Exercise 15.4

Chapter 15 Multiple Integrals 15.4 1E

Since there is no easy way to use only horizontal or vertical segments to integrate over, polar coordinates should be used to integrate.

The function has a radius of 4 and goes from 0 to 3π/2. Therefore, the solution is:

$$\int_0^{3\pi/2}\int_0^4 f\left(r\cos(\theta),\ r\sin(\theta)\right)r\ \mathrm{d}\ r\ \mathrm{d}\ \theta$$

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It is easier with rectangular coordinates.

$$R=\left\{f\left(x,y\right)\middle|0\le y\le 1-x^2,\ -1\le x\le 1\right\}$$

$$\iint_R f\left(x,y\right)\mathrm{dA}=\int_{-1}^{1}\int_0^{1-x^2} f(x,y)\ \mathrm{d}\ y\ \mathrm{d}\ x$$

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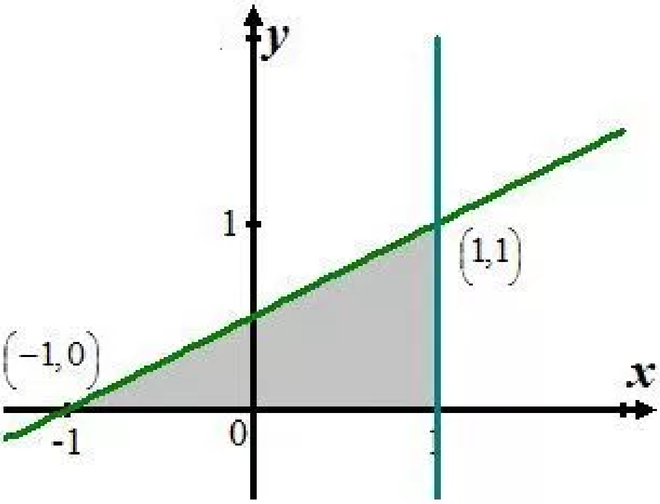
The objective is to write $\iint\limits_R f\left(x,y\right)dA$ as an iterated integral using rectangular or polar coordinates.

The region of integration is shown below:



Find the limits of integration. It is use to use rectangular coordinates to determine limits of integration.

From the diagram,

The value of x for the region changes from $-1 \text{ to } 1$.

The line passing through the points $(-1,0);(1,1)$.

Let $(x_1, y_1) = (-1,0);(x_2, y_2) = (1,1)$

The slope of line joining these points is,

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1-0}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

The equation of the line is,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= \frac{1}{2}(x+1) \\ y &= \frac{x+1}{2} \end{aligned}$$

This implies the value of y for the region changes from $0 \text{ to } \frac{x+1}{2}$.

So, the region of integration is in rectangular coordinates is,

$$R = \left\{(x, y) \mid \ -1 \le x \le 1, 0 \le x \le \frac{x+1}{2}\right\}$$

Therefore, the integral becomes,

$$\boxed{\iint_R f(x, y) dA = \int_{-1}^{1} \int_{0}^{\frac{x+1}{2}} f(x, y) dy dx}.$$

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The given region R can be descriubed as,

$$R = \{(x, y) \mid x \geq 0, 9 \leq x^2 + y^2 \leq 36\}.$$

Here, the given region is a the half-ring shaped region between the circles:
$x^2 + y^2 = 9$ and $x^2 + y^2 = 36$ lying on the fourth and first quadrants.

It's easy to use polar coordinates, to write it as an iterated integral.

In polar coordinate system,

$$x = r \cos\theta$$
$$y = r \sin\theta$$
$$x^2 + y^2 = r^2$$
$$dxdy = rdrd\theta$$

The given region R can be descriubed in polar coordinates as follows:

$$R = \left\{(r, \theta) \mid 3 \leq r \leq 6, \ \frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}\right\}.$$

Therefore, the iterated integral is,

$$\boxed{\iint_R f(x, y)\, dA = \int_{3\pi/2}^{5\pi/2} \int_3^6 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}.$$

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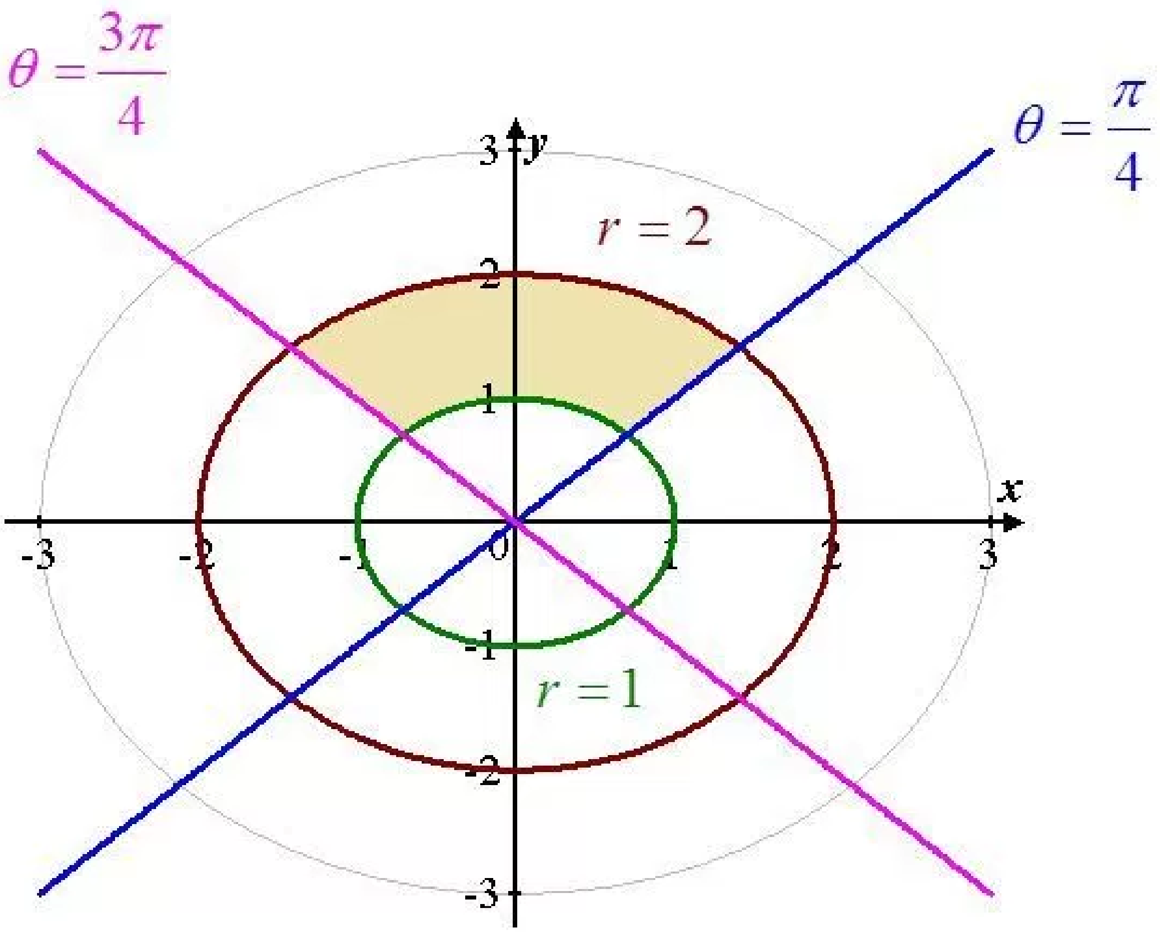
Consider the integral,

$$\int_{\pi/4}^{\frac{3\pi}{4}}\int_{1}^{2} r\, dr\, d\theta$$

From the integral, observe that the r limits are from 1 to 2.

And the θ limits are from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$.

Hence the region $D=\left\{(r,\theta)\middle|1\le r\le 2,\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}\right\}$ is shown below.



Now, evaluate the integral as:

$$\int_{\pi/4}^{\frac{3\pi}{4}}\int_{1}^{2} r\ dr\,d\theta = \int_{\pi/4}^{\frac{3\pi}{4}}\left[\frac{r^2}{2}\right]_1^2 d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{4}{2}-\frac{1}{2}\right)d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{3}{2}\right)d\theta$$

$$= \frac{3}{2}[\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{3}{2}\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)$$

$$= \frac{3}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4}$$

Therefore, the value of the integral $\int_{\pi/4}^{\frac{3\pi}{4}}\int_{1}^{2} r\ dr\,d\theta$ is $\boxed{\frac{3\pi}{4}}$.

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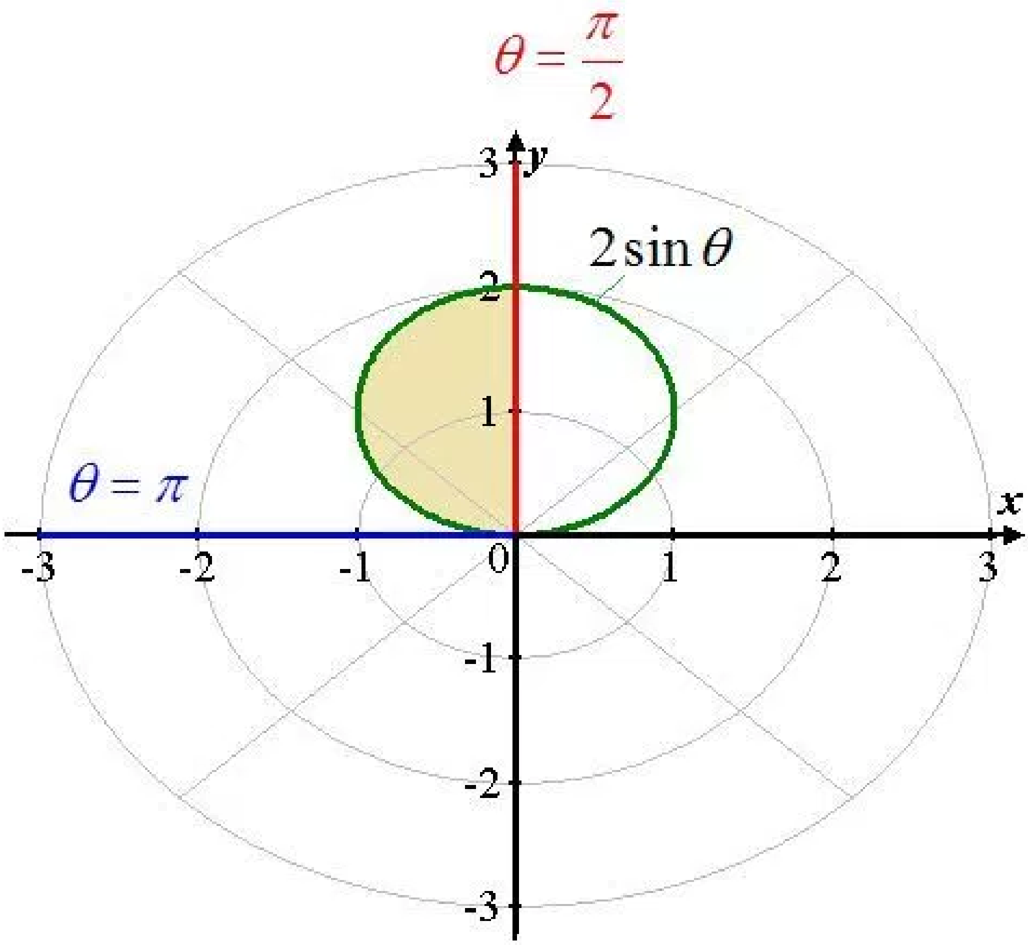
Consider the integral,

$$\int_{\pi/2}^{\pi} \int_{0}^{2\sin\theta} r\, dr\, d\theta$$

From the integral, observe that the r limits are from 0 to $2\sin\theta$.

And the θ limits are from $\frac{\pi}{2}$ to π.

Hence the region $D = \left\{ (r,\theta) \middle| 0 \le r \le 2\sin\theta, \frac{\pi}{2} \le \theta \le \pi \right\}$ is shown below.



Now, evaluate the integral as:

$$\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r\,dr\,d\theta = \int_{\pi/2}^{\pi}\left[\frac{r^2}{2}\right]_0^{2\sin\theta} d\theta$$

$$= \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta$$

$$= \int_{\pi/2}^{\pi}\left[1-\cos 2\theta\right]d\theta \qquad \text{Since } \cos 2\theta = 1-2\sin^2\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$$

$$= (\pi - 0) - \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{2}$$

Therefore, the value of the integral $\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r\,dr\,d\theta$ is $\boxed{\frac{\pi}{2}}$.

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Consider the integral $\iint_D x^2 y\,dA$

Since *D* is the top half of the disk with center the origin and radius 5.

So, $x = r\cos\theta$ and $y = r\sin\theta$.

In polar coordinates the area of a region is given by

$$\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

On substituting *x* with $r\cos\theta$ and *y* with , we get *x*2*y* as $r^3\cos^2\theta\sin\theta$.

The limits for *r* is obtained as (0, 5) and the limits for θ as $(0, \pi)$.

Now, evaluate the integral.

$$\begin{aligned}
\iint_D x^2 y\,dA &= \int_0^{\pi}\int_0^{5}\left(r^3\cos^2\theta\sin\theta\right) r\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^{5}\left(r^4\cos^2\theta\sin\theta\right) dr\,d\theta \\
&= \int_0^{\pi}\left(\frac{r^5}{5}\right)_0^5 \cos^2\theta\sin\theta\,d\theta \\
&= \int_0^{\pi}\left(\frac{5^5}{5}\right)\cos^2\theta\sin\theta\,d\theta \\
&= 625\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta
\end{aligned}$$

Suppose $u=\cos\theta$ then $du=-\sin\theta\,d\theta$

Limits: When $\theta=0 \Rightarrow u=1$

When $\theta=\pi \Rightarrow u=-1$

Therefore,

$$\begin{aligned}
\iint_D x^2 y\,dA &= 625\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta \\
&= 625\int_1^{-1}u^2(-du) \\
&= 625\int_{-1}^{1}u^2\,du \\
&= 625\left[\frac{u^3}{3}\right]_{-1}^{1}
\end{aligned}$$

$$=\frac{625}{3}\left[1-(-1)\right]$$

$$=\frac{1250}{3}$$

Therefore, $\iint_D x^2 y\,dA = \boxed{\frac{1250}{3}}$.

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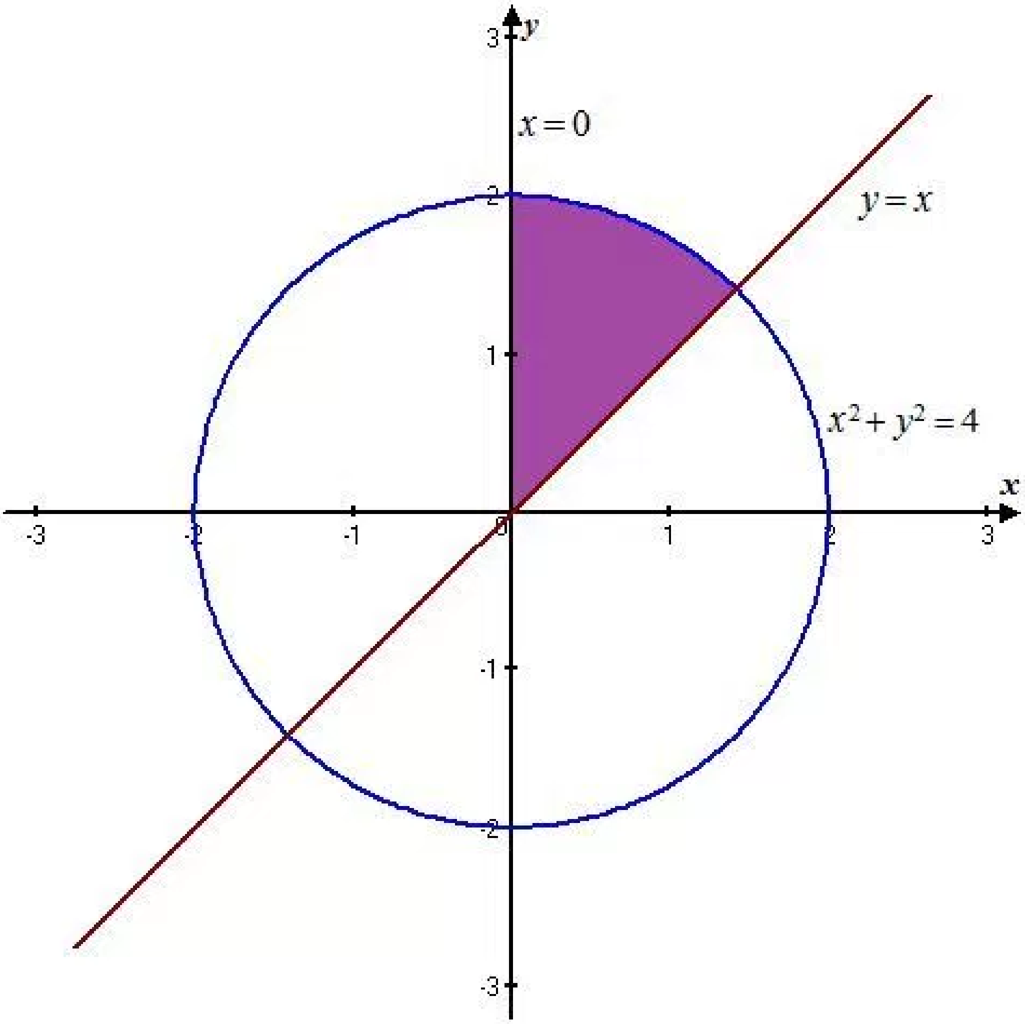
Consider the integral $\iint_R (2x - y)dA$

Here R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

Sketch the graph is shown below:



If f is continuous on a polar rectangle R is given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$.

