x)$ is a polynomial of degree <3, prove that
$$\begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}$$

4. Prove that for any A.P. $a_1, a_2, a_3, \ldots$ the determinant
$$\begin{vmatrix} a_p + a_{p+m} + a_{p+2m} & 2a_p + 3a_{p+m} + 4a_{p+2m} & 4a_p + 9a_{p+m} + 16a_{p+2m} \\ a_q + a_{q+m} + a_{q+2m} & 2a_q + 3a_{q+m} + 4a_{q+2m} & 4a_q + 9a_{q+m} + 16a_{q+2m} \\ a_r + a_{r+m} + a_{r+2m} & 2a_r + 3a_{r+m} + 4a_{r+2m} & 4a_r + 9a_{r+m} + 16a_{r+2m} \end{vmatrix} = 0$$

5. Let n and r be two positive integers such that $n \geq r + 2$ and
$\Delta(n, r) = \begin{vmatrix} {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \\ {}^{n+1}C_r & {}^{n+1}C_{r+1} & {}^{n+1}C_{r+2} \\ {}^{n+2}C_r & {}^{n+2}C_{r+1} & {}^{n+2}C_{r+2} \end{vmatrix}$. Show that
$\Delta(n,r) = \dfrac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1, r-1)$. Hence or otherwise, prove
that $\Delta(n, r) = \dfrac{{}^{n+2}C_3\, {}^{n+1}C_3 \ldots {}^{n-r+3}C_3}{{}^{r+2}C_3\, {}^{r+1}C_3 \ldots {}^3C_3}$.

6. Show that in general there are three values of t for which the following system of equations has a non-trivial solution:
   $(a-t)x + by + cz = 0$
   $bx + (c-t)y + az = 0$
   $cx + ay + (b-t)z = 0$
   Express the product of these values of t in the form of a determinant.

7. Let α_1, α_2 and β_1, β_2 be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has a non-trivial solution, then prove that $b^2pr = q^2ac$.

8. If A, B and C are the angles of a triangle, show that the system of equations $x \sin 2A + y \sin C + z \sin B = 0$, $x \sin C + y \sin 2B + z \sin A = 0$ and $x \sin B + y \sin A + z \sin 2C = 0$ possesses non-trivial solution. Hence, system has infinite solutions.

9. If $ax_1^2 + by_1^2 + cz_1^2 = ax_2^2 + by_2^2 + cz_2^2 = ax_3^2 + by_3^2 + cz_3^2 = d$, $ax_2x_3 + by_2y_3 + cz_2z_3 = ax_3x_1 + ax_3x_1 + by_3y_1 + cz_3z_1 = ax_1x_2 + by_1y_2 + cz_1z_2 = f$, then prove that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (d-f)\left\{\frac{(d+2f)}{abc}\right\}^{1/2}$$

10. Let $\Delta = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 \\ a_1b_2 + a_2b_1 & 2a_2b_2 & a_2b_3 + a_3b_2 \\ a_1b_3 + a_3b_1 & a_3b_2 + a_2b_3 & 2a_3b_3 \end{vmatrix}$. Expressing Δ as the product of two determinants, show that $\Delta = 0$.

Hence show that if $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$, then $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

11. If in a triangle, s denotes the semi-perimeter and a, b, c denote the lengths of sides, then prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$$

12. Evaluate $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$.

13. Prove that $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b)$.

14. Prove that $\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}$

$= (x^2 + y^2 + z^2)(a^2 + b^2 + c^2)(ax + by + cz)$.

15. If $\Delta(x) = \begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2+x & b_2+x & c_2+x \\ a_3+x & b_3+x & c_3+x \end{vmatrix}$, show that $\Delta''(x) = 0$ and that $\Delta(x) = \Delta(0) + Sx$, where S denotes the sum of all the cofactors of all the elements in $\Delta(0)$.

Objective Type

Solutions on page 7.42

Each question has four choices a, b, c and d, out of which *only one* answer is correct. Find the correrct answer.

1. If $p + q + r = 0 = a + b + c$, then the value of the determinant $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$ is

a. 0 **b.** $pa + qb + rc$
c. 1 **d.** none of these

2. If a determinant of order 3 × 3 is formed by using the numbers 1 or −1, then the minimum value of the determinant is

a. −2 **b.** −4
c. 0 **d.** −8

3. If $z = \begin{vmatrix} -5 & 3+4i & 5-7i \\ 3-4i & 6 & 8+7i \\ 5+7i & 8-7i & 9 \end{vmatrix}$, then z is

a. purely real **b.** purely imaginary
c. $a + ib$, where $a \neq 0, b \neq 0$ **d.** $a + ib$, where $b = 4$

4. If α, β, γ are the roots of $px^3 + qx^2 + r = 0$, then the value of the determinant $\begin{vmatrix} \alpha\beta & \beta\gamma & \gamma\alpha \\ \beta\gamma & \gamma\alpha & \alpha\beta \\ \gamma\alpha & \alpha\beta & \beta\gamma \end{vmatrix}$ is

a. p **b.** q
c. 0 **d.** r

5. When the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, then the constant term in that expression is

a. 1 **b.** 0
c. −1 **d.** 2

6. If $a = \cos\theta + i\sin\theta$, $b = \cos 2\theta - i\sin 2\theta$, $c = \cos 3\theta + i\sin 3\theta$ and if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then

a. $\theta = 2k\pi, k \in Z$ **b.** $\theta = (2k+1)\pi, k \in Z$
c. $\theta = (4k+1)\pi, k \in Z$ **d.** none of these

7. If $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$, then the line $ax + by + c = 0$ passes through the fixed point which is

a. (1, 2) **b.** (1, 1)
c. (−2, 1) **d.** (1, 0)

8. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then n equals

a. 1 **b.** −1
c. 2 **d.** −2

9. If $f(x) = a + bx + cx^2$ and α, β, γ are the roots of the equation $x^3 = 1$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is equal to

a. $f(\alpha)+f(\beta)+f(\gamma)$
b. $f(\alpha)f(\beta)+f(\beta)f(\gamma)+f(\gamma)f(\alpha)$
c. $f(\alpha)f(\beta)f(\gamma)$
d. $-f(\alpha)f(\beta)f(\gamma)$

10. If [ ] denotes the greatest integer less than or equal to the real number under consideration, and $-1 \le x < 0, 0 \le y < 1$, $1 \le z < 2$, then the value of the determinant

$$\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix} \text{ is}$$

a. $[x]$ b. $[y]$
c. $[z]$ d. none of these

11. Let $a, b, c \in R$ such that no two of them are equal and satisfy

$$\begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0,$$ then equation $24ax^2 + 4bx + c = 0$ has

a. at least one root in [0, 1]
b. at least one root in $\left[-\frac{1}{2}, \frac{1}{2}\right]$
c. at least one root in[−1, 0]
d. at least two roots in [0, 2]

12. If p, q, r are in A.P., then the value of determinant

$$\begin{vmatrix} a^2+a^{2n+1}+2p & b^2+2^{n+2}+3q & c^2+p \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix} \text{ is}$$

a. 1 b. 0
c. $a^2b^2c^2 - 2^n$ d. $(a^2 + b^2 + c^2) - 2^nq$

13. If $(x_1 - x_2)^2 + (y_1 - y_2)^2 = a^2$
$(x_2 - x_3)^2 + (y_2 - y_3)^2 = b^2$
$(x_3 - x_1)^2 + (y_3 - y_1)^2 = c^2$

and $k\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (a+b+c)(b+c-a)(c+a-b)$ $\times (a + b - c)$, then the value of k is

a. 1 b. 2
c. 4 d. none of these

14. The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is

a. $k(a + b)(b + c)(c + a)$
b. $k\,abc(a^2 + b^2 + c^2)$
c. $k(a - b)(b - c)(c - a)$
d. $k(a + b - c)(b + c - a)(c + a - b)$

15. If $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$, where a, b, c are all different, then the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$$ vanishes when

a. $a + b + c = 0$ b. $x = \frac{1}{3}(a+b+c)$
c. $x = \frac{1}{2}(a+b+c)$ d. $x = a + b + c$

16. The determinant $\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$ is equal to

a. $\begin{vmatrix} bx+ay & cx+by \\ b'x+a'y & c'x+b'y \end{vmatrix}$ b. $\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$
c. $\begin{vmatrix} bx+cy & ax+by \\ b'x+c'y & a'x+b'y \end{vmatrix}$ d. $\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$

17. If $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = ka^2b^2c^2$, then the value of k is

a. 2 b. 4
c. 0 d. none of these

18. If a, b and c are non-zero real numbers, then

$$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} \text{ is equal to}$$

a. abc b. $a^2b^2c^2$
c. $bc + ca + ab$ d. none of these

19. The value of $\begin{vmatrix} -1 & 2 & 1 \\ 3+2\sqrt{2} & 2+2\sqrt{2} & 1 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$ is equal to

a. zero b. $-16\sqrt{2}$
c. $-8\sqrt{2}$ d. none of these

20. Let $\{D_1, D_2, D_3, ..., D_n\}$ be the set of third-order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, ..., a_9$. Then

a. $\sum_{i=1}^{n} D_i = 1$ b. $\sum_{i=1}^{n} D_i = 0$
c. $D_i = D_j, \forall\, i, j$ d. None of these

21. If w is a complex cube root of unity, then value of

$$\Delta = \begin{vmatrix} a_1+b_1w & a_1w^2+b_1 & c_1+b_1\bar{w} \\ a_2+b_2w & a_2w^2+b_2 & c_2+b_2\bar{w} \\ a_3+b_3w & a_3w^2+b_3 & c_3+b_3\bar{w} \end{vmatrix}$$ is

a. 0 **b.** –1
c. 2 **d.** none of these

22. If x, y, z are in A.P., then the value of the determinant

$$\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$ is

a. 1 **b.** 0
c. $2a$ **d.** a

23. If $a+b+c=0$, one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is

a. $x=1$ **b.** $x=2$
c. $x=a^2+b^2+c^2$ **d.** $x=0$

24. In triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ \cot\frac{A}{2} & \cot\frac{B}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2}+\tan\frac{C}{2} & \tan\frac{C}{2}+\frac{A}{2} & \tan\frac{A}{2}+\tan\frac{B}{2} \end{vmatrix} = 0,$$ then the triangle must be

a. equilateral **b.** isosceles
c. obtuse angled **d.** none of these

25. If a, b, c, d, e and f are in G.P., then the value of

$$\begin{vmatrix} a^2 & d^2 & x \\ b^2 & e^2 & y \\ c^2 & f^2 & z \end{vmatrix}$$ depends on

a. x and y **b.** x and z
c. y and z **d.** independent of x, y and z

26. If $a^2+b^2+c^2=-2$ and $f(x)=\begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$,

then $f(x)$ is a polynomial of degree

a. 0 **b.** 1
c. 2 **d.** 3

27. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix}$ is equal to

a. 1 **b.** –1
c. 0 **d.** none of these

28. If $x \neq y \neq z$ and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then the value of xyz is

a. 1 **b.** 2
c. –1 **d.** –2

29. If $x \neq 0, y \neq 0, z \neq 0$ and $\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 1+3z \end{vmatrix} = 0$, then $x^{-1}+y^{-1}+z^{-1}$ is equal to

a. –1 **b.** –2
c. –3 **d.** none of these

30. The value of $\begin{vmatrix} yz & zx & xy \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$, where x, y, z are, respectively, p^{th}, $(2q)^{th}$ and $(3r)^{th}$ terms of an H.P., is

a. –1 **b.** 0
c. 1 **d.** none of these

31. If $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are 3-digit even natural numbers and $\Delta = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$, then Δ is

a. divisible by 2 but not necessarily by 4
b. divisible by 4 but not necessarily by 8
c. divisible by 8
d. none of these

32. The value of the determinant of n^{th} order, being given by

$$\begin{vmatrix} x & 1 & 1 & \cdots \\ 1 & x & 1 & \cdots \\ 1 & 1 & x & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$ is

a. $(x-1)^{n-1}(x+n-1)$ **b.** $(x-1)^n(x+n-1)$
c. $(1-x)^{-1}(x+n-1)$ **d.** none of these

33. If a, b, c are positive and are the p^{th}, q^{th} and r^{th} terms, respectively, of a G.P., then $\Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$ is

a. 0 **b.** $\log(abc)$
c. $-(p+q+r)$ **d.** none of these

34. If $f'(x) = \begin{vmatrix} mx & mx-p & mx+p \\ n & n+p & n-p \\ mx+2n & mx+2n+p & mx+2n-p \end{vmatrix}$, then $y=f(x)$ represents

a. a straight line parallel to x-axis
b. a straight line parallel to y-axis
c. parabola
d. a straight line with negative slope

35. If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 4 \\ x & 4 & 5 \end{vmatrix} = 0$, then '$x$' is equal to

a. 0 **b.** −9
c. 3 **d.** none of these

36. If $\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^{n+5} & x^{a+6} & x^{2n+5} \end{vmatrix} = 0, \forall x \in R$, where $n \in N$, then value of 'a' is

a. n **b.** $n - 1$
c. $n + 1$ **d.** none of these

37. If $\Delta = \begin{vmatrix} 3 & 4 & 5 & x \\ 4 & 5 & 6 & y \\ 5 & 6 & 7 & z \\ x & y & z & 0 \end{vmatrix} = 0$, then

a. x, y, z are in A.P. **b.** x, y, z are in G.P.
c. x, y, z are in H.P. **d.** none of these

38. Let $x < 1$, then value of $\begin{vmatrix} x^2+2 & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ is

a. non-negative **b.** non-positive
c. negative **d.** positive

39. Value of $\begin{vmatrix} x+y & z & z \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$, where x, y, z are non-zero real numbers, is equal to

a. xyz **b.** $2xyz$
c. $3xyz$ **d.** $4xyz$

40. Which of the following is not the root of the equation

$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$?

a. 2 **b.** 0
c. 1 **d.** −3

41. If $f(x) = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = 0$, then

a. $f'(x) = 0$ and $f''(x) = 0$ has one common root

b. $f(x) = 0$ and $f'(x) = 0$ has one common root

c. sum of roots of $f(x) = 0$ is $-3a$

d. none of these

42. Roots of the equation $\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$ are

a. independent of m and n

b. independent of a, b and c

c. depend on m, n and a, b, c

d. independent of m, n and a, b, c

43. If a, b, c are different, then the value of x satisfying $\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix} = 0$ is

a. c **b.** c
c. b **d.** 0

44. If $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = k \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, then the value of k is

a. 1 **b.** 2
c. 3 **d.** 4

45. Suppose $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$D' = \begin{vmatrix} a_1+pb_1 & b_1+qc_1 & c_1+ra_1 \\ a_2+pb_2 & b_2+qc_2 & c_2+ra_2 \\ a_3+pb_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix}$. Then

a. $D' = D$ **b.** $D' = D(1 - pqr)$
c. $D' = D(1 + p + q + r)$ **d.** $D' = D(1 + pqr)$

46. The value of the determinant

$\Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ is equal to

a. 1 **b.** 0
c. 2 **d.** 3

47. If $a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative, then

$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$ is

a. + ve **b.** $(ac - b)^2)(ax^2 + 2bx + c)$
c. −ve **d.** 0

(AIEEE, 2002)

48. If $a_1, a_2, \ldots, a_n, \ldots$ form a G.P. and $a_i > 0$, for all $i \geq 1$,

then $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is equal to

a. 0 b. 1
c. 2 d. 3

(AIEEE, 2005)

49. Let $\vec{a_r} = x_r\hat{i} + y_r\hat{j} + z_r\hat{k}$, $r = 1, 2, 3$ be three mutually perpendicular unit vectors, then the value of $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$ is equal to

a. zero b. ± 1
c. ± 2 d. none of these

50. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\pi/4 \le x \le \pi/4$ is

a. 0 b. 2
c. 1 d. 3

51. If $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda^2+1 & 2-\lambda & \lambda-3 \\ \lambda^2 -3 & \lambda+4 & 3\lambda \end{vmatrix}$, then p is equal to

a. −5 b. −4
c. −3 d. −2

52. If x, y, z are different from zero and $\Delta = \begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$, then the value of the expression $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$ is

a. 0 b. −1
c. 1 d. 2

53. If A, B, C are angles of a triangle, then the value of $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$ is

a. 1 b. −1
c. −2 d. − 4

54. For the equation $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = 0$,

a. There are exactly two distinct roots
b. There is one pair of equation real roots
c. There are three pairs of equal roots
d. Modulus of each root is 2

55. Let m be a positive integer and

$$\Delta_r = \begin{vmatrix} 2r & -1 & {}^mC_r & 1 \\ m^2 & -1 & 2^m & m+1 \\ \sin^2 & (m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} (0 \le r \le m).$$

Then the value of $\sum_{r=0}^{m} \Delta_r$ is given by

a. 0 b. $m^2 - 1$
c. 2^m d. $2^m \sin^2 (2^m)$

56. If $D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n} D_k = 56$, then n equals

a. 4 b. 6
c. 8 d. none of these

57. The value of $\sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^{n-2}C_{r-2} & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ -3 & 1 & 1 \\ 2 & -1 & 0 \end{vmatrix} (n > 2)$ is

a. $2n - 1 + (-1)^n$ b. $2n + 1 + (-1)^{n-1}$
c. $2n - 3 + (-1)^n$ d. none of these

58. The value of the determinant $\begin{vmatrix} {}^nC_{r-1} & {}^nC_r & (r+1)\ {}^{n+2}C_{r+1} \\ {}^nC_r & {}^nC_{r+1} & (r+2)\ {}^{n+2}C_{r+2} \\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)\ {}^{n+2}C_{r+3} \end{vmatrix}$ is

a. $n^2 + n - 1$ b. 0
c. ${}^{n+3}C_{r+3}$ d. ${}^nC_{r-1} + {}^nC_r + {}^nC_{r+1}$

59. $\Delta_1 = \begin{vmatrix} y^5z^6(z^3-y^3) & x^4z^6(x^3-z^3) & x^4y^5(y^3-x^3) \\ y^2z^3(y^6-z^6) & xz^3(z^6-x^6) & xy^2(x^6-y^6) \\ y^2z^3(z^3-y^3) & xz^3(x^3-z^3) & xy^2(y^3-x^3) \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} x & y^2 & z^3 \\ x^4 & y^5 & z^6 \\ x^7 & y^8 & z^9 \end{vmatrix}$. Then $\Delta_1\Delta_2$ is equal to

a. Δ_2^3 b. Δ_2^2
c. Δ_2^4 d. None of these

60. If $l_1^2 + m_1^2 + n_1^2 = 1$, etc. and $l_1l_2 + m_1m_2 + n_1n_2 = 0$, etc. and $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$, then

a. $|\Delta| = 3$ b. $|\Delta| = 2$
c. $|\Delta| = 1$ d. $\Delta = 0$

61. The value of the determinant

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix}$$ is

a. dependant on a_i, $i = 1, 2, 3, 4$
b. dependant on b_i, $i = 1, 2, 3, 4$
c. dependant on a_i, b_i, $i = 1, 2, 3, 4$
d. 0

62. The value of determinant $\begin{vmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$ is

a. always positive **b.** always negative
c. always zero **d.** cannot say anything

63. Value of $\begin{vmatrix} 1+x_1 & 1+x_1x & 1+x_1x^2 \\ 1+x_2 & 1+x_2x & 1+x_2x^2 \\ 1+x_3 & 1+x_3x & 1+x_3x^2 \end{vmatrix}$ depends upon

a. x only **b.** x_1 only
c. x_2 only **d.** none of these

64. If $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$

$= (1 + a^2 + b^2 + c^2)^3$, then the value of λ is

a. 8 **b.** 27
c. 1 **d.** –1

65. $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$. The value of $\lim_{x\to 0} \frac{f(x)}{x}$ is equal to

a. 1 **b.** –1
c. zero **d.** none of these

66. If the determinant $\begin{vmatrix} b-c & c-a & a-b \\ b'-c' & c'-a' & a'-b' \\ b''-c'' & c''-a'' & a''-b'' \end{vmatrix}$

$= m\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$, then the value of m is

a. 0 **b.** 2
c. –1 **d.** 1

67. Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$. Then the value of

$5A + 4B + 3C + 2D + E$ is equal to

a. zero **b.** –16
c. 16 **d.** –11

68. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

a. $\Delta_1 = 3(\Delta_2)^2$ **b.** $\frac{d}{dx}(\Delta_1) = 3\Delta_2$
c. $\frac{d}{dx}(\Delta_1) = 3(\Delta_2)^2$ **d.** $\Delta_1 = 3\Delta_2^{3/2}$

69. If $y = \sin mx$, then the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$,

where $y_n = \frac{d^n y}{dx^n}$, is

a. m^9 **b.** m^2
c. m^3 **d.** none of these

70. If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then $(a, b, c > 0)$

a. $abc > 1$ **b.** $abc > -8$
c. $abc < -8$ **d.** $abc > -2$

71. If $A_1, B_1, C_1, \ldots$ are, respectively, the cofactors of the elements $a_1, b_1, c_1, \ldots$ of the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta \neq 0$, then

the value of $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$ is equal to

a. $a_1^2\Delta$ **b.** $a_1\Delta$
c. $a_1\Delta^2$ **c.** $a_1^2\Delta^2$

72. Let $f(x) = \begin{vmatrix} 2\cos^2 x & \sin 2x & -\sin x \\ \sin 2x & 2\sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$. Then the value of

$\int_0^{\pi/2} [f(x) + f'(x)]dx$ is

a. π **b.** $\pi/2$
c. 2π **d.** $3\pi/2$

73. The number of positive integral solutions of the equation $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$ is

a. 0 **b.** 3
c. 6 **d.** 12

74. a, b, c are distinct real numbers, not equal to one. If $ax + y + z = 0$, $x + by + z = 0$ and $x + y + cz = 0$ have a non-trivial solution, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is equal to

a. -1 b. 1
c. zero d. none of these

75. If the system of linear equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $2x + 5y + \lambda z = \mu$ ($\lambda, \mu \in R$) has a unique solution, then
a. $\lambda \neq 8$ b. $\lambda = 8, \mu \neq 36$
c. $\lambda = 8, \mu = 36$ d. none of these

