11

Similarity

11.01. Introduction

Do you ever think how the distance of the moon, height of the Gaurishankar peak (mount Everest), Guru shikhar. (The hightest peak of mount Abu) has been measured? Will they measured by using measuring tape? Infact to measure all these distances and heights the concept of measurement was used. This indirect concept of measurment is based on the principle of similarity of figures. In this chapter we shall study about the similarity, specially similarity of triangles in detail.

11.02. Similar Figures

Remember, in class IX you have discussed the figures (congurent figures) of same size and of same measurment figures under which, you have seen that all the circles of same radius are congurent, all the squares of same side are congurent.Similarly all equilateral triangles of same length are congurent. Let us consider the following figures.

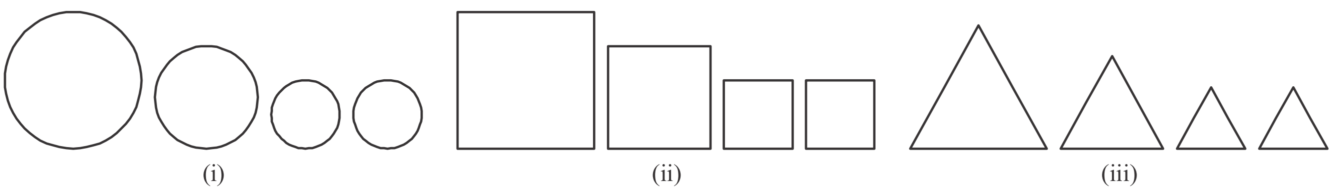


Fig. 11.1

Take two or more circles from the above fig. 11.1 (i). Are they congurent? Since all these do not have same radius, so they are not congurent. consider, Some of these are congurent while other are not. But all of them have same shape. So all of these figures are called similar. Two similar figures have the same shape but it is not essential having same size or measures. So all the circles are similar. Similarly in the fig. 11.1 (ii), (iii) all the squares and all the equilateral triangles can be called like similar like the circles mentioned above. At the bases of above explanation necessarily. We can say all the congurent figures are similar but all similar figures are not congurent.

Now, see the above figures 11.1 (i), (ii), (iii) and tell whether a circle and a square similar or a square and an equilateral triangle similar to each other? Definitely your answer will be negative as the shapes of these figures are not same. Now, what do you think about the figures of pentagons shown in the fig. 12.2? Are these similar? Although these two figures look similar but their similarity is doubtfull

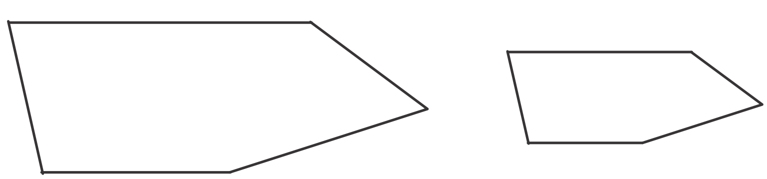


Fig. 12.2

Now see the fig. 11.3.

Fig. 11.3

In these figures, there are three pictures of great Indian mathematician Srinivas Ramanujan (22 December 1887-18 April 1920) in different size. Are all these pictures similar? Undoubtfully these are similar figures. Can you state after seeing these pictures why were you doubtfull about their similarity? So let's find a defination of smilarity so that we can decide whether the given figures are similar or not.

You will make the photocopy of Your documents *i.e.,* mark-sheet, birth certificate etc.

Similarly you would also have photographs your photo, stamp size, passport size or postcard size. All the photographs clicked at the same time are similar even they may be different in size. Draw a figure on a white paper and make it enlarged. Now you have two figures. Measure the sides and angles of these figures with the help of scale and protector and write them with names. See the fig. 11.4 given below.

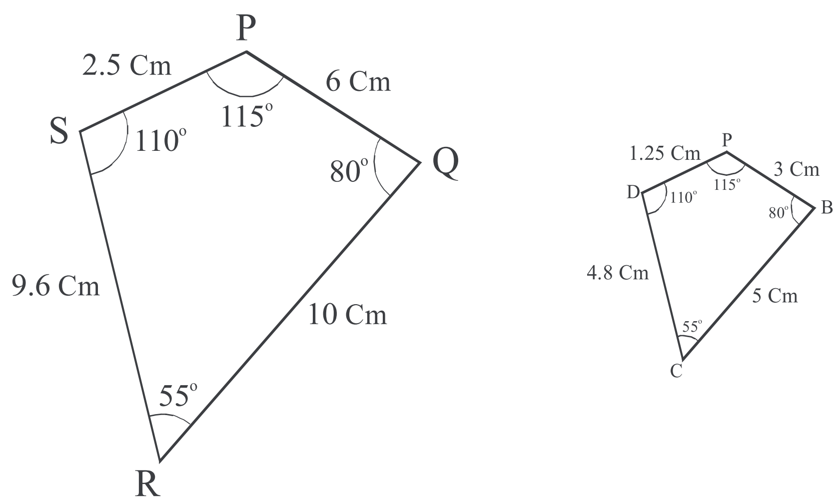


Fig. 11.4

Now compare the corresponding sides and angles of these two figures. You will get that the sides of larger figures have become doubled in the comparison of sides of smaller figure. In other words we can say that the ratio of sides of larger figure and that of smaller one is 2 : 1. In the same way corresponding angles of two figures are equal. There two results are considered the conclusion of similarity .Thus the conditions for the similarity of two polygons with equal sides are (i) The corresoponding angles are equal and (ii) The ratio of corresponding sides is equal.

In the fig. 11.4, there are two quadrilateral *ABCD* and *PQRS*, where vertex *A* is corresponding to vertex *P*, vertex *B* is corresponding to vertex *Q*, vertex *C* is corresponding to vertex *R* as well as vertex *D* is corresponding to vertex *S*. In short these correspondings can be expressed as $A \leftrightarrow P, B \leftrightarrow Q, C \leftrightarrow R$ and $D \leftrightarrow S$. So

(i) $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ and $\angle D = \angle S$.

(ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DA}{SP} = \frac{1}{2}$

Hence quadrilateral *ABCD* is similar to quadrilateral *PQRS*, For the pentagons *ABCDE* and *PQRST* see fig. 11.5. we get.

(i) $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R, \angle D = \angle S$ and $\angle E = \angle T$

(ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP} = \frac{1}{2}$

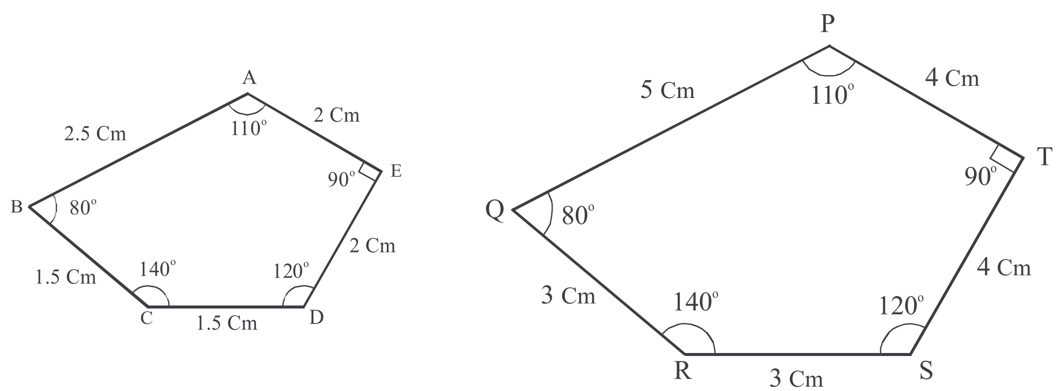


Fig. 11.5

Hence pentagon ABCDE is similar to pentagon PQRST

In the fig. 11.6 (i) corresponding angles of the square and rectangle are same but their sides are not in the same ratio. So they are not similar.

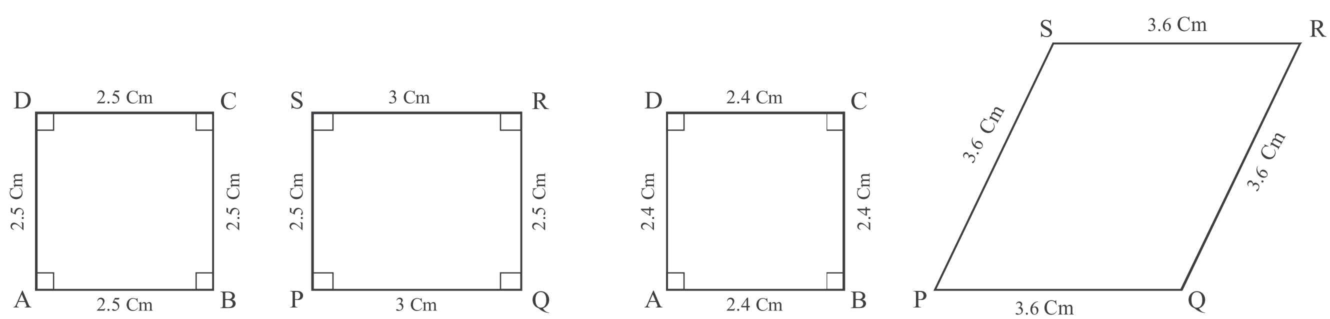


Fig. 11.6

Similarly in fig. 11.6 there is a square and a quadrilateral with opposite sides equal. Their corresponding sides are proportional but angles are not equal, hence both quadrialterals are not similar.

Exercise 11.1

1. **Fill in the blanks :**

   (i) All the circles are ....

   (ii) All the squares are ....

   (iii) All the equilateral triangles are ....

   (iv) The polygons having same number of sides are similar.

   (a) .... (b) ....

2. **State whether following statements are true and false.**

   (i) Two congruent figures are similar.

   (ii) Two similar figures are congruent.

   (iii) Two polygons having same ratio of their corresponding sides are similar.

   (iv) Two polygons are similar if their corresponding sides are proportional and corresponding angles are equal.

   (v) Two polygons having corresponding angles equal are similar.

3. **Give any two examples of simliar figures with their diagrams.**

Similarity of triangles and equiangular triangles

In this chapter we have studied about the conditions for the smilarity of two or more polygons. Since triangles are also polygons, so we can state the same conditions for the similarity of two triangles that is two triangles are similar, if

(i) their corresponding angles are equal and

(ii) their corresponding sides are in the same ratio

In fig. 11.7 ΔABC and ΔDEF are similar if

(i) $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

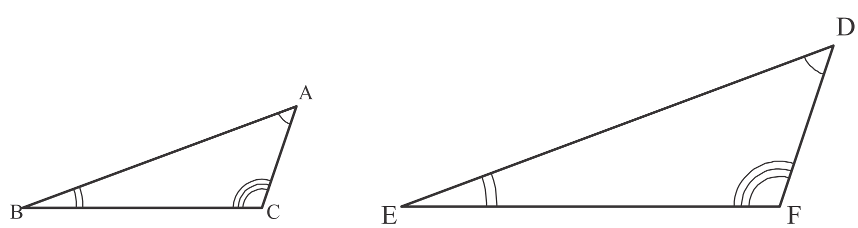


Fig. 11.7

Equiangular Triangles

If corresponding angles of two triangles are equal, then both triangles are called equiangular triangles.

Results related to the basic proportionality theorem :

(a) Draw an angle XAY, On its one arm AX mark five points P_1, P_2, D, P_3 and B at the equal distance. In this way we get $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$cm (If these points are marked at the same distance of 1 cm. it will be conventional)

(b) Take a point C on the side AY and meet B to C. Now form the point D, draw DE BC to meet AY at E. In this way we get a triangle.

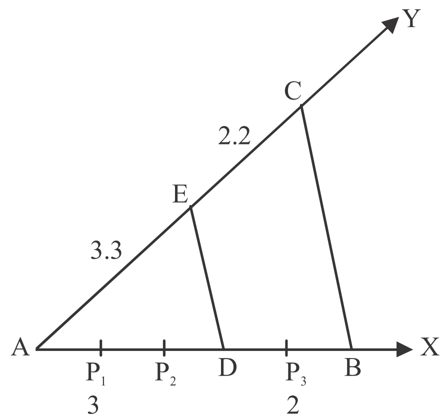


Fig. 11.8

According to the fig. 11.8.

$AD = AP_1 + P_1P_2 + P_2D = 3$ untis (since distance between to consecutive points is 1 cm)

$DB = DP_3 + P_3B = 2$ Units (2 cm)

$$\therefore \quad \frac{AD}{DB} = \frac{3}{2} \qquad \ldots (i)$$

Now measure the length of AE and AC (Here $AE = 3.3$ cm and $EC = 2.2$ cm)

$$\text{So} \quad \frac{AE}{EC} = \frac{3.3}{2.2} = \frac{3}{2} \qquad \ldots (ii)$$

Comparing (i) and (ii) we get

$$\frac{AD}{DB} = \frac{AE}{EC}$$

If in ΔABC, D and E are two points on the sides AB and AC such that $DE \parallel BC$ then we get,

$$\frac{AD}{DB} = \frac{AE}{EC}$$

This result was obtained by famous Greek Mathematician Thales first so it is also called the Thales theroem. This result is known as basic proportionality theroem.

Theorem 11.1 : (Basic proportionally theroem/Thales theorem)

If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.

Given : ΔABC, in which $DE \parallel BC$ and DE intersects AB and AC at D and E.

