Chapter 3

LOCUS

: EQUATION TO A LOCUS

36. WHEN a point moves so as always to satisfy a given condition, or conditions, the path it traces out is called its Locus under these conditions.

For example, suppose O to be a given point in the plane of the paper and that a point P is to move on the paper so that its distance from O shall be constant and equal to a. It is clear that all the positions of the moving point must lie on the circumference of a circle whose centre is O and whose radius is a. The circumference of this circle is therefore the "Locus" of P when it moves subject to the condition that its distance from O shall be equal to the constant distance a.

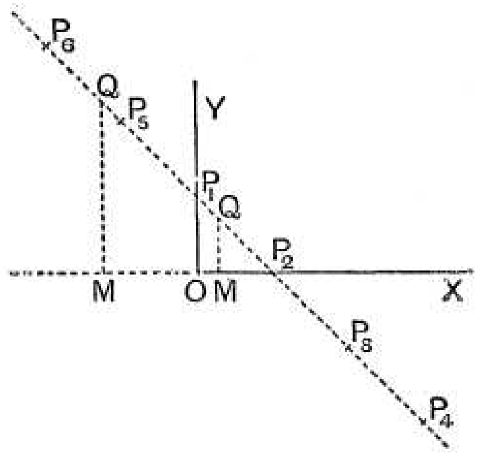
37. Again, suppose A and B to be two fixed points in the plane of the paper and that a point P is to move in the plane of the paper so that its distances from A and B are to be always equal. If we bisect AB in C and through it draw a straight line (of infinite length in both directions) perpendicular to AB, then any point on this straight line is at equal distances from A and B. Also there is no point, whose distances from A and B are the same, which does not lie on this straight line. This straight line is therefore the "Locus" of P subject to the assumed condition.

38. Again, suppose A and B to be two fixed points and that the point P is to move in the plane of the paper so that the angle APB is always a right angle. If we describe a circle on AB as diameter then P may be any point on the circumference of this circle, since the angle in a semi-circle is a right angle; also it could easily be shewn that APB is not a right angle except when P lies on this circumference. The "Locus" of P under the assumed condition is therefore a circle on AB as diameter.

39. One single equation between two unknown quantities x and y, *e.g.*

$$x + y = 1 \quad \text{.......................(1)},$$

cannot completely determine the values of x and y.



Such an equation has an infinite number of solutions.

Amongst them are the following:

$$\left.\begin{matrix} x = 0, \\ y = 1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 1, \\ y = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} x = \ \ 2, \\ y = -1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = \ \ 3, \\ y = -2 \end{matrix}\right\}, \dots$$

$$\left.\begin{matrix} x = -1, \\ y = \ \ 2 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -2, \\ y = \ \ 3 \end{matrix}\right\}, \dots$$

Let us mark down on paper a number of points whose coordinates (as defined in the last chapter) satisfy equation (1).

Let OX and OY be the axes of coordinates.

If we mark off a distance $OP_1 (= 1)$ along OY, we have a point P_1 whose coordinates (0, 1) clearly satisfy equation (1).

If we mark off a distance $OP_2 (= 1)$ along OX, we have a point P_2 whose coordinates (1, 0) satisfy (1).

Similarly the point P_3, $(2, -1)$, and P_4, $(3, -2)$, satisfy the equation (1).

Again, the coordinates $(-1, 2)$ of P_5 and the coordinates $(-2, 3)$ of P_6 satisfy equation (1).

On making the measurements carefully we should find that all the points we obtain lie on the line P_1P_2 (produced both ways).

Again, if we took *any* point Q, lying on P_1P_2, and draw a perpendicular QM to OX, we should find on measurement that the sum of its x and y (each taken with its proper sign) would be equal to unity, so that the coordinates of Q would satisfy (1).

Also we should find no point, whose coordinates satisfy (1), which does not lie on P_1P_2.

All the points, lying on the straight line P_1P_2, and no others are therefore such that their coordinates satisfy the equation (1).

This result is expressed in the language of Analytical Geometry by saying that (1) is the Equation to the Straight Line P_1P_2.

