EXERCISE: MISCELLANEOUS

Q No 1. Find a, b and n in the expansion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290, 30375 resp.

Sol: Here $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots {}^nC_n b^n$

or. $(a+b)^n = 1 \times a^n + \frac{n}{1} a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots + b^n$

or. $(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots b^n$

According to given conditions

$a^n = 729$ ----- (1)

$n a^{n-1} b = 7290$ ---- (2)

$\frac{n(n-1)}{2} a^{n-2} b^2 = 30375$ --- (3)

from (1) and (3)

$\frac{n(n-1)}{2} a^{2n-2} b^2 = 729 \times 30375$ --- (4) [on multiplying (1) and (3)]

Squarring (2), We get

$n^2 a^{2n-2} b^2 = 7290 \times 7290$ --- (5)

Dividing (4) by (5)

$$\frac{n-1}{2n} = \frac{729 \times 30375}{7290 \times 7290}$$

ie $\frac{n-1}{2n} = \frac{135}{162} \Rightarrow 162n - 162 = 135$

$\therefore$ $162n - 135n = 162 \Rightarrow 27n = 162$

$\Rightarrow n = 6$.

$\therefore$ From (1) $a^6 = 729$ ie $a^6 = 3^6$ ie $a = 3$

From (2) $6 \times 3^5 \times b = 7290 \Rightarrow b = 5$

$\therefore$ $a = 3$, $b = 5$, $n = 6$.

Q No 2 Find a if the coefficients of x^2 and x^3 in expansion of $(3+ax)^9$ are equal.

Sol. We have $(3+ax)^9 = {}^9C_0(3)^9 + {}^9C_1(3)^8(ax) + {}^9C_2(3)^7(ax)^2 + {}^9C_3(3)^6(ax)^3 + \dots + {}^9C_9(ax)^9$

Now

$$\text{Coeff. of } x^2 = \text{Coeff of } x^3$$

$$\therefore \; {}^9C_2(3)^7(a^2) = {}^9C_3(3)^6(a)^3$$

$$\frac{9\times 8}{1\times 2}\times 3 = \frac{9\times 8\times 7}{1\times 2\times 3}\, a$$

$$\Rightarrow \quad 3 = \frac{7a}{3} \quad \Rightarrow \quad a = \frac{9}{7}$$

Q No. 3: Find the coeff of x^5 in the product $(1+2x)^6(1-x)^7$ using Binomial Theorem.

Sol: $(1+2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6$

$$= 1+(6\times 2x)+\left(\frac{6\times 5}{1\times 2}\times 4x^2\right)+\left(\frac{6\times 5\times 4}{1\times 2\times 3}\times 8x^3\right)+\left(\frac{6\times 5}{1\times 2}\times 16x^4\right)$$
$$+\left(\frac{6}{1}\times 32x^5\right)+\left(1\times 64x^6\right)$$

$$= 1+12x+60x^2+160x^3+240x^4+192x^5+64x^6$$

Again $(1+x)^7 = {}^7C_0 + {}^7C_1(-x) + {}^7C_2(-x)^2 + {}^7C_3(-x)^3 + {}^7C_4(-x)^4 + {}^7C_5(-x)^5 + {}^7C_6(-x)^6 + {}^7C_7(-x)^7$

$$= 1-7x+\frac{7\times 6}{1\times 2}x^2-\frac{7\times 6\times 5}{1\times 2\times 3}x^3+\frac{7\times 6\times 5}{1\times 2\times 3}x^4-\frac{7\times 6}{1\times 2}x^5$$
$$+\frac{7}{1}x^6-1\times x^7$$

$$= 1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7.$$

$\therefore$ Coeff. of x^5 in $(1+2x)^6(1-x)^7$

$= \text{Coeff of } x^5 \text{ in } \left[1+12x+60x^2+160x^3+240x^4+192x^5+64x^6\right] \times$
$\left[1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7\right]$

$= -21+420-2100+3360-1680+192 = 171$

Q.No.4 If a and b are distinct integers, prove that $a-b$ is a factor of a^n-b^n, whenever 'n' is a positive integer.