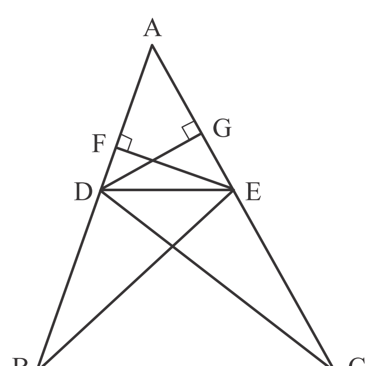


Fig. 11.9

To prove : $$\frac{AD}{DB} = \frac{AE}{EC}$$

Construction : Join B to E and C to D,

Draw $EF \perp BA$ and $DG \perp CA$

Proof : Since EF is the height of ΔADE and ΔABE.

$$\therefore \quad \text{Area of } \Delta ADE = \frac{1}{2} \text{base} \times \text{height}$$

$$= \frac{1}{2} \; AD \times EF$$

$$\text{and} \quad \text{Area of } \Delta DBE = \frac{1}{2} \text{base} \times \text{height}$$

$$= \frac{1}{2} DB \times EF$$

$$\therefore \quad \frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta DBE} = \frac{\frac{1}{2} AD \times EF}{\frac{1}{2} DB \times EF} = \frac{AD}{DB} \qquad \ldots \text{(i)}$$

$$\text{Similarly,} \quad \frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta DEC} = \frac{\frac{1}{2} AE \times DG}{\frac{1}{2} EC \times DG} = \frac{AE}{EC} \qquad \ldots \text{(ii)}$$

But ΔDBE and ΔDEC are at the same base DE and between lines $DE \,||\, BC$

$\therefore$ Area of ΔDBE = Area of ΔDEC . . . (iii)

From (i), (ii) and (iii) we get

$$\frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta DBE} = \frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta DEC}$$

or $$\frac{AD}{DB} = \frac{AE}{EC}$$ **Hence Proved**

With the help of basic proportionality theorem the following usefull results can also be obtained.

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ (ii) $\frac{AB}{DB} = \frac{AC}{EC}$

Theorem 11.2 : (Converse of the theorem 11.1)

If a line divides any two sides of a triangle in the same ratio, then this line is parallel to the third side.

Given : Line l intersects two sides BC and AC of a triangle ABC at D and E respectively.

To prove : $l \quad BC \Rightarrow DE \quad BC$

Proof : Assume, line DE is not parallel to side BC, then another line $DF \parallel BC$

$\because$ DF || BC

$\therefore \quad \frac{AD}{DB} = \frac{AF}{FC}$ ...(i)

(By basic proportionality theorem)

But $\quad \frac{AD}{DB} = \frac{AE}{EC}$ (Given) ...(ii)

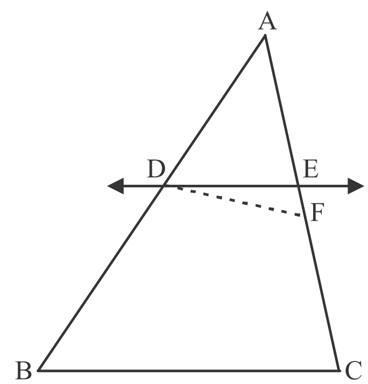


Fig. 11.10

So $\quad \frac{AF}{FC} = \frac{AE}{EC}$ from (i) and (ii)

Adding 1 to both the sides, we get

$$\frac{AF}{FC} + 1 = \frac{AE}{EC} + 1$$

or $\quad \frac{AF + FC}{FC} = \frac{AE + EC}{EC}$

or $\quad \frac{AC}{FC} = \frac{AC}{EC}$

or $\quad \frac{1}{FC} = \frac{1}{EC}$

or $\quad FC = EC$, this result can be obtained only when points F and E is coincides and also fall on DF and DE respectively

Hence DE || BC **Hence Proved**

Illustrative Examples

Examples 1 : In ΔABC, $DE \parallel BC$ and $\frac{AD}{DB} = \frac{3}{5}$. If $AC = 5.6$ unit, then find measure of AE.

Solution : In ΔABC, $DE \parallel BC$ (given)

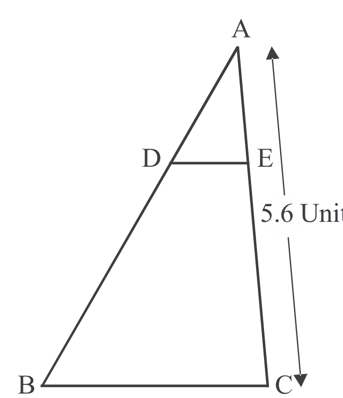


Fig. 11.11

$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}$

(By basic proportionality theorem)

or $\quad \frac{AD}{DB} = \frac{AE}{(AC - AE)}$

or $\quad \frac{3}{5} = \frac{AE}{5.6 - AE}$ $\left(\because \frac{AD}{DB} = \frac{3}{5} \text{ and } AC = 5.6\right)$

or $\quad 3\,(5.6 - AE) = 5AE$

or $\quad 16.8 - 3AE = 5AE$

or $\quad 5AE + 3AE = 16.8$

or $\quad 8AE = 16.8$

or $\quad AE = \dfrac{16.8}{8} = 2.1$ unit

Hence, the measure of $AE = 2.1$ unit.

Examnple 2 : In the given fig. 11.12 $DE \parallel BC$. If $AD = x$, $DB = x - 2$, $AE = x + 2$ and $EC = x - 1$, then find the value of x.

Solution : In ΔABC, $DE \parallel BC$

Thus $\quad \dfrac{AD}{DB} = \dfrac{AE}{EC}$ (By B.P.T)

or $\quad \dfrac{x}{x-2} = \dfrac{x+2}{x-1}$

or $\quad x(x-1) = (x+2)(x-2)$

or $\quad x^2 - x = x^2 - 4$

or $\quad x = 4$

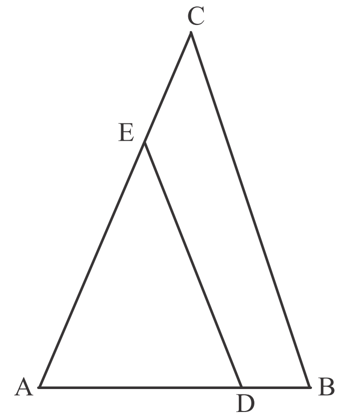


Fig. 11.12

Example 3 : In a trapezium $ABCD$, $AB \parallel DC$, E and F are points on AD and BC respectively such that $EF \parallel AB$. Prove that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

Solution : Join A and C such that interest EF at G.

Now, $\quad AB \parallel DC$ and $EF \parallel AB$ (Given)

$\therefore \quad EF \parallel DC$ (all parallel to the sameline are parallel)

In ΔADC, $EG \parallel DC$ (here $EF \parallel DC$ and EG is a part of EF)

So $\quad \dfrac{AE}{ED} = \dfrac{AG}{GC}$ (By basic proportionality theorem)

or $\quad \dfrac{AG}{CG} = \dfrac{AE}{ED}$ . . . (1)

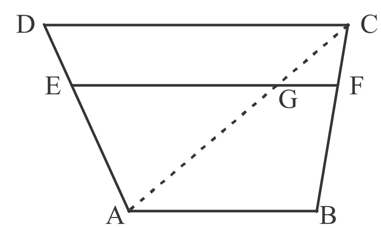


Fig. 11.13

Similarly, in ΔCAB, $\dfrac{CG}{AG} = \dfrac{CF}{BF}$

or $\quad \dfrac{AG}{CG} = \dfrac{BF}{CF}$ . . . (2)

From (i) and (ii), we get

$$\frac{AE}{ED} = \frac{BF}{FC}$$

Hence proved.

Example 4 : In quadrilateral $ABCD$, Points P, Q, R and S are on the sides AB, BC, CD and DA respecively in such a way that they trisect the vertics A and C, prove that $PQRS$ is a parallelogram.

Solution : To prove *PQRS* a parallelogram, we have to prove *PQ* || *SR* and *QR* || *PS*.

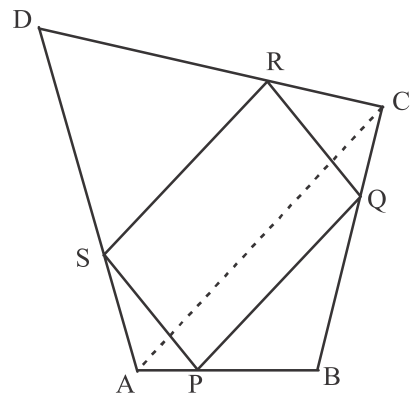


Fig. 11.14

Given : Points *P*, *Q*, *R* and *S* on the sides *AB*, *BC*, *CD* and *DA* in such a way that

$BP = 2\ PA$, $BQ = 2\ QC$, $DR = 2RC$ and $DS = 2SA$

Construction : Join *A* to *C*.

Proof : In ΔADC, $\frac{DS}{SA} = \frac{2SA}{SA} = 2$

and $\frac{DR}{RC} = \frac{2RC}{RC} = 2$ (given)

$\Rightarrow \quad \frac{DS}{SA} = \frac{DR}{RC} \Rightarrow$ SR || AC . . . (i)

$\frac{BP}{PA} = \frac{2PA}{PA} = 2$ (By basic proportionality theroem)

In ΔABC

$\frac{PQ}{QC} = \frac{2QC}{QC} = 2$ (given)

$\Rightarrow \quad \frac{BP}{PA} = \frac{BQ}{QC}$

∴ PQ || AC (By basic proportionality theroem) . . . (ii)

Now from (i) and (ii), we get

$SR \quad AC$ and $PQ \quad AC \Rightarrow SR \quad PQ$.

Similarly, joining *B* to *D*, we can also prove that *QR* || *PS*.

Hence *PQRS* is a parallelogram. **Hence proved.**

Example 5 : Diagonal of a quadrilateral *ABCD*, intersect each other at the point *O* such that $\frac{AO}{BO} = \frac{CO}{DO}$ prove that *ABCD* is a trapezium.

Solution :

Given : In ΔABCD, $\frac{AO}{BO} = \frac{CO}{DO}$ (See fig. 11.15)

To Prove : *ABCD* is a trapezium *e.g., AB* || *DC*.

Construction : Draw a line *OE*, such that *OE* || *AB*.

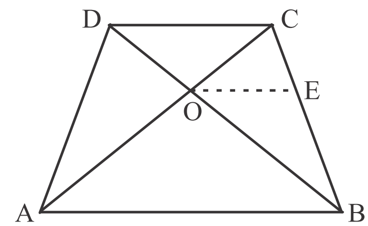


Fig. 11.15

Proof : $\frac{AO}{BO} = \frac{CO}{DO}$ (given)

or $\frac{AO}{CO} = \frac{BO}{DO}$ . . . (i)

In ΔABC, OE || AB

∴ $\frac{CO}{OA} = \frac{CE}{EB}$ (By B.P.T)

or $\frac{OA}{CO} = \frac{EB}{CE}$ . . . (ii)

from (i) and (ii), we get

$$\frac{BO}{OD} = \frac{EB}{CE}$$

or $\frac{BO}{OD} = \frac{BE}{EC}$

or OE || DC (by converse of BPT) . . . (iii)

and OE || AB (by construction) . . . (iv)

from (iii) and (iv), we get

AB || DC **Hence Proved.**

Hence ABCD is a trapezium.

Exercise 11.2

1. The points *D* and *E* are on the sides *AB* and *AC* in a *ΔABC*, Such that *DE* || *BC*, then
   (i) If *AD* =6 cm, *DB* = 9 cm and *AE* = 8 cm, then find the measure of *AC*.
   (ii) If $\frac{AD}{DB} = \frac{4}{13}$ and *AC* = 20.4 cm, find the length of *EC*
   (iii) If $\frac{AD}{DB} = \frac{7}{4}$ and *AE* = 6.3 cm, then find *AC*.
   (iv) If $AD = 4x - 3$, $AE = 8x - 7$, $BD = 3x - 1$ and $CE = 5x - 3$, then find the value of x.
2. Points *D* and *E* are on the sides *AB* and *AC* respectively in the *ΔABC* on the bases of the measures given below, state whether *DE* || *BC* or not :
   (i) *AB* = 12 cm, *AD* = 8 cm, *AE* = 12 cm and *AC* = 18 cm
   (ii) *AB* = 5.6 cm, *AD* = 1.4 cm, *AC* = 9.0 cm and *AE* = 1.8 cm
   (iii) *AD* = 10.5 cm, *BD* = 4.5 cm, *AC* = 4.8 cm and *AE* = 2.8 cm
   (iv) *AD* = 5.7 cm, *BD* = 9.5 cm, *AE* = 3.3 cm and *EC* = 5.5 cm

3. In the fig 11.16 points L. M and N are situsted on OA, OB and OC respectively such that $LM \parallel AB$ and $MN \parallel BC$, then show $LN \parallel AC$

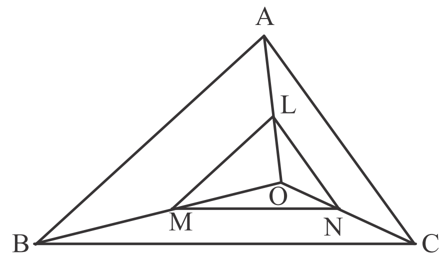


Fig. 11.16

4. In ΔABC, points D and E are respectively situated on AB and AC such that $BD = CE$. If $\angle B = \angle C$, then show that $DE \parallel BC$.