40. Consider again the equation

$$x^2 + y^2 = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Amongst an infinite number of solutions of this equation are the following:

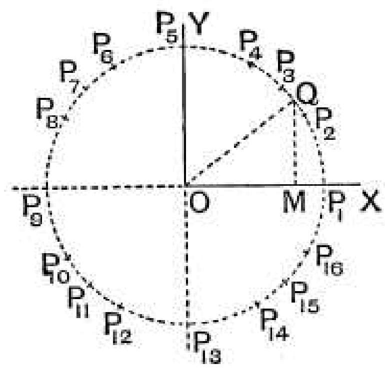
$$\left.\begin{matrix} x = 2, \\ y = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} x = \sqrt{3} \\ y = 1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = \sqrt{2} \\ y = \sqrt{2} \end{matrix}\right\}, \quad \left.\begin{matrix} x = 1 \\ y = \sqrt{3} \end{matrix}\right\},$$

$$\left.\begin{matrix} x = 0 \\ y = 2 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -1, \\ y = \sqrt{3} \end{matrix}\right\}, \quad \left.\begin{matrix} x = -\sqrt{2}, \\ y = \sqrt{2} \end{matrix}\right\}, \quad \left.\begin{matrix} x = -\sqrt{3}, \\ y = 1 \end{matrix}\right\},$$

$$\left.\begin{matrix} x = -2, \\ y = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -\sqrt{3}, \\ y = -1 \end{matrix}\right\}, \quad \left.\begin{matrix} x = -\sqrt{2}, \\ y = -\sqrt{2} \end{matrix}\right\}, \quad \left.\begin{matrix} x = -1, \\ y = -\sqrt{3} \end{matrix}\right\},$$

$$\left.\begin{matrix} x = 0, \\ y = -2 \end{matrix}\right\}, \quad \left.\begin{matrix} x = 1, \\ y = -\sqrt{3} \end{matrix}\right\}, \quad \left.\begin{matrix} x = \sqrt{2}, \\ y = -\sqrt{2} \end{matrix}\right\}, \text{ and } \left.\begin{matrix} x = \sqrt{3} \\ y = -1 \end{matrix}\right\}.$$

All these points are respectively represented by the points $P_1, P_2, P_3, \dots P_{16}$, and they will all be found to lie on the dotted circle whose centre is O and radius is 2.



Also, if we take any other point Q on this circle and its ordinate QM, it follows, since $OM^2 + MQ^2 = OQ^2 = 4$, that the x and y of the point Q satisfies (1).

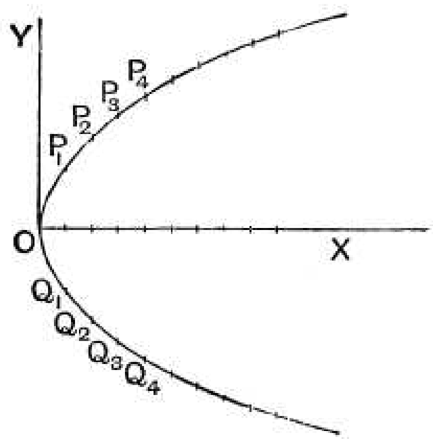
The dotted circle therefore passes through all the points whose coordinates satisfy (1).

In the language of Analytical Geometry the equation (1) is therefore the equation to the above circle.

41. As another example let us trace the locus of the point whose coordinates satisfy the equation

$$y^2 = 4x \quad \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

If we give x a negative value we see that y is impossible; for the square of a real quantity cannot be negative.



We see therefore that there are no points lying to the left of OY.

If we give x any positive value we see that y has two real corresponding values which are equal and of opposite signs.

The following values, amongst an infinite number of others, satisfy (1), viz.

$$\left.\begin{matrix} x=0, \\ y=0 \end{matrix}\right\}, \quad \left.\begin{matrix} x=1, \\ y=+2 \text{ or } -2 \end{matrix}\right\}, \quad \left.\begin{matrix} x=2, \\ y=2\sqrt{2} \text{ or } -2\sqrt{2} \end{matrix}\right\},$$

$$\left.\begin{matrix} x=4 \\ y=+4 \text{ or } -4 \end{matrix}\right\}, \quad \dots \quad \left.\begin{matrix} x=16, \\ y=8 \text{ or } -8 \end{matrix}\right\}, \quad \dots \quad \left.\begin{matrix} x=+\infty, \\ y=+\infty \text{ or } -\infty \end{matrix}\right\}.$$

The origin is the first of these points and P_1 and Q_1, P_2 and Q_2, P_3 and Q_3, ... represent the next pairs of points.

