Chapter : 3. BINARY OPERATIONS

Exercise : 3A

Question: 1

Let * be a binary

Solution:

To find: 4*5

a*b = 3a + 4b - 2

Here a = 4 and b = 5

⇒ 4*5 = 3 × 4 + 4 × 5 - 2 = 12 + 20 - 2 = 30

⇒ 4*5 = 30

Question: 2

The binary operat

Solution:

To find: (2*3)*4

Given: a*b = 2a + b

⇒ 2*3 = 2 × 2 + 3 = 7

Now 7*4 = 2 × 7 + 4 = 14 + 4 = 18

⇒ (2*3)*4 = 18

Question: 3

Let * be a binary

Solution:

To find: value of x

Given: a*b = $\frac{ab}{5}$

⇒ x*5 = $\frac{5x}{5}$ = x

Now (2*x) = $\frac{2x}{5}$

⇒ $\frac{2x}{5}$ = 10⇒ x = 25

Question: 4

Let *: R × R

Solution:

To find: (- 5)*(2*0)

Given: a*b = a + 4b^2

⇒ (2*0) = 2 + 4 × 0^2 = 2

Now (- 5)*2 = - 5 + 4 × 2^2 = - 5 + 16 = 11

Question: 5

Let be a binary o

Solution:

To find: 3*5 and 5*3

Given:a*b = $(2a - b)^2$

⇒ 3*5 = $(6 - 5)^2 = 1$

Now 5*3 = $(10 - 3)^2 = 49$

⇒ 3*5 is not equal to 5*3

Question: 6

Let * be a binary

Solution:

To find: LCM of 20 and 16

Prime factorizing 20 and 16 we get.

$20 = 2^2 \times 5$

$16 = 2^4$

⇒ LCM of 20 and 16 = $2^4 \times 5 = 80$

(i) To find LCM highest power of each prime factor has been taken from both the numbers and multiplied.

So it is irrelevant in which order the number are taken as their prime factors will remain the same.

So LCM(a,b) = LCM(b,a)

So * is commutative

(ii) Let us assume that * is associative

⇒ LCM[LCM(a,b),c] = LCM[a,LCM(b,c)]

Let the prime factors of a be p_1,p_2

Let the prime factors of b be p_2,p_3

Let the prime factors of c be p_3,p_4

Let the higher factor of p_i be q_i for i = 1,2,3,4

LCM (a,b) = $p_1^{q1} \times p_2^{q2} \times p_3^{q3}$

LCM[LCM(a,b),c] = $p_1^{q1} \times p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

LCM (b,c) = $p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

LCM[a,LCM(b,c)] = $p_1^{q1} \times p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

⇒ * is associative

Question: 7

If * be the binar

Solution:

To find: 2*4

Given: a*b = $a + 3b^2$

⇒ 2*4 = $(2 + 3 \times 4^2) = 2 + 48 = 50$

Question: 8

Show that * on Z

Solution:

To prove: * is not a binary operation

Given: a and b are defined on positive integer set

And a*b = |a - b|

⇒ a*b = (a - b), when a>b

= b - a when b>a

= 0 when a = b

But 0 is neither positive nor negative.

So 0 does not belong to Z^+ .

So a*b = |a - b| does not belong to Z^+ for all values of a and b

So * is not a binary operation.

Hence proved

Question: 9

Let * be a binary

Solution:

To prove: * is neither commutative nor associative

Let us assume that * is commutative

⇒ $a^b = b^a$ for all a,b $\in$ N

This is valid only for a = b

For example take a = 1, b = 2

$1^2 = 1$ and $2^1 = 2$

So * is not commutative

Let us assume that * is associative

⇒ $(a^b)^c = a^{b^c}$ for all a,b,c $\in$ N

⇒ $a^{bc} = a^{b^c}$ for all a,b,c $\in$ N

This is valid in the following cases:

(i) a = 1

(ii) b = 0

(iii) $bc = b^c$

For example, let a = 2,b = 1,c = 3

$a^{bc} = 2^{(1 \times 3)} = 2^3 = 8$

$a^{b^c} = 2^{1^3} = 2$

So * is not associative

Question: 10

Let a * b = 1 cm

Solution:

To find: (i)

LCM of 12 and 16

Prime factorizing 12 and 16 we get.