5. In the fig. 11.17. If $DE \parallel BC$ and $CD \parallel EF$, then prove $AD^2 = AB \times AF$.

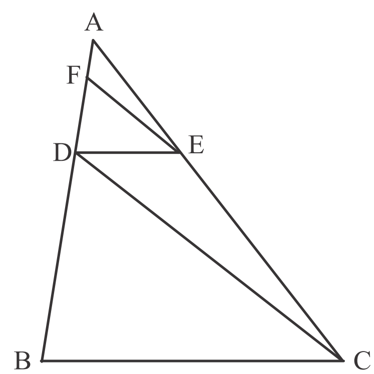


Fig. 11.17

6. In the fig. 11.18 if $EF \parallel DC \parallel AB$, then prove that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

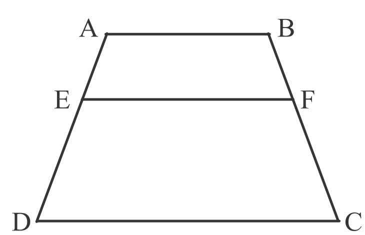


Fig. 11.18

7. $ABCD$ is a parallelogram in which point P is situated on the side BC. If DP and AB are increased then they meet at L then prove

(i) $\dfrac{DP}{PL} = \dfrac{DC}{BL}$ (ii) $\dfrac{DL}{DP} = \dfrac{AL}{DC}$

8. Point D and E are situated in the side AB in a ΔABC such that $AD = BE$. If $DP \parallel BC$ and $EQ \parallel AC$, then prove that $PQ \parallel AB$.

9. In trapezium $ABCD$, diagonals intersects each other at O and $AB \parallel DC$. Show that $\frac{AO}{BO} = \frac{CO}{DO}$

10. If the points D and E are situated on the sides AB and AC respectively in a triangle ABC. Such that $BD = CE$, then prove that ΔABC is an isosceles triangle.

11.4. Bisector of Internal and External Angles of a Triangle

We have studied about the basic proportionaity theorem and the related results. Now if any line divides the angles of a triangle, what results may be obtained. Let's study the theorems and their results given below.

Theorem 11.3

The internal bisector of an angle of a triangle divides the opposite side intertnally in the ratio of the sides containing the angle.

Given : AD is the bisector of $\angle A$ in ΔABC and so $\angle 1 = \angle 2$.

To Prove : $\frac{BD}{DC} = \frac{AB}{AC}$

Construction : Draw a line $CE \parallel DA$ and extend it to meet at E on produced BA.

Proof : $CE \parallel DA$ and AC and BE are transversals.

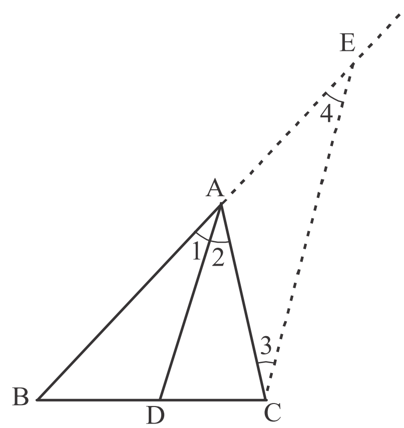


Fig. 11.19

$\therefore$ $\angle 2 = \angle 3$ (alternate angles) . . . (i)

$\angle 1 = \angle 4$ (corresponding angles) . . . (ii)

But $\angle 1 = \angle 2$ (given)

from (i) and (ii), we get

$\angle 3 = \angle 4$

$\therefore$ In ΔACE

AE = AC . . . (iii)

In ΔBCE, DA || CE

$\therefore$ $\frac{BD}{DC} = \frac{BA}{AE}$ (By B.P.T)

or $\frac{BD}{DC} = \frac{BA}{AC}$ from (iii)

Hence $\frac{BD}{DC} = \frac{BA}{AC}$ **Hence Proved.**

Theorem 11.4 : (Converse of theorem 11.3)

If a line through one vertex of a triangle divides the opposite sides in the ratio of other two sides, then the line bisects the angle at the vertex.

Given : In ΔABC, D is a point on BC such that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

To prove : AD is bisector of $\angle A$ and $\angle 1 = \angle 2$

Construction : Extend BA to E and such that $AE = AC$ and join E to C.

Proof : In ΔACE we have

$AE = AC$ (By construction)

$\therefore \angle 3 = \angle 4$ . . . (i)

$\because \dfrac{BD}{DC} = \dfrac{AB}{AC}$ (given)

$\therefore \dfrac{BD}{DC} = \dfrac{AB}{AE}$ ($\because AE = AC$ by construction)

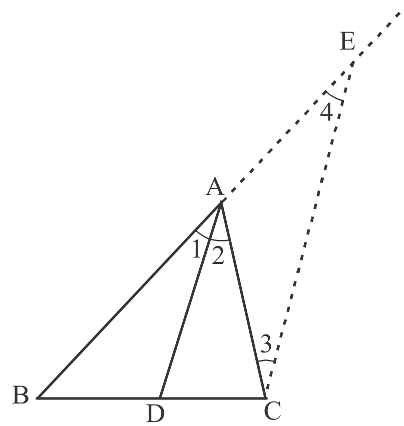


Fig. 11.20

Similarly, In ΔABC, If $\dfrac{BD}{DC} = \dfrac{AB}{AE}$ (conversed BPT)

$DA \parallel CE$ thus $\angle 1 = \angle 4$ (corresponding angles)

and $\angle 2 = \angle 3$ (alternate angles)

But $\angle 3 = \angle 4$ from (i)

$\therefore \angle 1 = \angle 2$

Hence AD is the bisector of $\angle A$ **Hence proved.**

Theorem 11.5

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

Given : In ΔABC, AD is the bisector of the exterior $\angle A$ and intersects BC produced in D.

To prove: $\dfrac{BD}{CD} = \dfrac{AB}{AC}$

Construction : Draw $CE \parallel DA$ meeting AB at E.

Proof : $CE \parallel DA$ (by constructions)

$\angle 1 = \angle 3$ (alternate angles) . . . (i)

$\angle 2 = \angle 4$ (Corresponding angles) . . . (ii)

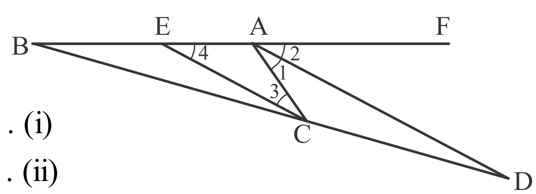


Fig. 11.21

DA and BF are transversal

$\angle 1 = \angle 2$ (given)

$\therefore \angle 3 = \angle 4$ . . .(iii)

In ΔAEC, AE = AC (Opposite sides of equal angles)

Now in ΔBAD, EC || AD

$\therefore \dfrac{BD}{CD} = \dfrac{BA}{EA}$ (properties of BPT)

$\Rightarrow \dfrac{BD}{CD} = \dfrac{BA}{AC}$ from (iii)

$\Rightarrow \dfrac{BD}{CD} = \dfrac{AB}{AC}$ **Hence proved.**

11.5. Similarity of Triangles

In the section 11.3, we havce studied that two triangles are similar if :

(i) their corresponding angles are equal and

(ii) corresponding sides are proportional. In the fig. 11.22. We have two triangles ABC and DEF in which

(i) $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

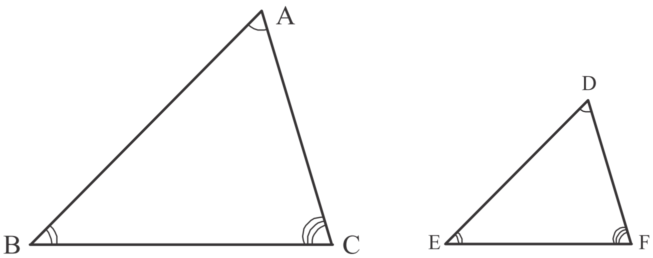


Fig. 11.22

Then ΔABC and ΔDEF are similar.

In the figures. you can see that A corresponds to D, B corresponds to E and C corresponds to F. Symbolicaly. we write the similarityof these two triangles as ΔABC ~ ΔDEF and read it as triangle ABC is similar to triangle DEF." Remember symbol '   ' stands for **'is similar to'.** Recall that you have used the symbol "≅" for 'is congurent to' in your previous class.

It must be remembered that the similarly of two triangles should also be expressed symbolically, using correct correspondence of their vertices. For example, for the ΔABC and DEF of fig. 11.22, we can not be written ΔABC ~ ΔEFD or ΔABC ~ ΔFED. We can write them as. ΔABC ~ ΔDEF or ΔBAC ~ ΔEDF or ΔBCA ~ ΔEFD.

In our previous class. You learnt about some criteria for congurency of two triangles involving only three pairs of corresponding parts of two triangles. Similarly here, we try certain criteria for similarity of two triangles involving relationship between some number of corresponding parts of the two triangles instead of all the six pairs of corresponding parts. Let us perform the following activity and see what result we obtain.

First of all we draw two line segment *BC* = 6 cm and *EF* = 4 cm. Then at the points *B* and *E* of line segments respectively *BC* and *EF* construct the angles measure 65° each and on the points *C* and *F* construct the angles 45°. In this way two triangles respectively *ABC* and *DEF* are obtained (see fig 11.23). Can you measure the third angle of these triangles? Since, sum of three internal angles of a triangle is 180°

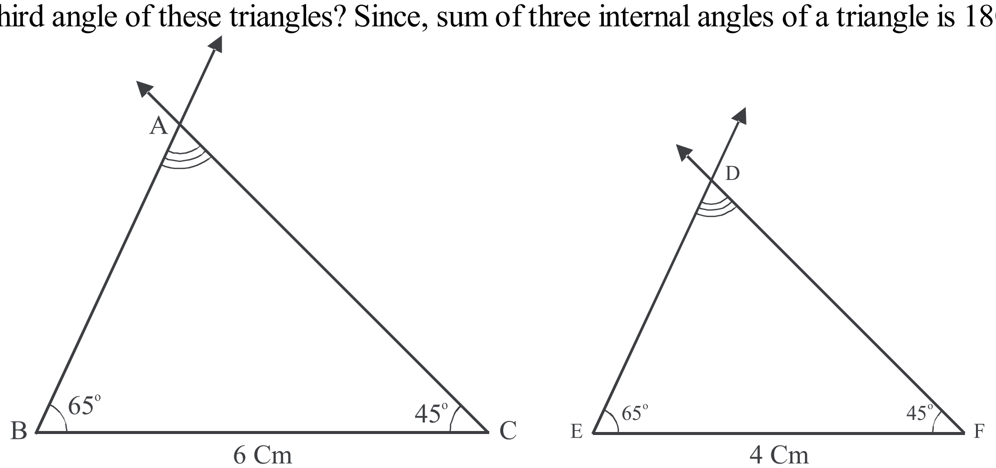


Fig. 11.23

$\angle A = 180° - (\angle B + \angle C) = 180° - (65° + 45°)$

$\angle A = 180° - 110° = 70°$

$\angle D = 180° - (\angle E + \angle F) = 180 - (65° + 45°)$

$\angle D = 180° - 110° = 70°$

At the basis of above figures, we get

$\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$

In this way we conclude that corresponding angles of ΔABC and ΔDEF are equal thus two triangle are equiangular. Now we measure the sides of these triangles and find out the ratio between corresponding sides.

$$\frac{BC}{EF} = \frac{6}{4} = 1.5, \quad \frac{AB}{DE} = \frac{4.5}{3} = 1.5 \quad \text{and} \quad \frac{AC}{DF} = \frac{5.85}{3.9} = 1.5$$

Here, AB = 4.5 cm, DE = 3 cm, AC = 5.85 cm and DF = 3.9 cm.

Thus we get $\frac{BC}{EF} = \frac{AB}{DE} = \frac{AC}{DF}$

By constructing several pairs of triangle having their corresponding angles equal, you can repeat this activity as many times as you wish but every time, you will find the same result. This activity leads us to the criterion " The ratio of corresponding sides of two equiangular triangles is always same."

Now according to the defination of similar triangles.If corresponding angles of two triangles ABC and DEF are equal and the ratio of their corresponding sides is also equal. Then two triangles are similar hence $\Delta ABC \sim \Delta DEF$.

Theorem 11.6 (AAA Criterion)

Two equiangular triangles are similar.

Given : ΔABC and ΔDEF such that

$\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Cut $DP = AB$ and $DQ = AC$ and join P to Q.

Proof : $AB = DP$ and $AC = DQ$ (by construction)

$\angle A = \angle D$ (given)

$\therefore$ $\Delta ABC \cong \Delta DPQ$ (SAS criterion)

$\therefore$ $\angle B = \angle DPQ$ $\therefore$ $\angle C = \angle DQP$

But $\angle B = \angle E$ and $\angle C = \angle F$ (given)

so $\angle DPQ = \angle E$ and $\angle DQP = \angle F$ (Corresponding angles)

$\therefore$ PQ || EF

Thus, $\frac{DP}{PE} = \frac{DQ}{QF}$ (Using the B.P.T.)

$\Rightarrow$ $\frac{PE}{DP} = \frac{QF}{DQ}$

$$\Rightarrow \quad \frac{PE}{DP}+1=\frac{QF}{DQ}+1$$

$$\Rightarrow \quad \frac{PE+DP}{DP}=\frac{QF+DQ}{DQ}$$

$$\Rightarrow \quad \frac{DE}{DP}=\frac{DF}{DQ}$$

$$\Rightarrow \quad \frac{DP}{DE}=\frac{DQ}{DF}$$

$$\Rightarrow \quad \frac{AB}{DE}=\frac{AC}{DF}$$

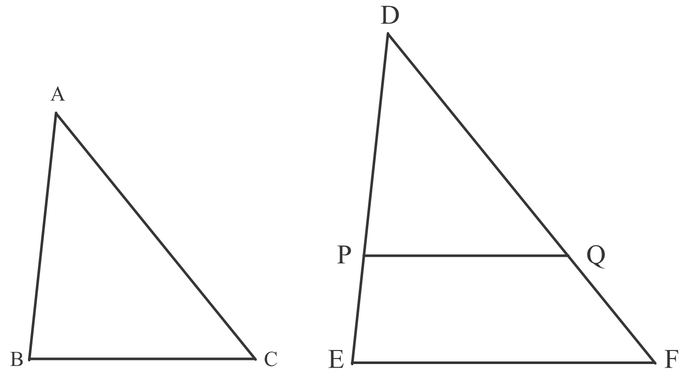


Fig. 11.24

Similarly, we can find

$$\frac{AB}{DE}=\frac{BC}{EF}$$

Thus $$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

Hence, ΔABC and ΔDEF follow the criterion of similarity of two triangles.