Then

$$\begin{aligned}2x - y &= 2r\cos\theta - r\sin\theta \\ &= 2(2\cos\theta - \sin\theta)\end{aligned}$$

From figure, the limits for integration of r as $(0,2)$ and the limits for θ as $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Substitute the known values in $\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$ and evaluating the integral, obtain that

$$\begin{aligned}\iint_R (2x-y)\,dA &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 r(2\cos\theta - \sin\theta)\, r\, dr\, d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta - \sin\theta)\left(\frac{r^3}{3}\right)_0^2 d\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta - \sin\theta)\left(\frac{8}{3}\right) d\theta\end{aligned}$$

$$= \frac{8}{3}(2\sin\theta + \cos\theta)_{\pi/4}^{\pi/2}$$

$$= \frac{8}{3}\left[(2+0) - \left(2\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{8}{3}\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$= \frac{4}{3}\left(4-3\sqrt{2}\right)$$

$$= \boxed{\frac{16}{3} - 4\sqrt{2}}$$

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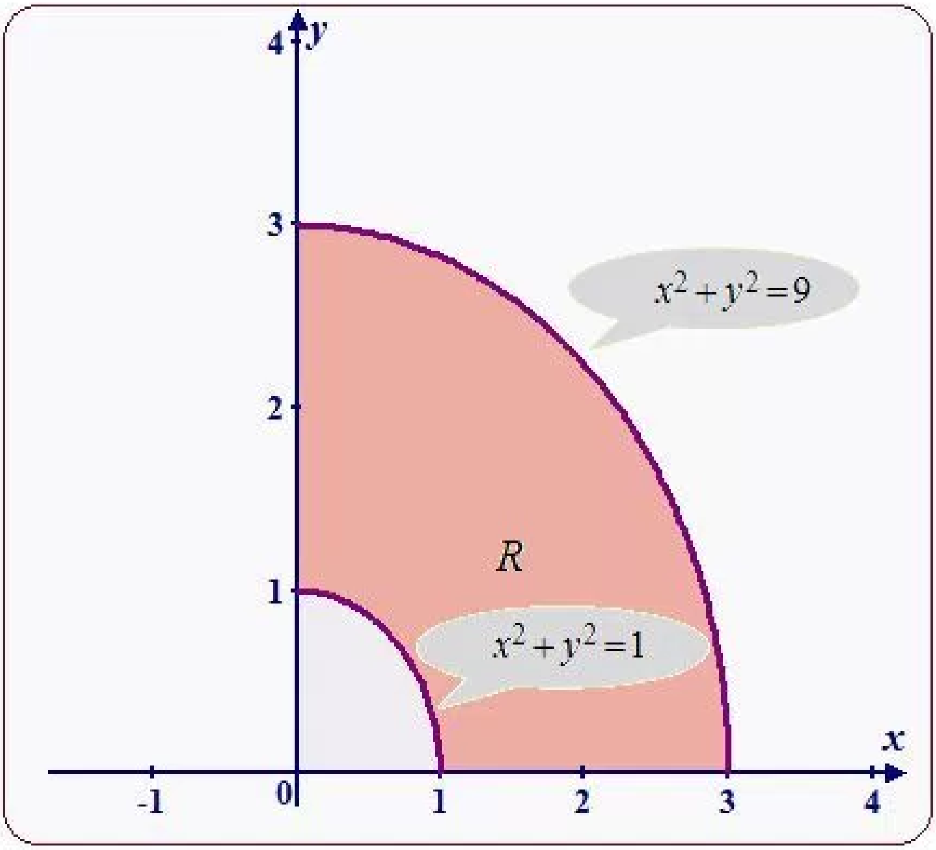
Consider the integral $\iint_R \sin(x^2+y^2)\,dA$, here R is the region in the first quadrant between the circles with center the origin and radii 1 and 3.

To evaluate the integral by changing the polar coordinates, use the theorem change to polar coordinates in a double integral.

Sketch the graph of the region:



The description of the region R in terms of rectangular coordinates is

$$R=\left\{(r,\theta)\mid 1\le r\le 3, 0\le\theta\le\frac{\pi}{2}\right\}.$$

Since the region is enclosed

Change to polar coordinates in a double integral:

If f is continuous on a polar rectangle R is given by $0\le a\le r\le b, \alpha\le\theta\le\beta,$ where

$0\le\beta-\alpha\le 2\pi,$

Then

$$\iint_R f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Let $x=r\cos\theta$ and $y=r\sin\theta$, then

$$\begin{aligned}x^2+y^2&=r^2\cos^2\theta+r^2\sin^2\theta\\&=r^2\end{aligned}$$

Thus, $\sin\left(x^2+y^2\right)=\sin r^2.$

The limits for integration of r is $(1,3)$ and the limits for θ is $\left(0,\frac{\pi}{2}\right).$

Plug in the values to the formula $\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ and evaluate the integral.

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_1^3 r\sin r^2\,dr\,d\theta&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_1^9(\sin u\,du)\,d\theta\quad\left(\text{Taking } r^2=u, r\,dr=\frac{1}{2}du\right)\\&=\frac{1}{2}\int_0^{\pi/2}(-\cos u)_1^9\,d\theta\end{aligned}$$

$$\begin{aligned}&=\frac{1}{2}\int_0^{\pi/2}(\cos 1-\cos 9)\,d\theta\\&=\frac{1}{2}(\cos 1-\cos 9)[\theta]_0^{\pi/2}\\&=\frac{1}{2}(\cos 1-\cos 9)\left[\frac{\pi}{2}-0\right]\\&=\frac{\pi}{4}(\cos 1-\cos 9)\end{aligned}$$

Therefore, the value of the integral $\iint_R \sin\left(x^2+y^2\right)dA,$ is $\boxed{\frac{\pi}{4}\cos 1-\frac{\pi}{4}\cos 9}.$

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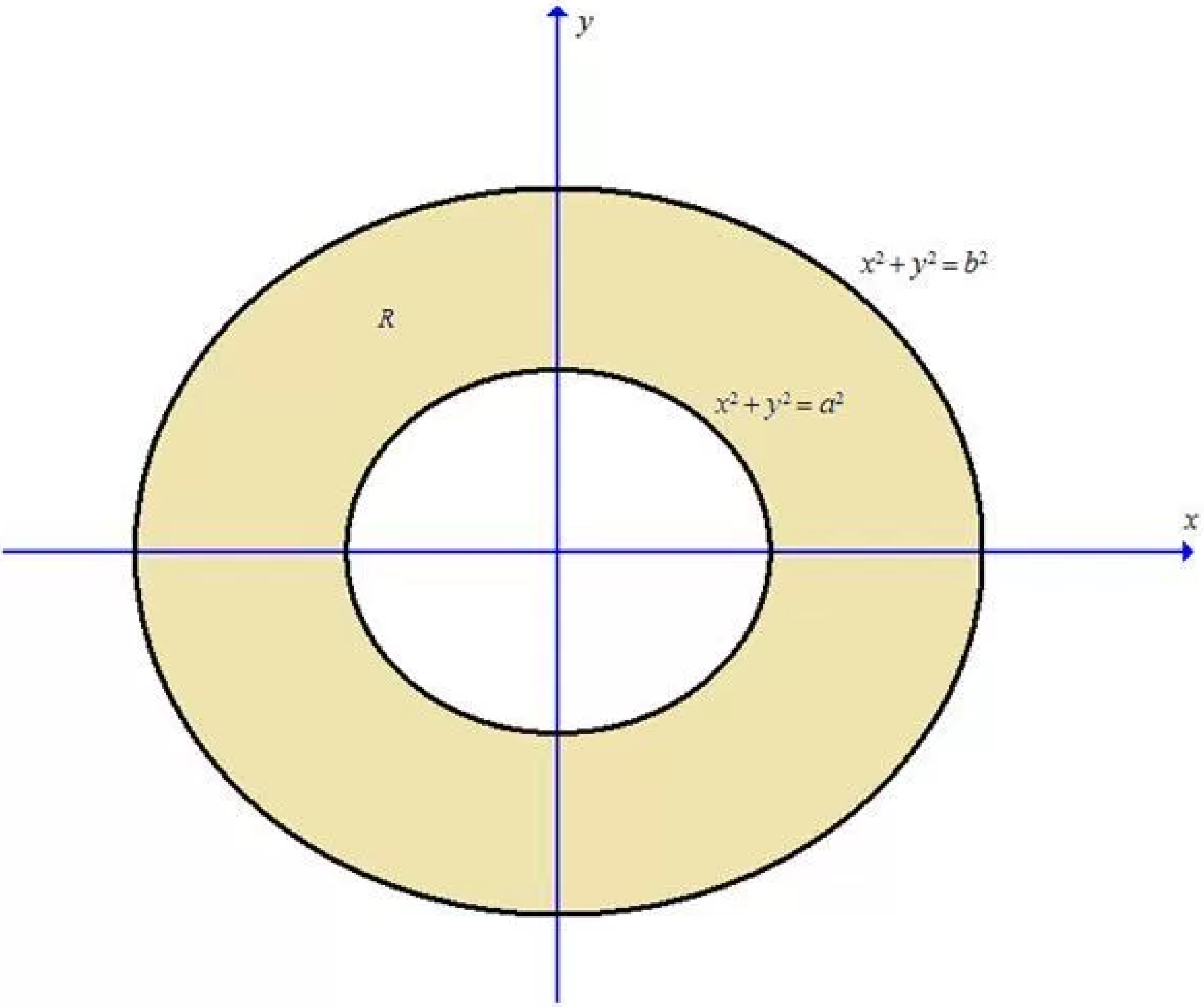
Consider the integral $\iint_R \frac{y^2}{x^2+y^2}\,dA$

Where *R* is the region that lies between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $0<a<b$.

The region is shown in the following figure:



Let $x=r\cos\theta, y=r\sin\theta$ then $x^2+y^2=r^2, dA=r\,dr\,d\theta$

R can be expressed as $R=\{(r,\theta)\,|\,0<a\le r\le b, 0\le\theta\le 2\pi\}$.

If *f* is continuous on a polar rectangle *D* given by

$D=\{(r,\theta)\,|\,0\le a\le r\le b, \alpha\le\theta\le\beta, 0\le\beta-\alpha\le 2\pi\}$

Then $\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Therefore,

$$\iint_R \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_a^b \frac{r^2\sin^2\theta}{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b r\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{2}r^2\sin^2\theta\right]_a^b d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{2}b^2\sin^2\theta - \frac{1}{2}a^2\sin^2\theta\right]d\theta$$

$$= \frac{1}{2}\left[b^2-a^2\right]\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= \frac{1}{4}\left[b^2-a^2\right]\int_0^{2\pi}2\sin^2\theta\,d\theta$$

Continuing the above step,

$$\iint_R \frac{y^2}{x^2+y^2}\,dA = \frac{1}{4}\left[b^2-a^2\right]\int_0^{2\pi}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left[b^2-a^2\right]\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{4}\left[b^2-a^2\right]\left[2\pi-0-0+0\right]$$

$$= \frac{\pi}{2}\left[b^2-a^2\right]$$

Therefore, $\iint_R \frac{y^2}{x^2+y^2}\,dA = \boxed{\frac{\pi}{2}\left[b^2-a^2\right]}$.

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Consider the following double integral:

$$\iint_D e^{-x^2-y^2}\,dA$$

Here, D is the region bounded by the semicircle $x=\sqrt{4-y^2}$ and the $y-$ axis

Consider

$$x=\sqrt{4-y^2}$$

Squaring on both sides

$$x^2=4-y^2$$

$$x^2+y^2=4$$

Use the Polar Coordinates,

$$x=r\cos\theta$$

$$y=r\sin\theta$$

$$x^2+y^2=r^2$$

$$dA=dxdy=rdrd\theta$$

In polar form $r^2=4$

$r=2$ Since radius is positive

$x=\sqrt{4-y^2}$ So x is never negative.

Therefore, the region is a semicircle that lies in the first and fourth quadrants

Therefore $\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$

$$D=\left\{(r,\theta):-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le r\le 2\right\}$$

Evaluate the integral is as follows:

$$\iint e^{-x^2-y^2}\,dA = \iint e^{-\left(x^2+y^2\right)}dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2}\, r\,dr\,d\theta$$

Let $r^2 = t$

$$2r\,dr = dt$$

$$r\,dr = \frac{dt}{2}$$

If $r = 0$ then $t = 0$

If $r = 2$ then $t = 4$

Substitute in above integral.

$$\iint e^{-x^2-y^2}\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^4 e^{-t}.\frac{dt}{2}\,d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{e^{-t}}{-1}\right]_0^4 d\theta$$

$$= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(e^{-4}-e^{0}\right)d\theta$$

$$= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(e^{-4}-1\right)d\theta$$

$$= -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-\left(1-e^{-4}\right)\right)d\theta$$

Continuation on above:

$$\iint e^{-x^2-y^2}\,dA = \frac{1}{2}\left(1-e^{-4}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$\iint e^{-x^2-y^2}\,dA = \frac{1}{2}\left(1-e^{-4}\right)\left[\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(1-e^{-4}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(1-e^{-4}\right)\pi$$

$$= \boxed{\frac{\pi}{2}\left(1-e^{-4}\right)}$$

Therefore, the value of the integral is $\boxed{\frac{\pi}{2}\left(1-e^{-4}\right)}$

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Given $\iint_D \cos\sqrt{x^2+y^2}\,dA$

If f is continuous on a polar rectangle R is given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then $\iint_R f(x, y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

We have $x = r\cos\theta$ and $y = r\sin\theta$. Then, $\cos\left(\sqrt{x^2+y^2}\right) = \cos r$.

We get the limits for integration of r as $(0, 2)$ and the limits for θ as $(0, 2\pi)$.

Substitute the known values in $\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ and evaluate the integral.

$$\int_0^{2\pi}\int_0^2 r\cos r\,dr\,d\theta = \int_0^{2\pi}\left[r(\sin r) - \int_0^2 \sin r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left[(r\sin r)_0^2 - \int_0^2 \sin r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left[(r\sin r)_0^2 - (-\cos r)_0^2\right]d\theta$$

$$= \int_0^{2\pi}\left[(r\sin r)_0^2 + (\cos r)_0^2\right]d\theta$$

$$= (2\pi)(2\sin 2 + \cos 2 - 0 - \cos 0)$$

$$= (2\pi)(2\sin 2 + \cos 2 - 1)$$

$$= 4\pi\sin 2 + 2\pi\cos 2 - 2\pi$$

Therefore, the integral evaluates to $\boxed{4\pi\sin 2 + 2\pi\cos 2 - 2\pi}$.