76. If α, β, γ are the angles of a triangle and the system of equations

$\cos(\alpha - \beta)x + \cos(\beta - \gamma)y + \cos(\gamma - \alpha)z = 0$

$\cos(\alpha + \beta)x + \cos(\beta + \gamma)y + \cos(\gamma + \alpha)z = 0$

$\sin(\alpha + \beta)x + \sin(\beta + \gamma)y + \sin(\gamma + \alpha)z = 0$

has non-trivial solutions, then triangle is necessarily
a. equilateral b. isosceles
c. right angled d. acute angled

77. Given $a = x/(y - z)$, $b = y/(z - x)$ and $c = z/(x - y)$, where x, y and z are not all zero, then the value of $ab + bc + ca$ is
a. 0 b. 1
c. -1 d. none of these

78. If $pqr \neq 0$ and the system of equations

$(p + a)x + by + cz = 0$

$ax + (q + b)y + cz = 0$

$ax + by + (r + c)z = 0$

has a non-trivial solution, then value of $\frac{a}{p} + \frac{b}{q} + \frac{c}{r}$ is
a. -1 b. 0
c. 1 d. 2

79. The system of equations

$\alpha x - y - z = \alpha - 1$

$x - \alpha y - z = \alpha - 1$

$x - y - \alpha z = \alpha - 1$

has no solution if α is
a. either -2 or 1 b. -2
c. 1 d. not -2

80. The set of equations $\lambda x - y + (\cos\theta)z = 0$, $3x + y + 2z = 0$, $(\cos\theta)x + y + 2z = 0$, $0 \leq \theta < 2\pi$, has non-trivial solution(s)
a. for no value of λ and θ
b. for all values of λ and θ
c. for all values of λ and only two values of θ
d. for only one value of λ and all values of θ

81. If a, b, c are non-zeros, then the system of equations $(\alpha + a)x + \alpha y + \alpha z = 0$, $\alpha x + (\alpha + b)y + \alpha z = 0$, $\alpha x + \alpha y + (\alpha + c)z = 0$ has a non-trivial solution if
a. $\alpha^{-1} = -(a^{-1} + b^{-1} + c^{-1})$ b. $\alpha^{-1} = a + b + c$
c. $\alpha + a + b + c = 1$ d. none of these

82. If $c < 1$ and the system of equations $x + y - 1 = 0$, $2x - y - c = 0$ and $bx + 3by - c = 0$ is consistent, then the possible real values of b are
a. $b \in \left(-3, \frac{3}{4}\right)$ b. $b \in \left(-\frac{3}{2}, 4\right)$
c. $b \in \left(-\frac{3}{4}, 3\right)$ d. none of these

83. If a, b, c are in G.P. with common ratio r_1 and α, β, γ are in G.P. with common ratio r_2, and equations $ax + \alpha y + z = 0$, $bx + \beta y + z = 0$, $cx + \gamma y + z = 0$ have only zero solution, then which of the following is not true?
a. $r_1 \neq 1$ b. $r_2 \neq 1$
c. $r_1 \neq r_2$ d. none of these

84. If a, b, c are non-zero real numbers and if the equations $(a - 1)x = y + z$, $(b - 1)y = z + x$, $(c - 1)z = x + y$ have a non-trivial solution, then $ab + bc + ca$ equals
a. $a + b + c$ b. abc
c. 1 d. none of these

Multiple Correct Answers Type

Solutions on page 7.53

Each question has four choices a, b, c and d, out of which *one or more* answers are correct. Find the correct answer.

1. Which of the following has/have value equal to zero?

a. $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ b. $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$

c. $\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$ d. $\begin{vmatrix} 2 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$

2. If $g(x) = \begin{vmatrix} a^{-x} & e^{x\log_e a} & x^2 \\ a^{-3x} & e^{3x\log_e a} & x^4 \\ a^{-5x} & e^{5x\log_e a} & 1 \end{vmatrix}$, then
a. graphs of $g(x)$ is symmetrical about origin
b. graphs of $g(x)$ is symmetrical about Y-axis
c. $\left.\frac{d^4g(x)}{dx^4}\right|_{x=0} = 0$
d. $f(x) = g(x) \times \log\left(\frac{a-x}{a+x}\right)$ is an odd function

3. If $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then a factor of Δ is
a. $a + b + x$
b. $x^2 - (a - b)x + a^2 + b^2 + ab$
c. $x^2 + (a + b)x + a^2 + b^2 - ab$
d. $a + b - x$

4. If $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$ then
a. Δ is independent of θ b. Δ is independent of ϕ
c. Δ is a constant d. $\left.\frac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = 0$

5. If $\phi(\alpha, \beta) = \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$, then

a. $f(300, 200) = f(400, 200)$ **b.** $f(200, 400) = f(200, 600)$
c. $f(100, 200) = f(200, 200)$ **d.** none of these

6. The determinant $\Delta = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$ is divisible by

a. x **b.** x^2
c. x^3 **d.** none of these

7. If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then $f(2x) - f(x)$ is divisible by

a. x **b.** a
c. $2a + 3x$ **d.** x^2

8. $\Delta = \begin{vmatrix} 1 & 1+ac & 1+bc \\ 1 & 1+ad & 1+bd \\ 1 & 1+ae & 1+be \end{vmatrix}$ is independent of

a. a **b.** b
c. c, d, e **d.** none of these

9. If $g(x) = \dfrac{f(x)}{(x-a)(x-b)(x-c)}$, where $f(x)$ is a polynomial of degree < 3, then

a. $\int g(x)dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + k$

b. $\dfrac{dg(x)}{dx} = \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$

c. $\dfrac{dg(x)}{dx} = \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

d. $\int g(x)dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} + k$

10. If $\Delta(x) = \begin{vmatrix} x^2+4x-3 & 2x+4 & 13 \\ 2x^2+5x-9 & 4x+5 & 26 \\ 8x^2-6x+1 & 16x-6 & 104 \end{vmatrix} = ax^3 + bx^2 + cx + d$, then

a. $a = 3$ **b.** $b = 0$
c. $c = 0$ **d.** None of these

11. If $f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & \cos\theta \\ \cos\theta & \sin\theta & \sin\theta \end{vmatrix}$, then

a. $f(\theta) = 0$ has exactly 2 real solutions in $[0, \pi]$

b. $f(\theta) = 0$ has exactly 3 real solutions in $[0, \pi]$

c. range of function $\dfrac{f(\theta)}{1-\sin 2\theta}$ is $\left[-\sqrt{2}, \sqrt{2}\right]$

d. range of function $\dfrac{f(\theta)}{\sin 2\theta - 1}$ is $[-3, 3]$

12. If $f(\theta) = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cos B & 1 \\ \sin^2 C & \cos C & 1 \end{vmatrix}$, then

a. $\tan A + \tan B + C$ **b.** $\text{Cot } A \cot B \cot C$
c. $\sin^2 A + \sin^2 B + \sin^2 C$ **d.** 0

13. The roots of the equation $\begin{vmatrix} {}^xC_r & {}^{n-1}C_r & {}^{n-1}C_{r-1} \\ {}^{x+1}C_r & {}^nC_r & {}^nC_{r-1} \\ {}^{x+2}C_r & {}^{n+1}C_r & {}^{n+1}C_{r-1} \end{vmatrix} = 0$ are

a. $x = n$ **b.** $x = n + 1$
c. $x = n - 1$ **d.** $x = n - 2$

14. If $f(x) = \begin{vmatrix} 3 & 3x & 3x^2+2a^2 \\ 3x & 3x^2+2a^2 & 3x^3+6a^2x \\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$, then

a. $f'(x) = 0$
b. $y = f(x)$ is a straight line parallel to x-axis
c. $\int_0^2 f(x)dx = 32a^4$
d. none of these

15. If $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ xz-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$, then

a. $r^2 = x + y + z$ **b.** $r^2 = x^2 + y^2 + z^2$
c. $u^2 = yz + zx + xy$ **d.** $u^2 = xyz$

16. Let $f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$ where the symbols have their usual meanings. Then $f(n)$ is divisible by

a. $n^2 + n + 1$ **b.** $(n + 1)!$
c. $n!$ **d.** none of these

17. If a, b, c are non-zero real numbers such that

$\begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix} = 0$, then

a. $\frac{1}{a}+\frac{1}{b\omega}+\frac{1}{c\omega^2}=0$ b. $\frac{1}{a}+\frac{1}{b\omega^2}+\frac{1}{c\omega}=0$

c. $\frac{1}{a\omega}+\frac{1}{b\omega^2}+\frac{1}{c}=0$ d. none of these

18. The values of $k \in R$ for which the system of equations $x + ky + 3z = 0$, $kx + 2y + 2z = 0$, $2x + 3y + 4z = 0$ admits of non-trivial solution is

a. 2 b. 5/2

c. 3 d. 5/4

19. If determinant $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is

a. positive b. independent of θ

c. independent of ϕ d. none of these

Reasoning Type

Solutions on page 7.55

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. Statement 1: If A, B and C are the angles of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$ then triangle may not be equilateral.

Statement 2: If any two rows of a determinant are the same, then the value of that determinant is zero.

2. Consider the system of the equations $kx + y + z = 1$, $x + ky + z = k$ and $x + y + kz = k^2$.

Statement 1: System of equations has infinite solutions when $k = 1$.

Statement 2: If the determinant $\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ k^2 & 1 & k \end{vmatrix} = 0$, then $k = -1$.

3. Consider the determinant $f(x) = \begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}$.

Statement 1: $f(x) = 0$ has one root $x = 0$.

Statement 2: The value of skew-symmetric determinant of odd-order is always zero.

4. Statement 1: If the system of equations $\lambda x + (b-a)y + (c-a)z = 0$, $(a-b)x + \lambda y + (c-b)z = 0$ and $(a-c)x + (b-c)y + \lambda z = 0$ has a non-trivial solution, then the value of λ is 0.

Statement 2: The value of skew-symmetric matrix of order 3 is zero.

5. Statement 1: $\Delta = \begin{vmatrix} my+nz & mq+nr & mb+nc \\ kz-mx & kr-mp & kc-ma \\ -nx-ky & -np-kq & -na-kb \end{vmatrix}$ is equal to 0.

Statement 2: The value of skew-symmetric matrix of order 3 is zero.

6. Statement 1: If $bc + qr = ca + rp = ab + pq = -1$,

then $\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix} = 0 \quad (abc, pqr \neq 0)$.

Statement 2: If system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ has non-trivial solutions, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

7. Consider the system of equation $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$.

Statement 1: If the system has infinite number of solutions, then $\mu = 10$.

Statement 2: The determinant $\begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = 0$ for $\mu = 10$.

8. Consider the determinant $\Delta = \begin{vmatrix} a_1+b_1x^2 & a_1x^2+b_1 & c_1 \\ a_2+b_2x^2 & a_2x^2+b_2 & c_2 \\ a_3+b_3x^2 & a_3x^2+b_3 & c_3 \end{vmatrix} = 0$,

where $a_i, b_i, c_i \in R$ $(i = 1, 2, 3)$ and $x \in R$.

Statement 1: The values of x satisfying $\Delta = 0$ are $x = 1, -1$.

Statement 2: If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then $\Delta = 0$.

Linked Comprehension Type

Solutions on page 7.56

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

$f(x) = \begin{vmatrix} x+c_1 & x+a & x+a \\ x+b & x+c_2 & x+a \\ x+b & x+b & x+c_3 \end{vmatrix}$ and $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$

1. Coefficient of x in $f(x)$ is

a. $\frac{g(a)-f(b)}{b-a}$ b. $\frac{g(-a)-g(-b)}{b-a}$

c. $\frac{g(a)-g(b)}{b-a}$ d. none of these

2. Which of the following is not a constant term in $f(x)$?

a. $\frac{bg(a)-ag(b)}{(b-a)}$ b. $\frac{bg(a)-af(-b)}{(b-a)}$

c. $\frac{bf(-a)-ag(b)}{(b-a)}$ d. none of these

3. Which of the following is not true?

a. $f(-a) = g(a)$ b. $f(-a) = g(-a)$

c. $f(-b) = g(b)$ d. none of these

For Problems 4–6

Consider the function $f(x) = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$

4. Which of the following is true?

a. $f(x) = 0$ and $f'(x) = 0$ have one positive common root.

b. $f(x) = 0$ and $f'(x) = 0$ have one negative common root.

c. $f(x) = 0$ and $f'(x) = 0$ have no common root.

d. None of these.

5. Which of the following is true?

a. $f(x)$ has one +ve point of maxima.

b. $f(x)$ has one –ve point of minima.

c. $f(x) = 0$ has three distinct roots.

d. Local minimum value of $f(x)$ is zero.

6. In which of the following interval $f(x)$ is strictly increasing?

a. $(-\infty, \infty)$ b. $(-\infty, 0)$

c. $(0, \infty)$ d. None of these

For Problems 7–9

Given that the system of equations $x = cy + bz$, $y = az + cx$, $z = bx + ay$ has non-zero solutions and at least one of the a, b, c is a proper fraction.

7. $a^2 + b^2 + c^2$ is

a. >2 c. >3

d. <3 d. <2

8. abc is

a. > –1 b. >1

c. <2 d. <3

9. System has solution such that

a. $x:y:z \equiv (1-2a^2):(1-2b^2):(1-2c^2)$

b. $x:y:z \equiv \frac{1}{1-2a^2}:\frac{1}{1-2b^2}:\frac{1}{1-2c^2}$

c. $x:y:z \equiv \frac{a}{1-a^2}:\frac{b}{1-b^2}:\frac{c}{1-c^2}$

d. $x:y:z \equiv \sqrt{1-a^2}:\sqrt{1-b^2}:\sqrt{1-c^2}$

For Problems 10–12

Consider the system of equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

10. The system has unique solution if

a. $\lambda \neq 3$ b. $\lambda = 3, \mu = 10$

c. $\lambda = 3, \mu \neq 10$ d. none of these

11. The system has infinite solutions if

a. $\lambda \neq 3$ b. $\lambda = 3, \mu = 10$

c. $\lambda = 3, \mu \neq 10$ d. none of these

12. The system has no solution if

a. $\lambda \neq 3$ b. $\lambda = 3, \mu = 10$

c. $\lambda = 3, \mu \neq 10$ d. none of these

For Problems 13–15

Let α, β be the roots of the equation $ax^2 + bx + c = 0$. Let $S_n = \alpha^n + \beta^n$ for $n \geq 1$ and $\Delta = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

13. If $\Delta < 0$, then the equation $ax^2 + bx + c = 0$ has

a. positive real roots b. negative real roots

c. equal roots d. imaginary roots

14. If a, b, c are rational and one of the roots of the equation is $1+\sqrt{2}$, then the value of Δ is

a. 8 b. 12

c. 30 d. 32

15. If $\Delta > 0$, then

a. $f(1) > 0$ b. $f(1) < 0$

c. $f(1) = 0$ d. cannot say anything about $f(1)$

For Problems 16–18

Let $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$ and the equation $px^3 + qx^2 + rx + s = 0$ has roots a, b, c, where $a, b, c \in R^+$.

16. The value of Δ is

a. r^2/p^2 b. r^3/p^3

c. $-s/p$ d. none of these

17. The value of Δ is

a. $\leq 9r^2/p^2$ b. $\geq 27s^2/p^2$

c. $\leq 27s^3/p^3$ d. none of these

18. If $\Delta = 27$ and $a^2 + b^2 + c^2 = 2$, then

a. $3p + 2q = 0$ b. $4p + 3q = 0$

c. $3p + q = 0$ d. none of these

For Problems 19–21

Consider the polynomial function

$$f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}, \; a, b \text{ being positive integers.}$$

19. The constant term in $f(x)$ is

a. 2   **b.** 1
c. –1   **d.** 0

20. The coefficient of x in $f(x)$ is

a. 2^a **b.** $2^a - 3 \times 2^b + 1$
c. 0   **d.** none of these

21. Which of the following is true?

a. All the roots of the equation $f(x) = 0$ are positive.
b. All the roots of the equation $f(x) = 0$ are negative.
c. At least one of the equation $f(x) = 0$ is repeating one.
d. None of these.

For Problems 22–24

If $x > m, y > n, z > r$ $(x, y, z > 0)$ such that $\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0$.

22. The value of $\dfrac{x}{x-m} + \dfrac{y}{y-n} + \dfrac{z}{z-r}$ is

a. 1   **b.** –1
c. 2   **d.** –2

23. The value of $\dfrac{m}{x-m} - 1 + \dfrac{y}{y-n} - 1 + \dfrac{r}{z-r}$ is

a. – 2   **b.** –4
c. 0   **d.** –1

24. The greatest value of $\dfrac{xyz}{(x-m)(y-n)(z-r)}$ is

a. 27   **b.** $\dfrac{8}{27}$
c. $\dfrac{64}{27}$ **d.** none of these

For Problems 25–27

Suppose $f(x)$ is a function satisfying the following conditions:

(i) $f(0) = 2, f(1) = 1$,

(ii) f has a minimum value at $x = 5/2$

(iii) For all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

25. The value of $f(2)$ is

a. 1/4   **b.** 1/2
c. –1   **d.** 3

26. $f(x) = 0$ has

a. both roots positive   **b.** both roots negative
c. roots of opposite sign   **d.** imaginary roots

27. Range of $f(x)$ is

a. $[7/16, \infty)$ **b.** $(-\infty, 15/16]$
c. $[3/4, \infty)$ **d.** none of these

Matrix-Match Type

Solutions on page 7.59

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, s, b→r, c→p, q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1.

Column I	Column II
a. Coefficient of x in $f(x) = \begin{vmatrix} x & (1+\sin x)^3 & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$.	**p.** 10
b. Value of $\begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$ is	**q.** 0
c. If a, b, c are in A.P. and $f(x) = \begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}$, then $f'(0)$ is	**r.** –12
d. If $\begin{vmatrix} x & 2 & x \\ 1 & x & 6 \\ x & x & x+1 \end{vmatrix} = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then a_0 is	**s.** –2

2.

Column I	Column II
a. The value of the determinant $\begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$ is	**p.** 1
b. If one of the roots of the equation $\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$ is $x + 2$, then sum of all other five roots is	–6

c. The value of $\begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i \end{vmatrix}$ is	**r.** 2
d. If $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$, then $f(\pi/3)$	**s.** −2

3.

Column I	Column II
a. $\begin{vmatrix} 1/c & 1/c & -(a+b)/c^2 \\ -(b+c)/a^2 & 1/a & 1/a \\ -b(b+c)/a^2c & (a+2b+c)/ac & -b(a+b)/ac^2 \end{vmatrix}$ is	**p.** independent of a
b. $\begin{vmatrix} \sin a\cos b & \sin a\sin b & \cos a \\ \cos a\cos b & \cos a\sin b & -\sin a \\ -\sin a\sin b & \sin a\cos b & 0 \end{vmatrix}$ is	**q.** independent of b
c. $\begin{vmatrix} \frac{1}{\sin a\cos b} & \frac{1}{\sin a\sin b} & \frac{1}{\cos a} \\ \frac{-\cos a}{\sin^2 a\cos b} & \frac{-\cos a}{\sin^2 a\sin b} & \frac{\sin a}{\cos^2 a} \\ \frac{\sin b}{\sin a\cos^2 b} & \frac{-\cos b}{\sin a\sin^2 b} & 0 \end{vmatrix}$ is	**r.** independent of c
d. If a, b, and c are the sides of a triangle and A, B and C are the angles opposite to a, b, and c, respectively, then $\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$	**s.** dependent on a, b

Integer Type

Solutions on page 7.60

1. Let α, β, γ are the real roots of the equation $x^3 + ax^2 + bx + c = 0$ ($a, b, c \in R$ and $a \neq 0$). If the system of equations (in u, v and w) given by
$\alpha u + \beta v + \gamma w = 0$
$\beta u + \gamma v + \alpha w = 0$
$\gamma u + \alpha v + \beta w = 0$
has non-trivial solutions, then the value of a^2/b is.

2. If $a_1, a_2, a_3, 5, 4, a_6, a_7, a_8, a_9$ are in H.P., and $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ then the value of [D] is (where [ ] represents the greatest integer function)

3. If $\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$
$= -k(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$,
then the value of $(k)^{1/2}$ is.

4. Absolute value of sum of roots of the equation $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$ is.

5. The value of $|\alpha|$ for which the system of equation
$\alpha x + y + z = \alpha - 1$
$x + \alpha y + z = \alpha - 1$
$x + y + \alpha z = \alpha - 1$
has no solution, is.

6. Sum of values of p for which, the equations: $x + y + z = 1$; $x + 2y + 4z = p$ and $x + 4y + 10z = p^2$ have a solution is.

7. Three distinct points $P(3u^2, 2u^3)$; $Q(3v^2, 2v^3)$ and $R(3w^2, 2w^3)$ are collinear then $uv + vw + wu$ is equal to.

8. Let $D_1 = \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ a & b & a-b \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & c & a+c \\ b & d & b+d \\ a & c & a+b+c \end{vmatrix}$ then the value of $\left|\frac{D_1}{D_2}\right|$ is where $b \neq 0$ and $ad \neq bc$,

9. If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, then the value of $(\Delta_{max.})/2$ is.

10. If $\begin{vmatrix} x & x+y & x+y+z \\ 2x & 3x+2y & 4x+3y+2z \\ 3x & 6x+3y & 10x+6y+3z \end{vmatrix} = 64$, then the real value of x is.

11. If $a_1, a_2, a_3, \ldots, a_{12}$ are in A.P. and
$\Delta_1 = \begin{vmatrix} a_1a_5 & a_1 & a_2 \\ a_2a_6 & a_2 & a_3 \\ a_3a_7 & a_3 & a_4 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} a_2a_{10} & a_2 & a_3 \\ a_3a_{11} & a_3 & a_4 \\ a_3a_{12} & a_4 & a_5 \end{vmatrix}$
then $\Delta_2 : \Delta_1 =$

12. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+4} \\ y^n & y^{n+2} & y^{n+4} \\ z^n & z^{n+2} & z^{n+4} \end{vmatrix} = \left(\frac{1}{y^2}-\frac{1}{x^2}\right)\left(\frac{1}{z^2}-\frac{1}{y^2}\right)\left(\frac{1}{x^2}-\frac{1}{z^2}\right)$
then $-n$ is.