$20 = 2^2 \times 3$

$16 = 2^4$

⇒ LCM of 20 and 16 = $2^4 \times 3 = 48$

(ii) To find LCM highest power of each prime factor has been taken from both the numbers and multiplied.

So it is irrelevant in which order the number are taken as their prime factors will remain the same.

So LCM(a,b) = LCM(b,a)

So * is commutative.

(iii)let $x \in N$ and x*1 = lcm(x,1) = x = lcm(1,x)

1 is the identity element.

(iv)let there exist y in n such that x*y = e = y*x

Here e = 1,

Lcm(x,y) = 1

This happens only when x = y = 1.

1 is the invertible element of n with respect to *.

Question: 11

Let Q be the set

Solution:

(i)Let a = 1, b = $2 \in Q^+$

a*b = $\frac{1}{2}(1 + 2) = 1.5 \in Q^+$

* is closed and is thus a binary operation on Q^+

(ii) a*b = $\frac{1}{2}(1 + 2) = 1.5$

And b*a = $\frac{1}{2}(2 + 1) = 1.5$

Hence * is commutative.

(iii)let c = 3.

(a*b)*c = 1.5*c = $\frac{1}{2}(1.5 + 3) = 2.75$

a*(b*c) = a*$\frac{1}{2}(2 + 3)$ = 1*2.5 = $\frac{1}{2}(1 + 2.5) = 1.75$

hence * is not associative.

Question: 12

Show that the set

Solution:

For a set to be closed for addition,

For any 2 elements of the set,say a and b, a + b must also be a member of the given set, where a and b may be same or distinct

In the given problem let a = 1 and b = 1

a + b = 2 which is not in the given in set

So the set is not closed for addition.

Hence proved.

Question: 13

Show that * on R

Solution:

let a = 1,b = 0$\in$R - { - 1}

a*b = $\frac{1}{0+1}$ = 1

And b*a = $\frac{0}{1+1}$ = 0

Hence * is not commutative.

Let c = 3.

(a*b)*c = 1*c = $\frac{1}{3+1}$ = $\frac{1}{4}$

a*(b*c) = a*$\frac{0}{3+1}$ = 1*0 = $\frac{1}{0+1}$ = 1

Hence * is not associative.

Question: 14

For all a, b

Solution:

a*b = a - b if a>b

= - (a - b) if b>a

b*a = a - b if a>b

= - (a - b) if b>a

So a*b = b*a

So * is commutative

To show that * is associative we need to show

(a*b)*c = a*(b*c)

Or ||a - b| - c| = |a - |b - c||

Let us consider c>a>b

Eg a = 1,b = - 1,c = 5

LHS:

|a - b| = |1 + 1| = 2

||a - b| - c| = |2 - 5| = 3

RHS

|b - c| = | - 1 - 5| = 6

|a - |b - c|| = |1 - 6| = | - 5| = 5

As LHS is not equal to RHS * is not associative

Question: 15

For all a, b

Solution:

let a = 1,b = 2$\in$N

$a*b = 1^3 + 2^3 = 9$

And $b*a = 2^3 + 1^3 = 9$

Hence * is commutative.

Let c = 3

$(a*b)*c = 9*c = 9^3 + 3^3$

$a*(b*c) = a*(2^3 + 3^3) = 1*35 = 1^3 + 35^3$

$(a*b)*c \neq a*(b*c)$

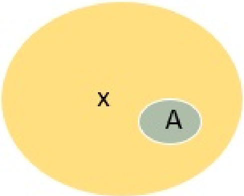
Hence * is not associative.

Question: 16

Let X be a nonemp

Solution:

e is the identity of * if e*a = a



From the above Venn diagram,

$A*X = A \cap X = A$

$X*A = X \cap A = A$

⇒ X is the identity element for binary operation *

Let B be the invertible element

⇒ A*B = X

⇒ $A \cap B = X$

This is only possible if A = B = X

Thus X is the only invertible element in P(X)

Hence proved.