$\therefore$ $\Delta ABC \sim \Delta DEF$ **Hence proved,**

If two angles of a triangle are respectively equal to two angles of another triangle, then by the angle sum property of a triangle their third angles will also be equal. Thus AAA similarly criterion can also be stated as AA similarity.

If it is possible that if corresponding sides of two triangles are proportion then their corresponding angles are equal. Let us check with the help of an activity given below.

Activity : Draw two triangles *ABC* and *DEF* such that *AB* = 3cm, *BC* = 6 cm and *CA* = 8 cm, also *DE* = 4.5 cm *EF* = 9 cm. and *FD* = 12 cm and measure the degree of angles with the help of a protector. Now compare their corresponding angles.

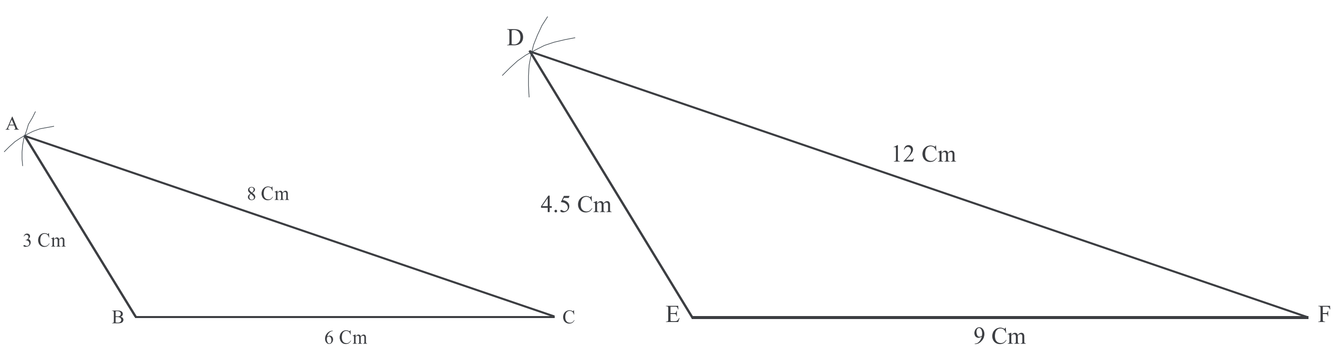


Fig. 11.25

Here, $$\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}=\frac{2}{3}$$ (ratio of the corresponding sides)

When angles are measured, we get $\angle A = \angle D = 40°, \angle B = \angle E = 120°, \angle C = \angle F = 20°$. It means the ratio of the corresponding sides of two triangles is same. their corresponding angles will also be equal. Therefore $\Delta ABC \sim \Delta DEF$.

You can repeat this activity by drawing several pairs of triangles having their sides in the same ratio. And every time you will find their corresponding angles are equal.

Let us prove this result of similarity through the theorem given below.

Theorem 11.7 : (SSS similarity criterion)

If the ratio of the corresponding sides of two triangles is equal, then the triangles are similar.

Given : In ΔABC and ΔDEF, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Cut $DP = AB$ and $DQ = AC$ and join P to Q.

Proof : $\frac{AB}{DE} = \frac{AC}{DF}$ (given)

$\Rightarrow$ $\frac{DP}{DE} = \frac{DQ}{DF}$ (By construction)

$\Rightarrow$ $PQ \parallel EF$ (Using converse of B.P.T))

$\therefore$ By AA similarity criterion

$\Delta DPQ \sim \Delta DEF$ . . . (i)

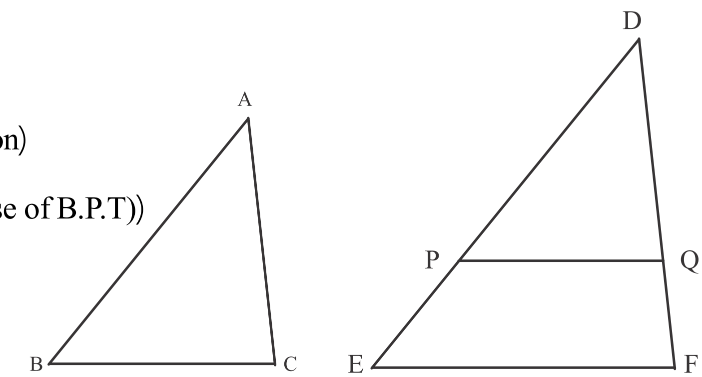


Fig. 11.26

$\Rightarrow$ $\frac{DP}{DE} = \frac{PQ}{EF}$

$\Rightarrow$ $\frac{AB}{DE} = \frac{PQ}{EF}$ ($\because$ $AB = DP$ by construction)

But $\frac{AB}{DE} = \frac{BC}{EF}$ (given)

$\therefore$ $\frac{PQ}{EF} = \frac{BC}{EF}$

$\Rightarrow$ $PQ = BC$

Similarly, in ΔABC and ΔDPQ

$AB = DP, BC = PQ$, and $AC = DQ$

From SSS similaity criterion,

$\Delta ABC \cong \Delta DPQ$ . . . (ii)

From (i) and (ii) we get

$\Delta ABC \cong \Delta DPQ$ or $\Delta DPQ \sim \Delta DEF$ (congurent ΔS are also similar)

So $\Delta ABC \sim DPQ$ and $\Delta DPQ \sim \Delta DEF$

Hence, $\Delta ABC \sim \Delta DEF$ **Hence proved,**

Theorem 11.8 : (SAS similarity criterion)

If two corresponding sides in two triangles are proportional and the angles between them are also equal, the two triangles are similar.

Given : In ΔABC and ΔDEF, $\frac{AB}{DE} = \frac{AC}{DF}$ and $\angle A = \angle D$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Cut DP from DE and DQ from DF such that $DP = AB$ and $DQ = AC$ in ΔDEF

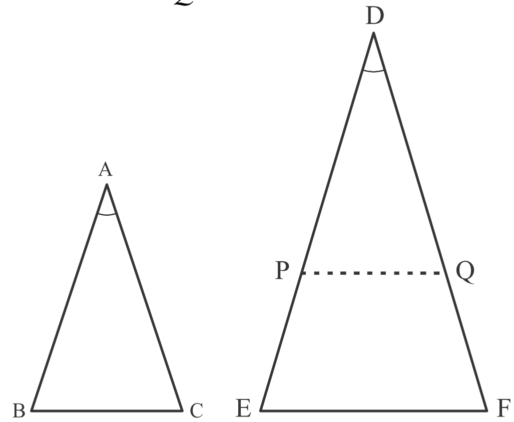


Fig. 11.27

Proof : In ΔABC and ΔDPQ

$AB = DP$

and $AC = DQ$ (by construction)

$\angle A = \angle D$ (given)

So, by SAS criterion of congurency

$\Delta ABC \cong \Delta DPQ$ . . . (i)

Now $\frac{AB}{DE} = \frac{AC}{DF}$ (given)

$\Rightarrow \frac{DP}{DE} = \frac{DQ}{DF}$ ($\because AB=DP$ and $AC=DQ$ by construction)

$\Rightarrow$ PQ || EF (converse of Thales theorem)

$\Rightarrow \angle DPQ = \angle E$ and $\angle DQP = \angle F$ (corresponding angles)

From AA similarity

$\therefore \Delta DPQ \sim \Delta DEF$ . . . (ii)

from (i) and (ii), we get

$\Delta ABC \cong \Delta DPQ$ and $\Delta DPQ \sim \Delta DEF$

(All congurent triangles are similar)

$\Rightarrow \Delta ABC \sim \Delta DEF$ **Hence proved**

Illustrative Examples

Example 1 : Find out the simillar pair in the following pairs of triangles given below, write the symbolic notation with similatriy criterion for them.

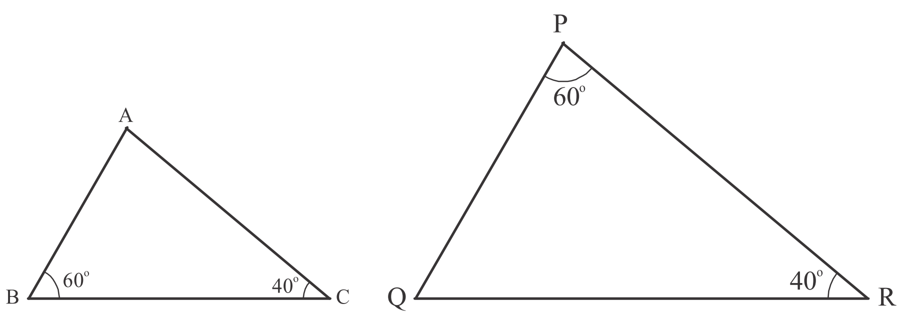


Fig. 11.28 (i)

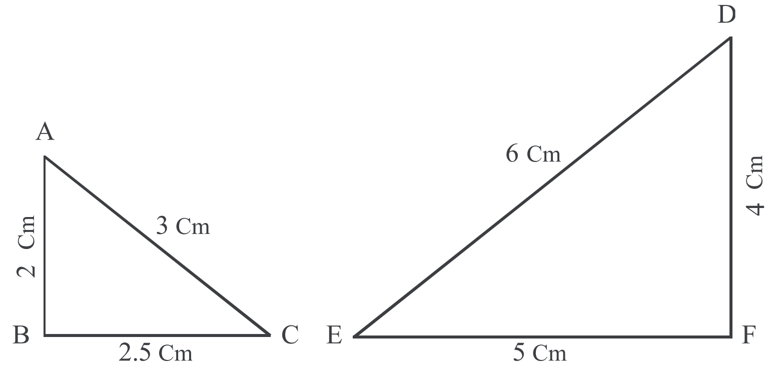


Fig. 11.28 (ii)

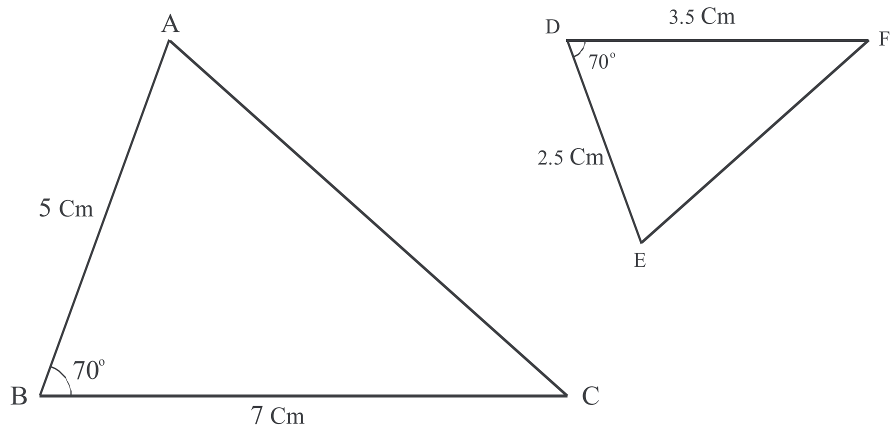


Fig. 11.28 (iii)

Solution : (i) $\Delta BCA \sim \Delta PQR$

$\because$ $\angle B = \angle P = 60°$, $\angle C = \angle R = 40°$

$\therefore$ $\angle A = 180° - (60° + 40°) = \angle Q = 80°$

$\therefore$ ΔBCA ~ ΔPRQ (By AAA similarity criterion)

(ii) In ΔABC and ΔDEF

$$\frac{AB}{DF} = \frac{BC}{FE} = \frac{CA}{ED} = \frac{1}{2}$$

$\therefore$ $\Delta ABC \sim \Delta DEF$ (By SSS similarity criterion)

(iii) In ΔABC and ΔEDF

$$\frac{AB}{DE} = \frac{BC}{DF} = 2 \text{ and } \angle ABC = \angle EDF = 70°$$

ΔABC ~ ΔEDF (By *SAS* Similarity criterion)

Example 2 : Compare ΔABC and ΔDEF and find the degree measures $\angle D, \angle E$ and $\angle F$.

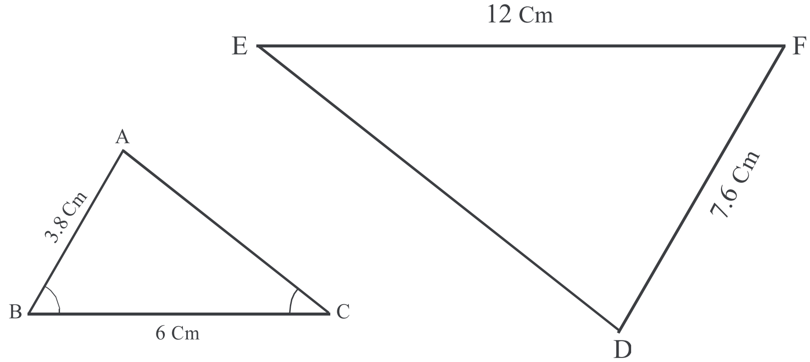


Fig. 11.29

Solution : In ΔABC and ΔDEF

$$\frac{AB}{DF} = \frac{BC}{FE} = \frac{CA}{ED} = \frac{1}{2}$$

∴ By SSS similarity criterion

ΔABC ~ ΔDEF

$\Rightarrow \quad \angle A = \angle D, \ \angle B = \angle F$ and $\angle C = \angle E$

$\Rightarrow \quad \angle F = 60°, \angle E = 40°$

$\Rightarrow \quad \angle D = 180° - (60° + 40°) = \angle A = 80°$

Hence $\quad \angle D = 80°, \angle E = 40°$ and $\angle F = 60°$

Example 3 : In the fig. 11.30, if $OA.OB = OC.OD$ then show that $\angle A = \angle C$ and $\angle B = \angle D$.