Sol. Since $a = a-b+b = (a-b)+b$

$\therefore \quad a^n = [(a-b)+b]^n$

$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}\cdot b + {}^nC_2(a-b)^{n-2}b^2 + \cdots$

$\cdots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_n b^n$

$\therefore a^n = {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \cdots {}^nC_{n-1}(a-b)b^{n-1} + b^n \quad [\because {}^nC_n = 1]$

$\therefore a^n - b^n = {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + {}^nC_2(a-b)^{n-2}b^2 + \cdots {}^nC_{n-1}(a-b)b^{n-1}$

$= (a-b)\left[{}^nC_0(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \cdots {}^nC_{n-1}b^{n-1}\right]$

$= (a-b) \times$ Some integer

$\Rightarrow a^n-b^n$ is divisible by $a-b$ for all +ve integer.

i.e. $a-b$ is factor of a^n-b^n.

Q.No.5 Evaluate $(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$

Sol. $(a+b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5ab^5 + {}^6C_6b^6$

$= 1\times a^6 + \frac{6}{1}a^5b + \frac{6\times5}{2\times1}a^4b^2 + \frac{6\times5\times4}{1\times2\times3}a^3b^3 + \frac{6\times5}{1\times2}a^2b^4 + \frac{6}{1}ab^5 + 1\times b^6$

$= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 \quad \cdots(1)$

Similarly on changing b to $(-b)$ we get

$(a-b)^6 = a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 \quad \cdots(2)$

Subtracting (2) from (1), we get

$(a+b)^6-(a-b)^6 = 12a^5b+40a^3b^3+12ab^5$

Put $a=\sqrt{3}$, $b=\sqrt{2}$

$\therefore (\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6 = 12(\sqrt{3})^5(\sqrt{2}) + 40(\sqrt{3})^3(\sqrt{2})^3 + 12(\sqrt{3})(\sqrt{2})^5$

$= 12\times 9\times\sqrt{3}\times\sqrt{2} + 40\times 3\sqrt{3}\times 2\sqrt{2} + 12\times\sqrt{3}\times 4\sqrt{2}$

$= 108\sqrt{6} + 240\sqrt{6} + 48\sqrt{6} = 396\sqrt{6}.$

Q No. 6 Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$.

Sol.: We know that

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

and $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

[Refer to Q No 1 Ex. 8.1]

$\therefore (x+y)^4 + (x-y)^4 = 2\left[x^4 + 6x^2y^2 + y^4\right]$

Put $x = a^2$ and $y = \sqrt{a^2-1}$.

$\therefore \left[a^2+\sqrt{a^2-1}\right]^4 + \left(a^2-\sqrt{a^2-1}\right)^4 = 2\left[(a^2)^4 + 6(a^2)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$

$= 2\left[a^8 + 6a^4(a^2-1) + (a^2-1)^2\right]$

$= 2\left[a^8 + 6a^6 - 6a^4 + a^4 + 1 - 2a^2\right] = 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right)$

$\therefore \left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4 = 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2.$

Q No. 7 Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Sol: $(0.99)^5 = [1-0.01]^5 = {}^5C_0 - {}^5C_1(0.01) + {}^5C_2(0.01)^2 - {}^5C_3(0.01)^3 + \cdots$

$\approx 1 - \frac{5}{1}\times 0.01 + \frac{5\times 4}{1\times 2}\times(0.0001)$ [Using first 3 terms as asked in question]

$= 1 - 0.05 + 0.001$

$= 0.951$

$\therefore (0.99)^5 \approx 0.951.$

Q No. 8: Find n, if the ratio of the fifth term from begining to the fifth term from the end in expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6} : 1$

Sol: In expansion of $\left(2^{1/4}+\frac{1}{3^{1/4}}\right)^n$

$$T_5 = T_{4+1} = {}^nC_4 \left(2^{1/4}\right)^{n-4}\left(\frac{1}{3^{1/4}}\right)^4 = {}^nC_4\ 2^{\frac{n-4}{4}} \times \frac{1}{3}.$$

Now Power of Binomial = n.

$\therefore$ No. of terms = $n+1$

Now 5th term from end has $(n+1)-5 = (n-4)$ terms before it

$\therefore$ 5th term from end is $(n-3)$rd term from begining

$\therefore$ 5th term from end = T_{n-3}

Altesnalively Note: rth term from end in $(x+y)^n$

= rth term from begining in $(y+x)^n$

$\therefore$ 5th term from end in expansion of $\left(2^{1/4}+\frac{1}{3^{1/4}}\right)^n$

= 5th term from begining in $\left(\frac{1}{3^{1/4}}+2^{1/4}\right)^n$.

$$= {}^nC_4\left(\frac{1}{3^{1/4}}\right)^{n-4}\left(2^{1/4}\right)^4 = {}^nC_4 \times 2 \times \frac{1}{3^{\frac{n-4}{4}}}.$$