If we took a large number of values of x and the corresponding values of y, the points thus obtained would be found all to lie on the curve in the figure.

Both of its branches would be found to stretch away to infinity towards the right of the figure.

Also, if we took any point on this curve and measured with sufficient accuracy its x and y the values thus obtained would be found to satisfy equation (1).

Also we should not be able to find any point, not lying on the curve, whose coordinates would satisfy (1).

In the language of Analytical Geometry the equation (1) is the equation to the above curve. This curve is called a Parabola and will be fully discussed in Chapter 10.

42. If a point move so as to satisfy any given condition it will describe some definite curve, or locus, and there can always be found an equation between the x and y of *any* point on the path.

This equation is called the equation to the locus or curve. Hence

Def. Equation to a curve. *The equation to a curve is the relation which exists between the coordinates of* **any** *point on the curve, and which holds for no other points except those lying on the curve.*

43. Conversely to every equation between x and y it will be found that there is, in general, a definite geometrical locus.

Thus in Art. 39 the equation is $x + y = 1$, and the definite path, or locus, is the straight line P_1P_2 (produced indefinitely both ways).

In Art. 40 the equation is $x^2 + y^2 = 4$, and the definite path, or locus, is the dotted circle.

Again the equation $y = 1$ states that the moving point is such that its ordinate is always unity, *i.e.* that it is always at a distance 1 from the axis of x. The definite path, or locus, is therefore a straight line parallel to OX and at a distance unity from it.

44. In the next chapter it will be found that if the equation be of the first degree (*i.e.* if it contain no products, squares, or higher powers of x and y) the locus corresponding is always a straight line.

If the equation be of the second or higher degree, the corresponding locus is, in general, a curved line.

45. We append a few simple examples of the formation of the equation to a locus.

Ex. 1. *A point moves so that the algebraic sum of its distances from two given perpendicular axes is equal to a constant quantity a; find the equation to its locus.*

Take the two straight lines as the axes of coordinates. Let (x, y) be any point satisfying the given condition. We then have $x+y=a$.

This being the relation connecting the coordinates of any point on the locus is the equation to the locus.

It will be found in the next chapter that this equation represents a straight line.

Ex. 2. *The sum of the squares of the distances of a moving point from the two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.*

Let (x, y) be any position of the moving point. Then, by Art. 20, the condition of the question gives

$$\{(x-a)^2+y^2\}+\{(x+a)^2+y^2\}=2c^2,$$

i.e.

$$x^2+y^2=c^2-a^2.$$

This being the relation between the coordinates of any, and every, point that satisfies the given condition is, by Art. 42, the equation to the required locus.

This equation tells us that the square of the distance of the point (x, y) from the origin is constant and equal to c^2-a^2, and therefore the locus of the point is a circle whose centre is the origin.

Ex. 3. *A point moves so that its distance from the point* $(-1, 0)$ *is always three times its distance from the point* $(0, 2)$.

Let (x, y) be *any* point which satisfies the given condition. We then have

$$\sqrt{(x+1)^2+(y-0)^2}=3\sqrt{(x-0)^2+(y-2)^2},$$

so that, on squaring,

$$x^2+2x+1+y^2=9(x^2+y^2-4y+4),$$

i.e.

$$8(x^2+y^2)-2x-36y+35=0.$$

This being the relation between the coordinates of each, and every, point that satisfies the given relation is, by Art. 42, the required equation.

It will be found, in a later chapter, that this equation represents a circle.

EXAMPLES IV

By taking a number of solutions, as in Arts. 39—41, sketch the loci of the following equations :

1. $2x+3y=10$.
2. $4x-y=7$.
3. $x^2-2ax+y^2=0$.
4. $x^2-4ax+y^2+3a^2=0$.
5. $y^2=x$.
6. $3x=y^2-9$.
7. $\frac{x^2}{4}+\frac{y^2}{9}=1$.

A and B being the fixed points $(a, 0)$ and $(-a, 0)$ respectively, obtain the equations giving the locus of P, when

8. $PA^2-PB^2=$ a constant quantity $=2k^2$.
9. $PA=nPB$, n being constant.
10. $PA+PB=c$, a constant quantity.
11. $PB^2+PC^2=2PA^2$, C being the point $(c, 0)$.
12. Find the locus of a point whose distance from the point (1, 2) is equal to its distance from the axis of y.