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Here given that $1 \le x^2 + y^2 \le 4$

That is in polar form $1 \le r^2 \le 4$

$$\Rightarrow 1 \le r \le 2$$

And $0 \le y \le x \Rightarrow 0 \le \theta \le \pi/4$

Therefore $\iint_R \arctan\left(y/x\right) dA$

$$= \int_0^{\pi/4}\int_1^2 \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_1^2 \tan^{-1}(\tan\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_1^2 \theta . r . dr\, d\theta$$

$$= \int_0^{\pi/4} \left.\frac{r^2}{2}\right]_1^2 \theta\, d\theta$$

i.e. $\iint_R \arctan\left(y/x\right) dA = \int_0^{\pi/4}\left(\frac{2^2}{2} - \frac{1}{2}\right)\theta\, d\theta$

$$= \frac{3}{2}\int_0^{\pi/4}\theta\, d\theta$$

$$= \frac{3}{2} . \left.\frac{\theta^2}{2}\right]_0^{\pi/4}$$

$$= \frac{3}{4}\left(\left(\frac{\pi}{4}\right)^2 - 0\right)$$

$$= \frac{3}{4} . \frac{\pi^2}{16}$$

$$= \boxed{\frac{3\pi^2}{64}}$$

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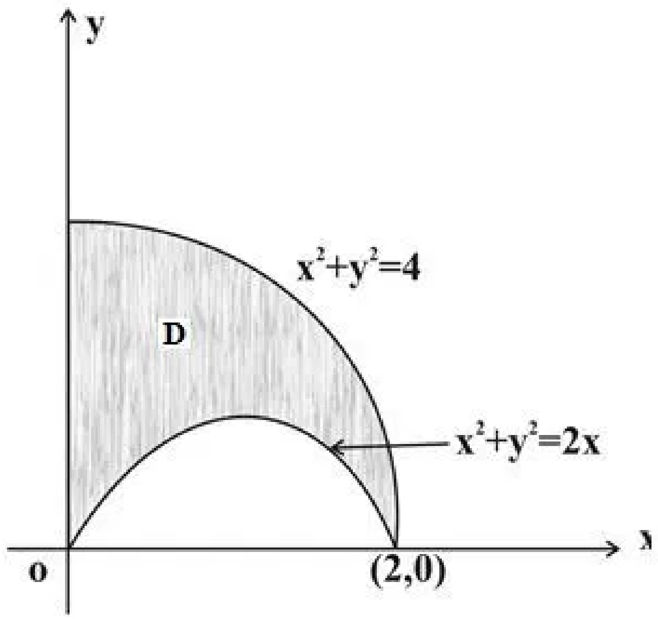
Consider the following equations of circles:

$x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

The objective is to evaluate the integral $\iint_D x\,dA$ where D is the region between the circles in the first quadrant.

The region D is as shown below:



Change the equation of circle $x^2 + y^2 = 2x$ to polar co – ordinates as follows:

Put $x = r\cos\theta, y = r\sin\theta$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 2r\cos\theta$$

$$r^2 = 2r\cos\theta \qquad \text{since } \cos^2\theta + \sin^2\theta = 1$$

$$r = 2\cos\theta$$

Change the equation of circle $x^2 + y^2 = 4$ to polar co – ordinates as follows:

Put $x = r\cos\theta, y = r\sin\theta$

$$r^2\cos^2\theta + r^2\sin^2\theta = 4$$

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = 4$$

$$r^2 = 4 \qquad \text{since } \cos^2\theta + \sin^2\theta = 1$$

$$r = 2$$

Compute the value of integral is as follows:

$$\iint_D x\ dA = \int_0^{\pi/2}\int_{2\cos\theta}^{2}(r\cos\theta)r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_{2\cos\theta}^{2} r^2\cos\theta dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left(\frac{r^3}{3}\right)_{r=2\cos\theta}^{r=2} d\theta$$

$$= \int_0^{\pi/2}\frac{\cos\theta}{3}\left(8-8\cos^3\theta\right)d\theta$$

$$\iint_D xdA = \frac{8}{3}\int_0^{\pi/2}\left(\cos\theta-\cos^4\theta\right)d\theta$$

$$= \frac{8}{3}\left[\sin\theta-\frac{1}{4}\left(\cos^3\theta\sin\theta+\frac{3}{2}\theta+\frac{3}{2}\sin\theta\cos\theta\right)_0^{\pi/2}\right]$$

$$= \frac{8}{3}\left[1-\frac{1}{4}\left(0+\frac{3\pi}{4}+0\right)\right]$$

$$= \boxed{\frac{8}{3}-\frac{\pi}{2}}$$

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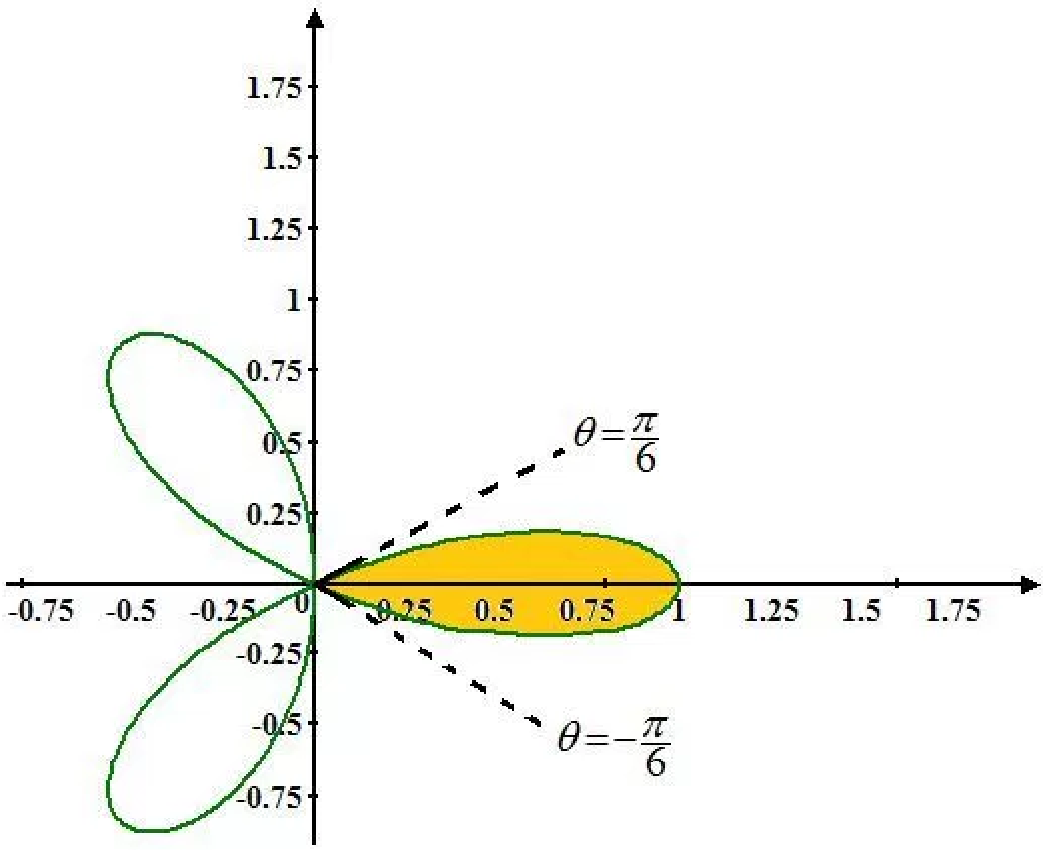
Consider the polar curve:

$$r = \cos 3\theta$$

Sketch the region to identify the limits of integration.

Figure showing the sketch of the polar curve:



Observe from the sketch, the loop is given by,

$$D = \left\{ (r, \theta) \mid -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}, 0 \le r \le \cos 3\theta \right\}$$

Use double integral to calculate the area of the loop.

$$A(D) = \iint_D dA$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} \left[r^2 \right]_0^{\cos 3\theta} d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta\, d\theta$$

Use the identity $\cos^2 3\theta = \frac{1}{2}(1+\cos 6\theta)$ and solve further.

$$A(D) = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1+\cos 6\theta)\, d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta + \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 6\theta d\theta$$

$$= \frac{1}{4}[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} + \frac{1}{24}[\sin 6\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

Simplify further.

$$= \frac{1}{4}\left(\frac{\pi}{6}+\frac{\pi}{6}\right) + \frac{1}{24}(\sin \pi - \sin(-\pi))$$

$$= \frac{\pi}{12} + \frac{1}{24}(\sin \pi + \sin \pi)$$

$$= \frac{\pi}{12} + \frac{1}{24}(0+0)$$

$$= \frac{\pi}{12}$$

Therefore area enclosed by one loop of the rose is $\boxed{\frac{\pi}{12}}$

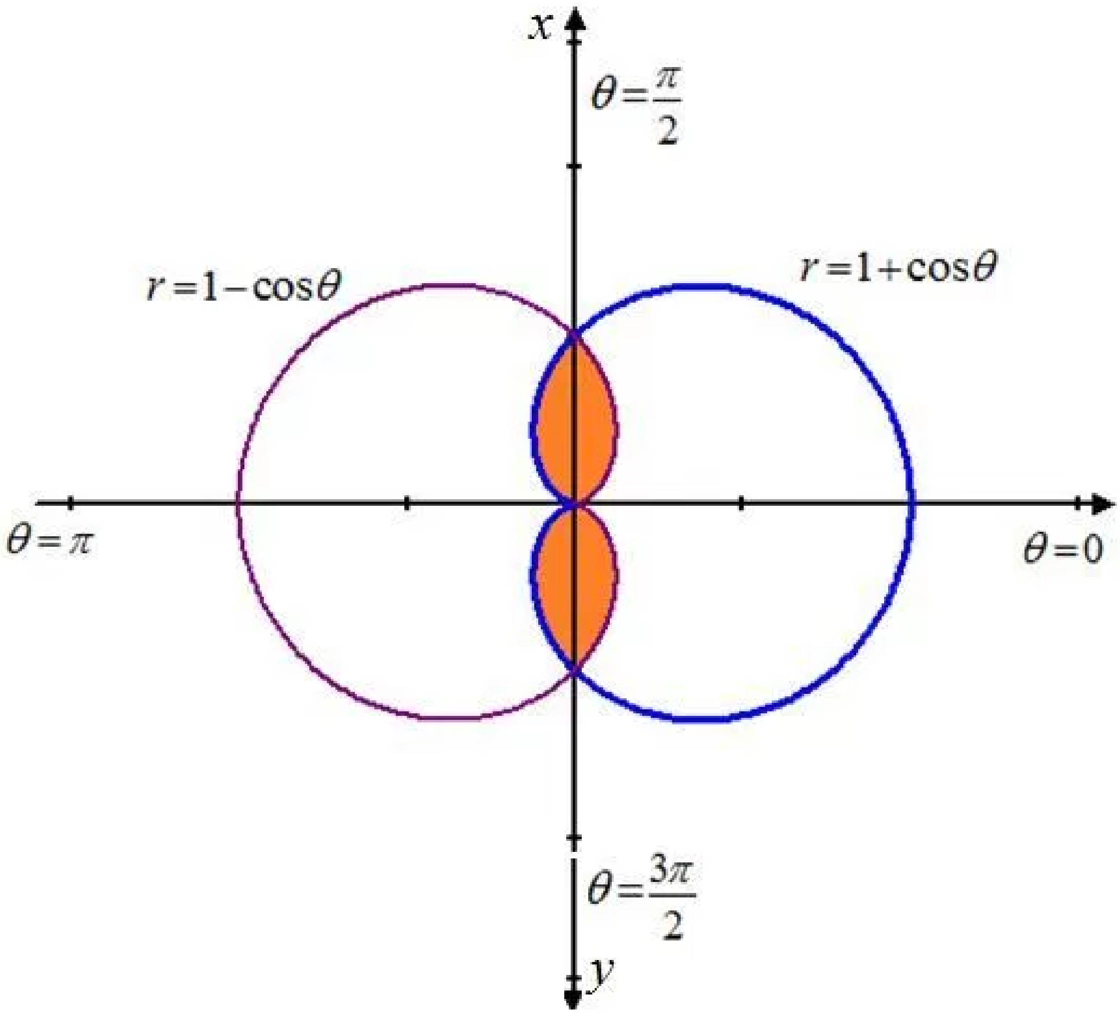
Chapter 15 Multiple Integrals 15.4 16E

Consider the following cardioids

$r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

Its need to find the area of the region enclosed by both of the cardioids

The graph helps identify the region of intersection of the two cardioids.



The desired region is the portion in the middle shaped like a figure eight that is common to both cardioids.

Note that $1 + \cos(\theta) \le 1 - \cos(\theta)$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Therefore, the area can be computed as:

$$A = \int_{-\pi/2}^{\pi/2} \int_0^{1-\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{3\pi/2} \int_0^{1+\cos\theta} r\,dr\,d\theta$$

However the region is symmetric both across the *x*-axis and across the *y*-axis.

Use this symmetry to simplify the formula for the area.

$$A = 4\int_0^{\pi/2} \int_0^{1-\cos\theta} r\,dr\,d\theta$$

Evaluate the integral as follows:

$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\left(\int_0^{1-\cos\theta} r\,dr\right)d\theta$$
$$= 4\int_0^{\pi/2}\left(\frac{1}{2}r^2\right)_0^{1-\cos\theta} d\theta$$
$$= 4\cdot\frac{1}{2}\cdot\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$
$$= 2\int_0^{\pi/2}\left(1-2\cos\theta+\cos^2\theta\right)d\theta$$
$$= 2\int_0^{\pi/2}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$
$$= 2\int_0^{\pi/2}\left(1-2\cos\theta+\frac{1}{2}+\frac{\cos 2\theta}{2}\right)d\theta$$
$$= 2\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

Continuation to the above

$$A = 2\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)_0^{\pi/2}$$
$$= 2\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right)_0^{\pi/2}$$
$$= 2\left\{\left(\frac{3}{2}\frac{\pi}{4}-2\sin\frac{\pi}{2}+\frac{1}{4}\sin 2\pi\right)-\left(\frac{3}{2}\cdot 0-2\sin 0+\frac{1}{4}\sin 0\right)\right\}$$
$$= 2\left\{\left(\frac{3}{2}\pi-2\cdot 1+\frac{1}{4}\cdot 0\right)-\left(0-2\cdot 0+0\right)\right\}$$
$$= 2\left(\frac{3\pi}{4}-2\right)$$
$$= \frac{3\pi}{2}-4$$

Hence, the required area is $A = \left(\frac{3\pi}{2}-4\right)$ sq units

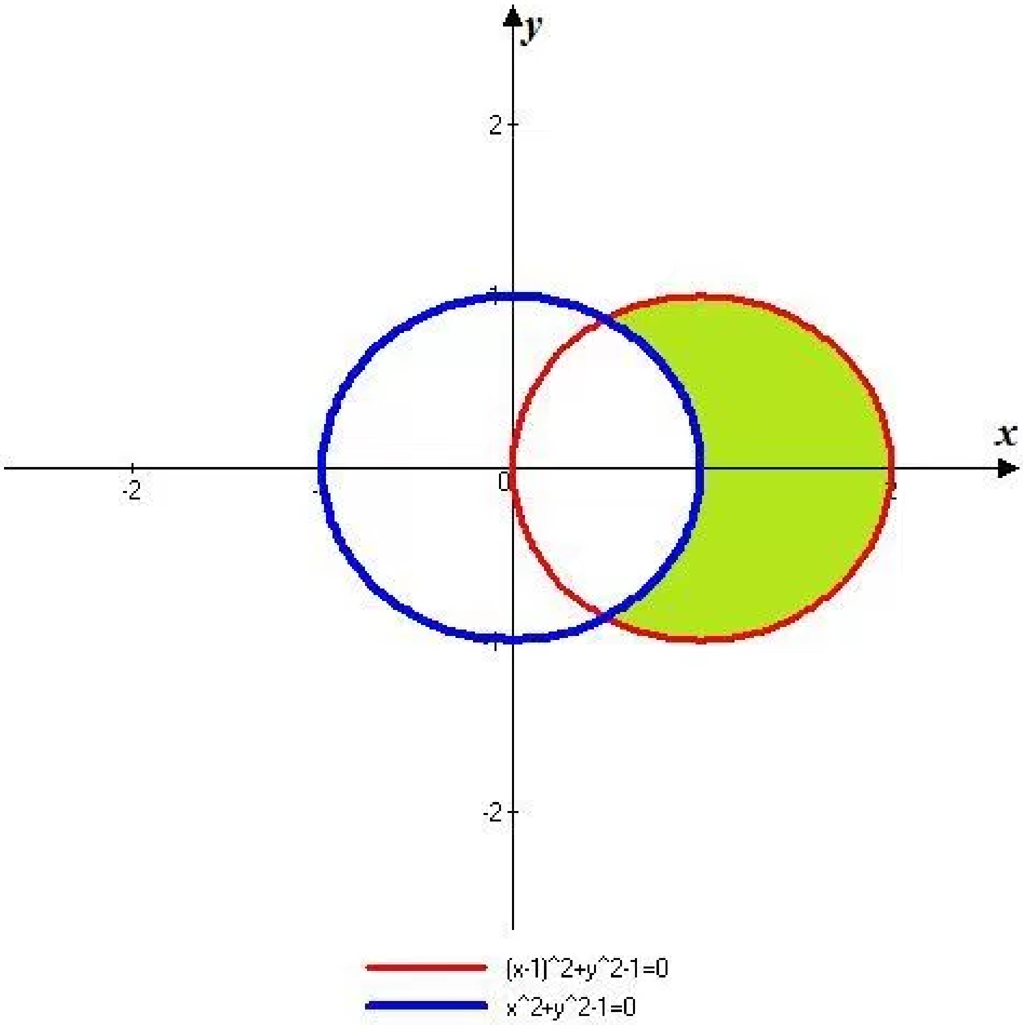
Chapter 15 Multiple Integrals 15.4 17E

Consider the region bounded by the following curves.