13. Given $A = \begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix}$, $B = \begin{vmatrix} f & 2d & e \\ 2n & 4l & 2m \\ c & 2a & b \end{vmatrix}$, then the value of B/A is.

14. The value of $\begin{vmatrix} 2x_1y_1 & x_1y_2+x_2y_1 & x_1y_3+x_3y_1 \\ x_1y_2+x_2y_1 & 2x_2y_2 & x_2y_3+x_3y_2 \\ x_1y_3+x_3y_1 & x_2y_3+x_3y_2 & 2x_3y_3 \end{vmatrix}$ is.

15. If $(1 + ax + bx^2)^4 = a_0 + a_1x + a_2x^2 + \cdots + a_8x^8$, where $a, b, a_0, a_1, \ldots, a_8 \in R$ such that $a_0 + a_1 + a_2 \neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix} = 0$ then the value of $5\frac{a}{b}$ is.

Archives

Solutions on page 7.62

Subjective Type

1. For what value of k do the following system of equations posses a non-trivial (i.e., not all zero) solution over the set of rationals Q?

$x + ky + 3z = 0$
$3x + ky - 2z = 0$
$2x + 3y - 4z = 0$

For that value of k, find all the solutions for the system. **(IIT-JEE, 1979)**

2. Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative. **(IIT-JEE, 1981)**

3. Without expanding a determinant at any stage, show that

$$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = xA + B,$$ where A and B are determinants of order 3 not involving x. **(IIT-JEE, 1982)**

4. Show that the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ has at least one solution for any real number λ. Find the set of solutions if $\lambda = -5$. **(IIT-JEE, 1983)**

5. Show that $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$. **(IIT-JEE, 1985)**

6. Consider the system of linear equations in x, y and z given by $(\sin 3\theta)x - y + z = 0$, $(\cos 2\theta)x + 4y + 3z = 0$, $2x + 7y + 7z = 0$. Find the values of θ for which the system has a non-trivial solution. **(IIT-JEE, 1986)**

7. Let the three-digit numbers $A28$, $3B9$ and $62C$, where A, B, C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is also divisible by the same integer k. **(IIT-JEE, 1990)**

8. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$. **(IIT-JEE, 1991)**

9. For a fixed positive integer n, if $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$, then show that $\left[\frac{\Delta}{(n!)^3} - 4\right]$ is divisible by n. **(IIT-JEE, 1992)**

10. Let λ and α be real. Find the set of all values of λ for which the system of linear equations $\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$, $x + (\cos \alpha)y + (\sin \alpha)z = 0$, $-x + (\sin \alpha)y - (\cos \alpha)z = 0$. **(IIT-JEE, 1993)**

11. For all values of A, B, C and P, Q, R show that

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$ **(IIT-JEE, 1994)**

12. Let $a > 0$, $d > 0$. Find the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$ **(IIT-JEE, 1996)**

13. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ where a, b and c are, respectively, the p^{th}, q^{th} and r^{th} terms of a harmonic progression. **(IIT-JEE, 1997)**

14. Prove that for all values of θ, the value of the determinant

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$ is 0. **(IIT-JEE, 2000)**

Objective Type

Fill in the blanks

1. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$ be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is ______________. **(IIT-JEE, 1981)**

2. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is ______________. **(IIT-JEE, 1981)**

3. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the value of determinant chosen is positive is ______________. **(IIT-JEE, 1982)**

4. Given that $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$. The other two roots are ______________ and ______________. **(IIT-JEE, 1983)**

5. The system of equation

   $\lambda x + y + z = 0$

   $-x + \lambda y + z = 0$

   $-x - y + \lambda z = 0$

   will have a non-zero solution if real values of λ are given by ______________. **(IIT-JEE, 1984)**

6. The value of the determinant $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is ______________. **(IIT-JEE, 1988)**

7. For positive numbers x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is ______________. **(IIT-JEE, 1993)**

8. If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{ f(\theta) : 0 \le \theta < \dfrac{\pi}{2} \right\}$ is ______________. **(IIT-JEE, 2011)**

True or false

1. The determinants $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identically equal. **(IIT-JEE, 1983)**

2. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$, then the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_1, b_2), (a_3, b_3) must be congruent. **(IIT-JEE, 1985)**

Multiple choice questions with one correct answer

1. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with values -1. Then

   a. C is empty
   b. B has as many elements as C
   c. $A = B \cup C$
   d. B has twice as many elements as elements as C

   (IIT-JEE, 1981)

2. If ω ($\neq 1$) is a cube root of unity, then value of the determinant $\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$ is

   a. 0 **b.** 1
   c. i **d.** ω

   (IIT-JEE, 1995)

3. Let a, b, c be the real numbers. Then following system of equations in x, y and z, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, has

   a. no solution **b.** unique solution
   c. infinitely many solutions **d.** finitely many solutions

   (IIT-JEE, 1995)

4. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$ if

   a. x, y, z are in A.P. **b.** x, y, z are in G.P.
   c. x, y, z are in H.P. **d.** xy, yz, zx are in A.P.

   (IIT-JEE, 1995)

5. The parameter, on which the value of the determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend, is

a. a b. p

c. d d. x

(IIT-JEE, 1997)

6. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)\ x(x-1) \end{vmatrix}$ then $f(500)$ is equal to

a. 0 b. 1

c. 500 d. −500

(IIT-JEE, 1999)

7. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution then the possible values of k are

a. −1, 2 b. 1, 2

c. 0, 1 d. −1, 1

(IIT-JEE, 2000)

8. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

a. 3ω b. $3\omega(\omega - 1)$

c. $3\omega^2$ d. $3\omega(1 - \omega)$

(IIT-JEE, 2002)

9. The number of values of k for which the system of equations $(k + 1)x + 8y = 4k$; $kx + (k + 3)y = 3k - 1$ has infinitely many solutions is

a. 0 b. 1

c. 2 d. infinite

(IIT-JEE, 2002)

10. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

a. −1 b. 1

c. 0 d. no real values

11. Given $2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ then the value of λ such that the given system of equation has no solution is

a. −3 b. 1

c. 0 d. 3

(IIT-JEE, 2004)

12. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

a. $x = 3, y = 1$ b. $x = 1, y = 3$

c. $x = 0, y = 3$ d. $x = 0, y = 0$

(IIT-JEE, 1998)

Multiple choice questions with one or more than one correct answer

1. The determinant $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$, if

a. a, b, c are in A.P.

b. a, b, c are in G.P.

c. a, b, c are in H.P.

d. α is a root of the equation $ax^2 + bx + c = 0$

e. $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

(IIT-JEE, 1986)

Matrix-match type

This question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matchs are a→p, a→s, b→q, b→r, c→q, d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1. Consider the following linear equations:

$ax + by + cz = 0$

$bx + cy + az = 0$

$cx + ay + bz = 0$

Match the expressions/statements in column I with expressions/statements in column II. **(IIT-JEE, 2007)**

Column I	Column II
a. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	p. the equations represent planes meeting only at a single point
b. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	q. the equations represent the line $x = y = z$
c. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	r. the equations represent identical planes
d. $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	s. the equations represent the whole of the three-dimensional space

Integer type

1. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1. Applying $R_3 \to R_3 - R_2$, we have

$$\Delta = \begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 6x^2a+2a^3 & 6x^2b+2b^3 & 6x^2c+2c^3 \end{vmatrix}$$

Applying $R_3 \to \frac{1}{2}R_3$, then $R_2 \to R_2 + R_3$, we get

$$\Delta = 2\begin{vmatrix} x^2-a^2 & x^2-b^2 & x^2-c^2 \\ x^3+3xa^2 & x^3+3xb^2 & x^3+3xc^2 \\ 3x^2a+a^3 & 3x^2b+b^3 & 3x^2c+c^3 \end{vmatrix}$$

Applying $R_2 \to (1/x) R_1$ and then $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we obtain

$$\Delta = 2x\begin{vmatrix} x^2-a^2 & a^2-b^2 & a^2-c^2 \\ x^2+3a^2 & 3(b^2-a^2) & 3(c^2-a^2) \\ 3x^2a+a^3 & 3x^2(b-a)+b^3-a^3 & 3x^2(c-a)+c^3-a^3 \end{vmatrix}$$

$$= 2x(b-a)(c-a)\begin{vmatrix} x^2-a^2 & -(a+b) & -(a+c) \\ x^2+3a^2 & 3(b+a) & 3(c+a) \\ 3x^2a+a^3 & 3x^2+b^2+ab+a^2 & 3x^2+c^2+a^2+ac \end{vmatrix}$$

Applying $R_2 \to R_2 + 3R_1$, and $R_3 \to R_3 + aR_1$, we get

$$\Delta = 2x(b-a)(c-a)\times\begin{vmatrix} x^2-a^2 & -(a+b) & -(a+c) \\ 4x^2 & 0 & 0 \\ 4x^2a & 3x^2+b^2 & 3x^2+c^2 \end{vmatrix}$$

Expanding along R_2, we have

$$\Delta = 8x^3(b-a)(c-a)\{(a+b)(3x^2+c^2)-(a+c)(3x^2+b^2)\}$$
$$= 8x^3(b-a)(c-a)\{3x^2(b-c)+ac^2+bc^2-ab^2-cb^2\}$$
$$= 8x^3(b-a)(c-a)\{3x^2(b-c)+bc(c-b)+a(c^2-b^2)\}$$
$$= 8x^3(b-a)(c-a)(b-c)\{3x^2-(bc+ac+ab)\}$$

As a, b, c are distinct, $\Delta = 0$ gives $x = 0$ or $x^2 = 1/3(bc+ca+ab)$. If $ab+bc+ca \le 0$, the only real root is $x = 0$. If $ab+bc+ca > 0$, roots are $x = 0$, $\pm\sqrt{\frac{1}{3}(bc+ca+ab)}$.

2. Applying $C_3 \to C_3 - (C_2 - C_1)$ and $C_4 \to C_4 - (C_1 + C_2)$, we get

$$\Delta = \begin{vmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ a-b & b-c & a+c-2b & 0 \\ x & y & z+x-y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a & c & 0 \\ c & b & 0 \\ a-b & b-c & a+c-2b \end{vmatrix} \quad \text{(expanding along } C_4\text{)}$$

$= (a+c-2b)(ab-c^2)$ (expanding along C_3)

$\therefore \quad \Delta = 0$

$\Rightarrow \quad a+c-2b = 0$

or $\quad ab-c^2 = 0$

$\Rightarrow \quad a, b, c$ are in A.P. or a, c, b are in G.P.

3. We have $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$$\text{L.H.S.} = \frac{1}{(a-b)(b-c)(c-a)} \times\left[\frac{f(a)(c-b)}{(x-a)}+\frac{f(b)(a-c)}{(x-b)}+\frac{f(c)(b-a)}{(x-c)}\right]$$

(expanding along C_3)

From R.H.S. by partial fractions, we get

$$\text{R.H.S.} = \frac{f(x)}{(x-a)(x-b)(x-c)}$$
$$= \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c} \quad \text{[since degree of } f(x) \text{ is less than 3]}$$

Then,

$$A = \left[\frac{f(x)}{(x-b)(x-c)}\right]_{x=a} = \frac{f(a)}{(a-b)(a-c)}$$

Similarly,

$$B = \frac{f(b)}{(b-a)(b-c)}$$

and

$$C = \frac{f(c)}{(c-a)(c-b)}$$

$$\therefore \quad \text{R.H.S.} = \frac{1}{(a-b)(b-c)(c-a)} \times\left[\frac{(c-b)f(a)}{(x-b)}+\frac{(a-c)f(b)}{(x-b)}+\frac{(b-a)f(c)}{(x-c)}\right]$$

Hence, L.H.S. = R.H.S.

4.
$$D = \begin{vmatrix} a_p+a_{p+m}+a_{p+2m} & 2a_p+3a_{p+m}+4a_{p+2m} & 4a_p+9a_{p+m}+16a_{p+2m} \\ a_q+a_{q+m}+a_{q+2m} & 2a_q+3a_{q+m}+4a_{q+2m} & 4a_q+9a_{q+m}+16a_{q+2m} \\ a_r+a_{r+m}+a_{r+2m} & 2a_r+3a_{r+m}+4a_{r+2m} & 4a_r+9a_{r+m}+16a_{r+2m} \end{vmatrix}$$

$$= \begin{vmatrix} a_p + a_{p+m} + a_{p+2m} & a_{p+m} + 2a_{p+2m} & 5a_{p+m} + 12a_{p+2m} \\ a_q + a_{q+m} + a_{q+2m} & a_{q+m} + 2a_{q+2m} & 5a_{q+m} + 12a_{q+2m} \\ a_r + a_{r+m} + a_{r+2m} & a_{r+m} + 2a_{r+2m} & 5a_{r+m} + 12a_{r+2m} \end{vmatrix}$$

[Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - 4C_1$]

$$= \begin{vmatrix} a_p - a_{p+2m} & a_{p+m} + 2a_{p+2m} & 2a_{p+2m} \\ a_q - a_{q+2m} & a_{q+m} + 2a_{q+2m} & 2a_{q+2m} \\ a_r - a_{r+2m} & a_{r+m} + 2a_{r+2m} & 2a_{r+2m} \end{vmatrix}$$

[Applying $C_1 \to C_1 - C_2$, $C_3 \to C_3 - 5C_2$]

Applying $C_2 \to C_2 - C_3$, then $C_1 \to C_1 + \frac{1}{2}C_3$, then taking 2 common from C_3, we get

$$D = 2\begin{vmatrix} a_p & a_{p+m} & a_{p+2m} \\ a_q & a_{q+m} & a_{q+2m} \\ a_r & a_{r+m} & a_{r+2m} \end{vmatrix}$$

$$= 2\begin{vmatrix} a_p + a_{p+2m} - 2a_{p+m} & a_{p+m} & a_{p+2m} \\ a_q + a_{q+2m} - 2a_{q+m} & a_{q+m} & a_{q+2m} \\ a_r + a_{r+2m} - 2a_{r+m} & a_{r+m} & a_{r+2m} \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2C_2$]

$$= 2\begin{vmatrix} 0 & a_{p+m} & a_{p+2m} \\ 0 & a_{q+m} & a_{q+2m} \\ 0 & a_{r+m} & a_{r+2m} \end{vmatrix}.$$

(as a_p, a_{p+m}, a_{p+2m} are in A.P. like others)

$= 0$

5. We know that

$${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$\therefore\quad \Delta(n, r) = \begin{vmatrix} \frac{n}{r}\,{}^{n-1}C_{r-1} & \frac{n}{r+1}\,{}^{n-1}C_r & \frac{n}{r+2}\,{}^{n-1}C_{r+1} \\ \frac{n+1}{r}\,{}^{n}C_{r-1} & \frac{n+1}{r+1}\,{}^{n}C_r & \frac{n+1}{r+2}\,{}^{n}C_{r+1} \\ \frac{n+2}{r}\,{}^{n+1}C_{r-1} & \frac{n+2}{r+1}\,{}^{n+1}C_r & \frac{n+2}{r+2}\,{}^{n+1}C_{r+1} \end{vmatrix}$$

$$= \frac{n(n+1)(n+2)}{r(r+1)(r+2)}\Delta(n-1,\ r-1)$$

$$= \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1,\ r-1) \qquad (1)$$

Repeating the process, we have

$$\Delta(n, r) = \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\frac{{}^{n+1}C_3}{{}^{r+1}C_3}\Delta(n-2,\ r-2)$$

$$= \frac{{}^{n+2}C_3}{{}^{r+2}C_3}\frac{{}^{n+1}C_3}{{}^{r+1}C_3}\frac{{}^{n}C_3}{{}^{r}C_3}\cdots\frac{{}^{n-r+3}C_3}{{}^{3}C_3}\Delta(n-r, 0) \qquad (2)$$

Now,

$$\Delta(n-r, 0) = \begin{vmatrix} {}^{n-r}C_0 & {}^{n-r}C_1 & {}^{n-r}C_2 \\ {}^{n-r+1}C_0 & {}^{n-r+1}C_1 & {}^{n-r+1}C_2 \\ {}^{n-r+2}C_0 & {}^{n-r+2}C_1 & {}^{n-r+2}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & n-r & \frac{1}{2}(n-r)(n-r-1) \\ 1 & n-r+1 & \frac{1}{2}(n-r+1)(n-r) \\ 1 & n-r+2 & \frac{1}{2}(n-r+2)(n-r+1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & n-r & \frac{1}{2}(n-r)(n-r-1) \\ 0 & 1 & n-r \\ 0 & 1 & n-r+1 \end{vmatrix}$$

$(R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_1)$

$$= n-r+1-n+r = 1 \qquad (3)$$

Hence from (2) and (3), we get

$$\Delta(n, r) = \frac{({}^{n+2}C_3)({}^{n+1}C_3)\cdots({}^{n-r+3}C_3)}{({}^{r+2}C_3)({}^{r+1}C_3)\cdots({}^{3}C_3)}$$

6. The given system of equations will have a non-trivial solution if the determinant of coefficients

$$\Delta = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix} = 0 \qquad (1)$$

$\Delta = 0$ is a cubic equation (equation of degree 3) in t so it has in general 3 solutions. Let t_1, t_2 and t_3 be the solutions and

$$\Delta = p_0t^3 + p_1t^2 + p_2t + p_3 \qquad (2)$$

Clearly, coefficient of t^3 is $p_0 = -1$. So

$t_1t_2t_3 = -\frac{p_3}{(-1)} = p_3$ [constant term in the expansion of Δ, i.e. $\Delta(t = 0)$]

Hence,

$$t_1t_2t_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

7. Clearly $\alpha_1 + \alpha_2 = -b/a$, $\alpha_1\alpha_2 = c/a$ and $\beta_1 + \beta_2 = -q/p$, $\beta_1\beta_2 = r/p$.

System of equations $\alpha_1 y + \alpha_2 z = 0$, $\beta_1 y + \beta_2 z = 0$ has a non-trivial solution. So,

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0, \text{ i.e., } \alpha_1\beta_2 - \alpha_2\beta_1 = 0$$

or $$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$$

$$\Rightarrow \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$$

$$\Rightarrow \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$$

$$\Rightarrow \quad \frac{-b/a}{-q/p} = \sqrt{\frac{c/a}{r/p}}$$

$$\Rightarrow \quad b^2pr = q^2ac$$

8. Determinant of coefficients is

$$\Delta = \begin{vmatrix} 2\sin A\cos A & \sin(A+B) & \sin(A+C) \\ \sin(A+B) & 2\sin B\cos B & \sin(B+C) \\ \sin(A+C) & \sin(B+C) & 2\sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A\cos A + \sin A\cos A & \sin A\cos B + \sin B\cos A & \sin A\cos C + \sin C\cos A \\ \sin A\cos B + \sin B\cos A & 2\sin B\cos B & \sin B\cos C + \sin C\cos B \\ \sin A\cos C + \sin C\cos A & \sin B\cos C + \sin C\cos B & 2\sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A\cos A + \sin A\cos A & \sin A\cos B + \sin B\cos A & \sin A\cos C + \sin C\cos A \\ \sin A\cos B + \sin B\cos A & \sin B\cos B + \sin B\cos B & \sin B\cos C + \sin C\cos B \\ \sin A\cos C + \sin C\cos A & \sin B\cos C + \sin C\cos B & \sin C\cos C + \sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A & \cos A & 0 \\ \sin B & \cos B & 0 \\ \sin C & \cos C & 0 \end{vmatrix} \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}$$

$$= 0$$

9. Let, $D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$\therefore \quad D^2 = D \times D$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} ax_1^2 + by_1^2 + cz_1^2 & ax_1x_2 + by_1y_2 + cz_1z_2 & ax_3x_1 + by_3y_1 + cz_3z_1 \\ ax_1x_2 + by_1y_2 + cz_1z_2 & ax_2^2 + by_2^2 + cz_2^2 & ax_2x_3 + by_2y_3 + cz_2z_3 \\ ax_3x_1 + by_3y_1 + cz_3z_1 & ax_2x_3 + by_2y_3 + cz_2z_3 & ax_3^2 + by_3^2 + cz_3^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} d+2f & f & f \\ d+2f & d & f \\ d+2f & f & d \end{vmatrix} \qquad [C_1 \to C_1 + C_2 + C_3]$$

$$= \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$$

$$= \frac{(d+2f)}{abc}\begin{vmatrix} 1 & f & f \\ 0 & d-f & 0 \\ 1 & f & d \end{vmatrix}$$

$$= \frac{(d+2f)}{abc}(d-f)\begin{vmatrix} 1 & f \\ 1 & d \end{vmatrix} \qquad [R_2 \to R_2 - R_1]$$

$$= \frac{(d+2f)}{abc}(d-f)(d-f)$$

$$= \frac{(d+2f)(d-f)^2}{abc}$$

$$\Rightarrow D = (d-f)\left\{\frac{d+2f}{abc}\right\}^{1/2} = \text{R.H.S.}$$

10. $$\Delta = \begin{vmatrix} a_1b_1 + b_1a_1 & a_1b_2 + a_2b_1 & a_1b_3 + b_1a_3 \\ a_2b_1 + b_2a_1 & a_2b_2 + a_2b_2 & a_2b_3 + a_3b_2 \\ a_3b_1 + b_3a_1 & a_3b_2 + b_3a_2 & a_3b_3 + a_3b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix} = 0 \qquad (1)$$

(row by row multiplication)

Now,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$
$= (lx + my + n)(l'x + my + n')$
$= ll'x^2 + (lm' + ml')xy + mm'y^2 + (ln' + l'n)x + (mn' + m'n)y + nn'$

Comparing the coefficients, we get

$a = ll'$, $h = \frac{1}{2}(lm' + ml')$, $b = mm'$, $g = \frac{1}{2}(ln' + l'n)$,
$f = \frac{1}{2}(mn' + m'n)$, $c = nn'$