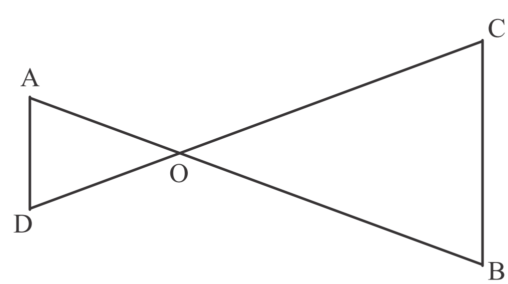


Fig. 11.30

Solution : In ΔAOD and ΔBOC, we have

$OA.OB = OC.OD$ (given)

∴ $\quad \frac{OA}{OD} = \frac{OC}{OB}$ . . .(i)

and $\quad \angle AOD = \angle COB$ (vertically opposite angles) . . . (ii)

From (i) and (ii), we get

$\Delta AOD \sim \Delta COB$

∴ $\quad \angle A = \angle C$ and $\angle D = \angle B$

(Corresponding angles of similar triangles) **Hence proved.**

Example 4 : In the fig. 11.31 *QA* and *PB* are perpendicular to *AB*. If *AB* = 16 cm, $OQ = 5\sqrt{3}$ cm and $OP = 3\sqrt{13}$ cm, then find the length of *AO* and *BO*.

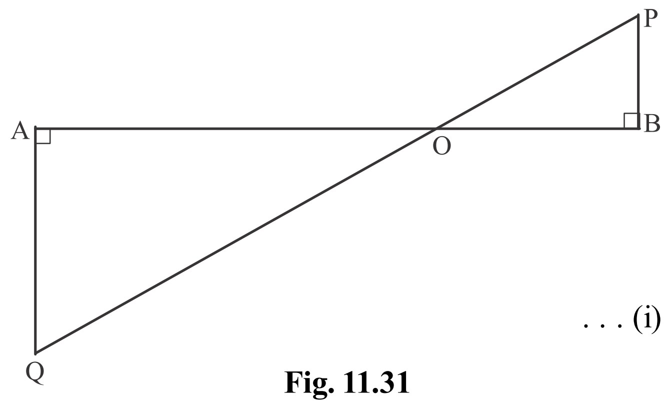


Fig. 11.31

Solution : In ΔAOQ and ΔBOP

$\angle OAQ = \angle OBP = 90°$

$\angle AOQ = \angle BOP$

(vertically opposite angles)

∴ By *AAA* similarity criterion.

$$\frac{AO}{BO} = \frac{OQ}{OP} = \frac{AQ}{BP} \qquad \text{. . . (i)}$$

But AB = AO + BO = 16 cm

Let AO = *x* ⇒ BO = 16 – *x*.

$$\frac{x}{16-x} = \frac{OQ}{OP} \qquad \text{(form (i))}$$

$$\Rightarrow \quad \frac{x}{16-x} = \frac{5\sqrt{13}}{3\sqrt{13}}$$

$$\Rightarrow \quad 3x = 80 - 5x$$

$$\Rightarrow \quad 8x = 80$$

$$\Rightarrow \quad x = 10$$

∴ *AO* = 10 cm and *BO* = 16 – 10 = 6 cm.

Hence *AO* = 10 cm and *BO* = 6 cm.

Example 5 : In the adjoining fig. 11.32, $\angle ADE = \angle B$ and *AD* = 3.8 cm, *AE* = 3.6 cm, *BE* = 2.1 cm and *BC* = 4.2 cm. find *DE*.

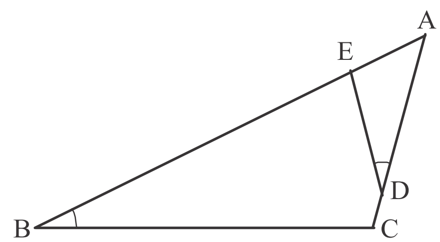


Fig. 11.32

Solution : In triangles *ADE* and *ABC*

$\angle ADE = \angle B$ (given) and

$\angle A = \angle A$ (common)

∴ By *AA* similarity criterion

$\Delta ADE \sim \Delta ABC$

$$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$$

or $$\frac{AD}{AE+EB} = \frac{DE}{BC}$$

or $$\frac{3.8}{3.6+2.1} = \frac{DE}{4.2}$$

or $$DE = \frac{3.8 \times 4.2}{5.7} = \frac{15.96}{5.7} = 2.8$$

Hence, the length of *DE* = 2.8 cm

Example 6 : In the given figure 11.33 $ABCD$ is a trapezium in which $AB \parallel DC$. If $\Delta AED \sim \Delta BEC$ then prove that $AD = BC$

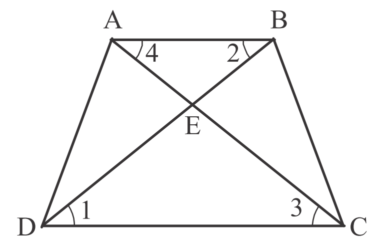


Fig. 11.33

Solution : In the triangles ΔEDC and ΔEBA

$\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (alternate angles)

$\therefore$ by AAA similarity criterion

$\Delta EDC \sim \Delta EBA$

So $\frac{ED}{EB} = \frac{EC}{EA}$

or $\frac{ED}{EC} = \frac{EB}{EA}$ . . . (i)

Since, $\Delta AED \sim \Delta BEC$

$\therefore \frac{AE}{BE} = \frac{ED}{EC} = \frac{AD}{BC}$ . . . (ii)

From (i) and (ii) $\frac{EB}{EA} = \frac{AE}{BE}$

$\Rightarrow (BE)^2 = (AE)^2$

$\Rightarrow BE = AE$

Substituting this value in (ii)

$\frac{AE}{AE} = \frac{AD}{BC}$

$\Rightarrow \frac{AD}{BC} = 1$

or $AD = BC$ **Hence Proved.**

Example 7 : In the parallelogram $ABCD$, M is the mid point of CD and joining line B to M intersects AC at L. If BM meets AD produced at E, then prove that $EL = 2\ BL$.

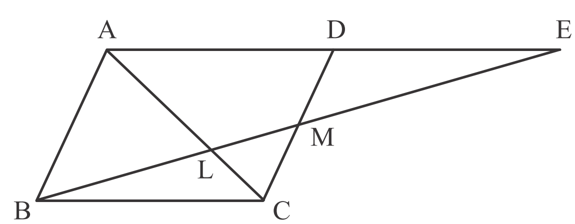


Fig. 11.34

Solution : In ΔBMC and ΔEMD

$MC = MD$ ($\because M$ is the mid point of CD)

$\angle CMB = \angle DME$ (vertically opposite angles)

$\angle MCB = \angle MDE$ (alternate angles)

$\therefore$ By ASA congruency

$\Delta BMC \cong \Delta EMD$

$\therefore$ BC = ED AD = BC ($\because ABCD$ is a parallelogram)

and AE = AD + DE

or AE = BC + BC

or AE = 2BC . . . (1)

Now, in ΔAEL and ΔCBL

$\angle ALE = \angle CLB$ (vertically opposite angles)

$\angle EAL = \angle BCL$ (alernate angles)

So. By AA simialrity criterion

$$\frac{EL}{BL} = \frac{AE}{CB}$$

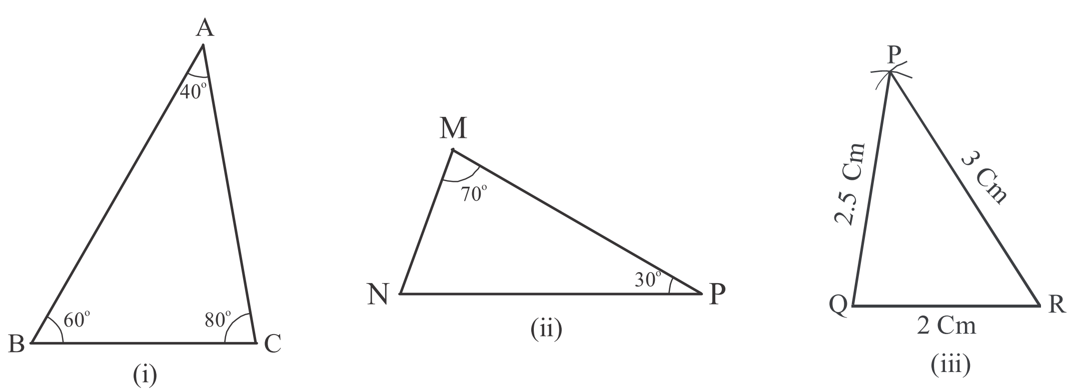
$$\Rightarrow \quad \frac{EL}{BL} = \frac{2BC}{BC}$$

$$\Rightarrow \quad \frac{EL}{BL} = 2$$

$\Rightarrow$ EL = 2 BL **Hence proved.**

Exercise 11.3

1. In ΔABC and ΔPQR, $\frac{AB}{PQ} = \frac{BC}{QR}$, State any two angles of two triangles which must be equal so that both triangles may be similar. Justify your answer.
2. In ΔABC and ΔDEF, $\angle A = \angle D, \angle B = \angle F$, state whether $\Delta ABC \sim \Delta DEF$? Give proper reason in support to your answer.
3. In $\Delta ABC \quad \Delta FDE$, state whether $\frac{AB}{DE} = \frac{BC}{EF}$ and $\frac{CA}{FD}$ is true or not? Justify your answer.
4. If two sides of a triangle are proportational to two sides of another triangles also one angle of first triangle is equal to one angle of other triangle. Then both triangles are similar. Is this statement true? Give reasons to support your answer.
5. What do you understand by equiangular triangles? What relation may be among them?
6. Find out the similar triangles out of the following figure 11.35 given below and write their similarity in symbolic notation.



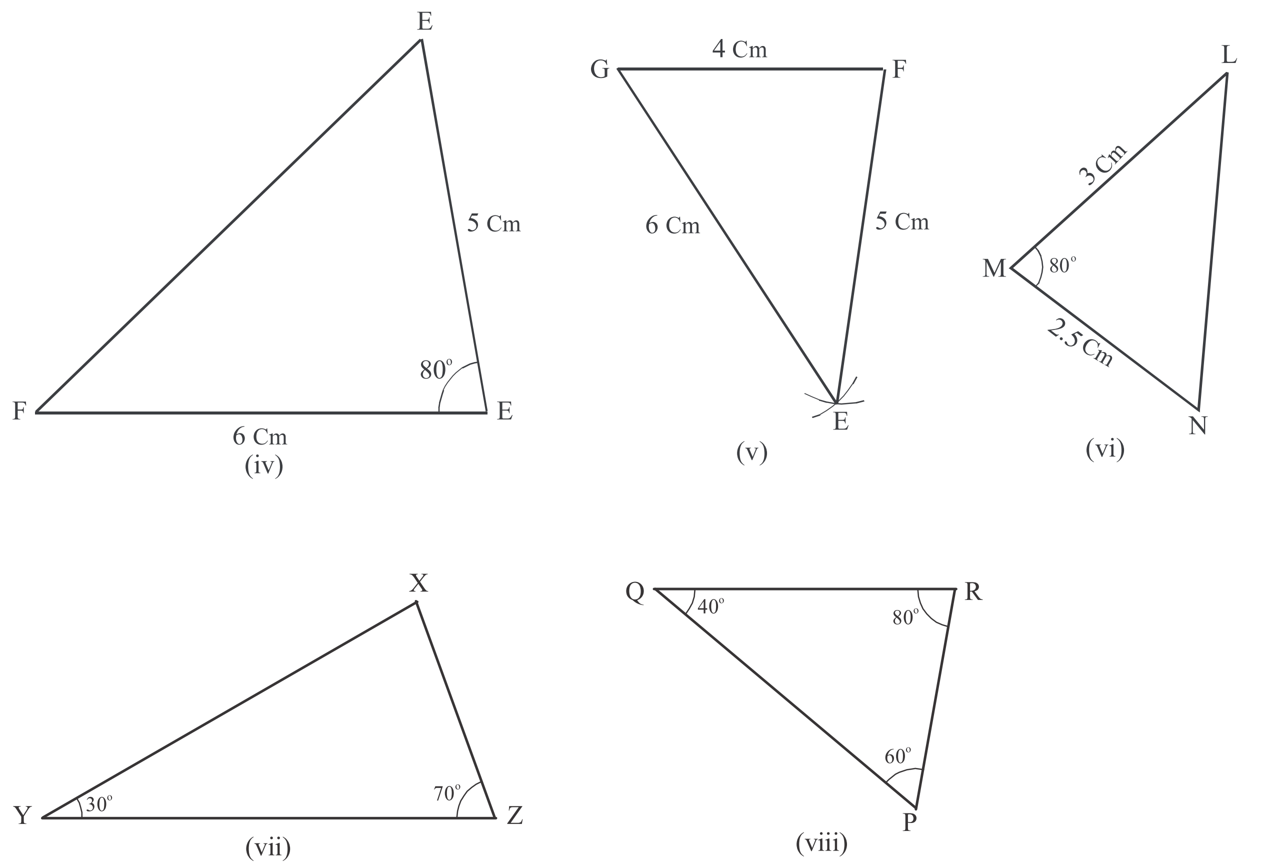


Fig. 11.35

7. In the adjoiniung fig. 11.36 $\Delta PRQ \sim \Delta TRS$, state which angles should be equal to each other in the given pair of similar triangles?