$(x-1)^2+y^2=1$ and $x^2+y^2=1$

The objective is to find the area of the region bounded by inside circle $(x-1)^2+y^2=1$ and outsides the circle $x^2+y^2=1$.

The region is shown below.



Using polar coordinates to find the area of the region, the transformation formulas are

$x^2+y^2=r^2$ and $x=r\cos\theta,\ y=r\sin\theta$

We can write

$$(x-1)^2+y^2=1\Rightarrow x^2+y^2=2x$$

Thus,

$$r^2=2r\cos\theta$$
$$r=2\cos\theta$$

And

$$r^2=1$$

Thus, the region of integration is $D=\left\{1\le r\le 2\cos\theta,\ -\frac{\pi}{3}\le\theta\le\frac{\pi}{3}\right\}$.

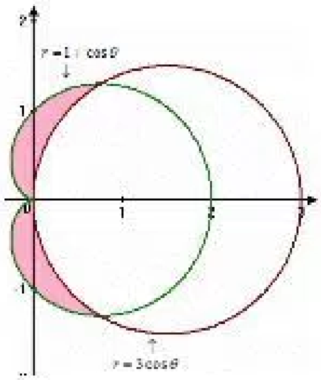
The area of the region can be obtained by the following integral.

$$V=\iint_D dA$$
$$=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\,dr\,d\theta$$
$$=2\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\,dr\,d\theta$$
$$=2\times\frac{1}{2}\int_0^{\frac{\pi}{3}}\left[r^2\right]_1^{2\cos\theta}d\theta$$
$$=\int_0^{\frac{\pi}{3}}\left[4\cos^2\theta-1\right]d\theta$$
$$=\int_0^{\frac{\pi}{3}}\left[\frac{4}{2}(1+\cos 2\theta)-1\right]d\theta$$
$$=\left[\left(2\theta+\frac{2\sin 2\theta}{2}\right)-\theta\right]_0^{\frac{\pi}{3}}$$
$$=\frac{2\pi}{3}+\sin\frac{2\pi}{3}-\frac{\pi}{3}$$
$$=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$$

Therefore, the area of the region is $\boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}$.

Chapter 15 Multiple Integrals 15.4 18E

We are required to find the area of the region inside the cardioid $r = 1+\cos\theta$ and outside the circle $r = 3\cos\theta$



To find the points of intersection of the given curves, we equate the curves and solve.

$$1+\cos\theta = 3\cos\theta$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

So, $\theta = \frac{\pi}{3}$, and $\frac{5\pi}{3} + 2n\pi$

The required area is $\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

Observe that the region shaded is symmetric about $x-$axis. So, we find the area of the upper half and multiply it with 2 to get the area of the entire shaded region.
While the region enclosed lies between the lower curves $r = 3\cos\theta$ and the upper curve $r = 1+\cos\theta$ between $\theta = \frac{\pi}{3}$, and $\theta = \frac{\pi}{2}$ and from $\theta = \frac{\pi}{2}$ to $\theta = \pi$, only the curve $r = 1+\cos\theta$ exists, the area of the region is $2\left(\int_{\theta=\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{r=3\cos\theta}^{1+\cos\theta} 1\,r\,dr\,d\theta + \int_{\frac{\pi}{2}}^{\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta\right)$

$$= 2\left\{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\left((1+\cos\theta)^2 - (3\cos\theta)^2\right)d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta\right\}$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos\theta - 3 - 4\cos 2\theta)\,d\theta + \int_{\frac{\pi}{2}}^{\pi}\left(\frac{3}{2} + \frac{\cos 2\theta}{2} + 2\cos\theta\right)d\theta$$

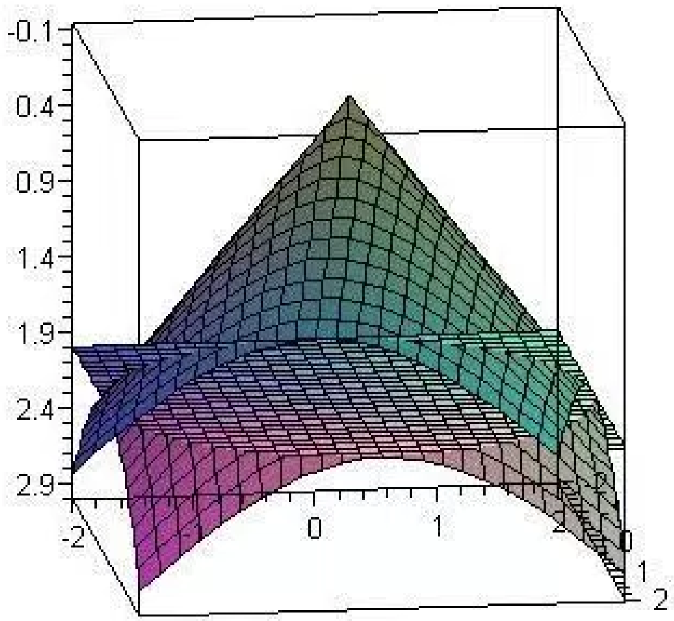
$$= 2\left\{\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right\} - 3\left\{\frac{\pi}{2} - \frac{\pi}{3}\right\} - 2\left\{\sin\pi - \sin\frac{2\pi}{3}\right\}$$

$$+ \frac{3}{2}\left\{\pi - \frac{\pi}{2}\right\} + 2\left\{\sin\pi - \sin\frac{\pi}{2}\right\} + \frac{1}{4}\{\sin 2\pi - \sin\pi\}$$

$$= \boxed{\frac{\pi}{4}}$$

Chapter 15 Multiple Integrals 15.4 19E

The region under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \le 4$ is shown by the cone and the plane $z = 2$ as



The conversion from a Cartesian volume integral to a polar one is as follows:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \quad \ldots\ldots (1)$$

where the region over which we are integrating is

$$D = \{(r,\theta) \mid a \le \theta \le b, g(\theta) \le r \le h(\theta)\}$$

The function to be integrated is $z = f(x,y) = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$

Also, the region is above the over which we are integrating is the disk $x^2 + y^2 \le 4$, which is a disk of radius 2. The limits of integration are therefore $\theta = 0$ to 2π for and $r = 0$ to 2

Using these details in (1), we get $\int_0^{2\pi} \int_0^2 (r)\,r\,dr\,d\theta$

$$= \frac{r^3}{3})_0^2 \theta)_0^{2\pi}$$

$$= \boxed{\frac{16\pi}{3}}$$

Chapter 15 Multiple Integrals 15.4 20E

Consider the paraboloid $z = 18 - 2x^2 - 2y^2$.

It is required to use polar coordinates to find the volume of the given solid below the paraboloid and above the $xy -$plane.

Recall the following:

The change of polar coordinates formula for double integrals is as follows:

$$\iint_D f(x, y) = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$$

Here R is the region in rectangular coordinates,

$0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, and $0 \le \beta - \alpha \le 2\pi$.

The **fundamental theorem** of calculus states the following:

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Here $F(x)$ is the antiderivative of $f(x)$.

The rectangular and polar coordinates are related as follows:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Rewrite the paraboloid as,

$$\begin{aligned} z &= 18 - 2\left(x^2 + y^2\right) \\ &= 18 - 2r^2 \end{aligned}$$

If $z = 0$, then the intersection with the *xy*-plane is,

$$0 = 18 - 2r^2$$
$$r = 3$$

This is a circle of radius 3 units.

From this information, the region of integration is the circle of radius 3 centred at the origin,

Write the region in polar coordinates as,

$$D = \{(r, \theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}.$$

So write the integral as a polar integral.

$$\int_0^{2\pi} \int_0^3 \left(18 - 2r^2\right) r\ dr\ d\theta = \int_0^{2\pi} \int_0^3 18r - 2r^3\ dr\ d\theta$$

First evaluate the integral with respect to r.

$$\int_0^3 18r - 2r^3\, dr = \left[9r^2 - \frac{1}{2}r^4\right]_{r=0}^{r=3}$$
$$= \left[9(3)^2 - \frac{1}{2}(3)^4\right] - [0]$$
$$= \frac{81}{2}$$

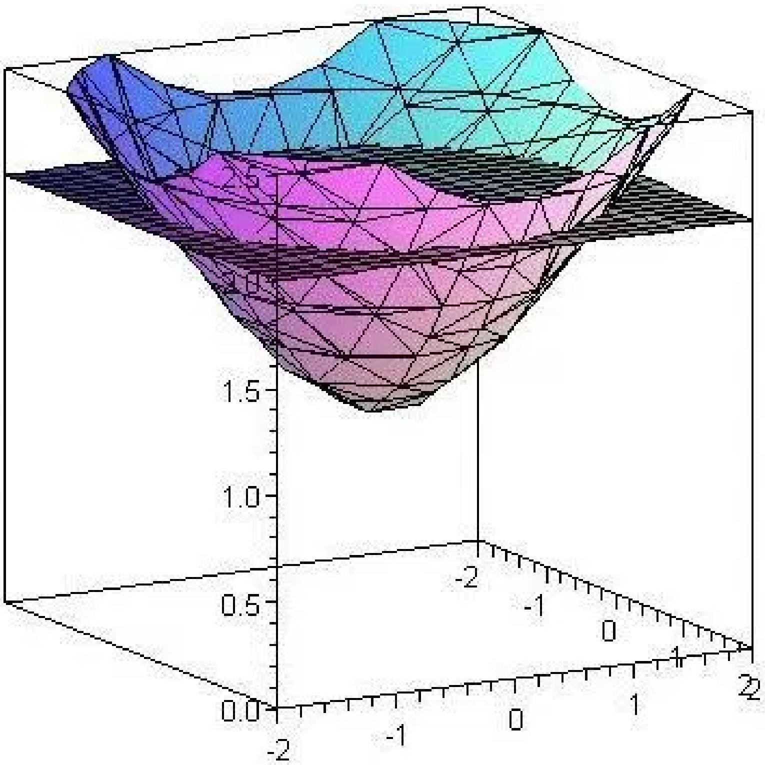
Evaluate the integral with respect to θ.

$$\int_0^{2\pi} \frac{81}{2}\, d\theta = \frac{81}{2}\left[\int_0^{2\pi} d\theta\right]$$
$$= \frac{81}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$
$$= 81\pi$$

Hence, the volume of the given solid is $\boxed{81\pi}$.

Chapter 15 Multiple Integrals 15.4 21E

One part of the hyperboloid $z^2 - x^2 - y^2 = 1$ intersected by the plane $z = 2$ is given by



The conversion from a Cartesian volume integral to a polar one is as follows:

$$\iint_D f(x,y)dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta \qquad \ldots\ldots (1)$$

where the region over which we are integrating is

$$D = \{(r,\theta) \mid a \le \theta \le b, g(\theta) \le r \le h(\theta)\}$$

To find the limits when the surface intersects the plane, we substitute $z = h_1(x,y) = 2$ in $h_2(x,y) = z^2 - x^2 - y^2 = 1$ to give $x^2 + y^2 = 3$

This is a circle of radius $\sqrt{3}$

We now change all the rectangular system into polar coordinates.

The given function is $z = f(x,y) = h_2(x,y) - h_1(x,y) = 2 - \sqrt{1+x^2+y^2}$

So, $f(r\cos\theta, r\sin\theta) = 2 - \sqrt{1+r^2}$ where r is the radius varies from 0 through $\sqrt{3}$ and θ varies from 0 through 2π

We now substitute all these details in (1) to get $\int_0^{2\pi}\int_0^{\sqrt{3}}\left(2-\sqrt{1+r^2}\right)r\,dr\,d\theta$

Suppose $1+r^2 = t$, then we get $r\,dr = \frac{dt}{2}$ and when $r = 0$, we get $t = 1$

When $r = \sqrt{3}$, we get $t = 4$

So, the above integral becomes $\int_0^{2\pi}\int_1^4 \frac{2-t^{\frac{1}{2}}}{2}\,dt\,d\theta$

$$= t\}_1^4 \theta\}_0^{2\pi} - \frac{1}{2}\times\frac{2}{3}\times t^{\frac{3}{2}}\}_1^4 \theta\}_0^{2\pi}$$

$$= 6\pi - \frac{1}{3}[8-1]2\pi$$

$$= 6\pi - \frac{14}{3}\pi$$

$$= \boxed{\frac{4\pi}{3}}$$

Chapter 15 Multiple Integrals 15.4 22E

The boundary circle has equation $x^2 + y^2 = 4$ and the sphere is $x^2 + y^2 + z^2 = 16$

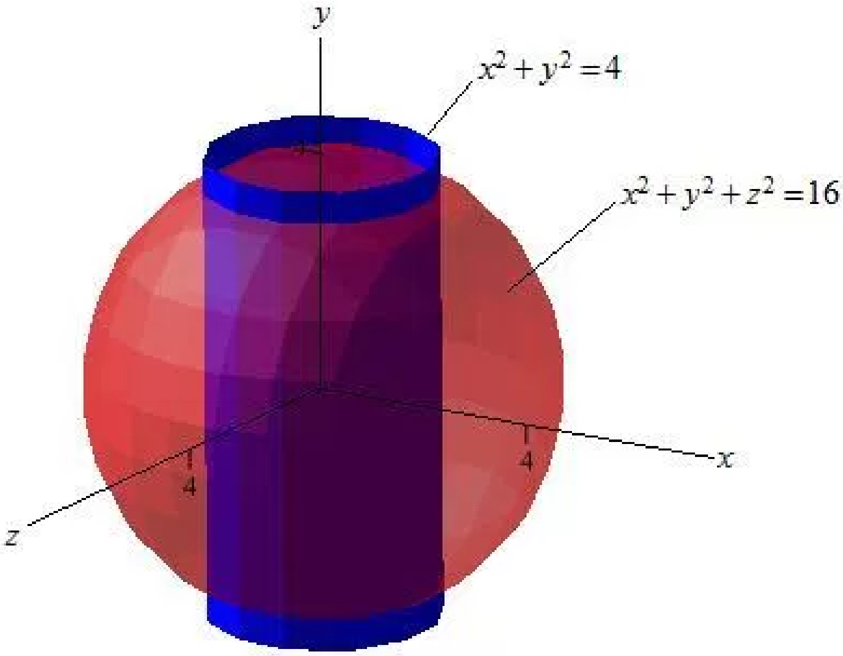
Or $z^2 = 16 - (x^2 + y^2)$

Or $z = \pm\sqrt{16 - (x^2 + y^2)}$

Changing to polar co – ordinates the region of integration becomes

$R = \{(r, \theta) : 0 \le \theta \le 2\pi,\ 2 \le r \le 4\}$

The required region is inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$ which can be shown in the following diagram as follows.