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} ll' & \frac{1}{2}(lm' + ml') & \frac{1}{2}(ln' + l'n) \\ \frac{1}{2}(lm' + ml') & mm' & \frac{1}{2}(mn' + m'n) \\ \frac{1}{2}(ln' + l'n) & \frac{1}{2}(mn' + m'n') & nn' \end{vmatrix}$$

$$= \frac{1}{8}\begin{vmatrix} 2ll' & lm'+l'm & ln'+l'n \\ lm'+l'm & 2mm' & mn'+m'n \\ ln'+l'n & mn'+m'n & 2nn' \end{vmatrix}$$

$= 0$ [From (1)]

11. Putting $s-a=\alpha$, $s-b=\beta$, $s-c=\gamma$, we get

$a = 2s-b-c = (s-b)+(s-c) = \alpha+\beta$

Similarly, $b=\gamma+\alpha$, $c=\alpha+\beta$. Also,

$\alpha+\beta+\gamma = 3s-(a+b+c) = 3s-2s = s$

$$\therefore \quad \Delta = \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma+\alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha+\beta)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (\beta+\gamma)^2 & \alpha^2-(\beta+\gamma)^2 & \alpha^2-(\beta+\gamma)^2 \\ \beta^2 & (\gamma+\alpha)^2-\beta^2 & 0 \\ \gamma^2 & 0 & (\alpha+\beta)^2-\gamma^2 \end{vmatrix}$$

$[C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$

$$= (\alpha+\beta+\gamma)^2 \times \begin{vmatrix} (\beta+\gamma)^2 & \alpha-\beta-\gamma & \alpha-\beta-\gamma \\ \beta^2 & \gamma+\alpha-\beta & 0 \\ \gamma^2 & 0 & \alpha+\beta-\gamma \end{vmatrix}$$

[Taking $(\alpha+\beta+\gamma)$ common from C_2 and C_3]

$$= 2(\alpha+\beta+\gamma)^2 \times \begin{vmatrix} \beta\gamma & -\gamma & -\beta \\ \beta^2 & \gamma+\alpha-\beta & 0 \\ \gamma^2 & 0 & \alpha+\beta-\gamma \end{vmatrix}$$

$[R_1 \to R_1 - R_2 - R_3$ and then $R_1 \to \frac{1}{2}R_1]$

Now applying $C_2 \to \beta C_2 + C_1$ and $C_3 \to \gamma C_3 + C_1$, we get

$$\Delta = \frac{2(\alpha+\beta+\gamma)^2}{\beta\gamma} \times \begin{vmatrix} \beta\gamma & 0 & 0 \\ \beta^2 & \beta(\gamma+\alpha) & \beta^2 \\ \gamma^2 & \gamma^2 & \gamma(\alpha+\beta) \end{vmatrix}$$

$$= \frac{2(\alpha+\beta+\gamma)^2}{\beta\gamma}\beta\gamma\,[(\beta\gamma+\beta\alpha)(\gamma\alpha+\gamma\beta)-\beta^2\gamma^2]$$

$= 2(\alpha+\beta+\gamma)^2\,[(\alpha\beta\gamma^2+\beta^2\gamma^2+\alpha^2\beta\gamma+\alpha\beta^2\gamma-\beta^2\gamma^2]$

$= 2(\alpha+\beta+\gamma)^3\,\alpha\beta\gamma$

$= 2s^3(s-a)(s-b)(s-c)$

12.
$$\Delta = \begin{vmatrix} x & \frac{1}{2}x(x-1) & \frac{1}{6}x(x-1)(x-2) \\ y & \frac{1}{2}y(y-1) & \frac{1}{6}y(y-1)(y-2) \\ z & \frac{1}{2}z(z-1) & \frac{1}{6}z(z-1)(z-2) \end{vmatrix}$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x-1 & x^2-3x+2 \\ 1 & y-1 & y^2-3y+2 \\ 1 & z-1 & z^2-3z+2 \end{vmatrix}$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x & x^2-3x+2 \\ 1 & y & y^2-3y+2 \\ 1 & z & z^2-3z+2 \end{vmatrix} \quad [R_2 \to R_2 + R_1]$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad [R_3 \to R_3 + 3R_2 - 2R_1]$$

$$= \frac{1}{12}xyz(x-y)(y-z)(z-x)$$

13. Let, $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$

Putting $a+b=0$ or $b=-a$, we get

$$\Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$$

Expanding along R_1,

$\Delta = -2a\{-4ac-(c-a)^2\} - 0 + (a+c)\{0-2a(c+a)\}$

$= 2a(c+a)^2 - 2a(c+a)^2$

$= 0$

Hence $a+b$ is a factor of Δ. Similarly $b+c$ and $c+a$ are the factors of Δ.

On expansion of determinant, we can see that each term of the determinant is a homogeneous expression in a, b, c of degree 3 and also R.H.S. is a homogeneous expression of degree 3.

$\therefore \quad \Delta = k(a+b)(b+c)(c+a)$

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a)$$

Putting $a=0$, $b=1$, $c=2$, we get

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(0+1)(1+2)(2+0)$$

$\Rightarrow \quad 0 - 1(-4-6) + 2(3+4) = 6k$

$\Rightarrow \quad 24 = 6k$

$\therefore \quad k = 4$

Hence,

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$$

14. Applying $C_1 \to xC_1 + yC_2 + zC_3$ [to get the term $(x^2 + y^2 + z^2)$], we get

$$\Delta = \frac{1}{x}\begin{vmatrix} a(x^2+y^2+z^2) & ay+bx & cx+az \\ b(x^2+y^2+z^2) & by-cz-ax & bz+cy \\ c(x^2+y^2+z^2) & bz+cy & cz-ax-by \end{vmatrix}$$

Now taking $(x^2 + y^2 + z^2)$ common from C_1 and then applying $R_1 \to aR_1 + bR_2 + cR_2$ [to get the term $(a^2 + b^2 + c^2)$], we get

$$\Delta = \frac{(x^2+y^2+z^2)}{ax}$$

$$\times\begin{vmatrix} (a^2+b^2+c^2) & y(a^2+b^2+c^2) & z(a^2+b^2+c^2) \\ b & by-cz-ax & bz+cy \\ c & bz+cy & cz-ax-by \end{vmatrix}$$

Taking $(a^2 + b^2 + c^2)$ common from R_1 and then applying $C_2 \to C_2 - yC_1$ and $C_3 \to C_3 - zC_1$, we get

$$\Delta = \frac{(x^2+y^2+z^2)(a^2+b^2+c^2)}{ax}\times\begin{vmatrix} 1 & 0 & 0 \\ b & -cz-ax & cy \\ c & bz & -ax-by \end{vmatrix}$$

Now expanding along R_1, we get

$$\Delta = \frac{1}{ax}(x^2+y^2+z^2)(a^2+b^2+c^2) \times [aczx + bczy + a^2x^2 + abxy - bcyz]$$

$$= (x^2 + y^2 + z^2)(a^2 + b^2 + c^2)(ax + by + cz)$$

15. $$\Delta(x) = \begin{vmatrix} a_1+x & b_1+x & c_1+x \\ a_2+x & b_2+x & c_2+x \\ a_3+x & b_3+x & c_3+x \end{vmatrix}$$

$$\therefore \quad \Delta'(x) = \begin{vmatrix} 1 & b_1+x & c_1+x \\ 1 & b_2+x & c_2+x \\ 1 & b_3+x & c_3+x \end{vmatrix} + \begin{vmatrix} a_1+x & 1 & c_1+x \\ a_2+x & 1 & c_2+x \\ a_3+x & 1 & c_3+x \end{vmatrix} + \begin{vmatrix} a_1+x & b_1+x & 1 \\ a_2+x & b_2+x & 1 \\ a_3+x & b_3+x & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - xC_1$ and $C_3 \to C_3 - xC_1$ in the first determinant of R.H.S., $C_1 \to C_1 - xC_2$ and $C_3 \to C_3 - xC_2$ in the second determinant and $C_1 \to C_1 - xC_3$ and $C_2 \to C_2 - xC_3$ in the third determinant, we get

$$\Delta'(x) = \begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Now consider the cofactors of

$$\Delta(0) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which are $b_2c_3 - b_3c_2$, $c_2a_3 - c_3a_2$, $a_2b_3 - b_2a_3$, etc. Clearly,

$$\begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 1 & b_3 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2) + (c_1b_3 - c_3b_1) + (b_1c_2 - b_2c_1)$$

which is the sum of cofactors of the first row elements of $\Delta(0)$. Similarly,

$$\begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 1 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

are the sum of cofactors of secong row and third row elements, respectively, of $\Delta(0)$. Hence $\Delta'(x) = S$, where S denotes the sum of all cofactors of elements of $\Delta(0)$.

$\therefore \quad \Delta''(x) = 0$

Since $\Delta'(x) = S$, $\Delta(x) = Sx + k$. So,

$\Delta(0) = k$

Hence,

$\Delta(x) = xS + \Delta(0)$

Objective Type

1. a. $$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$$

$$= pqr(a^3 + b^3 + c^3 - 3abc) - abc(p^3 + q^3 + r^3 - 3pqr)$$

$$= pqr(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) - abc(p + q + r)(p^2 + q^2 + r^2 - pq - qr - pr)$$

$$= 0$$

2. b. Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Applying $C_2 \to C_2 - \frac{a_{12}}{a_{11}}C_1, C_3 \to C_3 - \frac{a_{13}}{a_{11}}C_1$, we get

$$D = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \frac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \frac{a_{13}}{a_{11}} a_{21}\right) \\ a_{31} & \left(a_{32} - \frac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \frac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

which has minimum value of -4.

3. b. $$z = \begin{vmatrix} -5 & 3+4i & 5-7i \\ 3-4i & 6 & 8+7i \\ 5+7i & 8-7i & 9 \end{vmatrix}$$

$$\Rightarrow \quad \bar{z} = \begin{vmatrix} -5 & 3-4i & 5+7i \\ 3+4i & 6 & 8-7i \\ 5-7i & 8+7i & 9 \end{vmatrix} = \begin{vmatrix} -5 & 3+4i & 5-7i \\ 3-4i & 6 & 8+7i \\ 5+7i & 8-7i & 9 \end{vmatrix} = z$$

(Taking transpose)

$\Rightarrow \quad z$ is purely real

4. c. Operation $C_1 \to C_1 + C_2 + C_3$ gives $(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$\begin{vmatrix} 1 & \beta\gamma & \gamma\alpha \\ 1 & \gamma\alpha & \alpha\beta \\ 1 & \alpha\beta & \beta\gamma \end{vmatrix}$$

From the given equation, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$. So, the value of determinant is 0.

5. c.

$$f(x) = \begin{vmatrix} 1-2\sin^2 x & \sin^2 x & 1-8\sin^2 x(1-\sin^2 x) \\ \sin^2 x & 1-2\sin^2 x & 1-\sin^2 x \\ 1-8\sin^2 x(1-\sin^2 x) & 1-\sin^2 x & 1-2\sin^2 x \end{vmatrix}$$

The required constant term is

$$f(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-1) = -1$$

6. a. $$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

$\Rightarrow \quad a + b + c = 0$ or $a = b = c$

If $a + b + c = 0$, we have

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$ and $\sin\theta - \sin 2\theta + \sin 3\theta = 0$

$\Rightarrow \quad \cos 2\theta(2\cos\theta + 1) = 0$ and $\sin 2\theta(1 - 2\cos\theta) = 0$ (i)

which is not possible as $\cos 2\theta = 0$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$. And $\cos\theta = -1/2$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$. Therefore, Eq. (i) does not hold simultaneously.

$\therefore \quad a + b + c \neq 0$

$\therefore \quad a = b = c$

or

$e^{i\theta} = e^{-2i\theta} = e^{3i\theta}$

which is satisfied only by $e^{i\theta} = 1$, i.e., $\cos\theta = 1$, $\sin\theta = 0$ so $\theta = 2k\pi$, $k \in Z$.

7. b. Applying $C_1 \to aC_1$ and then $C_1 \to C_1 + bC_2 + cC_3$, and taking $(a^2 + b^2 + c^2)$ common from C_1, we get

$$\Delta = \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$= \frac{(a^2+b^2+c^2)}{a}(-bc + a^2 + ab + ac + bc)$ (expanding along C_1)

$= (a^2 + b^2 + c^2)(a + b + c)$

Hence, $\Delta = 0 \Rightarrow a + b + c = 0$

Therefore, line $ax + by + c = 0$ passes through the fixed point (1, 1).

8. b. The degree of the determinant is $n + (n + 2) + (n + 3) = 3n + 5$ and the degree of the expression on R.H.S. is 2.

$\therefore \quad 3n + 5 = 2 \Rightarrow n = -1$

9. d. $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

$= -(a + b + c)(a + b\omega^2 + c\omega)(a + b\omega + c\omega^2)$

(where ω is cube roots of unity)

$= -f(\alpha) f(\beta) f(\gamma)$ $\quad [\because \alpha = 1, \beta = \omega, \gamma = \omega^2]$

10. c. $\because -1 \leq x < 0 \quad \therefore [x] = -1$

$0 \leq y < 1 \quad \therefore [y] = 0$

$1 \leq z < 2 \quad \therefore [z] = 1$

Hence, the given determinant is

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

11. a. Given determinant,

$2a(bc - 4a^2) + b(2ac - b^2) + c(2ab - c^2) = 0$

$\Rightarrow \quad 6abc - 8a^3 - b^3 - c^3 = 0$

$\Rightarrow \quad (2a + b + c)[(2a - b)^2 + (b - c)^2 + (c - 2a)^2] = 0$

$\Rightarrow \quad 2a + b + c = 0 \quad (\because b \neq c)$

Let $f(x) = 8ax^3 + 2bx^2 + cx$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{2} = \frac{2a+b+c}{2} = 0$

So, $f(x)$ satisfies the Rolle's theorem and hence,

$f'(x) = 0$ has at least one root in $\left[0, \frac{1}{2}\right]$.

12. b. The given determinant is

$$\begin{bmatrix} 2^{n+1} - 2^n + p & 2^{n+2} - 2^{n+1} + q & p+r \\ 2^n + p & 2^{n+1} & p+r \\ a^2 + 2^n + p & b^2 + 2^n + 2q & c^2 - r \end{bmatrix}$$

(Using $R_1 \to R_1 - R_3$ and $2q = p + r$)

$$\begin{bmatrix} 2^n(2-1) + p & 2^{n+1}(2-1) + q & p+r \\ 2^n + p & 2^{n+1} + q & p+r \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{bmatrix}$$

$$= \begin{bmatrix} 2^n + p & 2^{n+1} + q & p + r \\ 2^n + p & 2^{n+1} + q & p + r \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{bmatrix} = 0 \quad (\because R_1 \equiv R_2)$$

13. c. Consider the triangle with vertices $B(x_1, y_1)$, $C(x_2, y_2)$ and $A(x_3, y_3)$, and $AB = c$, $BC = a$ and $AC = b$. Then area of triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } 2s = a + b + c$$

Squaring and simplifying, we get

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

Hence, $k = 4$.

14. c. We have,

$$\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix} = \begin{vmatrix} ka & k^2 & 1 \\ kb & k^2 & 1 \\ kc & k^2 & 1 \end{vmatrix} + \begin{vmatrix} ka & a^2 & 1 \\ kb & b^2 & 1 \\ kc & c^2 & 1 \end{vmatrix} = 0 + k\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= k(a-b)(b-c)(c-a)$$

15. b. We have,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c) \quad (1)$$

Also,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} \text{(taking } a,b,c \text{ common from } R_1, R_2, R_3)$$

$$= \begin{vmatrix} bc & ac & ab \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} \text{(Multiplying } R_1 \text{ by } abc)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ac & ab \end{vmatrix}$$

Then,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(3x-a-b-c)$$

Now given that a, b, c are all different, then $D = 0$.

$$\therefore \quad x = \frac{1}{3}(a+b+c)$$

16. d. Let, $\Delta = \begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$

Then,

$$\Delta = \frac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix} \quad [\text{Applying } C_1 \to xC_1, C_3 \to yC_3]$$

$$= \frac{1}{xy}\begin{vmatrix} 0 & -xy & 0 \\ ax+by & b & bx+cy \\ a'x+b'y & b' & b'x+c'y \end{vmatrix}$$

$$[\text{Applying } C_1 \to C_1 + yC_2, C_3 \to C_3 + xC_2]$$

$$= \frac{1}{xy}xy\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix} \quad [\text{Expanding along } R_1]$$

$$= \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

17. b. $\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$

Applying $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$, we get

$$\Delta = \frac{1}{abc} \times \begin{vmatrix} b^2+c^2 & a^2b & a^2c \\ ab^2 & b(c^2+a^2) & cb^2 \\ ac^2 & bc^2 & c(a^2+b^2) \end{vmatrix}$$

Now, applying $C_1 \to \frac{1}{a}C_1$, $C_2 \to \frac{1}{b}C_2$, $C_3 \to \frac{1}{c}C_3$, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad [R_1 \to R_1 - R_2 - R_3]$$

$$= 2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix}$$

(Taking 2 common from R_1 and applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$)

Evaluating along R_1, we get

$$\Delta = 2[c^2(a^2b^2) - b^2(-a^2c^2)]$$
$$= 4a^2b^2c^2$$

Hence, $k = 4$.

18. d. Applying $R_1 \to aR_1, R_2 \to bR_2$ and $R_3 \to cR_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$

$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ac & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

Applying $C_3 \to C_3 + C_1$ and taking $(bc + ca + ab)$ common, we get

$$\Delta = abc(bc+ca+ab)\begin{vmatrix} bc & 1 & 1 \\ ac & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

19. b. Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} -4-2\sqrt{2} & -2\sqrt{2} & 0 \\ 4\sqrt{2} & 4\sqrt{2} & 0 \\ 3-2\sqrt{2} & 2-2\sqrt{2} & 1 \end{vmatrix}$$

$$= 1(-(4+2\sqrt{2}))\,4\sqrt{2} + 2\sqrt{2} \times 4\sqrt{2}$$
$$= -16\sqrt{2}$$

20. b. The total number of third-order determinants is 9! Since the number of determinants is even and in which there are 9!/2 pairs of determinants which are obtained by changing two consecutive rows,

so $\sum_{i=1}^{n} D_i = 0$.

21. a.
$$\Delta = \begin{vmatrix} a_1+b_1w & a_1w^2+b_1 & c_1+b_1\bar{w} \\ a_2+b_2w & a_2w^2+b_2 & c_2+b_2\bar{w} \\ a_3+b_3w & a_3w^2+b_3 & c_3+b_3\bar{w} \end{vmatrix}$$

Operating $C_2 \to wC_2$, we have

$$\Delta = \frac{1}{w}\begin{vmatrix} a_1+b_1w & a_1w^3+b_1w & c_1+b_1\bar{w} \\ a_2+b_2w & a_2w^3+b_2w & c_2+b_2\bar{w} \\ a_3+b_3w & a_3w^3+b_3w & c_3+b_3\bar{w} \end{vmatrix}$$

$$= \frac{1}{w}\begin{vmatrix} a_1+b_1w & a_1+b_1w & c_1+b_1\bar{w} \\ a_2+b_2w & a_2+b_2w & c_2+b_2\bar{w} \\ a_3+b_3w & a_3+b_3w & c_3+b_3\bar{w} \end{vmatrix} \quad (\because \omega^3 = 1)$$

$$= 0$$

22. b. Since x, y, z are in A.P., therefore, $x + z - 2y = 0$. Now,

$$\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2(x+z-2y) \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$

[Applying $R_1 \to R_1 + R_3 - 2R_2$]

$$= \begin{vmatrix} 0 & 0 & 0 \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix} \quad [\because x+z-2y=0]$$

$$= 0$$

23. d. Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$(a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix} = 0$$

$\therefore$ $x = a + b + c = 0$

24. b. Applying $C_1 \to C_1 - C_2, C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cot\frac{A}{2} - \cot\frac{B}{2} & \cot\frac{B}{2} - \cot\frac{C}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2} - \tan\frac{A}{2} & \tan\frac{C}{2} - \tan\frac{B}{2} & \tan\frac{A}{2} + \tan\frac{B}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \cot\frac{A}{2} - \cot\frac{B}{2} & \cot\frac{B}{2} - \cot\frac{C}{2} & \cot\frac{C}{2} \\ \dfrac{\cot\frac{A}{2} - \cot\frac{B}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}} & \dfrac{\cot\frac{B}{2} - \cot\frac{C}{2}}{\cot\frac{B}{2}\cot\frac{C}{2}} & \tan\frac{A}{2} + \tan\frac{B}{2} \end{vmatrix}$$

$$= \left(\cot\frac{A}{2} - \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right)$$

$$\times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \cot\frac{C}{2} \\ \tan\frac{A}{2}\tan\frac{B}{2} & \tan\frac{B}{2}\tan\frac{C}{2} & \tan\frac{A}{2}\tan\frac{B}{2} \end{vmatrix}$$

$$= \left(\cot\frac{A}{2} - \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right)\left(\tan\frac{C}{2} - \tan\frac{A}{2}\right)\tan\frac{B}{2}$$

Since $\Delta = 0$, therefore

$$\cot\frac{A}{2} = \cot\frac{B}{2} \text{ or } \cot\frac{B}{2} = \cot\frac{C}{2} \text{ or } \tan\frac{A}{2} = \tan\frac{C}{2}$$

Hence, the triangle is definitely isosceles.

25. d. Since a, b, c, d, e, f are in G. P. and if r is the common ratio of the G.P., then

$b = ar$

$c = ar^2$

$d = ar^3$
$e = ar^4$
$f = ar^5$

Therefore, given determinant is

$$\begin{vmatrix} a^2 & a^2r^6 & x \\ a^2r^2 & a^2r^8 & y \\ a^2r^4 & a^2r^{10} & z \end{vmatrix}$$

$$= a^2a^2r^6 = \begin{vmatrix} 1 & 1 & x \\ r^2 & r^2 & y \\ r^4 & r^4 & z \end{vmatrix}$$

$= a^4r^6 (0) = 0$ [$\because C_1, C_2$ are identical]

26. c. Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & 1+b^2x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & 1+x^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \quad [\because a^2+b^2+c^2=-2]$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix}$$

[Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (1) [(1 - x)^2 - 0]$

$= (1 - x)^2$

which is a polynomial of degree 2.

