CHAPTER

5.6

THE DISCRETE-TIME FOURIER TRANSFORM

Statement for Q.1–9:

Determine the discrete-time Fourier Transform for the given signal and choose correct option.

1. $x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & \text{otherwise} \end{cases}$

(A) $\dfrac{\sin 5\Omega}{\sin \Omega}$ (B) $\dfrac{\sin 4\Omega}{\sin \Omega}$

(C) $\dfrac{\sin 2.5\Omega}{\sin \Omega}$ (D) None of the above

2. $x[n] = \left(\dfrac{3}{4}\right)^n u[n-4]$

(A) $\dfrac{\left(\frac{3}{4}e^{-j\Omega}\right)^4}{1-\frac{3}{4}e^{-j\Omega}}$ (B) $\dfrac{\left(\frac{3}{4}e^{j\Omega}\right)^4}{1-\frac{3}{4}e^{j\Omega}}$

(C) $\dfrac{\left(\frac{3}{4}e^{-j\Omega}\right)^4}{1+\frac{3}{4}e^{j\Omega}}$ (D) None of the above

3. $x[n] = u[n-2] - u[n-6]$

(A) $e^{3j\Omega} + e^{3j\Omega} + e^{4j\Omega} + e^{5j\Omega}$ (B) $\dfrac{e^{-2j\Omega}(1-e^{3j\Omega})}{1-e^{j\Omega}}$

(C) $e^{-2j\Omega} + e^{-3j\Omega} + e^{-4j\Omega} + e^{-5j\Omega}$ (D) $\dfrac{e^{-2j\Omega}(1-e^{-3j\Omega})}{1-e^{-j\Omega}}$

4. $x[n] = a^{|n|}, \quad |a| < 1$

(A) $\dfrac{1-a^2}{1-2a\sin\Omega + a^2}$ (B) $\dfrac{1-a^2}{1-2a\cos\Omega + a^2}$

(C) $\dfrac{1-a^2}{1-2ja\sin\Omega + a^2}$ (D) None of the above

5. $x[n] = \left(\dfrac{1}{2}\right)^{-n} u[-n-1]$

(A) $\dfrac{e^{j\Omega}}{2-e^{-j\Omega}}$ (B) $\dfrac{2e^{j\Omega}}{2-e^{-j\Omega}}$

(C) $\dfrac{e^{j\Omega}}{2-e^{j\Omega}}$ (D) $\dfrac{2e^{j\Omega}}{2-e^{j\Omega}}$

6. $x[n] = 2\delta[4-2n]$

(A) $2e^{-j2\Omega}$ (B) $2e^{j2\Omega}$

(C) 1 (D) None of the above

7. $x[n] = u[n]$

(A) $\pi\delta(\Omega) - \dfrac{1}{1+e^{-j\Omega}}$ (B) $\dfrac{1}{1-e^{-j\Omega}}$

(C) $\pi\delta(\Omega) + \dfrac{1}{1-e^{-j\Omega}}$ (D) $\dfrac{1}{1+e^{-j\Omega}}$

8. $x[n] = \{-2, \ -1, \ \underset{\uparrow}{0}, \ 1, \ 2\}$

(A) $2j(2\sin 2\Omega + \sin\Omega)$ (B) $2(2\cos 2\Omega - \cos\Omega)$

(C) $-2j(2\sin 2\Omega + \sin\Omega)$ (D) $-2(2\cos 2\Omega - \cos\Omega)$

9. $x[n] = \sin\left(\dfrac{\pi}{2}n\right)$

(A) $\pi(\delta[\Omega - \pi/2] - \delta[\Omega + \pi/2])$

(B) $\dfrac{j}{2}(\delta[\Omega + \pi/2] - \delta[\Omega - \pi/2])$

(C) $2\pi(\delta[\Omega - \pi/2] - \delta[\Omega + \pi/2])$

(D) $j\pi(\delta[\Omega + \pi/2] - \delta[\Omega - \pi/2])$

Statement for Q.10–21:

Determine the signal having the Fourier transform given in question.