From given conditions

$$\frac{{}^nC_4\ 2^{\frac{n-4}{4}} \times \frac{1}{3}}{{}^nC_4 \times 2 \times \frac{1}{3^{\frac{n-4}{4}}}} = \frac{\sqrt{6}}{1}.$$

$\therefore\ 2^{\frac{n-4}{4}-1} \times 3^{\frac{n-4}{4}-1} = \sqrt{6}$ ie $2^{\frac{n-8}{4}} \times 3^{\frac{n-8}{4}} = 6^{\frac{1}{2}}.$

$\therefore\ (6)^{\frac{n-8}{4}} = 6^{1/2}$

$$\Rightarrow \frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8 = 2 \Rightarrow n = 10$$

Q.No. 9: Expand using Binomial Theorem.

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4;\quad x \neq 0.$$

Sol: $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4 = \left[\left(1+\frac{x}{2}\right)+\left(-\frac{2}{x}\right)\right]^4$

$= {}^4C_0\left(1+\frac{x}{2}\right)^4 + {}^4C_1\left(1+\frac{x}{2}\right)^3\left(-\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(-\frac{2}{x}\right)^2$

$+ {}^4C_3\left(1+\frac{x}{2}\right)\left(-\frac{2}{x}\right)^3 + {}^4C_4\left(-\frac{2}{x}\right)^4$

$= 1\times\left[{}^4C_0 + {}^4C_1\left(\frac{x}{2}\right) + {}^4C_2\left(\frac{x}{2}\right)^2 + {}^4C_3\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4\right]$

$+ \frac{4}{1}\times\left[{}^3C_0 + {}^3C_1\left(\frac{x}{2}\right) + {}^3C_2\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3\right]\left(-\frac{2}{x}\right)$

$+ \frac{4\times3}{2\times1}\left[{}^2C_0 + {}^2C_1\left(\frac{x}{2}\right) + {}^2C_2\left(\frac{x}{2}\right)^2\right]\left[\frac{4}{x^2}\right] + \frac{4}{1}\left(1+\frac{x}{2}\right)\left(-\frac{8}{x^3}\right) + 1\times\frac{16}{x^4}.$

$= 1\times\left[1+\frac{4}{1}\times\frac{x}{2} + \frac{4\times3}{1\times2}\times\frac{x^2}{4} + \frac{4}{1}\times\frac{x^3}{8} + 1\times\frac{x^4}{16}\right] + 4\times\left\{1+\frac{3}{1}\left(\frac{x}{2}\right)+\frac{3}{1}\left(\frac{x^2}{4}\right)+1\times\frac{x^3}{8}\right\}$

$\left(-\frac{2}{x}\right) + 6\left[1+2\times\frac{x}{2}+1\times\frac{x^2}{4}\right]\left[\frac{4}{x^2}\right] + 4\times\left(1+\frac{x}{2}\right)\left(-\frac{8}{x^3}\right) + 1\times\frac{16}{x^4}.$

$= 1+2x+\frac{3}{2}x^2+\frac{1}{2}x^3+\frac{1}{16}x^4-\frac{8}{x}-12-6x-x^2+\frac{24}{x^2}+\frac{24}{x}+6$

$-\frac{32}{x^3}-\frac{16}{x^2}+\frac{16}{x^4}.$

$= \frac{16}{x^4}-\frac{32}{x^3}+\frac{8}{x^2}+\frac{16}{x}-5-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}.$

Q No. 10 Find the expansion of $(3x^2-2ax+3a^2)^3$ using binomial theorem.

Sol: $(3x^2-2ax+3a^2)^3 = [3x^2-a(2x-3a)]^3$

$= {}^3C_0(3x^2)^3 - {}^3C_1(3x^2)^2(a(2x-3a)) + {}^3C_2(3x^2)a^2(2x-3a)^2$

$- {}^3C_3\, a^3(2x-3a)^3.$

$= 1 \times 27x^6 - \frac{3}{1} \times 9x^4 \times a(2x-3a) + \frac{3}{1} \times 3x^2 \times a^2(2x-3a)^2 - a^3(2x-3a)^3$

$= 27x^6 - 27x^4 a(2x-3a) + 9x^2a^2(4x^2+9a^2-12xa)$
$\quad - a^3(8x^3 - 36x^2a + 54xa^2 - 27a^3)$

$= 27x^6 - 54ax^5 + 81a^2x^4 + 36x^4a^2 + 81x^2a^4 - 108x^3a^3$
$\quad - 8x^3a^3 + 36x^2a^4 - 54xa^5 + 27a^6$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6.$

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