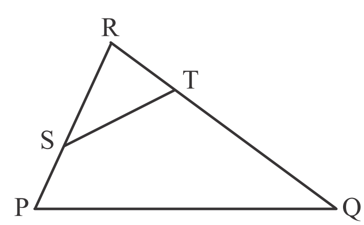


Fig. 11.36

8. You have to choose two similar triangles out of the following triangles which are similar to each other if $\angle CBE = \angle CAD$.

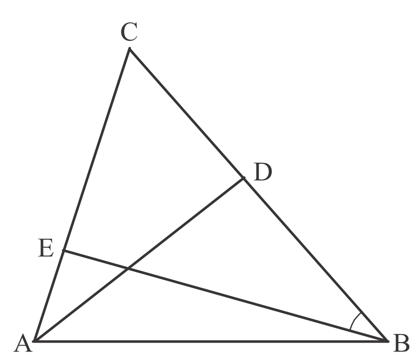


Fig. 11.37

9. In the fig. 11.38 given below PQ RS, then prove that $\Delta POQ \sim \Delta SOR$

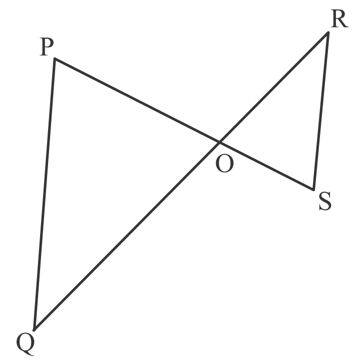


Fig. 11.38

10. A girl of height 90 cm is walking away from the base of a long bulbs put pole at a speed of 1.2 m/s. If bulb is put at the height of 3.6 m above the ground, find the length of her shadow after 4 minutes.
11. The length of the shadow of a vertical pole 12 m high from the ground is 8 m. At the same time the shadow of the vertical minaret is 56 meter long. Find the height of the minaret.
12. If perpendicular is drawn from vertex A to opposite side BD in a triangle ABC, we get $AD^2 = BD \times DC$, then prove that ABC is a right angled triangle.
13. Prove that four triangles obtained by joining the mid-points of three sides ofA triangle are similar to the original triangle.
14. According to the fig. if $AB \perp BC, DC \perp BC$ and $DE \perp AC$, then prove $\Delta CED \sim \Delta ABC$.

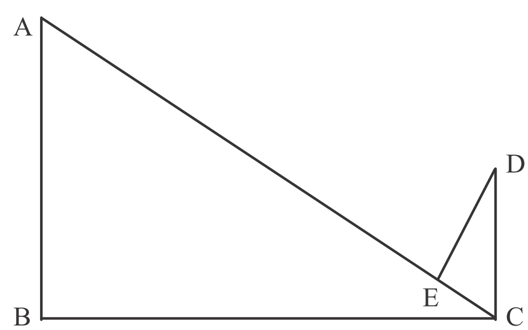


Fig. 11.39

15. In ΔABC, D is the mid-point BC. If a line is drawn in such a way that it bisects AD and cut AD and AC at E and X respectively, then prove that $\dfrac{EX}{BE} = \dfrac{1}{3}$.

11.5.2. Areas of Two Similar Triangles

You have learnt that in two similar triangles, the ratio of their corresponding sides is same. Now we shall study the relationship between the ratio of their area and the ratio of the corresponding sides. The area is always measured in square units, hence this ratio is the square of the ratio of their corresponding sides.

Theorem 11.13

The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Given : ΔABC ~ ΔDEF

To prove : $$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF}=\frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{AC^2}{DF^2}$$

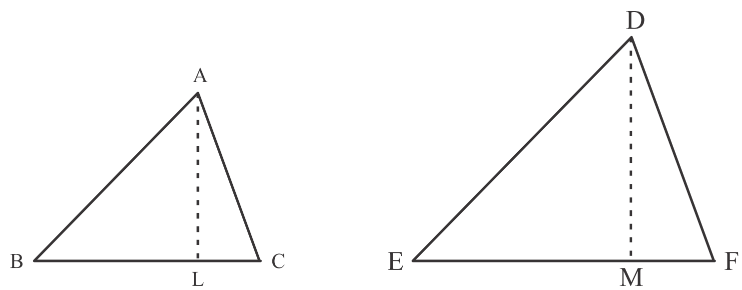


Fig. 11.40

Construction : Draw $AL \perp BC$ and $DM \perp EF$

Proof : ΔABC ~ ΔDEF (given)

$\therefore$ $\angle A=\angle D, \angle B=\angle E,$ and $\angle C=\angle F$ . . . (i)

Also $$\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}$$ . . . (ii)

Now in ΔALB and ΔDME

$\angle ALB=\angle DME$ (90° each)

$\angle B=\angle E$ (corresponding angles of similar triangles)

$\therefore$ $\Delta ALB \sim \Delta DME$ (AA similarity criterion)

$\Rightarrow$ $$\frac{AL}{DM}=\frac{AB}{DE}$$ . . . (iii)

from (ii) and (iii) we get

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}=\frac{AL}{DM}$$ . . . (iv)

Now, $$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF}=\frac{\frac{1}{2}BC\times AL}{\frac{1}{2}EF\times DM}$$ (area of triangle $=\frac{1}{2}\times$ base $\times$ height)

$$=\frac{BC}{EF}\times\frac{AL}{DM}=\frac{BC}{EF}\times\frac{BC}{EF}=\frac{BC^2}{EF^2}$$

But $$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

$$\Rightarrow \qquad \frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{AC^2}{DF^2}$$

$$\Rightarrow \qquad \frac{\text{area of } \Delta ABC}{\text{area of } \Delta DEF}=\frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{AC^2}{DF^2}$$

Hence Proved.

With the help of this theorem we can also find the other results.

Corollary 11.2. The ratio of the areas of two similar triangles is equal to the ratio of the square of the perpendiculars drawn form corresponding vertex to the opposite sides.

Corollary 11.3. The ratio of areas of two similar triangles is equal to the square of their corresponding medians.

Corollary 11.4. The ratio of the areas of two similar triangles is equal to the ratio of the square of their corresponding angles bisectors.

11.5.3. Justification of Bodhayan theorem (pythagoras theorem) By similarity criterion

In your earlier classes, you have learnt about the Bodhayan theorem. You have proved it in class IX and solved many related problems. Now, using the similarity criterion of triangles we shall prove it again.

Theorem 11.9

In a right triangle, the square of hypotenuse is equal to the sum of the square of remaining two sides.

Given : ΔABC right angled at B.

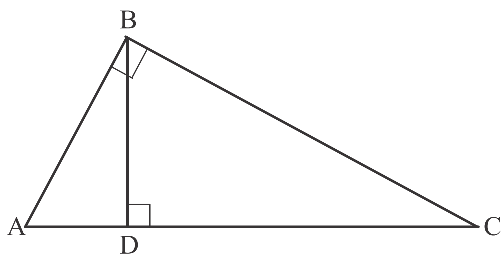


Fig. 11.41

To prove : $AC^2 = AB^2 + BC^2$

Construction : Draw $BD \perp AC$

Proof : In ΔADB and ΔABC

$\angle ADB = \angle ABC$ ($90°$ each, by construction)

$\angle A = \angle A$ (common)

By AA similarly criterion

$\Delta ADB \sim \Delta ABC$

$\Rightarrow \quad \frac{AD}{AB}=\frac{AB}{AC}$ (By BPT)

$\Rightarrow \quad AB^2 = AC \times AD$ .....(i)

In ΔABC and ΔCDB

$\angle ABC = \angle CDB$ ($90°$ each by construction)

$\angle C = \angle C$ (common)

By AA similarity criterion

$$\Delta ABC \sim \Delta BCD$$

$$\frac{BC}{AC} = \frac{CD}{CB} \qquad \text{(By BPT)}$$

$$\Rightarrow \quad \frac{BC}{AC} = \frac{DC}{BC}$$

$$\Rightarrow \quad BC^2 = AC \times DC \qquad .....(ii)$$

By adding (i) and (ii)

$$AB^2 + BC^2 = AC \times AD + AC \times DC$$

$$\Rightarrow \quad AB^2 + BC^2 = AC\ (AD + DC)$$

$$\Rightarrow \quad AB^2 + BC^2 = AC \times AC$$

$$\Rightarrow \quad AC^2 = AB^2 + BC^2$$

Hence Proved.

Let us prove to converse of this theorem by using same criterion.

Theorem 11.10

Converse of Bodhayan theorem :

In a triangle, if square of one side is equal to the sum of the square of the other sides, then the angle opposite the first side is a right angle.

In a triangle, if sum of the square of two sides is equal to the square of third side the triangle is right angled.

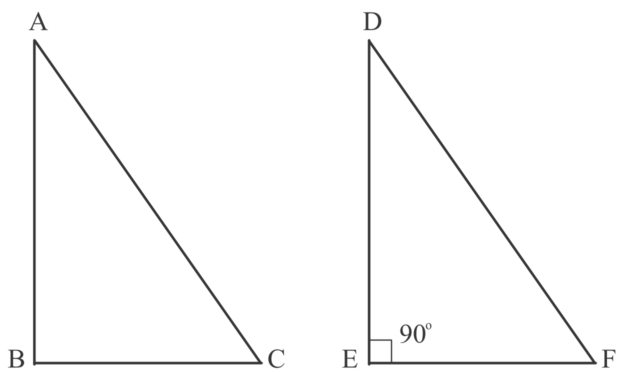


Fig. 11.42

Given : In a ΔABC $AC^2 = AB^2 + BC^2$

To prove : ΔABC is a right angled triangle.

Construction : Construct a ΔDEF, such that

$$DE = AB,\ EF = BC \text{ and } \angle E = 90°$$

Proof : $DF^2 = DE^2 + EF^2$ (By Bodhayan theorem)

$\Rightarrow \quad DF^2 = AB^2 + BC^2$ (By construction)

But $\quad AC^2 = AB^2 + BC^2$ (given)

$\therefore \quad AC^2 = DF^2$

$\Rightarrow \quad AC = DF$ ...(1)

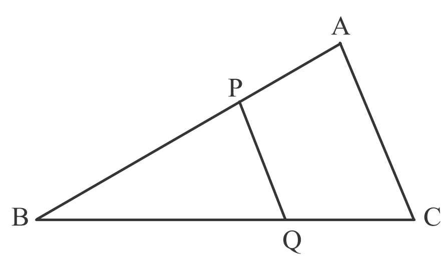


Fig. 11.43

Now, in ΔDEF

	$AB = DE,\ BC = EF$	(By construction)
and	$AC = DF$	(By i)
	$\Delta ABC \cong \Delta DEF$	
$\Rightarrow$	$\angle B = \angle D = 90°$	

Hence ΔABC is right angled.

11.5.4. Some Important results based on Bodhayan theorem

Theorem11.11

In an obtuse angled ΔABC in which $\angle B$ is the obtuse angle and $AD \perp BC$, then

$$AC^2 = AB^2 + BC^2 + 2BC \times BD$$

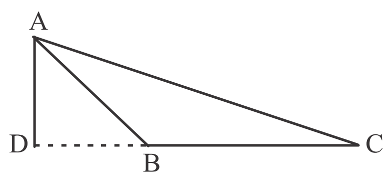


Fig. 11.43

In an obtuse triangle, the square of the side opposite to obtuse angle is equal to the sum of the squares of other two side plus twice the product of one side and the projection of other on first.

Given : ΔABC, obtuse angled at B.

To prove : $AC^2 = AB^2 + BC^2 + 2BC \times DB$

Proof : In $\Delta ADB,\ \angle D = 90°$ (given)

$\therefore$ $AB^2 = AD^2 + DB^2$ .....(i)

Now in ΔADC

$AC^2 = AD^2 + DC^2$

$\Rightarrow$ $AC^2 = AD^2 + (DB + BC)^2$

$\Rightarrow$ $AC^2 = AD^2 + DB^2 + BC^2 + 2DB \times BC$

$\Rightarrow$ $AC^2 = [AD^2 + DB^2] + BC^2 + 2DB \times BC$

$\Rightarrow$ $AC^2 = AB^2 + BC^2 + 2BC \times DB$ **Hence Proved**

Theorem 11.12 :

In an acute triangle ABC, $AD \perp BC$ then $AC^2 = AB^2 + BC^2 - 2BC \times BD$.

In an acute triangle, the square of the side opposite to an acute angle is equal to the sum of the squares of other two side minus twice the product of one side and the projection of other on first.

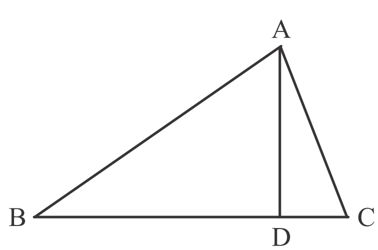


Fig. 11.44

Given : Acute ΔABC in which $AD \perp BC$

To prove : $AC^2 = AB^2 + BC^2 - 2BC \times BD$

Proof : In ΔABD

$$AB^2 = AD^2 + BD^2 \quad .....(i)$$

Similarly $AC^2 = AD^2 + DC^2$

$\Rightarrow$ $AC^2 = AD^2 + (BC - BD)^2$

$\Rightarrow$ $AC^2 = AD^2 + BC^2 + BD^2 - 2BC \times BD$ From (i)

$\Rightarrow$ $AC^2 = (AD^2 + BD^2) + BC^2 - 2BC \times BD$

$\Rightarrow$ $AC^2 = AB^2 + BC^2 - 2BC \times BD$

Hence $AC^2 = AB^2 + BC^2 - 2BC \times BD$ **Hence Proved.**

Corollary : In any triangle the sum of the square of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

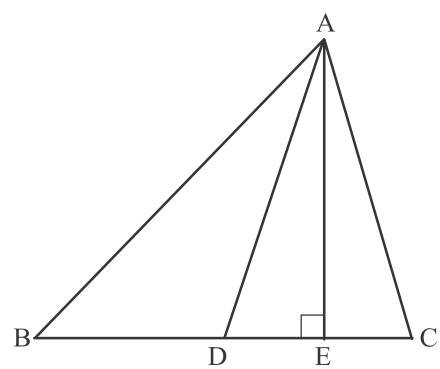


Fig. 11.45

Given : AD is median in the ΔABC.