27. a. $$\begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 + {}^{m+1}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 + {}^{m+1}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 - C_2]$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^mC_0 + {}^mC_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 + {}^mC_2 & {}^{m+1}C_1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ {}^mC_1 & {}^mC_0 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 & {}^{m+1}C_1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1]$$

$= {}^mC_0\,{}^{m+1}C_1 - {}^{m+1}C_0\,{}^mC_1$
$= m + 1 - m$
$= 1$

28. c. Since each element of C_1 is the sum of two elements, putting the determinant as sum of two determinants, we get

$$\Delta = \begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= xyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= -(xyz+1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$= -(xyz + 1)(x - y)(y - z)(z - x)(x + y + z)$

Since $\Delta = 0$, x, y, z all are distinct, we have $xyz + 1 = 0$ or $xyz = -1$.

29. c. $$\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 3+3z \end{vmatrix}$$

$$= xyz\begin{vmatrix} 1+\frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

$$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

$$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 0 & 0 \\ 1+\frac{1}{y} & 1 & -1 \\ 1+\frac{1}{z} & 0 & 2 \end{vmatrix}$$

$$= 2xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Hence, the given equation gives $x^{-1} + y^{-1} + z^{-1} = -3$.

30. b. Let a be the first term and d be the common difference of corresponding A.P. Then

$$\Delta = xyz\begin{vmatrix} 1/x & 1/y & 1/z \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= xyz\begin{vmatrix} a+(p-1)d & a+(2q-1)d & a+(3r-1)d \\ p & 2q & 3r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - aR_3$, $R_2 \to R_2 - R_3$ and then taking d common from R_1, we get

$$\Delta = xyzd\begin{vmatrix} (p-1) & (2q-1) & (3r-1) \\ (p-1) & (2q-1) & (3r-1) \\ 1 & 1 & 1 \end{vmatrix} = 0$$

31. a. As $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are even natural numbers, each of c_1, c_2, c_3 is divisible by 2. Let $c_i = 2k_i$ for $i = 1, 2, 3$. Thus,

$$\Delta = 2\begin{vmatrix} k_1 & a_1 & b_1 \\ k_2 & a_2 & b_2 \\ k_3 & a_3 & b_3 \end{vmatrix} = 2m$$

where m is some natural number. Thus, Δ is divisible by 2. That Δ may not be divisible by 4 can be seen by taking the three numbers as 112, 122 and 134. Note that

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 2$$

which is divisible by 2 but not by 4.

32. a. We have,

$$\begin{vmatrix} x & 1 & 1 & \cdots \\ 1 & x & 1 & \cdots \\ 1 & 1 & x & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} x & 1 & 1 & \cdots \\ (1-x) & (x-1) & 0 & \cdots \\ (1-x) & 0 & (x-1) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $\cdots$, $R_n \to R_n - R_1$]

$= x(x-1)^{n-1} + |(x-1)^{n-1} + (x-1)^{n-1} + \cdots + (x-1)^{n-1}\ (n-1) \text{ times}|$

[Expanding along R_1]

$= x(x-1)^{n-1} + (n-1)(x-1)^{n-1}$

$= (x-1)^{n-1}(x + n - 1)$

33. a. Let first term of G.P. is A and common ratio is R. Then,

$a = AR^{p-1} \Rightarrow \log a = \log A + (p-1)\log R$, etc.

$$\Rightarrow \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} \quad [C_1 \to C_1 - (\log A)C_3]$$

$$= \log R\begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

$$= \log R\begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} \quad (C_1 \to C_1 + C_3)$$

$= 0$

34. b. $R_3 \to R_3 - 2R_2$, hence two identical rows $\Rightarrow f(x) =$ constant.

35. b. In each determinant applying $R_1 \to R_1 + R_2 + R_3$ and then taking out $(x + 9)$ common, we get

$x + 9 = 0 \Rightarrow x = -9$

36. c. Taking x^5 common from last row, we get

$$x^5\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^n & x^{a+1} & x^{2n} \end{vmatrix} = 0, \forall x \in R$$

$\Rightarrow \ a + 1 = n + 2 \Rightarrow a = n + 1$

(as it will make first and third row is identical)

37. a. Applying $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & x+z-2y \\ 4 & 5 & 6 & y \\ 5 & 6 & 7 & z \\ x & y & z & 0 \end{vmatrix}$$

$$= -(x+z-2y)\begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ x & y & z \end{vmatrix} \quad \text{[Expanding along } R_1]$$

$$= -(x+z-2y)\begin{vmatrix} 0 & -1 & 6 \\ 0 & -1 & 7 \\ x-2y+z & y-z & z \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2C_2$ and $C_2 \to C_2 - C_3$]

$$= -(x+z-2y)^2\begin{vmatrix} -1 & 6 \\ -1 & 7 \end{vmatrix}$$

$= (x - 2y + z)^2$

Hence $\Delta = 0 \Rightarrow x, y, z$ are in A.P.

38. c. Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ reduce the determinant to

$$\begin{vmatrix} x^2-2x+1 & x-1 & 0 \\ 2x-2 & x-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$= (x-1)^3 - 2(x-1)^2 = (x-1)^2(x-1-2) = (x-1)^2(x-3)$, which is clearly negative for $x < 1$.

39. d. Applying $R_1 \to R_1 - (R_2 + R_3)$, we get

$$D = \begin{vmatrix} 0 & -2y & -2x \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -y & -x \\ x & y+z & x \\ y & y & z+x \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -y & -x \\ x & z & 0 \\ y & 0 & z \end{vmatrix} \qquad (R_2 \to R_2 + R_1 \text{ and } R_3 \to R_3 + R_1)$$

$$= 4xyz$$

40. b. Operation $R_1 \to R_1 - R_2$, gives

$$\Delta = \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 0 & -3(x+2) & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix}$$

$[R_2 \to R_2 - 2R_1, R_3 \to R_3 + 3R_1]$

$$= (x-2)\{-(3x+6)(x-1) - (x-1)(2x+9)\}$$

$$= -(x-2)(x-1)(5x+15)$$

Therefore, $\Delta = 0$ gives $x = 2, 1, -3$.

41. b. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+2a & a & a \\ x+2a & x & a \\ x+2a & a & x \end{vmatrix} = (x+2a)\begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = (x+2a)\begin{vmatrix} 0 & a-x & 0 \\ 0 & x-a & a-x \\ 1 & a & x \end{vmatrix} = (x-a)^2(x+2a)$$

42. a. $\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0 \; [R_1 \to R_1 - R_2, R_2 \to R_2 - R_3, R_3 \to R_3 - R_4]$

$$\Rightarrow \begin{vmatrix} x-a & m-x & 0 & 0 \\ 0 & x-b & n-x & 0 \\ 0 & 0 & x-c & a \\ a & b & c & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-a & m-x & 0 \\ 0 & x-b & n-x \\ 0 & 0 & x-c \end{vmatrix} = 0$$

$$\Rightarrow (x-a)\begin{vmatrix} (x-b) & n-x \\ 0 & (x-c) \end{vmatrix} = 0$$

$\Rightarrow (x-a)(x-b)(x-c) = 0 \Rightarrow$ roots are independent of m, n

43. d. Since for $x = 0$, the determinant reduces to the determinant of a skew-symmetric matrix of odd order which is always zero, hence $x = 0$ is the solution of the given equation.

44. b. We have,

$$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_1 \to C_1 + (C_2 + C_3)$ on L.H.S.]

$$\Rightarrow 2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -a & -b \\ a+b+c & -c & -a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$ on L.H.S.]

$$\Rightarrow \begin{vmatrix} a & -b & -c \\ c & -a & -b \\ b & -c & -a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$ on L.H.S.]

$$\Rightarrow 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$\therefore \quad k = 2$

45. d. $D' = \begin{vmatrix} a_1 + pb_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 + pb_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 + pb_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 + qc_1 & c_1 + ra_1 \\ pb_2 & b_2 + qc_2 & c_2 + ra_2 \\ pb_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix}$$

In the first determinant, apply $C_3 \to C_3 - rC_1$ and then $C_2 \to C_2 - qC_3$.

In second determinant take p common from C_1 and then apply $C_2 \to C_2 - C_1$. Then take q common from C_2 and apply $C_3 \to C_3 - C_2$. Finally taking r common from C_3, we have ultimately $D' = (1 + pqr)D$.

46. b. $\Delta = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} \qquad (R_3 \to R_3 - R_2, R_4 \to R_4 - R_3)$

$$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 15 & 21 & 27 & 33 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 5 & 7 & 9 & 11 \\ 5 & 7 & 9 & 11 \end{vmatrix} = 0 \qquad (R_4 \to R_4 - R_3)$$

47. c. Here $a > 0$ and $4b^2 - 4ac < 0$, i.e., $ac - b^2 > 0$.

$\therefore \quad ax^2 + 2bx + c > 0, \forall x \in R$

Now,

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ 0 & 0 & -(ax^2+2bx+c) \end{vmatrix}$$

[Operating $R_3 \to R_3 - xR_1 - R_2$]

$= -(ax^2 + 2bx + c)(ac - b^2)$

$= -(+\text{ve})(+\text{ve}) = -\text{ve}$

48. a. We have,

$a_{n+1}^2 = a_n a_{n+2}$

$\Rightarrow \quad 2 \log a_{n+1} = \log a_n + \log a_{n+2}$

Similarly,

$2 \log a_{n+4} = \log a_{n+3} + \log a_{n+5}$

$2 \log a_{n+7} = \log a_{n+6} + \log a_{n+8}$

Substituting these values in second column of determinant, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} \log a_n & \log a_n + \log a_{n+2} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+3} + \log a_{n+5} & \log a_{n+5} \\ \log a_{n+6} & \log_{n+6} + \log a_{n+8} & \log a_{n+8} \end{vmatrix}$$

$= \frac{1}{2}(0) = 0 \quad$ [Using $C_2 \to C_2 - C_1 - C_3$]

49. b.

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\Delta^2 = \begin{vmatrix} x_1 & y_1 & z_2 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1^2+y_1^2+z_1^2 & x_1x_2+y_1y_2+z_1z_2 & x_1x_3+y_1y_3+z_1z_3 \\ x_1x_2+y_2y_1+z_2z_1 & x_2^2+y_2^2+z_2^2 & x_2x_3+y_2y_3+z_2z_3 \\ x_3x_1+y_3y_1+z_3z_1 & x_2x_3+y_2y_3+z_2z_3 & x_3^2+y_3^2+z_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \Delta = \pm 1$$

50. c. Using $C_1 \to C_1 + C_2 + C_3$,

$$\Delta = \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}$$

$= (\sin x + 2\cos x)(\sin x - \cos x)^2+$

Thus, $\Delta = 0 \Rightarrow \tan x = -2$ or $\tan x = 1$

As $-\pi/4 \le x \le \pi/4$, we get $-1 \le \tan x \le 1$

$\therefore \quad \tan x = 1 \Rightarrow x = \pi/4$

51. b. We divide L.H.S. by λ^4 and C_1 by λ^2, C_2 by λ and C_3 by λ on the R.H.S. to obtain

$$p + q\left(\frac{1}{\lambda}\right) + r\left(\frac{1}{\lambda}\right)^2 + s\left(\frac{1}{\lambda}\right)^3 + t\left(\frac{1}{\lambda}\right)^4$$

$$= \begin{vmatrix} 1+3/\lambda & 1-1/\lambda & 1+3/\lambda \\ 1+1/\lambda^2 & 2/\lambda - 1 & 1-3/\lambda \\ 1-3/\lambda^2 & 1+4/\lambda & 3 \end{vmatrix}$$

Taking limit as $\lambda \to \infty$, we get

$$p = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -4$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

52. d. Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(c-z)\begin{vmatrix} -x & y \\ -x & 0 \end{vmatrix} + z\begin{vmatrix} a & b-y \\ -x & y \end{vmatrix} = 0$$

$\Rightarrow \quad (c - z)(xy) + z(ay + bx - xy) = 0$

$\Rightarrow \quad cxy - xyz + ayz + bxz - xyz = 0$

$\Rightarrow \quad ayz + bzx + cxy = 2xyz$

$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

53. d. Since $A + B + C = \pi$ and $e^{i\pi} = \cos \pi + i \sin \pi = -1$,

$e^{i(B+C)} = e^{i(\pi - A)} = -e^{iA}$ and $e^{-i(B+C)} = -e^{iA}$

By taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2 and R_3, respectively, we have

$$\Delta = -\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$= -\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

By taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2 and C_3, respectively, we have

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

54. c. $\Delta = (1 + x + x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1 + x + x^2)(x - 1)^2$

Therefore, $\Delta = 0$ has roots $1, 1, \omega, \omega, \omega^2, \omega^2$.

55. a. Using the sum property, we get

$$\sum_{r=0}^{m}\Delta_r = \begin{vmatrix} \sum_{r=0}^{m}(2r-1) & \sum_{r=0}^{m}{}^mC_r & \sum_{r=0}^{m}1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix},$$

But $\sum_{r=0}^{m}(2r-1)=\frac{1}{2}(m+1)(2m-1-1)=m^2-1,$

$\sum_{r=0}^{m}{}^mC_r = 2^m$ and $\sum_{r=0}^{m}1 = m+1$. Therefore,

$$\sum_{r=0}^{m}\Delta_r = \begin{vmatrix} m^2-1 & 2^m & m+1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = 0$$

56. d. $$\sum_{k=1}^{n}D_k = 56 \Rightarrow \begin{vmatrix} \sum_{k=1}^{n}1 & n & n \\ \sum_{k=1}^{n}2k & n^2+n+1 & n^2+n \\ \sum_{k=1}^{n}(2k-1) & n^2 & n^2+n+1 \end{vmatrix} = 56$$

$$\Rightarrow \begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix} = 56$$

Applying $C_3 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$, we get

$$\begin{vmatrix} n & 0 & 0 \\ n(n+1) & 1 & 0 \\ n^2 & 0 & n+1 \end{vmatrix} = 56 \Rightarrow n(n+1) = 56 \Rightarrow n = 7$$

57. a. Applying $C_1 \to C_1 + 2C_2 + C_3$, we get

$$S = \sum_{r=2}^{n}(-2)^r \begin{vmatrix} {}^nC_r & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \sum_{r=2}^{n}(-2)^r\,{}^nC_r$$

$$= \sum_{r=0}^{n}(-2)^r\,{}^nC_r - ({}^nC_0 - 2\,{}^nC_1)$$

$$= (1-2)^n - (1-2n) = 2n - 1 + (-1)^n$$

58. b. $$\Delta = \begin{vmatrix} {}^nC_{r-1} & {}^nC_r & (r+1)\,{}^{n+2}C_{r+1} \\ {}^nC_r & {}^nC_{r+1} & (r+2)\,{}^{n+2}C_{r+2} \\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)\,{}^{n+2}C_{r+3} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ and using ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$ in C_3, we get

$$\Delta = \begin{vmatrix} {}^{n+1}C_r & {}^nC_r & (n+2)\,{}^{n+1}C_r \\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & (n+2)\,{}^{n+1}C_{r+1} \\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & (n+2)\,{}^{n+1}C_{r+2} \end{vmatrix}$$

$$= (n+2)\begin{vmatrix} {}^{n+1}C_r & {}^nC_r & {}^{n+1}C_r \\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & {}^{n+1}C_{r+1} \\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & {}^{n+1}C_{r+2} \end{vmatrix}$$

$= 0$ (as C_1 and C_3 are identical)

59. a. The given determenant Δ_1 is obtained by corresponding co-factors of determinant Δ_2; hence $\Delta_1 = \Delta_2^2$. Now $\Delta_1\Delta_2 = \Delta_2^2\Delta_2 = \Delta_2^3$.

60. c. We have,

$$\Delta^2 = \Delta\,\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_1l_3+m_1m_3+n_1n_3 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} 1 \Rightarrow \Delta = \pm 1 \Rightarrow |\Delta| = 1$$

61. d. The given determinant, on simplification, gives

$$\Delta_1 = \begin{vmatrix} a_1^2 & -2a_1 & 1 & 0 \\ a_2^2 & -2a_2 & 1 & 0 \\ a_3^2 & -2a_3 & 1 & 0 \\ a_4^2 & --2a_4 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & b_1^2 & 0 \\ 1 & b_2 & b_2^2 & 0 \\ 1 & b_3 & b_3^2 & 0 \\ 1 & b_4 & b_4^2 & 0 \end{vmatrix}$$

$$= 0 \times 0 = 0$$

62. a. Determinant formed by the cofactors of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

$$\begin{vmatrix} bc-a^2 & ac-b^2 & ab-c^2 \\ ac-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

63. d. $$\begin{vmatrix} 1+x_1 & 1+x_1x & 1+x_1x^2 \\ 1+x_2 & 1+x_2x & 1+x_2x^2 \\ 1+x_3 & 1+x_3x & 1+x_3x^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & 0 \\ 1 & x_3 & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 & 0 \\ 1 & x & 0 \\ 1 & x^2 & 0 \end{vmatrix}$$

$$= 0$$

64. c. We observe that the elements in the pre-factor are the cofactors of the corresponding elements of the post-factor. Hence,

$$\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}^3 = [\lambda(\lambda^2+a^2+b^2+c^2)]^3 = (1+a^2+b^2+c^2)^3$$

$$\Rightarrow \quad \lambda = 1$$

Alternative solution:

Writing $a = 0, b = 0, c = 0$ on both sides, we get

$\lambda^6\lambda^3 = 1 \Rightarrow \lambda = 1$

65. c. $$f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sec^2 x & 1 & 0 \end{vmatrix}$$

$$\Rightarrow f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

Now, $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} f'(x)$ [as $f(0) = 0$]

$= f'(0) = 0$

66. a. Operating $C_1 \to C_1 + C_2 + C_3$ on the L.H.S. we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & c'-a' & a'-b' \\ 0 & c''-a'' & a''-b'' \end{vmatrix} = m \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

$\Rightarrow m = 0$

67. d. Let the given determinant be equal to $\Delta(x)$. Then,

$5A + 4B + 3C + 2D + E = \Delta(1) + \Delta'(1)$

Now, $\Delta(1) = 0$ as R_2 and R_3 are identical.

$$\Delta'(x) = \begin{vmatrix} 1 & 0 & 1 \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} + \begin{vmatrix} x & 2 & x \\ 2x & 1 & 0 \\ x & x & 6 \end{vmatrix} + \begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\Delta'(1) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 6 \\ 1 & 1 & 0 \end{vmatrix} = -17 + (12 + 1 - 1 - 6) = -11$$

68. b. $\Delta_1 = x(x^2 - ab) - b(ax - ab) + b(a^2 - ax)$

$= x^3 - 3abx + ab^2 + a^2 b$

$\frac{d}{dx}(\Delta_1) = 3x^2 - 3ab = 3(x^2 - ab) = 3\Delta_2$

69. d. We have $y = \sin mx$, therefore $y_1 = m\cos mx$, $y_2 = -m^2 \sin mx$, etc.

$$\therefore \Delta = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$$

$$= \begin{vmatrix} \sin mx & m\cos mx & -m^2 \sin mx \\ -m^3 \cos mx & m^4 \sin mx & m^5 \cos mx \\ -m^6 \sin mx & -m^7 \cos mx & m^8 \sin mx \end{vmatrix}$$

$$= m^{12} \begin{vmatrix} \sin mx & \cos mx & -\sin mx \\ -\cos mx & \sin mx & \cos mx \\ -\sin mx & -\cos mx & \sin mx \end{vmatrix} = 0$$

70. b. We have,

$$\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = abc - (a + b + c) + 2$$

$\therefore \Delta > 0 \Rightarrow abc + 2 > a + b + c$

$\Rightarrow abc + 2 > 3(abc)^{1/3}$ $\left[\because \text{A.M.} > \text{G.M.} \Rightarrow \frac{a+b+c}{3} > (abc)^{1/3}\right]$

$\Rightarrow x^3 + 2 > 3x$, where $x = (abc)^{1/3}$

$\Rightarrow x^3 - 3x + 2 > 0 \Rightarrow (x - 1)^2 (x + 2) > 0$

$\Rightarrow x + 2 > 0 \Rightarrow x > -2 \Rightarrow (abc)^{1/3} > -2 \Rightarrow abc > -8$

71. b. $B_2 = a_1c_3 - a_3c_1$, $C_2 = -(a_1b_3 - a_3b_1)$

$B_3 = -(a_1c_2 - a_2c_1)$, $C_3 = a_1b_2 - a_2b_1$

$$\therefore \begin{vmatrix} B_2 & C_2 \\ B_3 & C_2 \end{vmatrix} = \begin{vmatrix} a_1c_3 - a_3c_1 & -a_1b_3 + a_3b_1 \\ -a_1c_2 + a_2c_1 & a_1b_2 - a_2b_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1c_3 & -a_1b_3 \\ -a_1c_2 & a_1b_2 \end{vmatrix} + \begin{vmatrix} a_1c_3 & a_3b_1 \\ -a_1c_2 & -a_2b_1 \end{vmatrix} + \begin{vmatrix} -a_3c_1 & -a_1b_3 \\ a_2c_1 & a_1b_2 \end{vmatrix} + \begin{vmatrix} -a_3c_1 & a_3b_1 \\ a_2c_1 & -a_2b_1 \end{vmatrix}$$

$$= a_1^2 \begin{vmatrix} c_3 & -b_3 \\ -c_2 & b_2 \end{vmatrix} + a_1b_1 \begin{vmatrix} c_3 & a_3 \\ -c_2 & -a_2 \end{vmatrix} + a_1c_1 \begin{vmatrix} -a_3 & -b_3 \\ a_2 & b_2 \end{vmatrix} + b_1c_1 \begin{vmatrix} -a_3 & a_3 \\ a_2 & -a_2 \end{vmatrix}$$

$= a_1\{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$

$$= a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\Delta$$

72. a. Applying $C_1 \to C_1 - 2\sin x C_3$ and $C_2 \to C_2 + 2\cos x\, C_3$, we get

$$f(x) = \begin{vmatrix} 2 & 0 & -\sin x \\ 0 & 2 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

$= 2\cos^2 x + 2\sin^2 x = 2$

$\therefore f'(x) = 0$

$\therefore \int_0^{\pi/2} [f(x) + f'(x)] = dx = \int_0^{\pi/2} 2dx = \pi$

73. b. $$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$$

Multiplying R_1 by x, R_2 by y and R_3 by z, we get

$$\frac{1}{xyz} \begin{vmatrix} x^4+x & x^3y & x^3z \\ xy^3 & y^4+y & y^3z \\ xz^3 & yz^3 & z^4+z \end{vmatrix} = 11$$

Taking x, y, z common from C_1, C_2, C_3, respectively, we get

$$\begin{vmatrix} x^3+1 & x^3 & x^3 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$$

Using $R_1 \to R_1 + R_2 + R_3$, we have

$$(x^3 + y^3 + z^3 + 1) \begin{vmatrix} 1 & 1 & 1 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$$

Using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$(x^3+y^3+z^3+1)\begin{vmatrix}1 & 0 & 0\\ y^3 & 1 & 0\\ z^3 & 0 & 1\end{vmatrix}=11$$

Hence,

$x^3+y^3+z^3=10$

Therefore, the ordered triplets are (2, 1, 1), (1, 2, 1), (1, 1, 2).