10. $X(e^{j\Omega}) = \frac{1}{(1 - ae^{-j\Omega})^2}$, $|a| < 1$

(A) $(n-1)a^n u[n]$ (B) $(n+1)a^n u[n]$

(C) $na^n u[n]$ (D) None of the above

11. $X(e^{j\Omega}) = 8\cos^2 \omega$

(A) $(\delta[n+2] + 2\delta[n] + \delta[n-2])$

(B) $2(\delta[n+2] + 2\delta[n] + \delta[n-2])$

(C) $-4(\delta[n+2] + \delta[n] + \delta[n-2])$

(D) $\frac{1}{2}(\delta[n+2] + \delta[n] + \delta[n-2])$

12. $X(e^{j\Omega}) = \begin{cases} 2j, & 0 < \Omega \leq \pi \\ -2j, & -\pi < \Omega \leq 0 \end{cases}$

(A) $-\frac{4}{\pi n}\sin^2\left(\frac{\pi n}{2}\right)$ (B) $\frac{4}{\pi n}\sin^2\left(\frac{\pi n}{2}\right)$

(C) $\frac{8}{\pi n}\sin^2\left(\frac{\pi n}{2}\right)$ (D) $-\frac{8}{\pi n}\sin^2\left(\frac{\pi n}{2}\right)$

13. $X(e^{j\Omega}) = \begin{cases} 1, & \frac{\pi}{4} \leq |\Omega| < \frac{3\pi}{4} \\ 0, & 0 \leq |\Omega| < \frac{\pi}{4}, \ \frac{3\pi}{4} \leq |\Omega| \leq \pi \end{cases}$

(A) $\frac{2}{n}\left(\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right)\right)$

(B) $\frac{1}{\pi n}\left(\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right)\right)$

(C) $\frac{2}{n}\left(\cos\left(\frac{3\pi n}{4}\right) + \cos\left(\frac{\pi n}{4}\right)\right)$

(D) $\frac{1}{\pi n}\left(\cos\left(\frac{3\pi n}{4}\right) + \cos\left(\frac{\pi n}{4}\right)\right)$

14. $X(e^{j\Omega}) = e^{-\frac{j\Omega}{2}}$ for $-\pi \leq \Omega \leq \pi$

(A) $\pi\delta[n - 1/2]$ (B) $2\pi\delta[n - 1/2]$

(C) $\frac{(-1)^{n+1}}{\pi(n - \frac{1}{2})}$ (D) None of the above

15. $X(e^{j\Omega}) = \cos 2\Omega + j\sin 2\Omega$

(A) $2\pi\delta[n+2]$ (B) $\delta[n+2]$

(C) 0 (D) None of the above

16. $X(e^{j\Omega}) = j4\sin 4\Omega - 1$

(A) $4\pi\delta[n+4] - 4\pi\delta[n-4] - 2\pi\delta[n]$

(B) $2\delta[n+4] - 2\delta[n-4] - \delta[n]$

(C) $\delta[n+4] - \delta[n-4] - \delta[n]$

(D) None of the above

17. $X(e^{j\Omega}) = \frac{2}{-e^{-j2\Omega} + e^{-j\Omega} + 6}$

(A) $\frac{5}{2^{-n}}\left(1 + \left(\frac{-2}{3}\right)^{n+1}\right)u[n]$

(B) $2^{-n}\left(1 - \left(\frac{-2}{3}\right)^{n+1}\right)u[n]$

(C) $\frac{2^{-n}}{5}\left((-1)^n + \left(\frac{2}{3}\right)^{n+1}\right)u[n]$

(D) None of the above

18. $X(e^{j\Omega}) = \frac{2 + \frac{1}{4}e^{-j\Omega}}{-\frac{1}{8}e^{-j2\Omega} + \frac{1}{4}e^{-j\Omega} + 1}$