The volume of the required region will be $(V)=2\iint\limits_{4\le x^2+y^2=16}\sqrt{16-x^2-y^2}\;dA$

$$2\iint\limits_{4\le x^2+y^2=16}\sqrt{16-x^2-y^2}\;dA=2\int_0^{2\pi}\int_2^4\sqrt{16-r^2}\;r\,dr\,d\theta$$

$$=2\int_0^{2\pi}d\theta\int_2^4 r\sqrt{16-r^2}\;dr$$

$$=2(\theta)_0^{2\pi}\left[\frac{-1}{3}\left(16-r^2\right)^{3/2}\right]_2^4$$

$$=\frac{-2}{3}(2\pi)\left(0-12^{3/2}\right)$$

$$=\frac{4\pi}{3}\left(12\sqrt{12}\right)$$

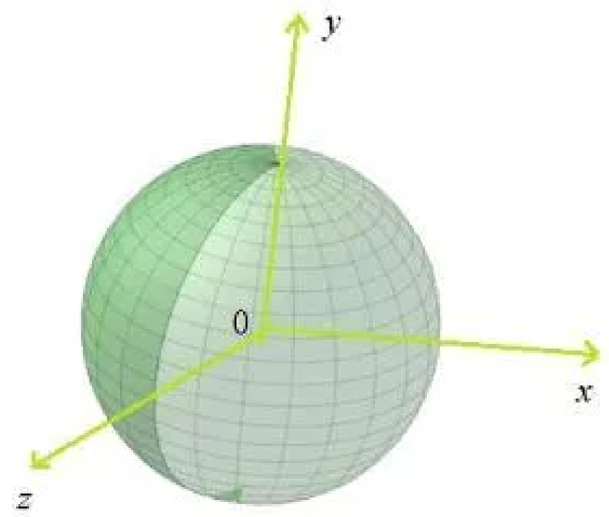
$$=\boxed{32\sqrt{3}\pi}$$

Chapter 15 Multiple Integrals 15.4 23E

Consider the sphere of radius a and the center origin as,

$$x^2+y^2+z^2=a^2$$
$$z=\pm\sqrt{a^2-\left(x^2+y^2\right)}$$

The sketch of the sphere $x^2+y^2+z^2=a^2$ is shown below:



Since the sphere is symmetric, we can determine the volume of the top half of the sphere and multiply it by 2.

So the volume of the sphere $x^2+y^2+z^2=a^2$ is,

$$V=2\iint_{x^2+y^2\le a} z\,dA$$
$$=2\iint_{x^2+y^2\le a}\sqrt{a^2-\left(x^2+y^2\right)}\,dA$$

The polar equations of the sphere are given as,

$$x=r\cos\theta, y=r\sin\theta, dA=dxdy=rdrd\theta$$
$$r:0\to a; \theta:0\to 2\pi$$

Then,

$$x^2+y^2=\left(r\cos\theta\right)^2+\left(r\sin\theta\right)^2$$
$$=r^2\cos^2\theta+r^2\sin^2\theta$$
$$=r^2$$

Use these equations; the volume of the sphere is,

$$V = 2\int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\, r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^a \left(a^2 - r^2\right)^{\frac{1}{2}} \cdot (-2r)\,dr\,d\theta \qquad \left[\text{Since} \int [f(x)]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= -\int_0^{2\pi}\left[\frac{2}{3}\left(a^2 - r^2\right)^{\frac{3}{2}}\right]_{r=0}^{a}$$

$$= -\int_0^{2\pi}\left[\frac{2}{3}\left(a^2 - a^2\right)^{\frac{3}{2}} - \frac{2}{3}\left(a^2 - 0\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\int_0^{2\pi} a^3\,d\theta$$

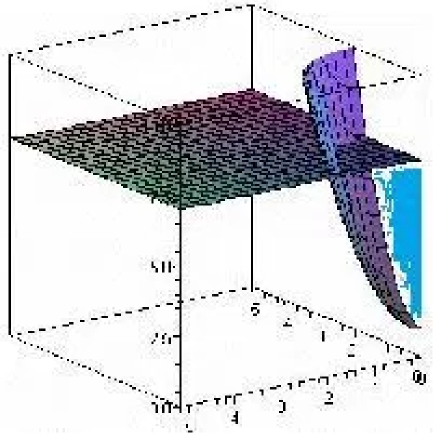
$$= \frac{2a^3}{3}(\theta)_0^{2\pi}$$

$$= \frac{4\pi a^3}{3}$$

So the volume of the sphere $x^2 + y^2 + z^2 = a^2$ is, $\boxed{\frac{4\pi a^3}{3}}$

Chapter 15 Multiple Integrals 15.4 24E

The portion of the paraboloid $z = 1 + 2x^2 + 2y^2$ when intersected by the plane $z = 7$ is given by



So, the upper surface is $z = h_1(x, y) = 7$ and the lower surface is

$z = h_2(x,y) = 1 + 2x^2 + 2y^2$

So, the function to be integrated is $f(x,y) = h_1(x,y) - h_2(x,y)$

$$= 6 - 2x^2 - 2y^2$$

To find the limits of integration, we equate the given surfaces and find the points of intersection.

i.e., $6 - 2x^2 - 2y^2 = 0$

This is nothing but the circle of radius $\sqrt{3}$ and angle is 0 through $\frac{\pi}{2}$ only while the entire region is in the 1st Octant.

The conversion from a Cartesian volume integral to a polar one is as follows:

$\iint\limits_D f(x,y)dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$ where the region over which we are integrating is $D = \{(r,\theta) \mid a \le \theta \le b, g(\theta) \le r \le h(\theta)\}$

Substituting the above details in this, we get

$$\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \left(6 - 2\left(x^2 + y^2\right)\right) r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \left(6 - 2r^2\right) r dr d\theta$$

$$= \frac{6r^2}{2}\Big)_0^{\sqrt{3}} \theta\Big)_0^{\pi/2} - \frac{2r^4}{4}\Big)_0^{\sqrt{3}} \theta\Big)_0^{\pi/2}$$

$$= 3 \times 3 \times \frac{\pi}{2} - \frac{9}{2} \times \frac{\pi}{2}$$

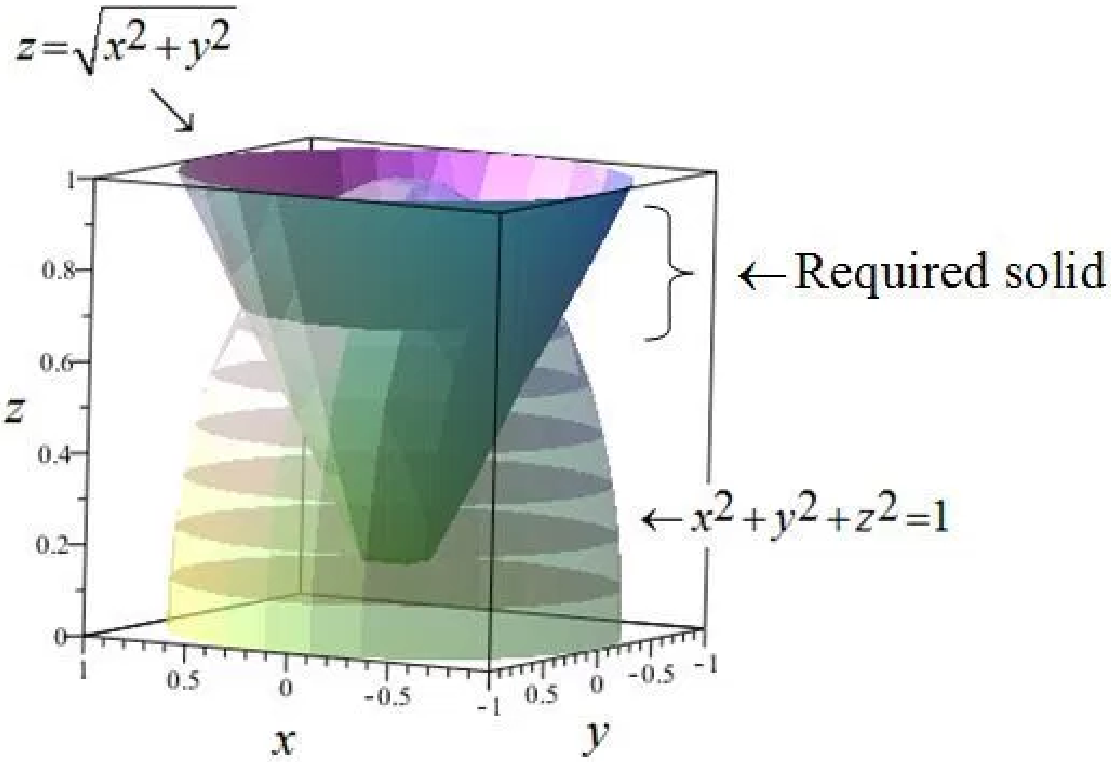
$$= \boxed{\frac{9\pi}{4}}$$

Chapter 15 Multiple Integrals 15.4 25E

Consider the cone $z=\sqrt{x^2+y^2}$ and sphere $x^2+y^2+z^2=1$.

Find the volume of the solid above by the cone and below by the sphere, use polar coordinates.

Sketch the solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$ is shown below:



Convert the rectangular coordinates to polar coordinates, use the following equations,

$$x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Rewrite the equation of the sphere $x^2 + y^2 + z^2 = 1$ as follows:

$$z^2 = 1 - \left(x^2 + y^2\right)$$
$$z = \sqrt{1 - \left(x^2 + y^2\right)}$$

Substitute $x = r\cos\theta, y = r\sin\theta,$ and $x^2 + y^2 = r^2$ in equation of the cone $z = \sqrt{x^2 + y^2}$.

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ &= \sqrt{r^2} \\ &= r \end{aligned}$$

Substitute $x^2 + y^2 = r^2$ in equation of the sphere $z = \sqrt{1 - \left(x^2 + y^2\right)}$.

$$z = \sqrt{1 - \left(x^2 + y^2\right)}$$
$$z = \sqrt{1 - r^2}$$

Find the point where the cone intersects the sphere by equating the formulas for z^2 and solving for r.

$$\begin{aligned} r^2 &= 1 - r^2 \\ 2r^2 &= 1 \\ r^2 &= \frac{1}{2} \\ r &= \sqrt{\frac{1}{2}} \end{aligned}$$

So the solid lies under the sphere and above the disk *D* given by

$$D = \left\{(r,\theta) \mid 0 \le r \le \frac{1}{\sqrt{2}},\ 0 \le \theta \le 2\pi\right\}.$$

Therefore, the volume of the solid is,

$$V = \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{2}}} \left(\sqrt{1-r^2} - r\right) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{2}}} \left(r\sqrt{1-r^2} - r^2\right) dr d\theta \;\cdots\cdots (1)$$

Let $u = 1 - r^2$, implies that $du = -2rdr$

$$-\frac{1}{2} du = rdr$$

For $r = 0$, then $u = 1$

For $r = \frac{1}{\sqrt{2}}$, then $u = \frac{1}{2}$

Substitute $u = 1 - r^2$, $-\frac{1}{2} du = rdr$ in (1),

$$V = -\frac{1}{2} \int_0^{2\pi} \int_1^{\frac{1}{2}} \left(\sqrt{u} - \sqrt{1-u}\right) du d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{-(1-u)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_1^{\frac{1}{2}} d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left[\frac{2u^{\frac{3}{2}}}{3} - \frac{-2(1-u)^{\frac{3}{2}}}{3} \right]_1^{\frac{1}{2}} d\theta$$

Continuous to the above step,

$$= -\frac{1}{2} \int_0^{2\pi} \left[\frac{2\left(\frac{1}{2}\right)^{\frac{3}{2}}}{3} - \frac{-2\left(1-\frac{1}{2}\right)^{\frac{3}{2}}}{3} - \frac{2(1)^{\frac{3}{2}}}{3} + \frac{-2(1-1)^{\frac{3}{2}}}{3} \right] d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left[\frac{2\left(\frac{1}{2}\right)^{\frac{3}{2}}}{3} + \frac{2\left(\frac{1}{2}\right)^{\frac{3}{2}}}{3} - \frac{2}{3} + \frac{0}{3} \right] d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left[\frac{4\left(\frac{1}{2}\right)^{\frac{3}{2}}}{3} - \frac{2}{3} \right] d\theta$$

$$=-\frac{1}{2}\int_0^{2\pi}\left[\frac{\sqrt{2}-2}{3}\right]d\theta$$

$$=-\frac{1}{2}\left(\frac{\sqrt{2}-2}{3}\right)\int_0^{2\pi}d\theta$$

$$=-\frac{1}{2}\left(\frac{\sqrt{2}-2}{3}\right)[\theta]_0^{2\pi}$$

$$=-\frac{1}{2}\left(\frac{\sqrt{2}-2}{3}\right)[2\pi-0]$$

$$=\frac{\left(2-\sqrt{2}\right)\pi}{3}$$

$$=\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

Hence, the required volume of the solid is $V=\boxed{\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)}$.

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If we put $z=3x^2+3y^2$ in the equation of parabolic $z=4-x^2-y^2$ we get $4\left(x^2+y^2\right)=4$ or $x^2+y^2=1$. This means that the two parabolic intersect in the circle $x^2+y^2=1$. So the solid lies under the parabolic $z=4-\left(x^2+y^2\right)$ and above the disk D given by $z=4-\left(x^2+y^2\right)$ and above parabolic $z=3\left(x^2+y^2\right)$ and under disk D.

$D=\{(r,\theta): 0\le\theta\le 2\pi,\ 0\le r\le 1\}$ In polar co – ordinates

Then the volume of the solid is

$$v=\iint_D\left[4-\left(x^2+y^2\right)\right]dA-\iint_D 3\left(x^2+y^2\right)dA$$

$$=\int_0^{2\pi}\int_0^1\left[4-r^2\right]r\,dr\,d\theta-\int_0^{2\pi}\int_0^1 3r^2\,r\,dr\,d\theta$$

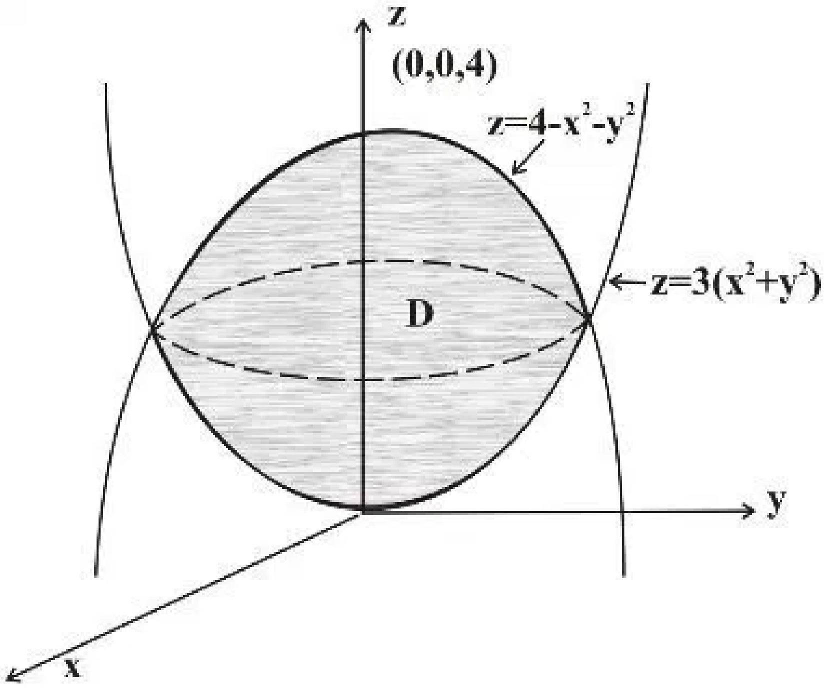
$$=\int_0^{2\pi}\int_0^1\left[4r-r^3\right]dr\,d\theta-3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^1 d\theta-\frac{3}{4}\int_0^{2\pi}\left(r^4\right)_0^1 d\theta$$

$$=\int_0^{2\pi}\left[2-\frac{1}{4}-0\right]d\theta-\frac{3}{4}(1-0)\int_0^{2\pi}d\theta$$

$$=\frac{7}{4}\int_0^{2\pi}1.\ d\theta-\frac{3}{4}\int_0^{2\pi}d\theta$$

$$=\frac{7}{4}(2\pi-0)-\frac{3}{4}(2\pi-0)$$



$$= \frac{7\pi}{2} - \frac{3\pi}{2}$$
$$= \boxed{2\pi}$$

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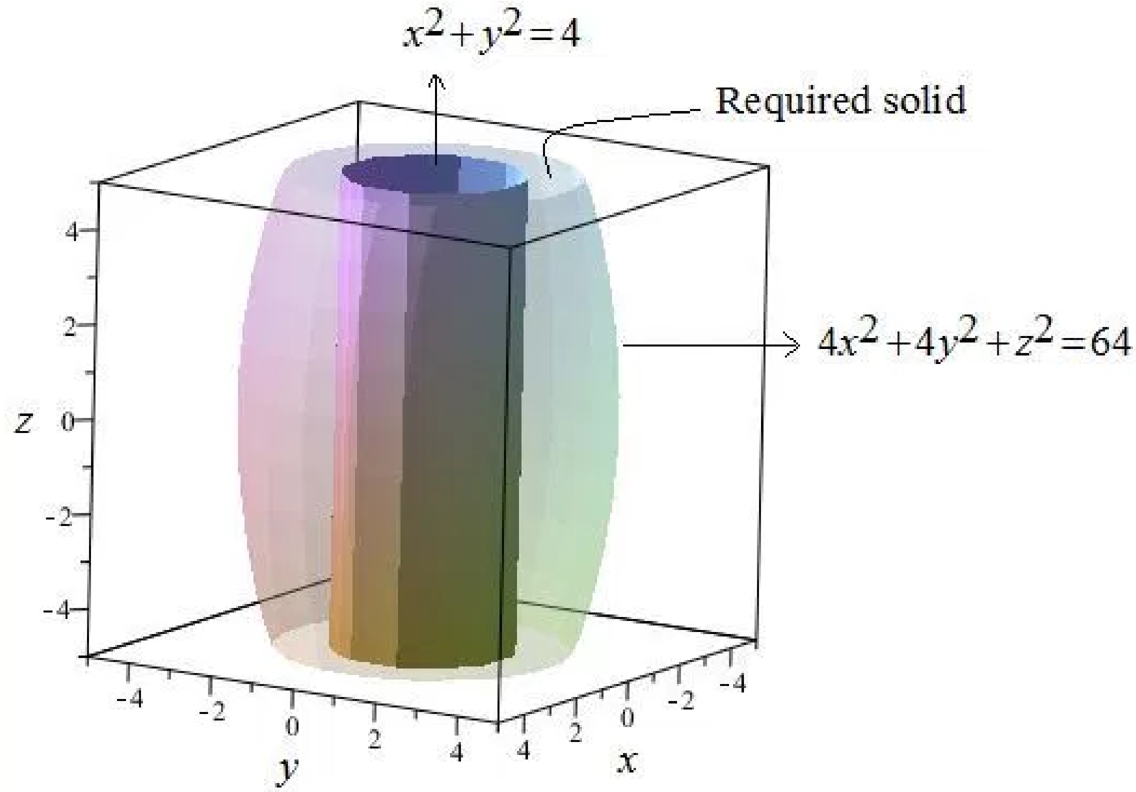
The solid formed by the cylinder and the ellipsoid is shown below:



The two parts of the required solid is symmetrical about the plane $z=0$ (xy-plane).