74. b. Since the system has non-trivial solution,

$$\therefore \quad \begin{vmatrix}a & 1 & 1\\ 1 & b & 1\\ 1 & 1 & c\end{vmatrix}=0$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$\Delta=\begin{vmatrix}a-1 & 1-b & 0\\ 0 & b-1 & 1-c\\ 1 & 1 & c\end{vmatrix}=0$$

$\Rightarrow \quad c(1-a)(1-b)+(1-b)(1-c)-(1-c)(a-1)=0$

Dividing throughout by $(1-a)(1-b)(1-c)$, we get

$$\frac{c}{1-c}+\frac{1}{1-c}+\frac{1}{1-b}=0$$

$$\Rightarrow \quad -1+\frac{1}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=0$$

$$\Rightarrow \quad \frac{1}{1-c}+\frac{1}{1-a}+\frac{1}{1-b}=1$$

75. a. The given system of linear equations has a unique solution if

$$\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 2 & 5 & \lambda\end{vmatrix}\neq 0$$

i.e., if $\lambda-8\neq 0$ or $\lambda\neq 8$.

76. b. Let, $\Delta=\begin{vmatrix}\cos(\alpha-\beta) & \cos(\beta-\gamma) & \cos(\gamma-\alpha)\\ \cos(\alpha+\beta) & \cos(\beta+\gamma) & \cos(\gamma+\alpha)\\ \sin(\alpha+\beta) & \sin(\beta+\gamma) & \sin(\gamma+\alpha)\end{vmatrix}$

It is clear that either $\alpha=\beta$ or $\beta=\gamma$ or $\gamma=\alpha$ is sufficient to make $\Delta=0$. It is not necessary that triangle is equilateral.

Also, isosceles triangle can be obtuse one.

77. c. $a=x/(y-z)\Rightarrow x-ay+az=0$ (1)

$b=y/(z-x)\Rightarrow bx+y-bz=0$ (2)

$c=z/(x-y)\Rightarrow -cx+cy+z=0$ (3)

Since x, y, z are not all zero, the above system has a non-trivial solution. So,

$$\Delta=\begin{vmatrix}1 & -a & a\\ b & 1 & -b\\ -c & c & 1\end{vmatrix}=0$$

$\therefore \quad 1+ab+bc+ca=0$

78. a. $\Delta=\begin{vmatrix}p+a & b & c\\ a & q+b & c\\ a & b & r+c\end{vmatrix}=0$

Applying $R_1\to R_2-R_1$ and $R_3\to R_3-R_1$, we get

$$\begin{vmatrix}p+a & b & c\\ -p & q & 0\\ -p & 0 & r\end{vmatrix}=0$$

$\Rightarrow \quad pqc+[q(p+a)+bp]r=0$

Dividing by pqr, we obtain

$$\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=-1$$

79. b. For no solution or infinitely many solutions

$$\begin{vmatrix}\alpha & -1 & -1\\ 1 & -\alpha & -1\\ 1 & -1 & -\alpha\end{vmatrix}=0$$

$\Rightarrow \quad \alpha(\alpha^2-1)-1(\alpha-1)+1(1-\alpha)=0$

$\Rightarrow \quad \alpha(\alpha^2-1)-2\alpha+2=0$

$\Rightarrow \quad \alpha(\alpha-1)(\alpha+1)-2(\alpha-1)=0$

$\Rightarrow \quad (\alpha-1)(\alpha^2+\alpha-2)=0$

$\Rightarrow \quad (\alpha-1)(\alpha+2)(\alpha-1)=0$

$\Rightarrow \quad (\alpha-1)^2(\alpha+2)=0$

$\Rightarrow \quad \alpha=1, 1, -2$

But for $\alpha=1$, there are infinite solutions. When $\alpha=-2$, we have

$-2x-y-z=-3$

$x+2y-z=-3$

$x-y+2z=-3$

Adding, we get $0=-9$, which is not true. Hence there is no solution.

80. a. $D=\cos\theta-\cos^2.+6>0$. Since $D>0$ only trivial solution is possible.

81. a The given system of equations will have a non-trivial solution if

$$\begin{vmatrix}\alpha+a & \alpha & \alpha\\ \alpha & \alpha+b & \alpha\\ \alpha & \alpha & \alpha+c\end{vmatrix}=0$$

Operating $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, we get

$$\begin{vmatrix}\alpha+a & \alpha & \alpha\\ -a & b & 0\\ -a & 0 & c\end{vmatrix}=0$$

$\Rightarrow \quad aab+c(\alpha b+ab+a\alpha)=0\Rightarrow \alpha(bc+ca+ab)+abc=0$

$$\Rightarrow \quad \frac{1}{\alpha}=-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad (\because a, b, c\neq 0)$$

82. c. The given system is consistent.

$$\therefore \quad \Delta=\begin{vmatrix}1 & 1 & -1\\ 2 & -1 & -c\\ -b & 3b & -c\end{vmatrix}=0$$

$\Rightarrow \quad c+bc-6b+b+2c+3bc=0$

$\Rightarrow \quad 3c+4bc-5b=0$

$$\Rightarrow \quad c=\frac{5b}{4b+3}$$

Now,

$c<1$

$\Rightarrow \quad \dfrac{5b}{4b+3} < 1$

$\Rightarrow \quad \dfrac{5b}{4b+3} - 1 < 0$

$\Rightarrow \quad \dfrac{b-3}{4b+3} < 0$

$\Rightarrow \quad b \in \left(-\dfrac{3}{4}, 3\right)$

83. c. As a, b, c are in G.P. with common ratio r_1 and α, β, γ are in G.P. having common ratio r_2, $a \neq 0$, $\alpha \neq 0$, $b = ar_1$, $c = ar_1^2$, $\beta = ar_2$, $\gamma = ar_2^2$.

Also the system of equations has only zero (trivial) solution.

$$\Delta = \begin{vmatrix} a & \alpha & 1 \\ b & \beta & 1 \\ c & \gamma & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow \quad a\alpha \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & 1 \\ r_1^2 & r_2^2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow \quad a\alpha(r_1 - 1)(r_2 - 1)(r_1 - r_2) \neq 0$

$\Rightarrow \quad r_1 \neq 1, r_2 \neq 1$ and $r_1 \neq r_2$

84. b. For non-trivial solution

$$\begin{vmatrix} a-1 & -1 & -1 \\ 1 & -(b-1) & 1 \\ 1 & 1 & -(c-1) \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} a-1 & -1 & 0 \\ 1 & -(b-1) & b \\ 1 & 1 & -c \end{vmatrix} = 0$$

$\Rightarrow \quad (a-1)(bc - c - b) + 1(-c - b) = 0$

$\Rightarrow \quad abc - ac - ab - bc + b + c - c - b = 0$

$\Rightarrow \quad ab + bc + ac = abc$

Multiple Correct Answers Type

1. a, b, c.

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix} \quad [R_3 \to R_3 - R_2 \text{ and } R_2 \to R_2 - R_1]$$

$= 0$

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

$[R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3]$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} \quad [\text{taking } abc \text{ common from } C_3]$$

$= 0$

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix} = \begin{vmatrix} a+b & 2a+b & 3a+b \\ a & a & a \\ 2a & 2a & 2a \end{vmatrix}$$

$[R_3 \to R_3 - R_2, R_2 \to R_2 - R_1]$

$= 0$

$$\begin{vmatrix} 2 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix} \quad [C_2 \to C_2 - 7C_3]$$

$$= \begin{vmatrix} 1 & 1 & 6 \\ 0 & 7 & 4 \\ 0 & 3 & 2 \end{vmatrix} \quad [C_1 \to C_1 - C_2]$$

$= 2$

2. a, c.

$$g(x) = \begin{vmatrix} a^{-x} & e^{\log_e a^x} & x^2 \\ a^{-3x} & e^{\log_e a^{3x}} & x^4 \\ a^{-5x} & e^{\log_e a^{5x}} & 1 \end{vmatrix} = \begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} \left(e^{\log a^x} = a^x\right)$$

$$\Rightarrow \quad g(-x) = \begin{vmatrix} a^x & a^{-x} & x^2 \\ a^{3x} & a^{-3x} & x^4 \\ a^{5x} & a^{-5x} & 1 \end{vmatrix} = -\begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix}$$

[interchanging 1st and 2nd columns]

$= -g(x)$

$\Rightarrow \quad g(x) + g(-x) = 0$

$\Rightarrow \quad g(x)$ is an odd function

Hence, the graph is symmetrical about origin. Also, $g_4(x)$ is an odd function [where $g_4(x)$ is fourth derivative of $g(x)$]. Hence,

$g_4(x) = -g_4(-x)$

$\Rightarrow \quad g_4(0) = -g_4(0)$

$\Rightarrow \quad g_4(0) = 0$

3. c, d.

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b-x & a & b \\ a+b-x & -x & a \\ a+b-x & b & -x \end{vmatrix} = (a+b-x)\begin{vmatrix} 1 & a & b \\ 1 & -x & a \\ 1 & b & -x \end{vmatrix}$$

$$= (a+b-x)\begin{vmatrix} 1 & a & b \\ 0 & -x-a & a-b \\ 0 & b-a & -x-b \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a+b-x)\,[(x+a)(x+b) + (a-b)^2]$ [expanding along C_1]

$= (a+b-x)[x^2 + (a+b)x + a^2 + b^2 - ab]$

4. b, d.

Applying $C_1 \to C_1 - (\cot\phi)\,C_2$, we get

$$\Delta = \begin{vmatrix} 0 & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta/\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$= -\dfrac{\sin\theta}{\sin\phi}$ [$-\sin\phi \sin^2\theta - \cos^2\theta \sin\phi$] [expanding along C_1]

$= \sin\theta$

which is independent of ϕ. Also,

$$\frac{d\Delta}{d\theta} = \cos\theta \Rightarrow \left.\frac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = \cos(\pi/2) = 0$$

5. a, c.

$$\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1+\sin\beta-\cos\beta \end{vmatrix}$$

[Applying $R_3 \to R_3 - R_1(\cos\beta) + R_2(\sin\beta)$]

$= (1 + \sin\beta - \cos\beta)(\cos^2\alpha + \sin^2\alpha) = 1 + \sin\beta - \cos\beta$ which is independent of α.

6. a, b,

$$\Delta = \frac{1}{a}\begin{vmatrix} a^3+ax & ab & ac \\ a^2b & b^2+x & bc \\ a^2c & bc & c^2+x \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$ and taking $a^2 + b^2 + c^2 + x$ common, we get

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & ab & ac \\ b & b^2+x & bc \\ c & bc & c^2+x \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$, we get

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & 0 & x \end{vmatrix}$$

$$= \frac{1}{a}(a^2+b^2+c^2+x)(ax^2) = x^2(a^2+b^2+c^2+x)$$

Thus Δ is divisible by x and x^2.

7. a, b, c.

Applying $R_3 \to R_3 - xR_2$ and $R_2 \to R_2 - xR_1$, we get

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} = a(a+x)^2$$

Hence,

$f(2x) - f(x) = a[(a + 2x)^2 - (a + x)^2] = a(a + 2x - a - x)(a + 2x + a + x) = ax(2a + 3x)$

8. a, b, c, Operating $C_1 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & ac & bc \\ 1 & ad & bd \\ 1 & ae & be \end{vmatrix} = ab\begin{vmatrix} 1 & c & c \\ 1 & d & d \\ 1 & e & e \end{vmatrix} ab(0) = 0$$

9. a, b.

By partial fractions, we have

$$g(x) = \frac{f(a)}{(x-a)(a-b)(a-c)} + \frac{f(b)}{(b-a)(x-b)(b-c)} + \frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$\Rightarrow g(x) = \frac{1}{(a-b)(b-c)(c-a)} \times \left[\frac{f(a)(c-b)}{(x-a)} + \frac{f(b)(a-c)}{(x-b)} + \frac{f(c)(b-a)}{(x-c)}\right]$$

$$\Rightarrow g(x) = \begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow \int g(x)dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + k$$

and

$$\frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & -f(a)(x-a)^{-2} \\ 1 & b & -f(b)(x-b)^{-2} \\ 1 & c & -f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

10. b, c.

$$\Delta'(x) = \begin{vmatrix} 2x+4 & 2x+4 & 13 \\ 4x+5 & 4x+5 & 26 \\ 16x-6 & 16x-6 & 104 \end{vmatrix} + \begin{vmatrix} x^2+4x-3 & 2 & 13 \\ 2x^2+5x-9 & 4 & 26 \\ 8x^2-6x+1 & 16 & 104 \end{vmatrix}$$

$= 0 + 2 \times 13 \times (0) = 0$

$\Rightarrow \Delta(x) = \text{constant} \Rightarrow a = 0, b = 0, c = 0$

11. a, c.

$f(\theta) = \sin^3\theta + \cos^3\theta - \cos\theta \sin\theta (\sin\theta + \cos\theta)$

$= (\sin\theta + \cos\theta)^3 - 4\sin\theta\cos\theta(\sin\theta + \cos\theta)$

$= (\sin\theta + \cos\theta)[1 - \sin 2\theta]$

Now,

$f(\theta) = 0$

$\Rightarrow \tan\theta = -1$ or $\sin 2\theta = 1$

$\Rightarrow f(\theta) = 0$ has 2 real solutions in $[0, \pi]$

Also, $\dfrac{f(\theta)}{1-\sin 2\theta} = \sin\theta + \cos\theta \in [-\sqrt{2}, \sqrt{2}]$

12. d. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin(B+A)\sin(B-A) & \dfrac{\sin(A-B)}{\sin A \sin B} & 0 \\ \sin(C+A)\sin(C-A) & \dfrac{\sin(A-C)}{\sin A \sin C} & 0 \end{vmatrix}$$

$$\left[\because \cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha \sin\beta}\right]$$

Expanding along C_3, we get

$$\Delta = \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin(B+A)}{\sin C}+\frac{\sin(C+A)}{\sin B}\right]$$
$$= \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin(\pi-C)}{\sin C}+\frac{\sin(\pi-B)}{\sin B}\right]$$
$$= \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin C}{\sin C}+\frac{\sin B}{\sin B}\right]=0$$

13. a, c.

$$\begin{vmatrix} {}^xC_r & {}^{n-1}C_r & {}^nC_r \\ {}^{x+1}C_r & {}^nC_r & {}^{n+1}C_r \\ {}^{x+2}C_r & {}^{n+1}C_r & {}^{n+2}C_r \end{vmatrix}=0 \qquad \text{(i)}$$

$$\Rightarrow \begin{vmatrix} \frac{x!}{r!(x-r)!} & \frac{(n-1)!}{r!(n-r-1)!} & \frac{n!}{r!(n-r)!} \\ \frac{(x+1)!}{r!(x+1-r)!} & \frac{n!}{r!(n-r)!} & \frac{(n+1)!}{r!(n-r+1)!} \\ \frac{(x+2)!}{r!(x+2-r)!} & \frac{(n+1)!}{r!(n+1-r)!} & \frac{(n+2)!}{r!(n-r+2)!} \end{vmatrix}=0$$

Taking $\frac{x!}{r!(x+2-r)!}$ common from C_1, we have quadratic equation in x.

Now in (i), if we put $x = n - 1$, C_1 and C_2 are the same, hence $x = n - 1$ is one root of the equation.

If we put $x = n$, then C_1 and C_3 are same. Hence, $x = n$ is the other root.

14. a, b, Applying $C_3 \to C_3 - xC_2$, $C_2 \to C_2 - xC_1$, we obtain

$$\Delta(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2+2a^2 & 4a^2x & 6a^2x^2+2a^2 \end{vmatrix}$$

Applying $C_3 \to C_3 - xC_2$, we get

$$\Delta(x) = 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2+2a^2 & 2x & x^2+2a^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 3C_3$, we get

$$\Delta(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -4a^2 & 2x & x^2+2a^2 \end{vmatrix} = 16a^6$$

15. b, c.

In the left-hand determinant, each element is the cofactor of the elements of the determinant

$$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = \Delta^* \text{ (say)}$$

Hence,

$$\Delta^{*2} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xz+yx+zy \\ \Sigma xy & \Sigma x^2 & \Sigma xy \\ \Sigma xy & \Sigma xy & \Sigma x^2 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix} \quad [\text{Since } x^2+y^2+z^2 = r^2, xy+yz+zx = u^2]$$

16. a, c.

$$f(n) = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} n & 1 & 1 \\ n! & nn! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix}$$

[Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$]

$= (n + 1)(n + 1)! - nn! = n![(n + 1)^2 - n] = n!(n^2 + n + 1)$

Thus, $f(n)$ is divisible by $n!$ and $n^2 + n + 1$.

17. a, b, c.

We have,

$$\begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix}=0$$

$\Rightarrow$ $(ab)^3 + (bc)^3 + (ca)^3 - 3(ab)(bc)(ca) = 0$

$\Rightarrow$ $(ab + bc\omega^2 + ca\omega)(ab\omega + bc\omega^2 + ca)(ab\omega^2 + bc\omega + ca) = 0$

$\Rightarrow$ $ab + bc\omega^2 + ca\omega = 0, ab\omega + bc\omega^2 + ca = 0, ab\omega^2 + bc\omega + ca = 0$

$\Rightarrow$ $\frac{1}{c\omega^2}+\frac{1}{a}+\frac{1}{b\omega}=0, \frac{1}{c\omega}+\frac{1}{a}+\frac{1}{b\omega^2}=0, \frac{1}{c}+\frac{1}{a\omega}+\frac{1}{b\omega^2}=0$

$\Rightarrow$ $\frac{1}{a}+\frac{1}{b\omega}+\frac{1}{c\omega^2}=0, \frac{1}{a}+\frac{1}{b\omega^2}+\frac{1}{c\omega}=0, \frac{1}{a\omega}+\frac{1}{b\omega^2}+\frac{1}{c}=0$

18. a, b. $\begin{vmatrix} 1 & k & 3 \\ k & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix}=0$

$\Rightarrow$ $8 + 4k + 9k - 12 - 4k^2 - 6 = 0$

$\Rightarrow$ $4k^2 - 13k + 10 = 0$

$\Rightarrow$ $4k^2 - 8k - 5k + 10 = 0$

$\Rightarrow$ $(2k - 5)(k - 2) = 0$

$\Rightarrow$ $k = 5/2, 2$

19. a, b.

Applying $R_1 \to R_1 + \sin\phi\,(R_2) + \cos\phi\,(R_3)$,

$$f(x) = \Delta = \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\phi & \sin\theta & \cos\phi \end{vmatrix}$$

$= (\cos 2\phi + 1)(\sin^2\theta + \cos^2\theta)$

$= (1 + \cos 2\phi)$

Hence, Δ is independent of θ.

Reasoning Type

1. a. $$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}=0 \qquad (1)$$

Then $A = B$ or $B = C$ or $C = A$, for which any two rows are same. For (1) to hold it is not necessary that all the three rows are same or $A = B = C$.

2. b. The system of equations $kx + y + z = 1, x + ky + z = k, x + y + kz = k^2$ is inconsistent if $\Delta = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} = 0$ and one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ k^2 & 1 & k \end{vmatrix}, \Delta_2 = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & k^2 & k \end{vmatrix} \Delta_3 = \begin{vmatrix} k & 1 & 1 \\ 1 & k & k \\ 1 & 1 & k^2 \end{vmatrix}$$

We have,

$\Delta = (k + 2)(k - 1)^2, \Delta_1 = -(k + 1)(k - 1)^2,$

$\Delta_2 = -k(k - 1)^2, \Delta_3 = (k + 1)^2(k - 1)^2$

The determinant given in statement 2 is $\Delta_1 = 0$, for which $k = 1$ or $k = -1$.

$k = 1$ makes all the determinants zero. But for $k = -1$, all the determinants are not zero.

Hence, both statements are true but statement 2 is not correct explanation of statement 1.

3. a. For $x = 0$, the determinant reduces to the determinant of a skew-symmetric matrix of odd order which is always zero. Hence, $x = 0$ is the solution of the given equation.

4. a. As the given system of equations has non-trivial solutions, hence

$$\begin{vmatrix} \lambda & b-a & c-a \\ a-b & \lambda & c-b \\ a-c & b-c & \lambda \end{vmatrix} = 0$$

When $\lambda = 0$, then the determinant becomes skew-symmetric of odd order, which is equal to zero. Thus, $\lambda = 0$.

5. a. $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix}$ where $\begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix}$ is skew symmetric.

$\therefore \quad \Delta = 0$

6. a. We are given that

$1 + bc + qr = 0$ (i)

$1 + ca + pr = 0$ (ii)

$1 + ab + pq = 0$ (iii)

The determinant in the question involves a column consisting the elements ap, bq and cr. So multiplying (i), (ii) and (iii) by ap, bq and cr, respectively, we get

$ap + abcp + apqr = 0$ (iv)

$bq + abcq + bpqr = 0$ (v)

$cq + abcr + cpqr = 0$ (vi)

Since abc and pqr occur in all the three equations, putting $abc = x$, $pqr = y$, we get the system

$ap + px + ay = 0$

$bq + qx + by = 0$ (vii)

$cr + rx + cy = 0$

System (vii) must have a common solution (i.e., system is consistent). So,

$$\begin{vmatrix} ap & p & a \\ bq & q & b \\ cr & r & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix} = 0$$

7. b. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3$. Now,

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & 3 \end{vmatrix} = \mu - 10$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & 3 \end{vmatrix} = 20 - 2\mu$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = \mu - 10$$

Clearly, for $\mu = 10$, all of $\Delta_1, \Delta_2, \Delta_3$ are zero.

