2. Matrices

- The various elementary operations or transformations on a matrix are as follows:
  - $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
  - $R_i \leftrightarrow kR_i$ *or* $C_i \leftrightarrow kC_j$, where k is a non-zero constant
  - $R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + kC_j$, where k is a constant.

For example, by applying $R_1 \rightarrow R_1 - 7R_3$ to the matrix $\begin{bmatrix} -9 & 5 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$, we obtain $\begin{bmatrix} -23 & 12 & 8 \\ 5 & 6 & 11 \\ 2 & -1 & 0 \end{bmatrix}$.

- If A and B are the square matrices of same order such that $AB = BA = I$, then B is called the inverse of A and A is called the inverse of B. i.e., $A^{-1} = B$ and $B^{-1} = A$
- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$
- If the inverse of a square matrix exists, then it is unique.
- If the inverse of a matrix exists, then it can be calculated either by using elementary row operations or by using elementary column operations.

Example: Find the inverse of the matrix: $A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$

Solution:
We know that $A = IA$. Therefore, we have

$$\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $R_1 \rightarrow \sin\theta\, R_1$ and $R_2 \rightarrow \cos\theta\, R_2$, we have

$$\begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ -\cos^2\theta & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - R_2$, we have

$$\begin{bmatrix} \sin^2\theta + \cos^2\theta & 0 \\ -\cos^2\theta & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ 0 & \cos\theta \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -\cos^2\theta & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ 0 & \cos\theta \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 + \cos^2\theta\, R_1$, we have

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \sin\theta\cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \sin\theta\cos^2\theta & \cos\theta\left(1 - \cos^2\theta\right) \end{bmatrix} A$$

- If A is a square matrix, then $A\,(adjA) = (adjA)\,A = |A|\,I$
- A square matrix A is said to be singular, if $|A| \neq 0$
- A square matrix A is said to be non-singular, if $|A| \neq 0$
- If A and B are square matrices of same order, then $|AB| = |A||B|$

Therefore, if A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of same order.

- If A is a non-singular matrix of order n, then $\{Ad\,jA|(ad\,jA)| = |A|^{n-1}$
- A square matrix A is invertible, if and only if A is non-singular and inverse of A is given by the formula:

$$A^{-1} = \frac{1}{|A|}(ad\,jA)$$

- The system of following linear equations $\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}$ can be written as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- A system of linear equations is said to be consistent, if its solution (one or more) exists.
- A system of linear equations is said to be inconsistent, if its solution does not exist.
- Unique solution of the equation $AX = B$ is given by $X = A^{-1}B$, where $|A| \neq 0$
- For a square matrix A in equation $AX = B$, if
  - $|A| \neq 0$, then there exists a unique solution
  - $|A| \neq 0$ and $(adjA)\,B \neq O$, then no solution exists
  - $|A| \neq 0$ and $(adjA)\,B \neq O$, then the system may or may not be consistent

Example 2:
Solve the following system of linear equations:

$$x - 3y + 4z = 12$$
$$2x + 2y - 3z = -7$$
$$6x - y + 2z = 13$$

Solution:
The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 2 & -3 \\ 6 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and} B = \begin{bmatrix} 12 \\ -7 \\ 13 \end{bmatrix}$$

Now, $|A| = 1[2\times2 - (-1)(-3)] + 3[2\times2 - 6(-3)] + 4[2\times(-1) - 6\times2] = 11 \neq 0$

Therefore, A is a non-singular matrix and hence, the given system of linear equations has only one solution.

Now,

$A_{11} = [2\times2-(-1)(-3)] = 1$
$A_{12} = -[2\times2-6(-3)] = -22$
$A_{13} = [2(-1)-6\times2] = -14$
$A_{21} = -[(-3)\times2-(-1)\times4] = 2$
$A_{22} = [1\times2-6\times4] = -22$
$A_{23} = -[1(-1)-6(-3)] = -17$
$A_{31} = [(-3)(-3)-4\times2] = 1$
$A_{32} = -[1(-3)-2\times4] = 11$
$A_{33} = [1\times2-2(-3)] = 8$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{11}\begin{bmatrix} 1 & 2 & 1 \\ -22 & -22 & 11 \\ -14 & -17 & 8 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 2 & 1 \\ -22 & -22 & 11 \\ -14 & -17 & 8 \end{bmatrix}\begin{bmatrix} 12 \\ -7 \\ 13 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 33 \\ 55 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$x = 1$, $y = 3$, and $z = 5$

Solution of System of Linear Equations (Method of Reduction)

The method of reduction can be used to solve a system of linear equations.

Steps involved:

- Write the given system of linear equations in the matrix equation form AX = B.
- Perform a suitable row transformation on matrix A to reduce it to an upper triangular matrix or a lower triangular matrix. The same row operations need to be simultaneously performed on matrix B.
- Rewrite the equations in the form of system of linear equations that can be solved by the elimination method.

Note: Matrix B is a column matrix, so elementary column transformations cannot be used to reduce matrix A to an upper triangular matrix or a lower triangular matrix.