Now, evaluate the total volume of the solid.

First, find the volume of the solid above the plane, $z=0$ (xy-plane), and multiply this by 2.

Suppose that D is the circular disk, is the projection of the required solid in xy-plane.

Thus, the total volume of the solid is given by the following:

$$V = 2\iint_D z\,dA$$

$$= 2\iint_D z\,dx\,dy$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\sqrt{64-4r^2}\,r\,dr\,d\theta$$

$$= \frac{2}{-8}\int_{\theta=0}^{2\pi}\left(\int_{r=0}^{2}\sqrt{64-4r^2}\,(-8r\,dr)\right)d\theta$$

$$= \frac{-1}{4}\left(\frac{2}{3}\right)\int_0^{2\pi}\left[\left(64-4r^2\right)^{3/2}\right]_{r=0}^{2}d\theta \text{ Use } \int\sqrt{f(r)}\,f'(r)\,dr = \frac{2}{3}\left(f(r)\right)^{\frac{3}{2}}$$

$$= \frac{-1}{6}\int_0^{2\pi}\left[\left((48)^{3/2}-(64)^{3/2}\right)\right]d\theta$$

Continue the above to get the following:

$$V = -\frac{2\pi}{6}\left[48\sqrt{48}-\left(8^2\right)^{\frac{3}{2}}\right]$$

$$= -\frac{\pi}{3}\left[48\sqrt{48}-64\times 8\right]$$

$$= -\frac{\pi}{3}\left[8\cdot 6\sqrt{16\cdot 3}-64\times 8\right]$$

$$= -\frac{8\pi}{3}\left(24\sqrt{3}-64\right)$$

$$= \frac{8\pi}{3}\left(64-24\sqrt{3}\right)$$

Hence the volume of the solid is $\boxed{\frac{8\pi}{3}\left(64-24\sqrt{3}\right)}$.

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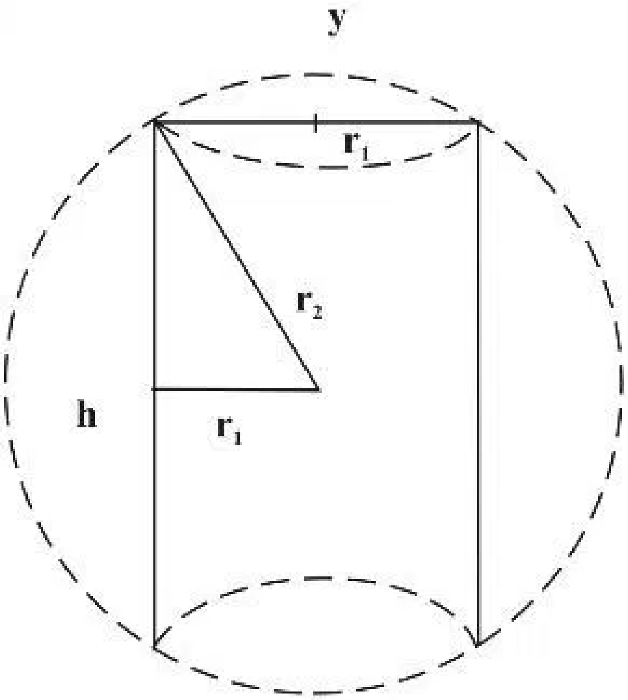

(A)

Here the region in xy – plane is the annular region

$$D = r_1^2 \le x^2 + y^2 \le r_2^2$$

And the desired volume is twice that above the xy – plane.

Then $\quad v = 2\iint_D \sqrt{r_2^2 - (x^2 + y^2)}\, dA$

By changing to polar co – ordinates the region becomes

$$\{(r,\theta):\ 0 \le \theta \le 2\pi,\ r_1 \le r \le r_2\}$$

Then $\quad v = 2\int_0^{2\pi}\int_{r_1}^{r_2} \sqrt{r_2^2 - r^2}\ r\, dr\, d\theta$

$$= 2\int_0^{2\pi}\left(\frac{-1}{2}\right)\frac{2}{3}\left[\left(r_2^2 - r^2\right)^{3/2}\right]_{r=r_1}^{r=r_2} d\theta$$

$$= -\frac{2}{3}\left[0 - \left(r_2^2 - r_1^2\right)^{3/2}\right]\int_0^{2\pi} 1.\ d\theta$$

$$= \frac{2}{3}\left(r_2^2 - r_1^2\right)^{3/2}(2\pi)$$

$$= \boxed{\frac{4\pi}{3}\left(r_2^2 - r_1^2\right)^{3/2}}$$

(B)

When h is the height of the ring then from the figure

$$r_2^2 = r_1^2 + \left(\frac{1}{2}h\right)^2$$

i.e. $$r_2^2 - r_1^2 = \frac{h^2}{4}$$

Then the volume becomes

$$v = \frac{4\pi}{3}\left(\frac{h^2}{4}\right)^{3/2}$$

i.e. $$v = \boxed{\frac{\pi}{6}h^3}$$

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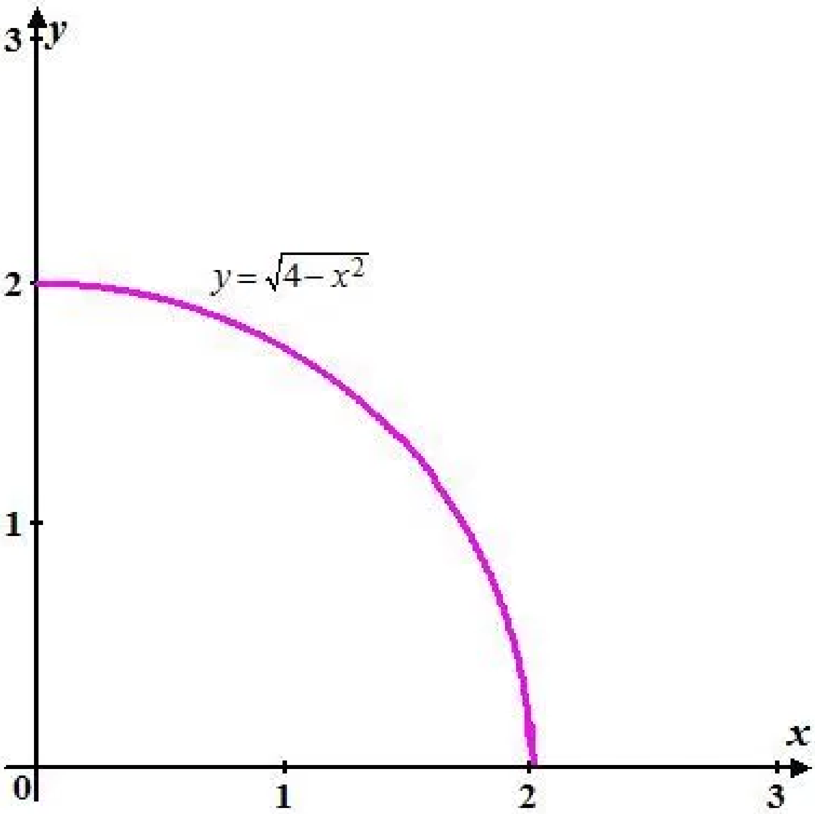
The objective is to evaluate the iterated integral by converting to polar coordinates:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx.$$

Sketch the region $0 \le y \le \sqrt{4-x^2}, 0 \le x \le 2.$



As $y=\sqrt{4-x^2}$ then $x^2+y^2=4$ and $0\le x\le 2.$

Therefore, the region of the integral is a semi-circle with radius 2 in the first quadrant.

Use polar coordinates $x=r\cos\theta, y=r\sin\theta$ to get $0\le r\le 2, 0\le\theta\le\frac{\pi}{2}.$

Also

$$\begin{aligned}x^2+y^2&=r^2\cos^2\theta+r^2\sin^2\theta\\&=r^2\left(\cos^2\theta+\sin^2\theta\right)\\&=r^2\end{aligned}$$

And $dxdy=rdrd\theta.$

Substitute all the values in $\int_0^2\int_0^{\sqrt{4-x^2}}e^{-x^2-y^2}dy\,dx.$

$$\begin{aligned}\int_0^2\int_0^{\sqrt{4-x^2}}e^{-x^2-y^2}dy\,dx&=\int_0^{\frac{\pi}{2}}\int_0^2 e^{-r^2}rdr\,d\theta\\&=\int_0^{\frac{\pi}{2}}\int_0^2 re^{-r^2}dr\,d\theta\end{aligned}$$

First integrate with respect to r using substitution method.

Take $r^2=t$ then $2rdr=dt.$

The limits of t are:

For $r=0,\quad t=0^2=0.$

For $r=2,\quad t=2^2=4.$

Then the integral is evaluated as:

$$\begin{aligned}\int_0^2 re^{-r^2}dr&=\int_0^4\frac{1}{2}e^{-t}dt\\&=\frac{1}{2}\int_0^4 e^{-t}dt\\&=\frac{1}{2}\left[-e^{-t}\right]_0^4\\&=\frac{1}{2}\left[-e^{-4}+1\right]\end{aligned}$$

Substitute the value in $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^2 re^{-r^2}\,dr\,d\theta$.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1-e^{-4}\right)d\theta$$

$$= \frac{1}{2}\left(1-e^{-4}\right)\int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$= \frac{1}{2}\left(1-e^{-4}\right)[\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(1-e^{-4}\right)\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{4}\left(1-e^{-4}\right)$$

Therefore, the value of the integral $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$ is $\boxed{\frac{\pi}{4}\left(1-e^{-4}\right)}$.

Chapter 15 Multiple Integrals 15.4 30E

The conversion from a Cartesian volume integral to a polar one is

$\iint\limits_D f(x,y)dA = \int_a^b \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$ ...... (1) where the region over which we are integrating is $D = \{(r,\theta)\,|\,a \le \theta \le b, g(\theta) \le r \le h(\theta)\}$

To find the limits of integration, we examine the region over which we are integrating.

Given integral is $\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2 y\,dx\,dy$

Given limits are $-\sqrt{a^2-y^2}$ to 0 for x and 0 to a for y. This region is a quarter of a circle of radius a. To see why, realize that the lower x limit is

$x = -\sqrt{a^2-y^2}$

$x^2 = a^2 - y^2$

$x^2 + y^2 = a^2$ which is the equation for a circle of radius a centered at the origin.

Further, we observe that the limits of x are in the negative region and the limits of y are positive.

So, the part of the given circle lies in the 2nd quadrant.

Consequently, in the polar coordinates the 2nd quadrant is denoted by $\frac{\pi}{2} \le \theta \le \pi$ and the radius is $0 \le r \le a$

$f(x,y) = x^2 y$ and so, $f(r\cos\theta, r\sin\theta) = r^3\cos^2\theta\sin\theta$

$$= r^3\sin\theta - r^3\sin^3\theta$$

Using these details in (1), we get $\int_{\pi/2}^{\pi}\int_{0}^{a}\left(r^3\sin\theta-r^3\sin^3\theta\right)rdrd\theta$

$$=\int_{\pi/2}^{\pi}\int_{0}^{a}r^4\sin\theta drd\theta+\int_{\pi/2}^{\pi}\int_{0}^{a}r^4\left(\frac{1}{4}\sin 3\theta-\frac{3}{4}\sin\theta\right)drd\theta$$

$$=\left(\frac{r^5}{5}\right)_0^a\left(-\cos\theta\right)_{\pi/2}^{\pi}+\left(\frac{r^5}{5}\right)_0^a\left(-\frac{1}{12}\cos 3\theta+\frac{3}{4}\cos\theta\right)_{\pi/2}^{\pi}$$

$$=\frac{a^5}{5}\left\{-\cos\pi+\cos\frac{\pi}{2}-\frac{1}{12}\cos 3\pi+\frac{1}{12}\cos\frac{3\pi}{2}+\frac{3}{4}\cos\pi-\frac{3}{4}\cos\frac{\pi}{2}\right\}$$

$$=\frac{a^5}{5}\left\{1+0+\frac{1}{12}+0-\frac{3}{4}-0\right\}$$

$$=\boxed{\frac{a^5}{15}}$$

Chapter 15 Multiple Integrals 15.4 31E

Consider the following iterated integral by converting to polar coordinates:

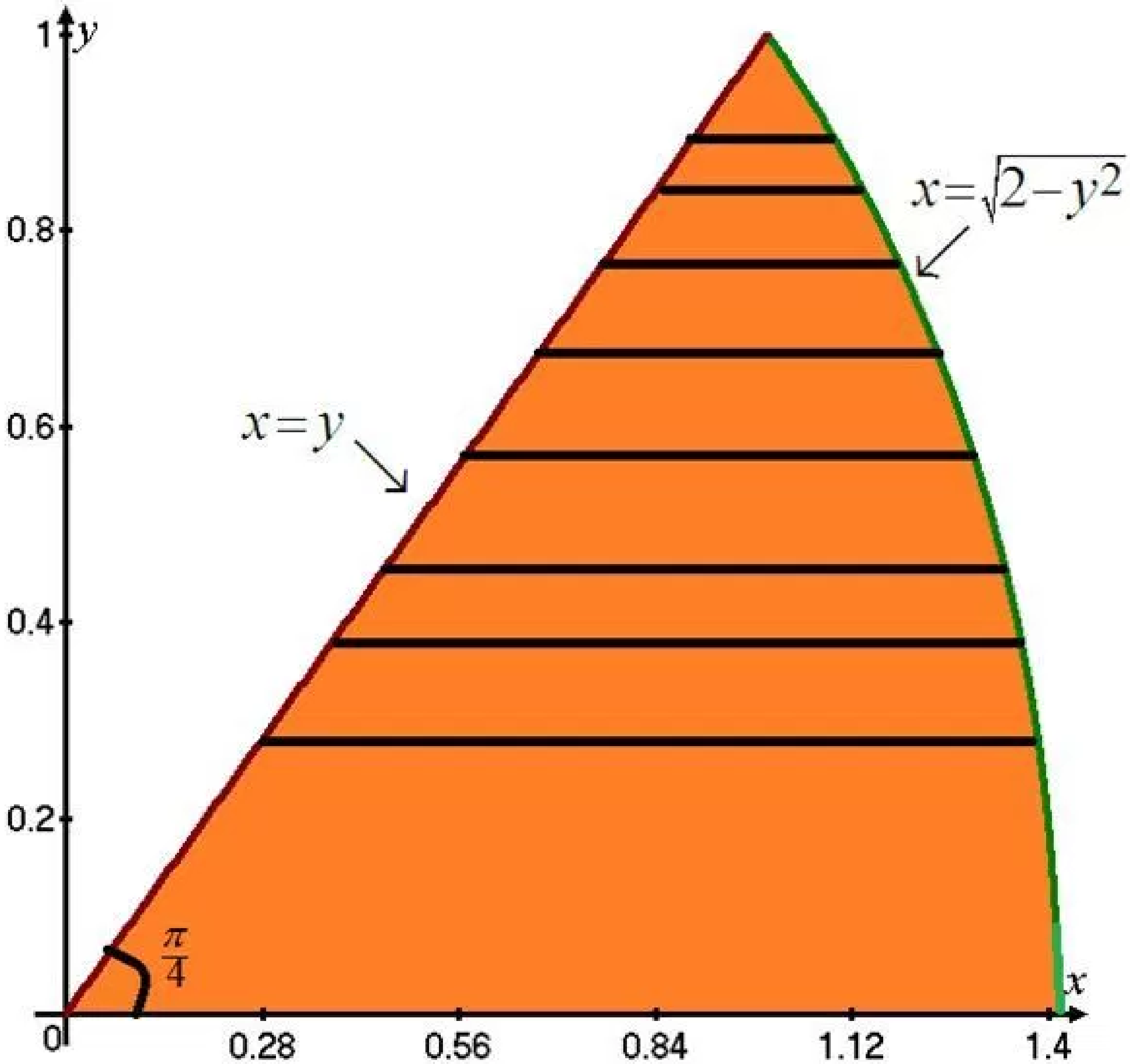
$$\int_{0}^{1}\int_{y}^{\sqrt{2-y^2}}(x+y)dxdy$$

Here, y ranges from 0 to 1 and

x ranges from $x=y$ to $x=\sqrt{2-y^2}$.