8. b. $\Delta = \Delta_1 \Delta_2$ where $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Hence, both the statements are true but statement 2 is not correct explanation of statement 1.

Linked Comprehension Type

For Problems 1–3

1. c, 2. d, 3. b.

Sol. In given determinant applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear. Let $f(x) = Px + Q$. Then

$f(-a) = -aP + Q, f(-b) = -bP + Q$

Then,

$$f(0) = 0 \times P + Q \Rightarrow Q = \frac{bf(-a) - af(-b)}{(b-a)} \quad (1)$$

Also,

$$f(-a) = \begin{vmatrix} c_1 - a & 0 & 0 \\ b-a & c_2 - a & 0 \\ b-a & b-a & c_3 - a \end{vmatrix}$$
$$= (c_1 - a)(c_1 - a)(c_3 - a)$$

Similarly,

$f(-b) = (c_1 - b)(c_2 - b)(c_3 - b)$

$g(x) = (c_1 - x)(c_2 - x)(c_3 - x) \Rightarrow g(a) = f(-a)$ and $g(b) = f(-b)$

Now from (1), we get

$$f(0) = \frac{bg(a) - ag(b)}{(b-a)}$$

For Problems 4–6

4. d, 5. d, 6. c.

Sol. $$\Delta = \frac{1}{a}\begin{vmatrix} a^3 + ax & ab & ac \\ a^2b & b^2 + x & bc \\ a^2c & bc & c^2 + x \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$ and taking $a^2 + b^2 + c^2 + x$ common, we get

$$\Delta = \frac{1}{a}(a^2 + b^2 + c^2 + x)\begin{vmatrix} a & ab & ac \\ b & b^2 + x & bc \\ c & bc & c^2 + x \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$, we get

$$\Delta = \frac{1}{a}(a^2 + b^2 + c^2 + x)\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & 0 & x \end{vmatrix}$$

$$= \frac{1}{a}(a^2 + b^2 + c^2 + x)(ax^2)$$
$$= x^2(a^2 + b^2 + c^2 + x)$$

Thus Δ is divisible by x and x^2. Also, graph of $f(x)$ is

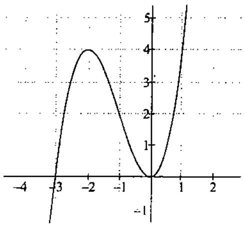

Fig. 7.2

For Problems 7–9

7. c, 8. a, 9. d.

Sol. The system of equations

$-x + cy + bz = 0$ (1)

$cx - y + az = 0$ (2)

$bx + ay - z = 0$ (3)

has a non-zero solution if

$$\Delta = \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow a^2 + b^2 + c^2 + 2abc - 1 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$ (4)

Then clearly the system has infinitely many solutions. From (1) and (2), we have

$$\frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2}$$

$$\therefore \quad \frac{x^2}{(ac+b)^2} = \frac{y^2}{(bc+a)^2} = \frac{z^2}{(1-c^2)^2}$$

or $$\frac{x^2}{(1-a^2)(1-c^2)} = \frac{y^2}{(1-b^2)(1-c^2)} = \frac{z^2}{(1-c^2)^2} \quad \text{[from (4)]}$$

or $$\frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2} \quad (5)$$

From (5), we see that $1 - a^2$, $1 - b^2$, $1 - c^2$ are all positive or all negative. Given that one of a, b, c is proper fraction, so

$1 - a^2 > 0, 1 - b^2 > 0, 1 - c^2 > 0$, which gives

$a^2 + b^2 + c^2 < 3$ (6)

Using (4) and (6), we get

$1 < 3 + 2abc$

or

$abc > -1$ (7)

For Problems 10–12

10. a, 11. b, 12. c.

Sol. $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 2\lambda + 3 + 2 - 2 - \lambda - 6 = \lambda - 3$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & \lambda \end{vmatrix} = 12\lambda + 3\mu + 20 - 2\mu - 10\lambda - 36$$

$$= 2\lambda + \mu - 16$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = 10\lambda + 18 + \mu - 10 - 3\mu - 6\lambda$$

$$= 4\lambda - 2\mu + 8$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & \mu \end{vmatrix} = 2\mu + 10 + 12 - 12 - \mu - 20$$

$$= \mu - 10$$

Thus the system has unique solutions if $\Delta \neq 0$ or $\lambda \neq 3$ and the system has infinite solutions if $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ or $\lambda = 3$ and $\mu = 10$. System has no solution if $\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero or $\lambda = 3$ and $\mu \neq 10$.

For Problems 13–15

13. d, 14. d, 15. d.

Sol. $$\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \quad \text{[multiplying row by row]}$$

$$= D^2 \text{ (say)}$$

Now,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}.$$

$= (1 - \alpha)(\alpha - \beta)(\beta - 1)$

$= (\beta - \alpha)\ [\alpha\beta - \alpha - \beta + 1]$

$= (\beta - \alpha)\left(\frac{c}{a} + \frac{b}{a} + 1\right) = \frac{(\beta - \alpha)}{a}(a + b + c)$

$\therefore \quad \Delta = D^2 = \frac{(\beta - \alpha)^2}{a^2}(a + b + c)^2$

$= \frac{1}{a^2}(a + b + c)^2\left[\frac{b^2}{a^2} - 4\frac{c}{a}\right]$

$= \frac{1}{a^4}(a + b + c)^2(b^2 - 4ac)$

If $\Delta < 0$, i.e., $b^2 - 4ac < 0$, then roots are imaginary.

If one root is $1 + \sqrt{2}$ and since coefficients are real, the other root is $1 - \sqrt{2}$. Hence the equation is $x^2 - 2x - 1 = 0$. Then the value of Δ is $(1 - 2 - 1)^2(4 - 4(1)(-1)) = 32$.

If $\Delta > 0$, i.e., $b^2 - 4ac > 0$, then roots are real and distinct but nothing can be said about $f(1)$.

For Problems 16–18

16. a, 17. b, 18. c.

Sol. Multiplying R_1, R_2, R_3 by a, b, c, respectively, and then taking a, b, c common from C_1, C_2 and C_3, we get

$$\Delta = \begin{vmatrix} -bc & ab + ac & ac + ab \\ ab + bc & -ac & bc + ab \\ ac + bc & bc + ac & -ab \end{vmatrix}$$

Now, using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, and then taking $(ab + bc + ca)$ common from C_2 and C_3, we get

$$\Delta = \begin{vmatrix} -bc & 1 & 1 \\ ab + bc & -1 & 0 \\ ac + bc & 0 & -1 \end{vmatrix} \times (ab + bc + ca)^2$$

Now, applying $R_2 \to R_2 + R_1$, we get

$$\Delta = \begin{vmatrix} -bc & 1 & 1 \\ ab & 0 & 1 \\ ac + bc & 0 & -1 \end{vmatrix}(ab + bc + ca)^2$$

Expanding along C_2, we get

$\Delta = (ab + bc + ca)^2\ [ac + bc + ab]$

$= (ab + bc + ca)^3$

$= (r/p)^3 = r^3/p^3$

Now given a, b, c are all positive, then

A.M. $\geq$ G.M.

$\Rightarrow \quad \frac{ab + bc + ac}{3} \geq (ab \times bc \times ac)^{1/3}$

$\Rightarrow \quad (ab + bc + ac)^3 \geq 27a^2b^2c^2$

$\Rightarrow \quad (ab + bc + ac)^3 \geq 27(s^2/p^2)$

If $\Delta = 27$, then $ab + bc + ca = 3$, and given that $a^2 + b^2 + c^2 = 3$, from $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we have $a + b + c = \pm 3$

$\Rightarrow \quad a + b + c = 3$ (since all the roots are positive)

$\Rightarrow \quad 3p + q = 0$

For Problems 19–21

19. d, 20. c, 21. c.

Let,

$$\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} = A + Bx + Cx^2 + \cdots$$

Putting $x = 0$, we get

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Now differentiating both sides with respect to x and putting $x = 0$, we get

$$B = \begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = 0$$

Hence coefficient of x is 0. Since $f(x) = 0$ and $f'(0) = 0$, $x = 0$ is a repeating root of the equation $f(x) = 0$.

For Problems 22–24

22. c, 23. d, 24. b.

$$\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} x - m & n - y & 0 \\ 0 & y - n & r - z \\ m & n & z \end{vmatrix} = 0$$

$\Rightarrow \quad (x - m)(y - n)z + (n - y)(r - z)m - n(r - z)(x - m) = 0$

Dividing by $(x - m)(y - n)(z - r)$, we have

$\frac{z}{z - r} + \frac{m}{x - m} + \frac{n}{y - n} = 0$

$\Rightarrow \quad \frac{z}{z - r} + \frac{m}{x - m} + \frac{n}{y - n} = 0$

$\Rightarrow \quad \frac{z}{z - r} + \frac{m}{x - m} + 1 + \frac{n}{y - n} + 1 = 2$

$\Rightarrow \quad \frac{z}{z - r} + \frac{x}{x - m} + \frac{y}{y - n} = 2 = 2$

$\Rightarrow \quad \frac{z}{z - r} - 1 + \frac{x}{x - m} - 1 + \frac{y}{y - n} - 1 = -1$

$\Rightarrow \quad \frac{m}{x - m} + \frac{n}{y - n} + \frac{r}{z - r} = -1$

Now,

A.M. $\geq$ G.M.

$\Rightarrow \quad \frac{\frac{z}{z - r} + \frac{x}{x - m} + \frac{y}{y - n}}{3} \geq \left(\frac{z}{(z - r)}\frac{x}{(x - m)}\frac{y}{(y - n)}\right)^{1/3}$

$\Rightarrow \quad \frac{z}{z - r}\frac{x}{x - m}\frac{y}{y - n} \leq \frac{8}{27}$

For Problems 25–27

25. b, 26. d, 27. a.

Sol. $$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$

$$f'(x) = \begin{vmatrix} -(b+1) & -(b+2) & 2ax+b+1 \\ (b+1) & (b+2) & -1 \\ b & b+1 & 2ax+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - R_2$, we get

$$f'(x) = \begin{vmatrix} 0 & 0 & 2ax+b \\ b+1 & b+2 & -1 \\ -1 & -1 & 2ax+b+1 \end{vmatrix}$$

$= (2ax + b)[-b - 1 + b + 2]$

$\therefore \quad f'(x) = 2ax + b$

$\therefore \quad f(x) = ax^2 + bx + c$

$f(0) = 2 \Rightarrow c = 2$

$f(1) = 1 \Rightarrow a + b + 2 = 1 \Rightarrow a + b = -1$

$f'(5/2) = 0 \Rightarrow 5a + b = 0$

$\Rightarrow \quad a = 1/4, b = -5/4$

Hence, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

Clearly, discriminant (D) of the equation $f(x) = 0$ is less than 0.

Hence, $f(x) = 0$ has imaginary roots. Also, $f(2) = 1/2$. And minimum value of $f(x)$ is

$$\frac{\frac{25}{16} - 4\frac{1}{4}(2)}{4\frac{1}{4}} = \frac{7}{16}$$

Hence, range of the $f(x)$ is $\left[\frac{7}{16}, \infty\right)$.

Matrix-Match Type

1. a→s; b→p; c→s; d→s.

a. Coefficient of x in $f(x)$ is coefficient of x in $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 2 \\ x^2 & 1 & 0 \end{vmatrix}$

Therefore, coefficient of x is –2.

b. Let $D = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$

$= (3 \cos \theta - \sin \theta)^2$

$\Delta_{max} = 10$

c. $f'(x) = 0$

$\Rightarrow \; f'(0) = 0$

d. $a_0 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 6 \\ 0 & 0 & 1 \end{vmatrix} = -2(1) = -2$

2. a→s; b→r; c→q, r; d→p.

a. The given determinant is $\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we have

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & x & x+1 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} \quad \text{[Applying } R_1 \to R_1 - R_2 \text{ and } R_3 \to R_3 - R_2]$$

$$= 2\begin{vmatrix} x & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{[Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_2]$$

$= -2$ [Expanding along R_3]

b. $$\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix}$$

Let $x^2 - 13 = t$. Then

$t^3 - 67t + 126 = 0$

$\Rightarrow \; t = -9, 2, 7 \Rightarrow x = \pm 2, \pm\sqrt{20}, \pm\sqrt{15}$

Hence sum of other five roots is 2.

c. $$\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3} + \sqrt{8}i & 3\sqrt{2} + \sqrt{6}i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}\,i & \sqrt{27} + 2i \end{vmatrix}$$

Taking $\sqrt{6}$ common from C_1, we get

$$\Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ \sqrt{2} & \sqrt{3}+2\sqrt{2}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{3} & \sqrt{2}+2\sqrt{3}i & 3\sqrt{3}+2i \end{vmatrix}$$

Applying $R_2 \to R_2 - \sqrt{2}R_1$ and $R_3 \to R_3 - \sqrt{3}R_1$, we get

$$\Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ 0 & \sqrt{3} & \sqrt{6}i - 2\sqrt{3} \\ 0 & \sqrt{2} & 2i - 3\sqrt{2} \end{vmatrix}$$

$$= \sqrt{6}\begin{vmatrix} \sqrt{3} & \sqrt{6}i - 2\sqrt{3} \\ \sqrt{2} & 2i - 3\sqrt{2} \end{vmatrix}$$

$$= \sqrt{6}\begin{vmatrix} \sqrt{3} & -2\sqrt{3} \\ \sqrt{2} & -3\sqrt{2} \end{vmatrix} \quad \text{[Applying } C_2 \to C_2 - \sqrt{2}i\, C_1]$$

$= \sqrt{6}\,(-3\sqrt{6} + 2\sqrt{6})$

$= -6$, which is an integer

d. $$f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 + (\sin\theta)R_3$ and $R_2 \to R_2 - (\cos\theta)R_3$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

$= \sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \; f(\pi/3) = 1$

3. a→p, q, r; b→q; c→s; d→p, q, r.

a. Multiplying C_1 by a, C_2 by b and C_3 by c, we obtain

$$\Delta = \frac{1}{abc}\begin{vmatrix} \frac{a}{c} & \frac{b}{c} & -\frac{a+b}{c} \\ -\frac{b+c}{a} & \frac{b}{a} & \frac{c}{a} \\ -\frac{b(b+c)}{ac} & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} 0 & \frac{b}{c} & -\frac{a+b}{c} \\ 0 & \frac{b}{a} & \frac{c}{a} \\ 0 & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac} \end{vmatrix}$$

This shows that Δ is independent of a, b and c.

b. Applying $C_1 \to C_1 - (\cot b)C_2$, we get

$$\Delta = \begin{vmatrix} 0 & \sin a \sin b & \cos a \\ 0 & \cos a \sin b & -\sin a \\ -\sin a/\sin b & \sin a \cos b & 0 \end{vmatrix}$$

$= -\frac{\sin a}{\sin b}$ [$-\sin b \sin^2 a - \cos^2 a \sin b$] [expanding along C_1]

$= \sin a$

c. Taking $1/\sin a \cos b$, $1/\sin a \sin b$, $1/\cos a$ common from C_1, C_2, C_3, respectively, we get

$$\Delta = \frac{1}{\sin^2 a \ \cos a \ \sin b \cos b}\Delta_1$$

where $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ -\cot a & -\cot a & \tan a \\ \tan b & -\cot b & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 0 & -\cot a & \tan a \\ 1/\sin b \cos b & -\cot b & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$\Delta = \frac{1}{\sin b \cos b}$ [tan a + cot a]

$$= \frac{1}{\sin a \cos a \sin b \cos b}$$

d. $\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & a\sin B & a\sin C \\ a\sin B & 1 & \cos A \\ a\sin C & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & 0 & 0 \\ \sin B & 1-\sin^2 B & \cos A - \sin B \sin C \\ \sin C & \cos A - \sin B \sin C & 1-\sin^2 C \end{vmatrix}$$

[Applying $C_2 \to C_2 - (\sin B)C_1$ and $C_3 \to C_3 - (\sin C)C_1$]

$= a^2[\cos^2 B \cos^2 C - (\cos A - \sin B \sin C)^2]$

$= a^2[\cos^2 B \cos^2 C - (\cos(B + C) + \sin B \sin C)^2]$

$= a^2[\cos^2 B \cos^2 C - \cos^2 B \cos^2 C]$

$= 0$

Integer Type

1.(3) Equation $x^3 + ax^2 + bx + c = 0$ has roots α, β, γ.

$\therefore \quad \alpha + \beta + \gamma = -a$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

Since the given system of equations has non-trivial solutions, so

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$$

$\Rightarrow \quad \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 0$

$\Rightarrow \quad (\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha] = 0$

$\Rightarrow \quad (\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)] = 0$

$\Rightarrow \quad -a[a^2 - 3b] = 0 \quad \Rightarrow \quad a^2/b = 3$

2.(2) We have $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$

Since $a_n = \frac{20}{n}$; $d = \frac{1}{20}$

Hence, $D = \begin{vmatrix} 20 & \frac{20}{2} & \frac{20}{3} \\ \frac{20}{4} & \frac{20}{5} & \frac{20}{6} \\ \frac{20}{7} & \frac{20}{8} & \frac{20}{9} \end{vmatrix} = \frac{(20)^3}{4\times 7}\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{4}{5} & \frac{2}{3} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix}$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$= \frac{(20)^3}{4\times 7}\begin{vmatrix} 0 & \frac{-3}{10} & \frac{-1}{3} \\ 0 & \frac{-3}{40} & \frac{-1}{9} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix} = \frac{50}{21}$$

$\Rightarrow \quad [D] = 2$

3.(8) Let $D = \begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$

$R_2 \to R_2 - R_3$

$$= \begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 - (\alpha+\beta-\gamma-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 - (\alpha+\beta-\gamma-\delta)^2 & 0 \\ (\gamma+\delta-\beta-\delta)^4 - (\alpha+\beta-\gamma-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 - (\alpha+\beta-\gamma-\delta)^2 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

$= 4(\beta-\delta)(\gamma-\alpha).\,4(\alpha-\delta)(\gamma-\beta)$

$$\times\begin{vmatrix} (\beta+\gamma-\alpha-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\gamma+\alpha-\beta-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$

$= 16(\beta-\delta)(\gamma-\alpha)(\alpha-\delta).4(\gamma-\delta)(\beta-\alpha)$

$$\begin{vmatrix} 1 & 0 & 0 \\ (\gamma+\alpha-\beta-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

$= -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$

4.(4) $\Delta = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$$\Delta = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ x+1 & x+1 & x+1 \\ x+2 & 2(x+2) & 6(x+2) \end{vmatrix} = 0$$

$\therefore \quad \Delta = (x+1)(x+2)\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 0$

$\therefore \quad \Delta = (x+1)(x+2)[(x+2).4-(2x+3).5+(3x+4).1] = 0$

$\Delta = (x+1)(x+2)(-3x-3) = 0$

or $(x+1)^2(x+2) = 0$

$\therefore \quad x = -1, -1, -2$

5.(2) System of equations

$\Rightarrow \quad \alpha x + y + z = \alpha - 1$ (1)

$x + \alpha y + z = \alpha - 1$ (2)

$x + y + \alpha z = \alpha - 1$ (3)

Since system has no solution.

Therefore, (1) $\Delta = 0$ and (2) $\alpha - 1 \neq 0$

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0, \alpha \neq 1$$

$R_1 \to R_1 - R_3,\ R_2 \to R_2 \to R_3$

$$\begin{vmatrix} \alpha-1 & 0 & 1-\alpha \\ 0 & \alpha-1 & 1-\alpha \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$\Rightarrow \quad (\alpha-1)[\alpha(\alpha-1)-(1-\alpha)] + (1-\alpha)[-(\alpha-1)] = 0$

$\Rightarrow \quad (\alpha-1)[\alpha(\alpha-1)+(\alpha-1)] + (\alpha-1)^2 = 0$

$\Rightarrow \quad (\alpha-1)^2[(\alpha+1)+1] = 0$

$\Rightarrow \quad \alpha = 1, 1, -2 \Rightarrow \alpha = 1, -2$

Since system has no solution, $\alpha \neq 1$.

$\therefore \quad \alpha = -2$

6.(3) $x + y + z = 1$ (1)

$x + 2y + 4z = p$ (2)

$x + 4y + 10z = p^2$ (3)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix}$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 0 & -1 & -3 \\ 0 & -2 & -6 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

Since $\Delta = 0$, solution is not unique solution.