Find the equation to the locus of a point which is always equidistant from the points whose coordinates are

13. (1, 0) and (0, −2).
14. (2, 3) and (4, 5).
15. $(a+b, a-b)$ and $(a-b, a+b)$.

Find the equation to the locus of a point which moves so that

16. its distance from the axis of x is three times its distance from the axis of y.
17. its distance from the point $(a, 0)$ is always four times its distance from the axis of y.
18. the sum of the squares of its distances from the axes is equal to 3.
19. the square of its distance from the point (0, 2) is equal to 4.
20. its distance from the point (3, 0) is three times its distance from (0, 2).
21. its distance from the axis of x is always one half its distance from the origin.
22. A fixed point is at a perpendicular distance a from a fixed straight line and a point moves so that its distance from the fixed point is always equal to its distance from the fixed line. Find the equation to its locus, the axes of coordinates being drawn through the fixed point and being parallel and perpendicular to the given line.

23. In the previous question if the first distance be (1), always half, and (2), always twice, the second distance, find the equations to the respective loci.

ANSWERS

1. 10. **2.** 1. **3.** 29. **4.** $2ac$.

5. a^2. **6.** $2ab \sin \frac{\phi_2 - \phi_3}{2} \sin \frac{\phi_3 - \phi_1}{2} \sin \frac{\phi_1 - \phi_2}{2}$.

7. $a^2 (m_2 - m_3)(m_3 - m_1)(m_1 - m_2)$.

8. $\frac{1}{2} a^2 (m_2 - m_3)(m_3 - m_1)(m_1 - m_2)$.

9. $\frac{1}{2} a^2 (m_2 - m_3)(m_3 - m_1)(m_1 - m_2) \div m_1 m_2 m_3$.

13. $20\frac{1}{2}$. **14.** 96.

SOLUTIONS/HINTS

8. $(x-a)^2 + y^2 - (x+a)^2 - y^2 = 2k^2$. $\therefore\ 2ax + k^2 = 0$.

9. $$(x-a)^2 + y^2 = n^2 \{(x+a)^2 + y^2\}.$$

$$\therefore\ (n^2 - 1)(x^2 + y^2 + a^2) + 2ax(n^2 + 1) = 0.$$

10. $$\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} = c.$$

We also have

$$-[(x-a)^2 + y^2] + [(x+a)^2 + y^2] = 4ax.$$

Hence, by division,

$$-\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} = \frac{4ax}{c}.$$

$$\therefore\ 2\sqrt{(x+a)^2 + y^2} = \frac{4ax}{c} + c.$$

$$\therefore\ 4\{x^2 + a^2 + y^2 + 2ax\} = \frac{16a^2x^2}{c^2} + c^2 + 8ax.$$

$$\therefore\ 4x^2(c^2 - 4a^2) + 4c^2y^2 = c^2(c^2 - 4a^2).$$

11. $$(x+a)^2 + y^2 + (x-c)^2 + y^2 = 2(x-a)^2 + 2y^2.$$

$$\therefore\ 6ax - a^2 - 2cx + c^2 = 0.$$

12. $$(x-1)^2 + (y-2)^2 = x^2.$$

13. $$(x-1)^2 + y^2 = x^2 + (y+2)^2.$$

14. $(x-2)^2+(y-3)^2=(x-4)^2+(y-5)^2.$

15. $\{x-\overline{a+b}\}^2+\{y-\overline{a-b}\}^2=\{x-\overline{a-b}\}^2+\{y-\overline{a+b}\}^2.$

$$\therefore\ -4bx=-4by.\quad \therefore\ x=y.$$

16. $y=3x.$ **17.** $(x-a)^2+y^2=16x^2.$

18. $x^2+y^2=3.$ **19.** $x^2+(y-2)^2=4.$

20. $(x-3)^2+y^2=9\{x^2+(y-2)^2\}.$

$$\therefore\ 8x^2+8y^2+6x-36y+27=0.$$

21. $y=\frac{1}{2}\sqrt{x^2+y^2}.\quad \therefore\ 3y^2=x^2.$

22. From a figure it is easily seen that the locus is given by $\sqrt{x^2+y^2}=a\sim y.\quad \therefore\ x^2+y^2=a^2+y^2-2ay.$

23. (1) $\sqrt{x^2+y^2}=\frac{1}{2}(a-y).$

(2) $\sqrt{x^2+y^2}=2(a-y).$