The region is between the line $x=y$ and arc of the circle $x^2+y^2=2$ as shown below:

The region is divided into horizontal lines for particular $y=y_0$, such that $0\le y_0\le 1$.



y
1
0.8
0.6
0.4
0.2
0
0.28
0.56
0.84
1.12
1.4
x
$x=\sqrt{2-y^2}$
$x=y$
$\frac{\pi}{4}$

The polar coordinates for region R is $x = r\cos\theta, y = r\sin\theta.$

The limits of r is 0 to radius of circle $\sqrt{2}.$

For limits of θ, observe that the region is part of the circle up to angular line $\theta = \frac{\pi}{4}.$

The line $y = x$ makes angle $\frac{\pi}{4}$ with $x-\text{axis}$ in positive direction (antilock wise direction).

θ is ranges from 0 to $\frac{\pi}{4}$

$dxdy = rdrd\theta$

$$\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)dxdy = \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{\sqrt{2}} (r\cos\theta + r\sin\theta) rdrd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{\sqrt{2}} r^2 (\cos\theta + \sin\theta) drd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \left[\frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \left(\frac{2\sqrt{2}}{3} - 0\right) d\theta$$

It can be further simplified as shown below:

$$= \frac{2\sqrt{2}}{3} \int_{\theta=0}^{\frac{\pi}{4}} (\cos\theta + \sin\theta) d\theta$$

$$= \frac{2\sqrt{2}}{3} [\sin\theta - \cos\theta]_{\theta=0}^{\frac{\pi}{4}}$$

$$= \frac{2\sqrt{2}}{3} \left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right] - \frac{2\sqrt{2}}{3} [\sin 0 - \cos 0]$$

$$= \frac{2\sqrt{2}}{3} \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \frac{2\sqrt{2}}{3} [0-1]$$

$$= \frac{2\sqrt{2}}{3} [0] - \frac{2\sqrt{2}}{3} [0-1]$$

$$= \frac{2\sqrt{2}}{3}$$

Therefore, $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)dxdy = \boxed{\frac{2\sqrt{2}}{3}}.$

Chapter 15 Multiple Integrals 15.4 32E

Consider the integral,

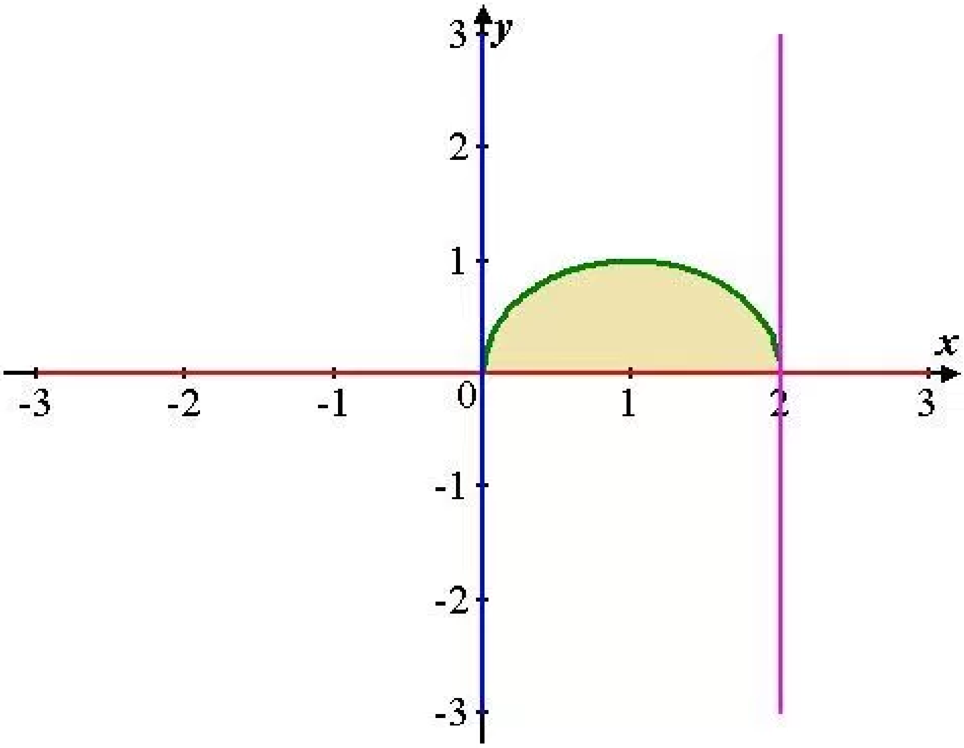
$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ dy\ dx$$

The objective is to evaluate the iterated integral by converting to polar coordinates.

To convert the integral into polar coordinates, we make the following substitutions:

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dxdy &= rdrd\theta \\ x^2+y^2 &= r^2 \end{aligned}\right\} \qquad \text{......}(1)$$

The region is shown below.



To find the limits, we look at the region first set up by the limits:

$$\begin{aligned} y &= \sqrt{2x-x^2} \\ y^2 &= 2x-x^2 \\ x^2+y^2 &= 2x \\ r^2 &= 2r\cos\theta \qquad \text{from}(1) \\ &\Rightarrow r = 2\cos\theta \end{aligned}$$

Therefore, the radius goes from 0 to $2\cos\theta$, which is the distance from the origin to a point on the circle. Also, the angle goes from 0 to $\frac{\pi}{2}$ as the circle is only in the first quadrant.

Hence the region in polar coordinates is $R = \left\{(r,\theta)\,|\,0 \le r \le 2\cos\theta, 0 \le \theta \le \frac{\pi}{2}\right\}$

Now, compute the integral in polar coordinates as follows:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r.r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}\left(r^3\right)_0^{2\cos\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} 8\cos^3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos^2\theta \cdot \cos\theta\right) d\theta$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\left(1-\sin^2\theta\right) \cdot \cos\theta\right) d\theta$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\cos\theta\right) d\theta$$

$$= \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3}\left(\sin\frac{\pi}{2} - \frac{1}{3}\sin^3\frac{\pi}{2}\right) - \frac{8}{3}\left(\sin 0 - \frac{1}{3}\sin^3 0\right)$$

$$= \frac{8}{3}\left(1 - \frac{1}{3}\right)$$

$$= \boxed{\frac{16}{9}}$$

Therefore, $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ dy\ dx = \boxed{\frac{16}{9}}$.

Chapter 15 Multiple Integrals 15.4 33E

Convert the integral to polar coordinates, and then evaluate.

On a disk of radius 1, the limits of integration in θ go all the way around the circle, from 0 to 2π, and the limits of integration in r are the origin to the circumference of the disk, or 0 to 1. We use the conversion $r^2 = x^2 + y^2$ to change the integrand to polar coordinates and enter these limits of integration; in converting the integral to polar coordinates we must add a factor of r:

$$\int_0^1 \int_0^{2\pi} \left(e^{\left(r^2\right)^2} r\right) d\theta dr$$

$$\int_0^1 \int_0^{2\pi} \left(e^{r^4} r\right) d\theta dr$$

Integrate in terms of θ to make a single integral in terms of r:

$$\int_0^1\int_0^{2\pi}\left(e^{r^4}r\right)d\theta dr$$

$$\int_0^1\left(e^{r^4}r\theta\right)\Big|_0^{2\pi}dr$$

$$\int_0^1\left(2\pi e^{r^4}r-0\right)dr$$

$$\boxed{\int_0^1\left(2\pi e^{r^4}r\right)dr}$$

We now have a single integral in terms of r, as the problem asks. As specified we will use the calculator to find an approximation, reaching the solution $\boxed{4.5951}$.

Chapter 15 Multiple Integrals 15.4 34E

Convert the integral to polar coordinates, and then evaluate.

The region of integration is the quarter of a disk of radius 1 that lies in the first quadrant. The limits of integration in r are the origin to the circumference of the disk, or 0 to 1. The limits in θ bound the region in the first quadrant, going from 0 to $\pi/2$.

We use the following conversions from rectangular to polar coordinates to convert the integrand:

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

Change the integrand to polar coordinates and enter the limits of integration; in converting the integral to polar coordinates we must add a factor of r:

$$\int_0^1\int_0^{\pi/2}\left[(r\cos\theta)(r\sin\theta)\left(\sqrt{1+r^2}\right)r\right]d\theta dr$$

$$\int_0^1\int_0^{\pi/2}\left[(r^3\cos\theta\sin\theta)\left(\sqrt{1+r^2}\right)\right]d\theta dr$$

Use the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$, rewritten as $\sin\theta\cos\theta = (1/2)(\sin 2\theta)$, to substitute and then integrate in terms of θ (note that the θ integration could also be done via substitution if desired):

$$\int_0^1\int_0^{\pi/2}\left[(r^3(1/2)(\sin 2\theta))\left(\sqrt{1+r^2}\right)\right]d\theta dr$$

$$=\frac{1}{2}\int_0^1\int_0^{\pi/2}\left[(\sin 2\theta)\left(r^3\sqrt{1+r^2}\right)\right]d\theta dr$$

$$=\frac{1}{2}\int_0^1\left[\frac{(-\cos 2\theta)}{2}\left(r^3\sqrt{1+r^2}\right)\right]\Bigg|_0^{\pi/2}dr$$

$$=\frac{1}{4}\int_0^1\left[(-\cos 2(\pi/2))\left(r^3\sqrt{1+r^2}\right)-(-\cos 2(0))\left(r^3\sqrt{1+r^2}\right)\right]dr$$

$$=\frac{1}{4}\int_0^1\left[(-\cos\pi)\left(r^3\sqrt{1+r^2}\right)-(-1)\left(r^3\sqrt{1+r^2}\right)\right]dr$$

$$= \frac{1}{4}\int_0^1 \left[(1)\left(r^3\sqrt{1+r^2}\right) + \left(r^3\sqrt{1+r^2}\right) \right] dr$$

$$= \frac{1}{4}\int_0^1 \left[2\left(r^3\sqrt{1+r^2}\right) \right] dr$$

$$= \boxed{\frac{1}{2}\int_0^1 \left(r^3\sqrt{1+r^2}\right) dr}$$

We now have a single integral in terms of r, as the problem asks. As specified we will use the calculator to find an approximation, reaching the solution $\boxed{0.1609}$.

Chapter 15 Multiple Integrals 15.4 35E

Consider the circular swimming pool with diameter 40 ft. depth is constant along east west lines and increases 2 ft at the south end to 7 ft at the north end.

The objective is find the volume of water in the pool.

If we set the top of the pool as the *xy*-plane with its centre as the origin, we can describe the region as $x^2 + y^2 \le 400$, or in polar coordinates, $D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 20\}$.

We can set the depth of the pool as the positive *z*-axis, and let the *y*-axis represent the position at the top of the pool as we move south to north, so $z = 2$ corresponds to $y = -20$, and $z = 7$ corresponds to $y = 20$. Thus we can represent the depth of the pool as the line

$$z - 7 = \frac{7-2}{20-(-20)}(y-20)$$

$$z = \frac{1}{8}y + \frac{9}{2}$$

which in polar coordinates is $z = \frac{1}{8}r\sin\theta + \frac{9}{2}$

This allows us to write the volume as a polar integral:

$$\int_0^{2\pi}\int_0^{20}\left(\frac{1}{8}r\sin\theta + \frac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^{20}\left(\frac{1}{8}r^2\sin\theta + \frac{9}{2}r\right) dr\,d\theta$$

First, evaluate the integral with respect to *r*:

$$\int_0^{20} \frac{1}{8}r^2\sin\theta + \frac{9}{2}r\,dr = \left[\frac{1}{24}r^3\sin\theta + \frac{9}{4}r^2\right]_{r=0}^{r=20}$$

$$= \frac{1}{24}(20)^3\sin\theta + \frac{9}{4}(20)^2 - 0$$

$$= \frac{1000}{3}\sin\theta + 900$$

Now evaluate the integral with respect to θ:

$$\int_0^{2\pi} \frac{1000}{3}\sin\theta + 900\, d\theta = \left[-\frac{1000}{3}\cos\theta + 900\theta\right]_{\theta=0}^{\theta=2\pi}$$
$$= \left[-\frac{1000}{3} + 900(2\pi)\right] - \left[-\frac{1000}{3}\right]$$
$$= 1800\pi \text{ ft}^3$$

Therefore, the volume of water in the pool is $\boxed{1800\pi \text{ ft}^3}$.

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(a)

The objective is to calculate the amount of water supplied use integration with polar coordinates.

Use the change of polar coordinates formula for double integrals:

Suppose f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$ then $\iint_R f(x,y) = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

The following rectangular to polar conversions are used:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Since the depth of e^{-r} feet per hour at a distance of r feet from the sprinkler, $z = -e^{-r}$ or to match the problem, $z = -e^{-R}$.

The region of integration is a punctured disk centered at the origin.

From the given information write the region in polar coordinates as $R = \{(r,\theta) \mid 0 < r \le 100,\ 0 \le \theta \le 2\pi\}$.

Since sprinkler distributes the water in a circular region of radius 100 ft.

The required polar integral as:

$$\int_0^{2\pi} \int_0^{100} -re^{-r}\, dr\, d\theta$$

Since the depth of e^{-r} feet per hour at a distance of r feet from the sprinkler,

So, consider $z = -e^{-r}$

Evaluating the integral as follows:

$$\int_0^{2\pi}\int_0^{100} -re^{-r}\ dr\ d\theta = \int_0^{2\pi} d\theta \int_0^{100} -re^{-r}\ dr$$
$$= [\theta]_0^{2\pi} \int_0^{100} -re^{-r}\ dr$$
$$= [2\pi - 0]\int_0^{100} -re^{-r}\ dr$$
$$= -2\pi \int_0^{100} re^{-r}\ dr$$

$$= -2\pi \lim_{t\to 0^+} \int_t^{100} re^{-r}\ dr$$
$$= -2\pi \lim_{t\to 0^+} \left[r\int e^{-r}\ dr - \int \frac{dr}{dr}\int e^{-r}\ drdr \right]_t^{100}$$ Use integration by parts
$$= -2\pi \lim_{t\to 0^+} \left[-re^{-r} + \int e^{-r}\ dr \right]_t^{100}$$
$$= 2\pi \lim_{t\to 0^+} \left[re^{-r} + e^{-r} \right]_t^{100}$$

Continuous to the above step,

$$= 2\pi \lim_{t\to 0^+} \left[100e^{-100} + e^{-100} - \left(te^{-t} + e^{-t} \right) \right]$$
$$= 2\pi \lim_{t\to 0^+} \left[101e^{-100} - e^{-t}(t+1) \right]$$
$$= 2\pi \left[101e^{-100} - e^{-0}(0+1) \right]$$
$$= 2\pi \left[101e^{-100} - 1 \right]$$

Hence, the required value of the integral $\int_0^{2\pi}\int_0^{100} -re^{-r}\ dr\ d\theta = \boxed{2\pi\left(101e^{-100} - 1\right)}$.