(A) $2^{-n+1}[1 + (-2)^{-n}]u[n]$

(B) $2^{-n}[1 + (-2)^{-n}]u[n]$

(C) $2^{-n+1}[(-1)^n + 2^{-n}]u[n]$

(D) $2^{-n}[(-1)^n + 2^{-n}]u[n]$

19. $X(e^{j\Omega}) = \frac{2e^{-j\Omega}}{1 - \frac{1}{4}e^{-j2\Omega}}$

(A) $2^{n-1}[1 + (-1)^n]u[n]$

(B) $2^{1-n}[1 + (-1)^n]u[n]$

(C) $2^{1-n}[1 - (-1)^n]u[n]$

(D) $2^{n-1}[1 - (-1)^n]u[n]$

20. $X(e^{j\Omega}) = \frac{1 - \frac{1}{3}e^{-j\Omega}}{1 - \frac{1}{4}e^{-j\Omega} - \frac{1}{8}e^{-2j\Omega}}$

(A) $\left(\frac{2}{9}\left(\frac{1}{2}\right)^n + \frac{7}{9}\left(-\frac{1}{4}\right)^n\right)u[n]$

(B) $\left(\frac{2}{9}\left(-\frac{1}{2}\right)^n + \frac{7}{9}\left(\frac{1}{4}\right)^n\right)u[n]$

(C) $\left(\frac{2}{9}\left(-\frac{1}{2}\right)^n - \frac{7}{9}\left(\frac{1}{4}\right)^n\right)u[n]$

(D) $\left(\frac{2}{9}\left(\frac{1}{2}\right)^n - \frac{7}{9}\left(-\frac{1}{4}\right)^n\right)u[n]$

$$x[n]=\left(-\frac{1}{2}\right)^n u[n]+\left(\frac{1}{4}\right)^n u[n]=2^{-n}[(-1)^n+2^{-n}]u[n]$$

19. (C) $X(e^{j\Omega})=\dfrac{2e^{-j\Omega}}{1-\frac{1}{4}e^{-j2\Omega}}=\dfrac{2}{1-\frac{1}{2}e^{-j\Omega}}-\dfrac{2}{1+\frac{1}{2}e^{-j\Omega}}$

$$x[n]=2\left(\frac{1}{2}\right)^n u[n]-2\left(-\frac{1}{2}\right)^n u[n]$$

$$=\frac{1}{2^{n-1}}[1-(-1)^n]u[n]=2^{1-n}[1-(-1)^n]u[n]$$

20. (A) $X(e^{j\Omega})=\dfrac{1-\frac{1}{3}e^{-j\Omega}}{1-\frac{1}{4}e^{-j\Omega}-\frac{1}{8}e^{-2j\Omega}}$

$$=\frac{\frac{2}{9}}{1-\frac{1}{2}e^{-j\Omega}}+\frac{\frac{7}{9}}{1+\frac{1}{4}e^{-j\Omega}}$$

$$x[n]=\frac{2}{9}\left(\frac{1}{2}\right)^n u[n]+\frac{7}{9}\left(-\frac{1}{4}\right)^n u[n]$$

21. (C) $X(e^{j\Omega})=\dfrac{(b-a)e^{j\Omega}}{e^{j2\Omega}-(a+b)e^{j\Omega}+ab}$

$$=\frac{(b-a)e^{-j\Omega}}{1-(a+b)e^{-j\Omega}+abe^{-j2\Omega}}=\frac{1}{1-be^{-j\Omega}}+\frac{-1}{1-ae^{-j\Omega}}$$

$x[n]=b^n u[n]+a^n u[-n-1]$.

22. (D) The signal must be read and odd. Only signal (h) is real and odd.

23. (A) The signal must be real and even. Only signal (c) and (e) are real and even.

24. (A) $Y(e^{j\Omega})=e^{j\alpha\Omega}X(e^{j\Omega})$, $\quad y[n]=x[n+\alpha]$

If $Y(e^{j\Omega})$ is real, then $y[n]$ is real and even (if $x[n]$ is real.). Therefore $x[n+\alpha]$ is even and $x[n]$ has to be symmetric about α.This is true for signal (a), (c), (e), (f) and (g).

25. (D) $\int_{-\pi}^{\pi} X(e^{j\Omega})\,d\Omega=2\pi x[0]$,

$x[0]=0$ is for signal (c), (f), (g) and (h).

26. (D) $X(e^{j\Omega})$ is always periodic with period 2π. Therefore all signals satisfy the condition.

27. (D) $X(e^{j0})=\sum_{n=-\infty}^{\infty} x[n]$, This condition is satisfied only if the samples of the signal add up to zero. This is true for signal (b) and (h).