The system will have infinite solutions if $\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ p & 2 & 4 \\ p^2 & 4 & 10 \end{vmatrix} = 0$$

$C_3 \to C_3 - C_2$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ p & 2 & 2 \\ p^2 & 4 & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(12-8) - 1(6p - 2p^2) = 0$

$\Rightarrow \quad 4 - 6p + 2p^2 = 0$

$\Rightarrow \quad 2(p^2 - 3p + 2) = 0$

$\Rightarrow \quad p^2 - 3p + 2 = 0$

$\Rightarrow \quad p = 1$ or 2

Also for these values of p, $\Delta_2, \Delta_3 = 0$

7.(0) $\begin{vmatrix} 3u^2 & 2u^3 & 1 \\ 3v^2 & 2v^3 & 1 \\ 3w^2 & 2w^3 & 1 \end{vmatrix} = 0$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} u^2-v^2 & u^3-v^3 & 0 \\ v^2-w^2 & v^3-w^3 & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} u+v & u^2+v^2+vu & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$\Rightarrow \begin{vmatrix} u-w & (u^2-w^2)+v(u-w) & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & u+w+v & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (v^2 + w^2 + vw) - (v + w)[(v + w) + u] = 0$

$\Rightarrow \quad v^2 + w^2 + vw - (v + w)^2 - u(v + w) = 0$

$\Rightarrow \quad uv + vw + wu = 0$

8.(2) Using $C_3 \to C_3 - (C_1 + C_2)$ in D_1 and D_2, we have

$\therefore \quad \dfrac{D_1}{D_2} = \dfrac{-2b(ad-bc)}{b(ad-bc)} = -2$

9.(5) $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$

Applying $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 0 & \sin\theta - 3\cos\theta & 0 \end{vmatrix}$$

$= -(\sin\theta - 3\cos\theta)(3\cos\theta - \sin\theta)$

$= (3\cos\theta - \sin\theta)^2$

Now, $-\sqrt{9+1} \le 3\cos\theta - \sin\theta \le \sqrt{9+1}$

$\Rightarrow \quad (3\cos\theta - \sin\theta)^2 \le 10.1$

$\Rightarrow \quad \Delta_{max.} = 10$

10.(4)

$$\Delta = x\begin{vmatrix} 1 & x+y & x+y+z \\ 2 & 3x+2y & 4x+3y+2z \\ 3 & 6x+3y & 10x+6y+3z \end{vmatrix}$$

$$= x^2\begin{vmatrix} 1 & 1 & x+y \\ 2 & 3 & 4x+3y \\ 3 & 6 & 10x+6y \end{vmatrix} \begin{vmatrix} C_3 \to C_3 - zC_1 \\ C_2 \to C_2 - yC_1 \end{vmatrix}$$

$$= x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} [C_3 \to C_3 - yC_2]$$

$= x^3(6 - 8 + 3) = 64$

$\Rightarrow x^3 = 64 \Rightarrow x = 4$

11.(1)

$$\Delta_1 = \begin{vmatrix} a_1^2 + 4a_1d & a_1 & d \\ a_2^2 + 4a_2d & a_2 & d \\ a_3^2 + 4a_3d & a_3 & d \end{vmatrix}, [C_1 \to C_1 - C_2]$$

where d is the common difference of A.P.

$$= d\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} + 4d\begin{vmatrix} a_1 & a_1 & d \\ a_2 & a_2 & d \\ a_3 & a_3 & d \end{vmatrix}$$

$= d(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = -2d^4$

Similarly, $\Delta_2 = -2d^4$.

12.(4) $\Delta = (xyz)^n \begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$

$= (xyz)^n (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$

Clearly when

$$n = -4, \Delta = \left(\frac{1}{y^2} - \frac{1}{x^2}\right)\left(\frac{1}{z^2} - \frac{1}{y^2}\right)\left(\frac{1}{x^2} - \frac{1}{z^2}\right)$$

13.(2)

$$B = 2.2\begin{vmatrix} f & d & e \\ n & l & m \\ c & a & b \end{vmatrix}$$

[Taking 2 common from R_2 and C_1]

$$= 2\begin{vmatrix} 2f & d & e \\ 2n & l & m \\ 2c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} 2c & a & b \\ 2f & d & e \\ 2n & l & m \end{vmatrix}$$

$[R_3 \leftrightarrow R_2$, then $R_2 \leftrightarrow R_1]$

$$= 2\begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix} = 2A$$

$[C_1 \leftrightarrow C_2$ and then $C_2 \leftrightarrow C_3]$

14.(0) $\Delta = \begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix} \begin{vmatrix} y_1 & x_1 & 0 \\ y_2 & x_2 & 0 \\ y_3 & x_3 & 0 \end{vmatrix} = 0.0 = 0$

15.(8) Putting $x = 0$, $a_0 = 1$

$(1 + ax + bx^2)^4 = (1 + ax + bx^2)(1 + ax + bx^2)$

$(1 + ax + bx^2)(1 + ax + bx^2)$

Clearly $a_0 = 1$, a_1 = coefficient of $x = a + a + a + a = 4a$

a_2 = coefficient of $x^2 = 4b + 6a^2$

Now $\Delta = -(a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2)$

$\because \quad a_0 + a_1 + a_2 \ne 0$

$\therefore \quad a_0 = a_1 = a_2$

$1 = 4a = 6a^2 + 4b \Rightarrow a = \frac{1}{4}, b = \frac{5}{32}$

Archives

Subjective Type

1. We should have

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(-4k + 6) - k(-12 + 4) + 3(9 - 2k) = 0$

$\Rightarrow \quad -2k + 33 = 0 \Rightarrow k = \frac{33}{2}$

Substituting $k = \frac{33}{2}$ and putting $x = m$ where $m \in Q$, we get the system as

$33y + 6z = -2m$ (1)

$33y - 4z = -6m$ (2)

$3y - 4z = -2m$ (3)

$(1) - (2) \Rightarrow 10z = 4m \Rightarrow z = \frac{2}{5}m$

$(1) \Rightarrow 33y = -2m - \frac{12m}{5} = -\frac{22m}{5}$

$\Rightarrow \quad y = -\frac{2m}{15}$

Therefore, the solution is $x = m, y = \frac{-2m}{15}, z = \frac{2m}{5}$.

2. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca]$

$= -\frac{1}{2}(a + b + c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$

$= -\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

As $a, b, c > 0$, therefore $a + b + c > 0$. Also $a \neq b \neq c$.

$\therefore \quad (a - b)^2 + (b - c)^2 + (c - a)^2 > 0$

Hence, the given determinant is –ve.

3. $\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = Ax + B$

On L.H.S. operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get

$\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ x - 1 & x - 2 & x + 1 \\ x + 3 & x - 2 & x + 1 \end{vmatrix}$

$= \begin{vmatrix} x^2 & x + 1 & x - 2 \\ 0 & x - 2 & x + 1 \\ 0 & x - 2 & x + 1 \end{vmatrix} + \begin{vmatrix} x & x & x - 2 \\ x - 1 & x - 2 & x + 1 \\ x + 3 & x - 2 & x + 1 \end{vmatrix}$

$= 0 + \begin{vmatrix} x & x & x - 2 \\ x - 1 & x - 2 & x + 1 \\ x + 3 & x - 2 & x + 1 \end{vmatrix}$

$= \begin{vmatrix} x & x + 1 & x - 2 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix}$ (Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$)

$= \begin{vmatrix} x & x & x \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix} x + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -2 & 3 \\ 4 & 0 & 0 \end{vmatrix}$

$= xA + B =$ R.H.S.

Hence proved.

4. $\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 7\lambda + 35$

If $7\lambda + 35 \neq 0$, i.e. $\lambda \neq -5$, system has a unique solution.

[$\because \Delta \neq 0 \Rightarrow$ unique solution]

But if $\lambda = -5$, we have $\Delta = 0$. Solution exists in this case if $\Delta_x = \Delta_y = \Delta_z = 0$. Now for $\lambda = -5$,

$\Delta_x = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0$

$\Delta_y = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0$

$\Delta_z = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$

Thus $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$, so there exists infinite number of solutions. Now eliminating x from the equations, we have

$7y - 13z = -9$

$7y - 13z = -9$

which are same, so putting

$z = k \in R$, $y = (13k - 9)/7$ and so $x = (4 - 5k)/7$, where k is any real number.

5. Applying $C_2 \to C_2 + C_1$, $C_3 \to C_2 + C_3$ and using ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$, we have

$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$

$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$ [Applying $C_3 \to C_3 + C_2$]

6. The system has a non-trivial solution if

$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

$\Rightarrow$ $7 \sin 3\theta + 7 \cos 2\theta - 6 + 7 \cos 2\theta - 8 = 0$

$\Rightarrow$ $\sin 3\theta + 2 \cos 2\theta = 2$

$\Rightarrow$ $3 \sin \theta - 4 \sin^3 \theta + 2 - 4 \sin^2 \theta = 2$

$\Rightarrow$ $\sin \theta (4 \sin^2 \theta + 4 \sin \theta - 3) = 0$

$\Rightarrow$ $\sin \theta (2 \sin \theta + 3)(2 \sin \theta - 1) = 0$

$\Rightarrow$ $\sin \theta = 0$; 1/2, since $\sin \theta \neq -3/2$

Hence, $\theta = n\pi$ or $n\pi + (-1)^n \pi/6$; $n \in Z$.

7. As $A28$, $3B9$ and $62C$ are divisible by k, there exists $m_1, m_2, m_3 \in Z$ such that

$100A + 20 + 8 = m_1k$, $300 + 10B + 9 = m_2k$ and $600 + 20 + C = m_3k$.

Now,

$\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

Applying $R_2 \to 100R_1 + R_2 + 10R_3$, we get

$\Delta = \begin{vmatrix} A & 3 & 6 \\ 100A + 20 + 8 & 300 + 10B + 9 & 600 + 20 + C \\ 2 & B & 2 \end{vmatrix}$

$$= \begin{vmatrix} A & 3 & 6 \\ m_1k & m_2k & m_3k \\ 2 & B & 2 \end{vmatrix}$$

$$= k\begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix} = k\Delta_1$$

As all elements of Δ_1 are integers, Δ_1 must be integer.

$\therefore$ $\Delta = k \times$ some integer

$\Rightarrow$ Δ is divisible by k.

8. Given,

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ reduce the determinant into

$$\begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$\Rightarrow$ $(p-a)(q-b)r + a(b-q)(c-r) - b(p-a)(c-r) = 0$

Dividing throughout by $(p-a)(q-b)(r-c)$, we get

$$\frac{r}{r-c} + \frac{a}{p-a} + \frac{b}{q-b} = 0$$

$$\Rightarrow \quad \frac{r}{r-c} + 1 + \frac{a}{p-a} + 1 + \frac{b}{q-b} = 2$$

9. Taking $n!$, $(n+1)!$, $(n+2)!$ common from R_1, R_2 and R_3, respectively, we get

$$\Delta = (n)!\,(n+1)!\,(n+2)! \times \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

$$\therefore \quad \frac{\Delta}{(n!)^3} = (n+1)^2(n+2) \times \begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix}$$

$[R_2 \to R_2 - R_1, R_3 \to R_3 - R_2]$

$= (n+1)^2(n+2)2 = 2[n^3 + 4n^2 + 5n + 2]$

$$\therefore \quad \frac{\Delta}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$$

Hence, $\dfrac{\Delta}{(n!)^3} - 4$ is divisible by n.

10. The given system has a non-trivial solution if

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

By expanding the determinant along first column, we get

$\lambda = \sin 2\alpha + \cos 2\alpha$

Now,

$-\sqrt{2} \le \sin 2\alpha + \cos 2\alpha \le \sqrt{2}$

$\Rightarrow$ $-\sqrt{2} \le \lambda \le \sqrt{2}$

For $\lambda = 1$,

$\sin 2\alpha + \cos 2\alpha = 1$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin 2\alpha + \frac{1}{\sqrt{2}}\cos 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(2\alpha - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(2n\pi \pm \frac{\pi}{4}\right)$$

$$\Rightarrow \quad 2\alpha = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}, \ n \text{ being an integer}$$

11.

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$= 0 \times 0 = 0$

12. Taking $\dfrac{1}{a(a+d)(a+2d)}$ common from R_1, $\dfrac{1}{(a+d)(a+2d)(a+3d)}$ from R_2 an $\dfrac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3, we have

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta'$$

where

$$\Delta' = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

$$= \begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)(a+3d) & 2d & a+d \\ (a+3d)(a+4d) & 2d & a+2d \end{vmatrix} \text{[Applying } C_2 \to C_2 - C_3]$$

$$= \begin{vmatrix} (a+d)(a+2d) & 2d & a \\ (a+2d)2d & 0 & d \\ (a+3d)2d & 0 & d \end{vmatrix} \text{[applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_2]$$

$= -2d[(a+2d)2d^2 - (a+3d)2d^2] = 4d^4$

Hence, $\Delta = 4d^4/[a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)]$.

13. Given that a, b, c are p^{th}, q^{th} and r^{th} terms of a H.P. Hence, $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are p^{th}, q^{th} and r^{th} terms of an A.P. So,

$$\therefore \quad \left.\begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned}\right\} \quad (1)$$

Now given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Substituting the values of $1/a$, $1/b$, $1/c$ from (i), we get

$$\Delta = abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Operating $R_1 \to R_1 - (A-D)R_3 - DR_2$, we get

$$\Delta = abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

14. Operating $R_1 \to R_1 + R_2 + R_3$ and using trigonometric identities, the given determinant becomes

$$\begin{vmatrix} \sin\theta+2\sin\theta\left(-\frac{1}{2}\right) & \cos\theta+2\cos\theta\left(-\frac{1}{2}\right) & \sin 2\theta+2\sin 2\theta\left(-\frac{1}{2}\right) \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$=\begin{vmatrix} 0 & 0 & 0 \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Objective Type

Fill in the blanks

1. Putting $\lambda = 0$, we have $t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = 0$ (skew-symmetric determinant)
2. Clearly for $x = -1$, $R_2 \equiv R_3$ and for $x = 2$, $R_1 \equiv R_3$. Hence roots are $x = -1, 2$.
3. With 0 and 1 as elements there are $2 \times 2 \times 2 \times 2 = 16$ determinants of order 2×2 out of which only $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ are the three determinants whose values are +ve. Therefore, the required probability is 3/16.
4. $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Operating $R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

$$\therefore \quad (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$(x+9)\begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(x+9)(x-2)(x-7) = 0$ (Expanding along R_1)

$\Rightarrow$ $x = -9, 2, 7$

5. The given homogeneous system of equations will have non-zero solutions if

$D = 0$

$$\Rightarrow \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda(\lambda^2+1) - 1(-\lambda+1) + 1(1+\lambda) = 0$

$\Rightarrow$ $\lambda^3 + 3\lambda = 0$

$\Rightarrow$ $\lambda(\lambda^2+3) = 0$, but $\lambda^2 + 3 \neq 0$ for real λ

$\Rightarrow$ $\lambda = 0$

6. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

[Multiplying R_1 by a, R_2 by b and R_3 by c]

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ [Applying $C_1 \leftrightarrow C_2$ and then $C_2 \leftrightarrow C_3$]

$= 0$

7. $D = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$=\begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$$

(Taking $\frac{1}{\log x}, \frac{1}{\log y}, \frac{1}{\log z}$ common from R_1, R_2 and R_3, respectively)

$$=\frac{1}{\log x\ \log y\ \log z}\begin{vmatrix}\log x & \log y & \log z\\ \log x & \log y & \log z\\ \log x & \log y & \log z\end{vmatrix}$$

$=0$

8. $R_1 \to R_1 + R_3$

$$\Rightarrow f(\theta)=\begin{vmatrix}0 & 0 & 2\\ -\tan\theta & 1 & \tan\theta\\ -1 & -\tan\theta & 1\end{vmatrix}$$

$= 2(\tan^2\theta + 1) = 2\sec^2\theta$

True or false

1. $\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}$

$$=\frac{1}{abc}\begin{vmatrix}a & a^2 & abc\\ b & b^2 & abc\\ c & c^2 & abc\end{vmatrix}\ \begin{pmatrix}R_1\to aR_1\\ R_2\to bR_2\\ R_3\to cR_3\end{pmatrix}$$

$$=\begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\ \begin{pmatrix}C_1\leftrightarrow C_3\\ C_2\leftrightarrow C_3\end{pmatrix}$$

Hence, the given statement is false.

2. $\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & 1\\ a_2 & b_2 & 1\\ a_3 & b_3 & 1\end{vmatrix}$

$$\Rightarrow \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}a_1 & b_1 & 1\\ a_2 & b_2 & 1\\ a_3 & b_3 & 1\end{vmatrix}$$

$\Rightarrow$ Area of Δ_1 = Area of Δ_2

where Δ_1 is the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) and Δ_2 is the triangle with vertices (a_1, b_1), (a_2, b_2) and (a_3, b_3).

But two triangles of same area may not be congruent. Hence, the given statement is false.

Multiple choice questions with one correct answer

1. b. For every 'det. with 1' ($\in B$) we can find a det. with value -1 by changing the sign of one entry of '1'. Hence there are equal number of elements in B and C.

Therefore, (b) is the correct option.

2. b $\begin{vmatrix}0 & 1+\omega+\omega^2 & 0\\ 1-i & -1 & \omega^2-1\\ -1 & -1+\omega-1 & -1\end{vmatrix}$

$$=\begin{vmatrix}0 & 0 & 0\\ 1-i & -1 & \omega^2-1\\ -1 & -i+\omega-1 & -1\end{vmatrix}\qquad [\because 1+\omega+\omega^2=0]$$

(Operating $R_1 \to R_1 - R_2 + R_3$)

3. b. Let $\frac{x^2}{a^2}=X, \frac{y^2}{b^2}=Y, \frac{z^2}{c^2}=Z$

Then the given system of equations is

$X + Y - Z = 1$
$X - Y + Z = 1$
$-X + Y + Z = 1$

Coefficient determinant is

$$A=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{vmatrix}$$

$= 1(-1-1) - 1(1+1) - 1(1-1)$
$= -4 \neq 0$

Hence, the given system of equations has unique solutions.

4. b. Given,

$$\begin{vmatrix}xp+y & x & y\\ yp+z & y & z\\ 0 & xp+y & yp+z\end{vmatrix}=0$$

Operating $C_1 \to C_1 - pC_2 - C_3$, we get

$$\begin{vmatrix}0 & x & y\\ 0 & y & z\\ -(xp^2+2py+z) & xp+y & yp+z\end{vmatrix}=0$$

$\Rightarrow (xz - y^2)(xp^2 + 2py + z) = 0$
$\Rightarrow xz - y^2 = 0$
$\Rightarrow y^2 = xz$

Hence, x, y, z are in G.P.

5. b. Let, $\Delta=\begin{vmatrix}1 & a & a^2\\ \cos(p-d)x & \cos px & \cos(p+d)x\\ \sin(p-d)x & \sin px & \sin(p+d)x\end{vmatrix}$

Expanding along first row, we have

$1[\cos px\sin(p+d)x - \cos(p+d)x\sin px]$
$-a[\cos(p-d)x\sin(p+d)x - \cos(p+d)x\sin(p-d)x]$
$+ a^2[\cos(p-d)x\sin px - \cos px\sin(p-d)x]$
$= \sin dx - a\sin 2dx + a^2\sin dx$

which is independent of p.

6. a. Taking x common from R_2 and $x(x-1)$ common from R_3, we get

$$f(x)=x^2(x-1)\begin{vmatrix}1 & x & x+1\\ 2 & x-1 & x+1\\ 3 & x-2 & x+1\end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$$f(x)=x^2(x-1)\begin{vmatrix}1 & x & 1\\ 2 & x-1 & 2\\ 3 & x-2 & 3\end{vmatrix}=0$$

Thus, $f(500) = 0$.

7. d. For the given homogeneous system of equations to have non-zero solution, determinant of coefficient matrix should be zero, i.e.,

$$\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 1(1+1) + k(-k+1) - (k+1) = 0$$

$\Rightarrow\ 2 - k^2 + k - k - 1 = 0$

$\Rightarrow\ k^2 = 1$

$\Rightarrow\ k = \pm 1$

8. b. Given that $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Also, $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$. Now given determinant is

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

[Using $\omega = -1 - \omega^2$ and $\omega^3 = 1$]

Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} \quad [\text{as } 1 + \omega + \omega^2 = 0]$$

Expanding along C_1, we get

$3(\omega^2 - \omega^4) = 3(\omega^2 - \omega)$

$= 3\omega(\omega - 1)$

9. b. For infinitely many solutions the two equations become identical. Hence,

$$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \Rightarrow k = 1$$

10. a. The given system is

$x + ay = 0$

$az + y = 0$

$ax + z = 0$

It is a system of homogeneous equations, therefore, it will have infinitely many solutions if determinant of coefficient matrix is zero. Therefore,

$$\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow\ 1(1 - 0) - a(0 - a^2) = 0$

$\Rightarrow\ 1 + a^3 = 0$

$\Rightarrow\ a^3 = -1$

$\Rightarrow\ a = -1$

11. d. Since the system has no solution

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow\ 2(-2\lambda + 1) + 1(\lambda + 1) + 2(3) = 0$

$\Rightarrow\ -4\lambda + 2 + \lambda + 1 + 6 = 0$

$\Rightarrow\ 3\lambda = 9$

$\Rightarrow\ \lambda = 3$

12. d. $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$ (given)

$$\Rightarrow\ -3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = x + iy$$

$\Rightarrow\ x + iy = 0 + i0$

$\Rightarrow\ x = y = 0$

Multiple choice questions with one or more than one correct answer

1. b. e. Given that

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$$

Operating $C_3 \to C_3 - C_1\alpha - C_2$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + b\alpha + b\alpha + c) \end{vmatrix} = 0$$

$$\Rightarrow\ (a\alpha^2 + 2b\alpha + c)\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & 1 \end{vmatrix} = 0$$

$\Rightarrow\ (ac - b^2)(a\alpha^2 + 2b\alpha + c) = 0$

$\Rightarrow$ either $ac - b^2 = 0$ or $a\alpha^2 + 2b\alpha + c = 0$

$\Rightarrow$ either a, b, c are in G.P. or $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

Hence, (b) and (e) are the correct answers.

Matrix-match type

1. a→r; b→q; c→p; d→s.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= -\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

a. If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow\ \Delta = 0$ and $a = b = c \neq 0$

Therefore, the equations represent identical planes.

b. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow\ \Delta = 0$

Therefore, the equations have infinitely many solutions.

$ax + by = (a + b)z$

$bx + cy = (b + c)z$

$\Rightarrow\ (b^2 - ac)y = (b^2 - ac)z \Rightarrow y = z$

$\Rightarrow\ ax + by + cy = 0 \Rightarrow ax = ay \Rightarrow x = y = z$

c. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$

$\Rightarrow\ \Delta \neq 0$

Therefore, the equations represent planes meeting at only one point.

d. $a+b+c=0$ and $a^2+b^2+c^2=ab+bc+ca$

$\Rightarrow$ $a=b=c=0$

Therefore, the equations represent whole of the three-dimensional space.

Integer type

1. (1)

$\omega=e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix}=0$$

$\Rightarrow$ $z\left[(z+\omega)^2(z+\omega)-1-\omega(z+\omega-1)+\omega^2(1-z-\omega^2)\right]=0$

$\Rightarrow$ $z^3=0$

$\Rightarrow$ $z=0$ is only solution.