(b)

The average value is defined as:

$$f_{avg} = \frac{1}{A(D)}\iint_D f\ dA$$

Over the region, the area of the punctured disk is:

$$A(D) = \int_0^{2\pi}\int_0^R r\ dr\ d\theta\ = \int_0^{2\pi}\left(\int_0^R rdr\ \right)d\theta$$

$$= \int_0^{2\pi}\left(\left[\frac{1}{2}r^2\right]_0^R\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}R^2 - \frac{1}{2}(0)^2\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}R^2 - 0\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{R^2}{2}\right)d\theta$$

$$= \left(\frac{R^2}{2}\right)\int_0^{2\pi} d\theta$$

$$= \left(\frac{R^2}{2}\right)[\theta]_0^{2\pi}$$

Continuous to the above step,

$$= \left(\frac{R^2}{2}\right)[2\pi - 0]$$

$$= 2\pi\left(\frac{R^2}{2}\right)$$

$$= \pi R^2$$

Hence, the required area is $A(D) = \boxed{\pi R^2}$

Compute the average value integral.

$$f_{avg} = \frac{1}{A(D)}\iint_D f\ dA$$

$$\frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R -re^{-r}\ dr\ d\theta = \frac{1}{\pi R^2}\int_0^{2\pi} d\theta\int_0^R -re^{-r}\ dr$$

$$= \frac{1}{\pi R^2}[\theta]_0^{2\pi}\int_0^R -re^{-r}\ dr$$

$$= \frac{1}{\pi R^2}[2\pi - 0]\int_0^R -re^{-r}\ dr$$

$$= \frac{2\pi}{\pi R^2}\int_0^R -re^{-r}\ dr$$

Continuous to the above step,

$$= \frac{2}{R^2}\int_0^R -re^{-r}\ dr$$

$$= -\frac{2}{R^2}\left[r\int e^{-r}\ dr - \int \frac{dr}{dr}\int e^{-r}\ drdr\right]_0^R$$ Use integration by parts

$$= -\frac{2}{R^2}\left[-re^{-r} + \int e^{-r}\ dr\right]_0^R$$

$$= -\frac{2}{R^2}\left[-re^{-r} - e^{-r}\right]_0^R$$

$$= \frac{2}{R^2}\left[re^{-r} + e^{-r}\right]_0^R$$

$$= \frac{2}{R^2}\left[Re^{-R} + e^{-R} - \left((0)e^{-0} + e^{-0}\right)\right]$$

$$= \frac{2}{R^2}\left[Re^{-R} + e^{-R} - 1\right]$$

Hence, the required average amount of the water per hour per square foot supplied to the region inside the circle of radius R is $f_{avg} = \boxed{\frac{2}{R^2}\left(e^{-R}(R+1)-1\right)}$

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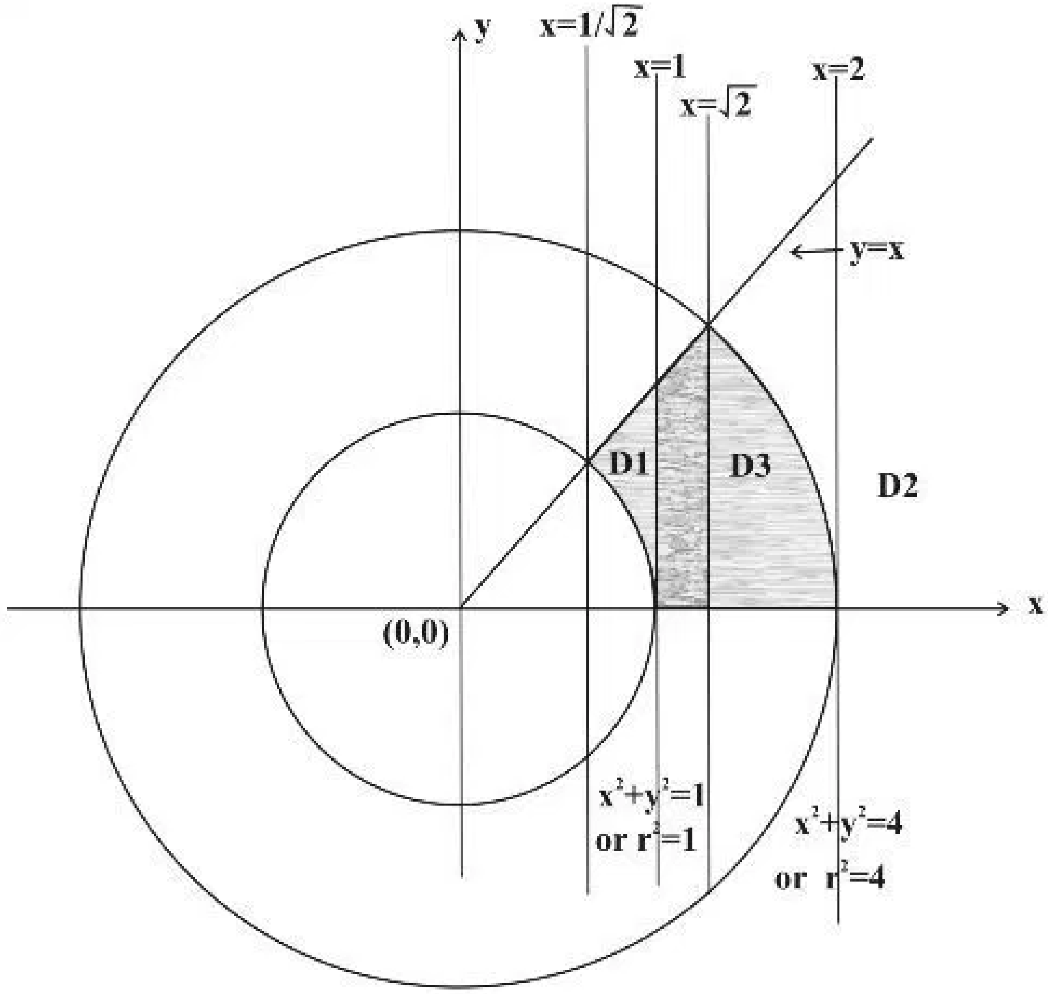

The given integrals are

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx+\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx+\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

$$=\iint_{D_1} xy\,dA+\iint_{D_2} xy\,dA+\iint_{D_3} xy\,dA$$

Where $D_1\left\{\frac{1}{\sqrt{2}}\le x\le 1,\ \sqrt{1-x^2}\le y\le x\right\}$

$$D_2=\left\{1\le x\le\sqrt{2},\ 0\le y\le x\right\}$$

$$D_3=\left\{\sqrt{2}\le x\le 2,\ 0\le y\le\sqrt{4-x^2}\right\}$$

On changing to polar co – ordinates we see that the integral is over the region D where D is the combination of given region D_1, D_2 and D_3 and

$$D=\left\{(x,\theta):\ 0\le\theta\le\frac{\pi}{4},\ 1\le r\le 2\right\}$$

Then the given integral becomes

$$\iint_D (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$$

$$=\int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\int_1^2 r^3\sin 2\theta dr\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\sin 2\theta\,d\theta\int_1^2 r^3\,dr$$

i.e. $\iint_D (r\cos\theta)(r\sin\theta)r\,dr\,d\theta=\frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/4}\left[\frac{r^4}{4}\right]_1^2$

$$=\frac{1}{2}\left[0+\frac{1}{2}\right]\left[\frac{16}{4}-\frac{1}{4}\right]$$

$$=\frac{1}{4}\left(\frac{15}{4}\right)$$

$$=\boxed{\frac{15}{16}}$$

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(A) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA$

$$=\lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}dA$$

$$=\lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}r\,dr\,d\theta$$

$$=\int_0^{2\pi}\lim_{a\to\infty}\left\{\int_0^a e^{-r^2}r\,dr\right\}d\theta$$

Now in, $\int_0^a e^{-r^2} r\,dr$

Put $\quad r^2 = t$

$\quad\quad 2r\,dr = dt$

$\Rightarrow \quad r\,dr = \frac{dt}{2}$

This integral will reduce to

$$\frac{1}{2}\int_0^{a^2} e^{-t}dt = \frac{1}{2}\left[-e^{-t}\right]_0^{a^2}$$
$$= \frac{1}{2}\left[-\frac{1}{e^{a^2}} + 1\right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{e^{a^2}}\right]$$

Therefore

$$\lim_{a\to\infty}\int_0^a e^{-r^2} r\,dr = \lim_{a\to\infty}\frac{1}{2}\left[1 - \frac{1}{e^{a^2}}\right]$$
$$= \frac{1}{2}$$

Now, $\int_0^{2\pi}\left\{\lim_{a\to\infty}\int_0^a e^{-r^2} r\,dr\right\}d\theta$

$$= \frac{1}{2}\int_0^{2\pi} d\theta$$
$$= \frac{1}{2}\left[\theta\right]_0^{2\pi}$$
$$= \pi$$

Hence $\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA = \pi}$

(B) We have $\iint_{D_a} e^{-(x^2+y^2)}dA = \int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta$

$$= \int_0^{2\pi}\left\{\int_0^a e^{-r^2} r\,dr\right\}d\theta$$

Now in $\int_0^a e^{-r^2} r dr$

Put $r^2 = t$

$2rdr = dt$

Or $rdr = \frac{dt}{2}$

Therefore $\int_0^a e^{-r^2} r dr = \frac{1}{2}\int_0^{a^2} e^{-t} dt$

$$= \frac{1}{2}\left[-e^{-t}\right]_0^{a^2}$$

$$= \frac{1}{2}\left[-\frac{1}{e^t}\right]_{a^2}^0$$

$$= \frac{1}{2}\left[1 - \frac{1}{e^{a^2}}\right]$$

Therefore $\int_0^{2\pi}\left\{\int_0^a e^{-r^2} r dr\right\} d\theta = \int_0^{2\pi} \frac{1}{2}\left(1 - \frac{1}{e^{a^2}}\right) d\theta$

$$= \frac{1}{2}\left(1 - \frac{1}{e^{a^2}}\right)\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^{a^2}}\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^{a^2}}\right) 2\pi$$

$$= \pi\left(1 - \frac{1}{e^{a^2}}\right)$$

Let D_{2a} is the disc with radius 2a and centre at region.

Therefore, $\int_{D_{2a}}\int e^{-(x^2+y^2)} dA = \int_0^{2\pi}\int_0^{2a} e^{-r^2} r dr \, d\theta$

$$= \pi\left(1 - \frac{1}{e^{4a^2}}\right)$$

Since S_a is the square with vertices

$(\pm a, \pm a)$ i.e. $-a \le x \le a$, $-a \le y \le a$.

Therefore it is clear that $D_a \subseteq S_a \subseteq D_{2a}$

Since $D_a \subseteq S_a \subseteq D_{2a}$

Therefore $\int_{D_a}\int e^{-(x^2+y^2)}dA \le \int_{S_a}\int e^{-(x^2+y^2)}dA \le \int_{D_{2a}}\int e^{-(x^2+y^2)}dA$

$$\Rightarrow \qquad \pi\left(1-\frac{1}{e^{a^2}}\right) \le \int_{S_a}\int e^{-(x^2+y^2)}dA \le \pi\left(1-\frac{1}{e^{4a^2}}\right)$$

As $a \to \infty$ both the terms $\frac{1}{e^{a^2}}$ and $\frac{1}{e^{4a^2}}$ tend to zero.

Therefore $\lim_{a\to\infty}\int_{D_a}\int e^{-(x^2+y^2)}dA \le \lim_{a\to\infty}\int_{S_a}\int e^{-(x^2+y^2)}dA \le \lim_{a\to\infty}\iint e^{-(x^2+y^2)}dA$

$$\Rightarrow \qquad \pi \le \lim_{a\to\infty}\int_{S_a}\int e^{-(x^2+y^2)}dA \le \pi$$

$$\Rightarrow \qquad \lim_{a\to\infty}\int_{S_a}\int e^{-(x^2+y^2)}dA = \pi \text{ .... } (i)$$

Now $\int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}dxdy$

$$= \int_{R^2}\int e^{-(x^2+y^2)}dA$$

$$= \lim_{a\to\infty}\int_{S_a}\int e^{-(x^2+y^2)}dA$$

$= \pi$ from (i)

Hence $\boxed{\int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \pi}$

(C) We have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}dA$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)$$

$$= \left(\int_{-A}^{A} e^{-x^2}dx\right)\left(\int_{-A}^{A} e^{-x^2}dx\right) \qquad \left[\textit{Since} \int_{-A}^{A} e^{-x^2}dx = \int_{-A}^{A} e^{-y^2}dy\right]$$

Therefore $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA}$

$$= \sqrt{\pi}$$

Hence $\boxed{\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}}$

(D) $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$

Put $x = \sqrt{2}t$

$dx = \sqrt{2}dt$

Therefore

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}dt$$

$$= \sqrt{2}\int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \sqrt{2}\int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= \sqrt{2}.\sqrt{\pi}$$ Since we have.

$$\int_{-\infty}^{\infty} e^{-x^2 dx} = \sqrt{\pi}$$

$$= \sqrt{2\pi}$$

Hence $\boxed{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}}$

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(A) $\int_{0}^{\infty} x^2 e^{-x^2} dx = \int_{0}^{\infty} x\left(xe^{-x^2}\right) dx$

$$= \lim_{t\to\infty}\int_{0}^{t} x\left(xe^{-x^2}\right) dx$$

Now to evaluate $\int xe^{-x^2} dx$

Put $x^2 = t$

$2xdx = dt$

$xdx = \frac{dt}{2}$

Therefore, $\int xe^{-x^2} dx = \frac{1}{2}\int e^{-t} dt$

$$= -\frac{1}{2}e^{-t}$$

$$= -\frac{1}{2}e^{-x^2}$$

Again $\int_0^\infty x^2 e^{-x^2} dx = \lim_{t\to\infty} \int_0^t x\left(xe^{-x^2}\right) dx$

$$= \lim_{t\to\infty}\left[\left[x\left(-\frac{1}{2}e^{-x^2}\right)\right]_0^t - \int_0^t 1\left(-\frac{1}{2}e^{-x^2}dx\right)\right]$$

$$= \lim_{t\to\infty}\left[\frac{-t}{2e^{t^2}}\right] + \lim_{t\to\infty}\frac{1}{2}\int_0^t e^{-x^2}dx$$

Now, $\lim_{t\to\infty}\left[\frac{-t}{2e^{t^2}}\right] = \lim_{t\to\infty}\left[\frac{-1}{2.e^{t^2}.2t}\right]$ [By L'hospital rule]

And given $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

$$\Rightarrow \quad 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$$

$$\Rightarrow \quad \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Therefore

$$\int_0^\infty x^2 e^{-x^2}dx = \lim_{t\to\infty}\left[\frac{-t}{2e^{t^2}}\right] + \lim_{t\to\infty}\frac{1}{2}\int_0^t e^{-x^2}dx$$

$$= 0 + \frac{1}{2}\int_0^\infty e^{-x^2}dx$$

$$= \frac{1}{2}.\frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{4}$$

Hence

$$\boxed{\int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{4}}$$

(B) $\int_0^\infty \sqrt{x}e^{-x}dx$

Put $x = t^2$

$\Rightarrow$ $dx = 2t\,dt$

Therefore $\int_0^\infty \sqrt{x}e^{-x}dx = \int_0^\infty \sqrt{t^2}e^{-t^2}\,2t\,dt$

$$= 2\int_0^\infty t^2 e^{-t^2}dt$$

$$= 2.\frac{\sqrt{\pi}}{4} \qquad \left[\text{since } \int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{4}\right]$$

$$= \frac{\sqrt{\pi}}{2}$$

Hence

$$\boxed{\int_0^\infty \sqrt{x}e^{-x}dx = \frac{\sqrt{\pi}}{2}}$$