Question: 17

A binary operatio

Solution:

To find: identity and inverse element

For a binary operation if a*e = a, then e s called the right identity

If e*a = a then e is called the left identity

For the given binary operation,

e*b = b

⇒ e + b = b

⇒ e = 0 which is less than 6.

b*e = b

⇒ b + e = b

⇒ e = 0 which is less than 6

For the 2^{nd} condition,

e*b = b

⇒ e + b - 6 = b

⇒ e = 6

But e = 6 does not belong to the given set (0,1,2,3,4,5)

So the identity element is 0

An element c is said to be the inverse of a, if a*c = e where e is the identity element (in our case it is 0)

a*c = e

⇒ a + c = e

⇒ a + c = 0

⇒ c = - a

a belongs to (0,1,2,3,4,5)

- a belongs to (0, - 1, - 2, - 3, - 4, - 5)

So c belongs to (0, - 1, - 2, - 3, - 4, - 5)

So c = - a is not the inverse for all elements a

Putting in the 2^{nd} condition

a*c = e

⇒ a + c - 6 = 0

⇒ c = 6 - a

$0 \le a < 6$

$\Rightarrow -6 \le -a < 0 \Rightarrow 0 \le 6 - a < 6 0 \le c < 5$

So c belongs to the given set

Hence the inverse of the element a is (6 - a)

Hence proved

Exercise : 3B

Question: 1

Define * on N by

Solution:

* is an operation as m*n = LCM (m, n) where m, n ∈ N. Let m = 2 and b = 3 two natural numbers.

m*n = 2*3

= LCM (2, 3)

= 6∈ N

So, * is a binary operation from $N \times N \to N$.

For commutative,

n*m = 3*2

= LCM (3, 2)

= 6∈ N

Since m*n = n*m, hence * is commutative operation.

Again, for associative, let p = 4

m*(n*p) = 2*LCM (3, 4)

= 2*12

= LCM (2, 12)

= 12∈ N

(m*n) *p = LCM (2, 3) *4

= 6*4

= LCM (6, 4)

= 12∈ N

As m*(n*p) = (m*n) *p, hence * an associative operation.

Question: 2

Define * on Z by

Solution:

* is an operation as a*b = a-b + ab where a, b ∈ Z. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 = \frac{1}{2} - 2 + \frac{1}{2}.2 \Rightarrow \frac{1-4}{2} + 1 = \frac{-3+2}{2} \Rightarrow \frac{-1}{2} \in Z$$

So, * is a binary operation from $Z \times Z \rightarrow Z$.

For commutative,

$$b*a = 2 - \frac{1}{2} + 2.\frac{1}{2} = \frac{4-1}{2} + 1 \Rightarrow \frac{3+2}{2} = \frac{5}{2} \in Z$$

Since a*b ≠ b*a, hence * is not commutative operation.

Again for associative,

a*(b*c) = a*(b-c+ bc)

= a- (b- c+ bc) +a (b- c+ bc)

= a-b+ c- bc+ ab- ac+ abc

(a*b) *c = (a-b+ ab) *c

= a- b+ ab-c+ (a- b+ ab) c

= a-b-c+ ab+ ac- bc+ abc

As a*(b*c) ≠ (a*b) *c, hence * not an associative operation.

Question: 3

Define * on Z by

Solution:

* is an operation as a*b = a+ b - ab where a, b ∈ Z. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 = \frac{1}{2} + 2 - \frac{1}{2}.2 \Rightarrow \frac{1+4}{2} - 1 = \frac{5-2}{2} \Rightarrow \frac{3}{2} \in Z$$

So, * is a binary operation from $Z \times Z \rightarrow Z$.

For commutative,

$b*a = 2+\frac{1}{2}-2.\frac{1}{2} = \frac{4+1}{2}-1 \Rightarrow \frac{5-2}{2} = \frac{3}{2} \in Z$

Since a*b = b*a, hence * is a commutative binary operation.

Again for associative,

a*(b*c) = a*(b+ c- bc)

= a+ (b+ c- bc) -a (b+ c- bc)

= a+ b+ c- bc- ab- ac+ abc

(a*b) *c = (a+ b- ab) *c

= a+ b- ab+ c- (a+ b- ab) c

= a+ b+ c- ab- ac- bc+ abc

As a*(b*c) = (a*b) *c, hence * an associative binary operation.