To Prove : $AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$

Construction : Draw $AE \perp BC$

Proof : Since $\angle AED = 90°$, therefore in ΔADE

$\angle ADE < 90° \Rightarrow \angle ADB > 90°$

Thus ΔADB is an obtuse angled triangle

and ΔADC is an acute angled triangle

$\therefore$ In obtuse ΔABD, Producing BD and draw $AE \perp BD$.

$AB^2 = AD^2 + BD^2 + 2BD \times DE$ (By theorem 11.11) . . .(i)

In acute ΔACD, $AE \perp CD$

$AC^2 = AD^2 + DC^2 - 2DC \times DE$

$\Rightarrow$ $AC^2 = AD^2 + BD^2 - 2BD \times DE$ [$\because$ CD = BD] . . . (2)

Adding (i) and (ii), we get

$AB^2 + AC^2 = AD^2 + BD^2 + 2BD \times DE + AD^2 + BD^2 - 2BD \times DE$

$\Rightarrow \quad AB^2 + AC^2 = 2AD^2 + 2BD^2$

$\Rightarrow \quad AB^2 + AC^2 = 2AD^2 + 2\left(\frac{BC}{2}\right)^2$

$\Rightarrow \quad AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$

Hence $\quad AB^2 + AC^2 = 2\left[AD^2 + \left(\frac{BC}{2}\right)^2\right]$

or $\quad AB^2 + AC^2 = 2(AD^2 + BD^2)$ **Hence Proved**

Illustrative Examples

Example 1 : A ladder 10 meter long reaches a window 8 m above the ground. Find the distane of the foot of the ladder form the base of the wall.

Solution : According to the fig. ΔABC is right angled triangle in which $\angle B = 90°$

Now,By the Bodhayan theorem

$AC^2 = AB^2 + BC^2$

$\Rightarrow \quad BC^2 = AC^2 - AB^2$

$\Rightarrow \quad BC^2 = 10^2 - 8^2$

$\Rightarrow \quad BC^2 = 100 - 64$

$\Rightarrow \quad BC^2 = 36$

$\Rightarrow \quad BC = \sqrt{36} = 6$ m.

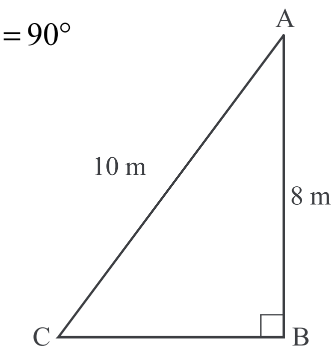


Fig.11.46

Example 2 : An aeroplane leaves an airport and flies due to north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart the two planes after $1\frac{1}{2}$ hours?

Solution : Speed of the first aeroplane = 1000 km/h

Distance covered by it in $1\frac{1}{2}$ hour to north $= 1000 \times \frac{3}{2} = 1500$ km.

Speed of the second areoplane = 1200 km/h

Distance covered by it in $1\frac{1}{2}$ hour to east and = $1200 \times \frac{3}{2} = 1800$ km.

In this way we obtain a right triangle as shown in the fig. 11.47. using Bodhayan theorem, we get.

$AB^2 = AC^2 + BC^2$

$AB^2 = 1500^2 + 1800^2$

$= 2250000 + 3240000$

$= 5490000$

$AB = \sqrt{5490000} = 2343.1$ km. (approx.)

Hence, the distance between the two areoplanes

= **2343.1 km.** (approx.)

Example 3 : If $\Delta ABC \sim \Delta DEF$ in which AB = 2.2 cm and DE = 3.3 cm find the ratio of the areas of ΔABC and ΔDEF.

Solution : We know that the ratio of the areas of two similar triangles is equal to squares of their corresponding sides.

So, $$\frac{area\ of\ \Delta ABC}{area\ of\ \Delta DEF} = \frac{AB^2}{DE^2} = \frac{(2.2)^2}{(3.3)^2} = \left(\frac{22}{33}\right)^2$$

$$\Rightarrow \quad \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Hence, the ratio of the areas of two triangles = 4 : 9

Example 4 : If areas of two similar triangles ABC abd PQR are 36 cm^2 and 43 cm^2 respectively, find the ratio of their corresponding sides.

Solution : We know that the ratio of corresponding sides of two similar triangles is equal to the ratio of their areas.

$$\therefore \quad \frac{AB^2}{PQ^2} = \frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta PQR}$$

$$\Rightarrow \quad \frac{AB^2}{PQ^2} = \frac{36}{49} \Rightarrow \frac{AB}{PQ} = \sqrt{\frac{36}{49}} = \frac{6}{7}$$

Hence, the ratio of the corresponding sides = 6 : 7

Example 5 : $\Delta ABC \sim \Delta PQR$ area ΔABC = 16 cm and area ΔPQR = 9 cm also if AB = 2.1 cm, then find the length of the side PQ.

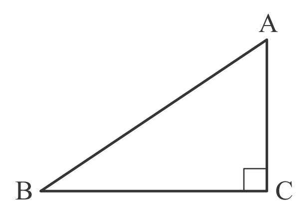


Fig. 11.47

Solution : We know, $$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta PQR} = \frac{(AB)^2}{(PQ)^2}$$

$$\Rightarrow \quad \frac{16}{9} = \frac{(2.1)^2}{PQ^2}$$

Taking square root of two sides, we get

$$\Rightarrow \quad \frac{4}{3} = \frac{2.1}{PQ}$$

$$\Rightarrow \quad PQ = \frac{2.1 \times 3}{4} = \frac{6.3}{4} = 1.575 \text{ cm.}$$

Hence, the length of PQ = 1.6 cm (approx)

Example 6 : In fig. 11.48, in $\Delta ABC,$ line parallel to BC intersects AB and AC at D and E respectively and divides AB in the ratio 1:2, find the ratio of the areas of trapezium BDEC and ΔADE formed.

Solution : Since $\ell \parallel BC$

So, $\angle ADE = \angle B$ and $\angle AED = \angle C$ (corresponding angles)

In ΔADE and ΔABC

$\angle ADE = \angle B$

and $\angle AED = \angle C$

$\therefore$ $\Delta ADE \sim \Delta ABC$ (By AA similarity criterion)

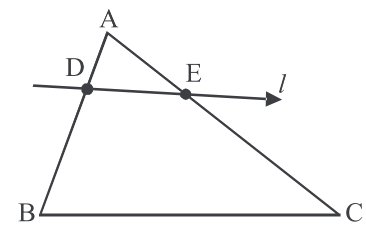


Fig. 11.48

So, $$\frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta ABC} = \frac{AD^2}{AB^2} \quad \ldots (i)$$

But $$\frac{AD}{AB} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{AD}{AD + DB} = \frac{1}{1+2} = \frac{1}{3} = \frac{AD}{AB} \quad \ldots (ii)$$

From (i) and (ii), we get

$$\frac{Area\ of\ \Delta ADE}{Area\ of\ \Delta ABC} = \frac{1^2}{3^2} = \frac{1}{9}$$

$\Rightarrow$ Area ΔABC = 9 × Area ΔADE . . . (iii)

But area trapezium $BDEC$ = area ΔABC – area ΔADE

from (iii) we get

Area trapezium $BDEC$ = 9 × area ΔADE – area ΔADE = 8 × area ΔADE

$$\Rightarrow \quad \frac{area\ trapezium\ BDEC}{area\ \Delta ADE} = \frac{8}{1}$$

Hence, the ratio of areas of trapezium BDEC and ΔADE = 8:1

Example 7 : In fig. 11.49, in a ΔABC, a line segment PQ is drawn parallel to side AC, which intersects AB such that $\frac{BP}{BA} = \frac{1}{\sqrt{2}}$, then prove that PQ, also divides the triangle ABC in the same ratio.

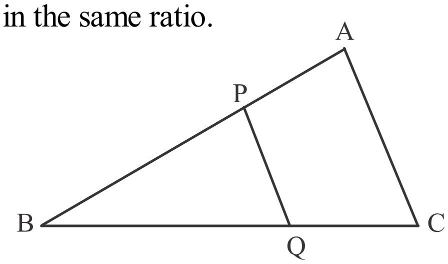


Fig. 11.49

Solution : **Given :** $PQ \parallel AC$

$\therefore$ $\angle A = \angle BPQ$

and $\angle C = \angle BQP$ (corresponding angles)

Also. $$\frac{BP}{BA} = \frac{1}{\sqrt{2}}$$

So, $\Delta BAC \sim \Delta BPQ$ (By AA similarity criterion)

To prove: area ΔBPQ = area trapezium $PACQ$

or area trapezium $PACQ$ $= \frac{1}{2} \Delta BAC$

= area of ΔBPQ (given)

Hence, we are to prove $2 \times$ area ΔBPQ = Area of ΔBAC

Proof : Since $\Delta BAC \sim \Delta BPQ \Rightarrow \Delta BPQ \sim \Delta BAC$

$$\therefore \quad \frac{Area\ of\ \Delta BPQ}{Area\ of\ \Delta BAC} = \frac{(1)^2}{(\sqrt{2})^2}$$

$$\Rightarrow \quad \frac{Area\ of\ \Delta BPQ}{Area\ of\ \Delta BAC} = \frac{1}{2}$$

$\Rightarrow \quad 2\times$ area ΔBPQ = area ΔBAC **Hence Proved**

Example 8 : Prove that if area of two simialr triangles are equal, they are congurent.

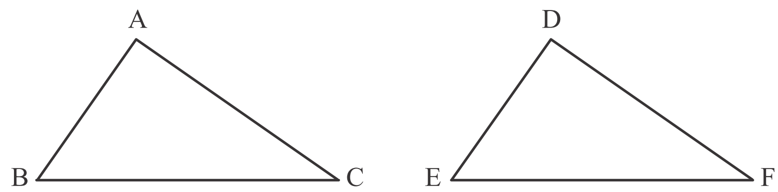


Fig. 11.50

Solution : **Given :** $\Delta ABC \sim \Delta DEF$

and area ΔABC = area ΔDEF

To prove : $\Delta ABC \cong \Delta DEF$

Proof : $\because$ $\Delta ABC \sim \Delta DEF$

$\therefore$ ΔABC and ΔDEF are equilateral triuangles

$$\frac{Area\ of\ \Delta ABC}{Area\ of\ \Delta DEF} = \frac{BC^2}{EF^2}$$

$\Rightarrow \quad 1 = \dfrac{BC^2}{EF^2}$ (Areas of two triangles are equal)

$\Rightarrow \quad BC^2 = EF^2$ or $BC = EF$ . . . (i)

Now in triangles ΔABC and ΔDEF, we have

$\angle B = \angle C$ (given)

$BC = EF$ (proved)

$\angle C = \angle F$ (given)

$\therefore$ $\Delta ABC \cong \Delta DEF$ (By *SAS* congurency)

Example 9 : ΔABC is a isoceles triangle with $\angle C = 90°$. Prove that $AB^2 = 2AC^2$.

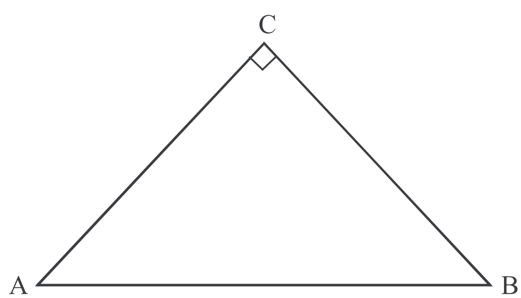


Fig. 11.51

Solution : We have ΔABC in which

$$\angle C = 90°, AC = BC$$

Now using Bodhayan theorem, we get

$$AB^2 = AC^2 + BC^2$$

$\Rightarrow$ $AB^2 = AC^2 + AC^2$ (since $BC = AC$)

$\Rightarrow$ $AB^2 = 2AC^2$ **Hence Proved.**

Example 10 : If D is a point on the side BC of a equilateral ΔABC such that $BD = \frac{1}{3}BC$ then prove that $9AD^2 = 7AB^2$.

Solution : Given : ΔABC is equilateral triangle

Construction : $AE \perp BC$ from A

To Prove : $9AD^2 = 7AB^2$

Proof : $\because$ In an equilateral triangle perpendicular drawn form a vertex bisects the opposite side.