28. (A) $X(e^{j0})=\sum_{n=-\infty}^{\infty} x[n]=6$

29. (A) $y[n]=x[n+2]$ is an even signal. Therefore $Y(e^{j\Omega})$ is real and even.

$Y(e^{j\Omega})=e^{j2\Omega}X(e^{j\Omega}) \quad \Rightarrow \quad X(e^{j\Omega})=e^{-j2\Omega}Y(e^{j\Omega})$,

Since $Y(e^{j\Omega})$ is real. This imply $\arg\{Y(e^{j\Omega})\}=0$

Thus $\arg\{X(e^{j\Omega})\}=-2\Omega$

30. (C) $\int_{-\pi}^{\pi} X(e^{j\Omega})\,d\Omega=2\pi x[0]=4\pi$

31. (A) $X(e^{j\pi})=\sum_{n=-\infty}^{\infty}(-1)^n x[n]=2$

32. (C) $\text{Ev}\{x[n]\} \xleftrightarrow{DTFT} \text{Re}\{X(e^{j\Omega})\}$

$$\text{Ev}\{x[n]=\frac{(x[n]+x[-n])}{2}$$

$$=\left\{-\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ 0,\ 0,\ 1,\ \underset{\uparrow}{2},\ 1,\ 0,\ 0,\ 1,\ \frac{1}{2},\ 0,\ -\frac{1}{2}\right\}$$

33. (D) $\int_{-\pi}^{\pi}|X(e^{j\Omega})|^2=2\pi\sum_{n=-\infty}^{\infty}|x[n]|^2=28\pi$

34. (C) $nx[n] \xleftrightarrow{DTFT} j\dfrac{d\,X(e^{j\Omega})}{d\Omega}$

$$\int_{-\pi}^{\pi}\left|\frac{dX(e^{j\Omega})}{d\Omega}\right|=2\pi\sum_{n=-\infty}^{\infty}|n|^2|x[n]|^2=316\pi$$

35. (A) $Y(e^{j\Omega})=e^{-j4\Omega}X(e^{j\Omega})$

$$y[n]=x[n-4]=(n-4)\left(\frac{3}{4}\right)^{|n-4|}$$

36. (C) Since $x[n]$ is real and odd, $X(e^{j\Omega})$ is purely imaginary. Thus $y[n]=0$.

37. (D) $X_2(e^{j\Omega})=X(e^{j2\Omega})$

$$X(e^{j2\Omega}) \xleftrightarrow{DTFT} x_2[n]=\begin{cases} x[n], & n \text{ even} \\ 0, & \text{otherwise}\end{cases}$$

$$y[n]=\begin{cases}-jn^2\left(\frac{3}{4}\right)^{|n|}, & n \text{ even} \\ 0, & \text{otherwise}\end{cases}$$

38. (B) $Y(e^{j\Omega}) = X(e^{j\Omega}) * X(e^{j(\Omega-\pi/2)})$

$y[n] = 2\pi x[n] x_1[n],\ x_1[n] = e^{j\pi n/2} x[n]$,

$$\Rightarrow \quad y[n] = 2\pi n^2 e^{j\pi n/2}\left(\frac{3}{4}\right)^{2|n|}$$

39. (C) $Y(e^{j\Omega}) = \dfrac{d}{d\Omega} X(e^{j\Omega})$

$$\Rightarrow \quad y[n] = -jnx[n] = -jn^2\left(\frac{3}{4}\right)^{|n|}$$

40. (B) $Y(e^{j\Omega}) = X(e^{j\Omega}) + X(e^{-j\Omega})$

$$\Rightarrow \quad y[n] = x[n] + x[-n] = 0$$

41. (C) For a real signal $x[n]$

$\text{od}\{x[n]\} \xleftrightarrow{DTFT} j\text{Im}\{X(e^{j\Omega})\}$

$j\text{Im}\{X(e^{j\Omega})\} = j\sin\Omega - j\sin 2\Omega,$

$= \frac{1}{2}\left(e^{j\Omega} - e^{-j\Omega} - e^{2j\Omega} + e^{-2j\Omega}\right)$