$\Rightarrow$ $BE = EC = \frac{1}{2}BC$ (By construction)

and $BD = \frac{1}{3}BC$ (given)

Also AB = BC = CA (given)

In right ΔABE, $AB^2 = AE^2 + BE^2$

$\Rightarrow$ $AE^2 = AB^2 - BE^2$

$\Rightarrow$ $AE^2 = AB^2 - \left(\frac{1}{2}BC\right)^2$ $\left[\because BE = \frac{1}{2}BC\right]$

$\Rightarrow$ $AE^2 = AB^2 - \frac{BC^2}{4}$

$\Rightarrow$ $AE^2 = \frac{4AB^2 - BC^2}{4}$ . . . (i)

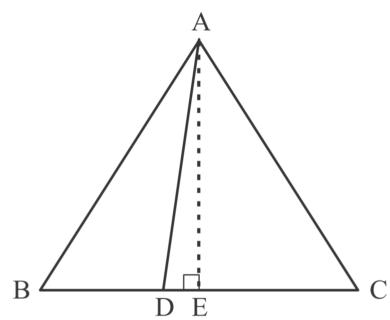


Fig. 11.52

In right angled ΔADE.

$\Rightarrow$ $AD^2 = AE^2 + DE^2$

$\Rightarrow$ $AE^2 = AD^2 - DE^2$

$\Rightarrow$ $AE^2 = AD^2 - (BE - BD)^2$

$\Rightarrow$ $AE^2 = AD^2 - \left(\frac{1}{2}BC - \frac{1}{3}BC\right)^2$ [$\because$ BE = $\frac{1}{2}$ BC एवं BD = $\frac{1}{3}$ BC]

$\Rightarrow$ $AE^2 = AD^2 - \left(\frac{BC}{6}\right)^2$

$\Rightarrow$ $AE^2 = \frac{36AD^2 - BC^2}{36}$ .....(ii)

Now from (i) and (ii) we get.

$$\frac{4AB^2 - BC^2}{4} = \frac{36AD^2 - AB^2}{36}$$

$\Rightarrow \quad \dfrac{4AB^2 - AB^2}{4} = \dfrac{36AD^2 - AB^2}{36}$ $\quad [\because$ AB = BC = CA]

$\Rightarrow \quad \dfrac{3AB^2}{4} = \dfrac{36AD^2 - AB^2}{36}$

$\Rightarrow \quad 27\ AB^2 = 36\ AD^2 - AB^2$

$\Rightarrow \quad 28\ AB^2 = 36\ AD^2$

$\Rightarrow \quad 7\ AB^2 = 9\ AD^2$ (Dividing bothside by 4)

Hence $9AD^2 = 7AB^2$ **Hence proved.**

Example 11 : In a right angled ΔABC, $\angle B = 90°$. If points D and E on the AB and BC respectively then prove $AE^2 + CD^2 = AC^2 + DE^2$ (see fig. 11.53)

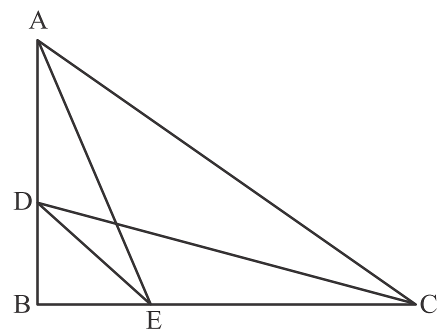


Fig. 11.53

Solution : ΔABE is a right angled with $\angle B = 90°$

$\therefore \quad AE^2 = AB^2 + BE^2$ . . . (i)

again in right angled $\Delta DBC, \angle B = 90°$

$CD^2 = BD^2 + BC^2$ . . . (ii)

On adding (i) and (ii) we get

$AE^2 + CD^2 = (AB^2 + BC^2) + (BE^2 + BD^2)$ . . . (iii)

Similarly in right angled triangles ΔABC and ΔDBE

$AC^2 = AB^2 + BC^2$ and $DE^2 = BE^2 + BD^2$ . . . (iv)

From $AE^2 + CD^2 = AC^2 + DE^2$ **Hence proved.**

Exercise 11.4

1. State whether the following statements are true or false? And justify your answer if possible.
   (i) If ratio of corresponding sides of two similar triangles then the ratio of their areas will be 4 : 9.
   (ii) If in two ΔABC and ΔDEF, in $\dfrac{area\,\Delta ABC}{area\,\Delta DEF} = \dfrac{AB^2}{DE^2} = \dfrac{9}{4}$ then $\Delta ABC \cong \Delta DEF$.

(iii) Ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

(iv) If $\Delta ABC \sim \Delta AXY$ and their areas are equal then the sides XY and BC may coinsides.

2. If $\Delta ABC \sim \Delta DEF$ also area ΔABC = 64 cm^2, area ΔDEF = 121 cm^2 and EF = 15.4 cm. find BC.

3. ΔABC and ΔDBC are at the same base BC. If AD and BC intersect each other at O. then prove.

$$\frac{Area\,\Delta ABC}{Area\,\Delta DBC} = \frac{AO}{DO}$$

4. Solve the following problems.

(i) In ΔABC ,DE || BC and AD : DB = 2 : 3, find the ratio of the areas of ΔADE and ΔABC

(ii) Perpendiculars PA and QB are drawn to two ends of a line segment AB. If P and Q are at the opposite sides of AB and their joining line intersects AB at O also PO = 5 cm, QO = 7 cm, area ΔPOB = 150 cm^2, find the area ΔQOA.

(iii) In the fig. 11.54 given below, find the value of x in the terms of a, b and c.

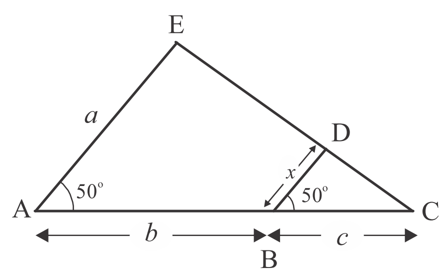


Fig. 11.54

5. In $\Delta ABC,\ \angle B = 90°$ and $BD \perp AC$ then prove $\Delta ADB \sim \Delta BDC$.

6. Prove that area of an equilateral triangle formed on one side of a square is equal to the half of the area of the equilateral triangles constructed on it diagonal.

Miscellaneous Exercise 11

1. In the fig. 11.55 $DE \parallel AD$, AD = 4 cm. DB = 6 cm, and AE = 5 cm, then measure of EC will be :

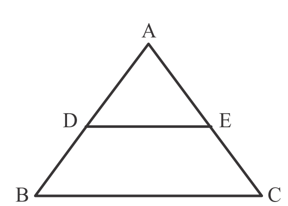


Fig. 11.55

(a) 6.5 cm. (b) 7.0 cm. (c) 7.5 cm. (d) 8.0 cm.

2. In the fig. 11.56 AD is the bisector of $\angle A$, if $AB = 6$ cm, $BD = 8$ cm, $DC = 6$ cm, then the length of AC will be :

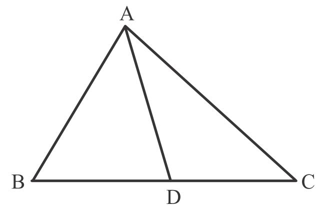


Fig. 11.56

(a) 4.0 cm (b) 4.5 cm (c) 5 cm (d) 5.5 cm.

3. In the fig. 11.57, if $DE \parallel BC$ the value of x is :

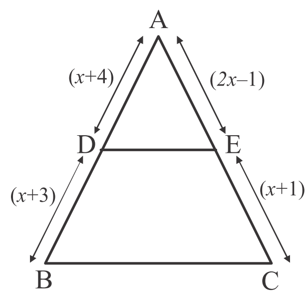


Fig. 11.57

(a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{3}$ (d) $\sqrt{7}$

4. In the fig. 11.58m if $AB = 3.4$ cm, $BD = 4$ cm, $BC = 10$ cm, then the measure of AC is.

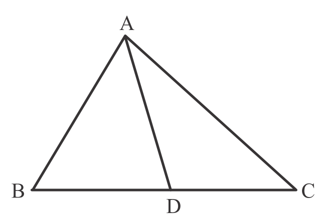


Fig. 11.58

(a) 5.1 cm (b) 3.4 cm (c) 6 cm (d) 5.3 cm.

5. The areas of two triangles are 25.25 cm^2 and 36 cm^2 respectively. If a median of smaller triangles is 10 cm, the corresponding median of larger triangle is :

(a) 12 cm (b) 15 cm (c) 10 cm (d) 18 cm

6. In a trapezium $ABCD$, $AB \parallel DC$ and its diagonals intersects each other at O.If $AB = 6$ cm, the ratio of the areas DAOB and DCOD is :

(a) 4 : 1 (b) 1 : 2 (c) 2 : 1 (d) 1 : 4

7. In DABC and DDEF, $\angle A = 50°, \angle B = 70°, \angle C = 60°, \angle D = 60°, \angle E = 70°$ and $\angle F = 50°$. State which of the following is true :

(a) $\Delta ABC \sim \Delta DEF$ (b) $\Delta ABC \sim \Delta EDF$ (c) $\Delta ABC \sim \Delta DEF$ (d) $\Delta ABC \sim \Delta FED$

8. If $\Delta ABC \sim \Delta DEF$, and $AB = 10$ cm, $DE = 8$ cm, then area of DABC : area of DDEF will be :
(a) 25 % 16 (b) 16 % 25 (c) 4 % 5 (d) 5 % 4

9. If DABC DDEF , also D and E points on the lines AB and AC such that $DE \parallel BC$ and AD = 8 cm, AB= 12 cm and AE = 12 cm, the measure of CE is :
(a) 6 cm (b) 18 cm (c) 9 cm (d) 15 cm.

10. The length of the shadow of a vertical rod of length 12 cm on the ground is 8 cm. If at the same time the length of the shadow of a minaret is 40 cm. The height of the minerat is :
(a) 60 m (b) 60 cm (c) 40 cm (d) 80 cm.

11. In a DABC, if point D is on the BC such that $\frac{AB}{AC} = \frac{BD}{DC}$ anda $\angle B = 70°, \angle C = 50°$. Find $\angle BAD$.

12. In DABC, $DE \parallel BC$ also $AD = 6$ cm. $DB = 9$ cm and $AE = 8$, find the length of AC.

13. In DABC, AD is the bisector of $\angle A$, also $AB = 8$ cm. BD = 5 cm and DC = 4 cm, find the length of AC.

14. If the ratio of the heights of two similar triangles is 4 : 9,find the ratio of their areas.

Important Points

1. It is not necessary that two similar figures have same shape and size.
2. Two polygons are similar if their corresponding sides are proportion and corresponding angles are equal.
3. Two triangles are similar if their corresponding sides are proportional and corresponding angles are equal.
4. **Thales theorem (basic proportional theorem) :** If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points, then the other two sides are divided in the same ratio.
5. If a line divides two sides of a triangle in the same ratio, then the line is parallel to the third.
6. Two straight line figures are similar if their corresponding angles are equal and corresponding sides are in the same ratio.
7. ***AAA* similarity criterion :** If corresponsing angles of two triangles are equal, then the triangle are similar.
8. ***AA* similarity criterion :** If two angles of a triangle are equal to two corresponding angles of another triangle, the triangle are simialr.
9. **SAS simiilarity criterion :** If an angle of a triangle is equal to one angle of another triangle and the sides including this angle are in the same ratio, then the triangles are similar.
10. The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.
11. The ratio of the area of two similar triangles is equal to the ratio of the square of their height.
12. In obtuse triangles ABC, $\angle B > 90°$ and $AD \perp BC$ then $AC^2 = AB^2 + BC^2 + 2\ BC \times BD$.
13. In acute triangle ABC $\angle B < 90°$ and $AD \perp BC$ then
$$AC^2 = AB^2 + BC^2 - 2BC \times BD$$

Answer Sheet

Exercise 11.1

1. (i) Similar, (ii) Similar, (iii) Equilateral (iv) (a) Corresponding angles are equal (b) Ratio of corresponding sides is equal.

2. (i) True, (ii) False (iii) False (since, it is not sufficient to be proportional of corresponding sides. (iv) True, (v) False.

Exercise 11.2

1. (i) 20 cm. (ii) 15.6 cm. (iii) 9.9 cm. (iv) $x = 1, \frac{-1}{2}$

2. (i) Parallel, (ii) Not parallel, (iii) Not parallel, (iv) Parallel.

Exercise 11.3

1. If $\angle A = \angle P$ and $\angle C = \angle R$, then $\angle B$ and $\angle Q$ will be automatically equal.

2. $\Delta ABC \sim \Delta DEF$. (since angles must be taken consecutively

   $\therefore$ $\Delta ABC \sim \Delta DFE$

3. The given ratio can not be expressed for $\Delta ABC \sim \Delta FDE$ the serial must be

$$\frac{AB}{FD} = \frac{BC}{DE} = \frac{CA}{EF}$$

4. The statement is not true, for similar triangles because one angle must be equal and the raito of the sides including this angle must be same.

5. In two equiangular triangles, corresponding angles are equal. If corresponding angles are equal, then both triangles are similar.

6. (i) and (viii) $\Delta ABC \sim \Delta QRP$, (ii) and (vii) $\Delta MPN \sim \Delta ZYX$, (iii) and (v) $\Delta PQR \sim \Delta EFG$, (iv) and (vi) $\Delta EDF \sim \Delta NML$

7. $\angle P = \angle RTS, \angle Q = \angle RST$

8. $\Delta ADC \sim \Delta BEC$

10. 1.6 m.

11. 84 m.

Exercise 11.4

1. (i) False the ratio of the squares of the sides i.e. 16 : 8.

   (ii) False because the ratio of the corresponding sides is $=\frac{3}{2}$ where for similarity the ratio is 1 : 1

   (iii) False

   (iv) True

11. 11.2 cm

4. (i) 4 : 25 (ii) 294 cm^2 (iii) $x = \frac{ac}{b+c}$

Miscellaneous Exercise 11

1. (c)	2. (b)	3. (d)	4. (a)	5. (a)	6. (a)	7. (d)
8. (a)	9. (a)	10. (a)	11. 20	12. 20 cm	13. 6.4 cm	14. 16: 81