Therefore $\text{od}\{x[n]\} = F^{-1}\{j\text{Im}\{X(e^{j\Omega})\}\}$

$= \frac{1}{2}(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$

$\text{Od}\{x[n]\} = \dfrac{x[n] - x[-n]}{2}$

Since $x[n] = 0$ for $n > 0$,

$x[n] = 2\text{od}\{x[n]\} = \delta[n+1] - \delta[n+2]$ For $n < 0$

Using Parseval's relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(e^{j\Omega})\right|^2 d\Omega = \sum_{n=-\infty}^{\infty}|x[n]|^2$$

$$3 = \sum_{n=-\infty}^{-1}|x[n]|^2 = (x[0])^2 + 2$$

$x[0] = \pm 1$, But $x[0] = 0$, Hence $x[0] = 1$

$x[n] = \delta[n] + \delta[n+1] - \delta[n+2]$

42. (C) $\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{DTFT} \dfrac{1}{1-\frac{1}{4}e^{-j\Omega}}\left(\frac{1}{4}\right)^n$

$$n\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{DTFT} j\frac{d}{d\Omega}\left(\frac{1}{1-\frac{1}{4}e^{-j\Omega}}\right) = \frac{\frac{1}{4}e^{-j\Omega}}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2}$$

$$\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=-\infty}^{\infty} x[n] = X(e^{j0}) = \frac{4}{9}$$

43. (A) For all pass system $\left|H(e^{j\Omega})\right| = 1$ for all Ω

$H(e^{j\Omega}) = \dfrac{b + e^{-j\Omega}}{1 - ae^{-j\Omega}}$, $\left|b + e^{-j\Omega}\right| = \left|1 - a\,e^{-j\Omega}\right|$

$1 + b^2 + 2b\cos\Omega = 1 + a^2 - 2a\cos\Omega$

This is possible only if $b = -a$.

44. (A) For $x[n] = \delta[n]$, $X(e^{j\Omega}) = 1$, $\dfrac{dX(e^{j\Omega})}{d\Omega} = 0$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\Omega})\, e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\Omega}\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n-1)} d\Omega = \frac{\sin\pi(n-1)}{\pi(n-1)}$$

45. (B) $H(e^{j\Omega}) = H_1(e^{j\Omega}) + H_2(e^{j\Omega})$

$$\frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}} = \frac{1}{1-\frac{1}{3}e^{-j\Omega}} + \frac{-2}{1-\frac{1}{4}e^{-j\Omega}}$$

$$H_2(e^{j\Omega}) = \frac{1}{1-\frac{1}{3}e^{-j\Omega}}, \qquad h_2[n] = \left(\frac{1}{3}\right)^n u[n]$$

46. (D) $H(e^{j\Omega}) = \dfrac{Y(e^{j\Omega})}{X(e^{j\Omega})}$,

$$\left(\frac{2}{3}\right)^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-\frac{2}{3}e^{-j\Omega}}$$

$$n\left(\frac{2}{3}\right)^n u[n] \xleftrightarrow{DTFT} j\frac{d}{d\Omega}\left(\frac{1}{1-\frac{2}{3}e^{-j\Omega}}\right) = \frac{\frac{2}{3}e^{-j\Omega}}{1-\frac{2}{3}e^{-j\Omega}}$$

$$H(e^{j\Omega}) = \frac{\frac{2}{3}e^{-j\Omega}}{1-\frac{2}{3}e^{-j\Omega}} = \frac{2\,e^{-j\Omega}}{3 - 2\,e^{-j\Omega}}$$

47. (B) $H(e^{j\Omega}) = \dfrac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \dfrac{\frac{2}{3}e^{-j\Omega}}{1-\frac{2}{3}e^{-j\Omega}}$

$$\Rightarrow \quad \left(1 - \frac{2}{3}e^{-j\Omega}\right)Y(e^{j\Omega}) = \frac{2}{3}e^{-j\Omega}X(e^{j\Omega})$$

$$\Rightarrow \quad y[n] - \frac{2}{3}y[n-1] = \frac{2}{3}x[n-1]$$

$$\Rightarrow \quad 3y[n] - 2y[n-1] = 2x[n-1].$$

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