Question: 4

Consider a binary

Solution:

(i) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

a*e = a+ e- ae (1)

e*a = e+ a- e a (2)

using a*e = a

a+ e- ae = a

e-ae = 0

e(1-a) = 0

either e = 0 or a = 1 as operation is on Q excluding 1 so a≠1, hence e = 0.

So identity element e = 0.

(ii) for a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 0

a+ b- ab = 0

b(1-a) = -a

$b = \frac{-a}{(1-a)} \Rightarrow \frac{a}{(a-1)}$

$a^{-1} = \frac{a}{(a-1)}$

Question: 5

Let Q_0

Solution:

(i) For commutative binary operation, a*b = b*a.

$a*b = \frac{ab}{4}$ and $b*a = \frac{ba}{4}$

as multiplication is commutative ab = ba so a*b = b*a. Hence * is commutative binary operation.

For associative binary operation, a*(b*c) = (a*b) *c

$$a*(b*c) = a*\frac{bc}{4} \Rightarrow \frac{a.\frac{bc}{4}}{4} = \frac{abc}{16}$$

$$(a*b)*c = \frac{ab}{4}*c \Rightarrow \frac{\frac{ab}{4}.c}{4} = \frac{abc}{16}$$

Since a*(b*c) = (a*b) *c, hence * is an associative binary operation.

(ii) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

$a*e = \frac{ae}{4}$ (1)

$e*a = \frac{ea}{4}$ (2)

using a*e = a

$$\frac{ae}{4} = a \Rightarrow \frac{ae}{4} - a = 0 \Rightarrow \frac{a}{4}(e-4) = 0$$

Either a = 0 or e = 4 as given a≠0, so e = 4.

Identity element e = 4.

(iii) For a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 4

$$\frac{ab}{4} = 4 \Rightarrow b = \frac{16}{a}$$

$$a^{-1} = \frac{16}{a}$$

Question: 6

On the set Q

Solution:

(i) * is an operation as $a*b = \frac{ab}{2}$ where a, b ∈ Q^+. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 \Rightarrow 1 \in Q^+$$

So, * is a binary operation from $Q^+ \times Q^+ \rightarrow Q^+$.

(ii) For commutative binary operation, a*b = b*a.

$$b*a = 2.\frac{1}{2} \Rightarrow 1 \in Q^+$$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c.

$$a*(b*c) = a*\frac{bc}{2} \Rightarrow \frac{a.\frac{bc}{2}}{2} = \frac{abc}{4}$$

$$(a*b)*c = \frac{ab}{2}*c \Rightarrow \frac{\frac{ab}{2}.c}{2} = \frac{abc}{4}$$

As a*(b*c) = (a*b) *c, hence * is an associative binary operation.

For a binary operation *, e identity element exists if a*e = e*a = a.

$a*e = \frac{ae}{2}$ (1)

$e*a = \frac{ea}{2}$ (2)

using a*e = a

$\frac{ae}{2} = a \Rightarrow \frac{ae}{2} - a = 0 \Rightarrow \frac{a}{2}(e-2) = 0$

Either a = 0 or e = 2 as given a≠0, so e = 2.

For a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 2

$\frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a}$

$a^{-1} = \frac{4}{a}$

Question: 7

(i) * is an operation as $a*b = \frac{1}{2}(a+b)$ where a, b ∈ Q^+. Let $a = 1$ and b = 2 two integers.

$a*b = \frac{1}{2}(1+2) \Rightarrow \frac{3}{2} \in Q^+$

So, * is a binary operation from $Q^+ \times Q^+ \to Q^+$.

(ii) For commutative binary operation, a*b = b*a.

$b*a = \frac{1}{2}(2+1) \Rightarrow \frac{3}{2} \in Q^+$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c.

$a*(b*c) = a*\frac{1}{2}(b+c) \Rightarrow \frac{1}{2}\left(a+\frac{b+c}{2}\right) = \frac{1}{4}(2a+b+c)$

$(a*b)*c = \frac{1}{2}(a+b)*c \Rightarrow \frac{1}{2}\left(\frac{a+b}{2}+c\right) = \frac{1}{4}(a+b+2c)$

As a*(b*c) ≠(a*b) *c, hence * is not associative binary operation.

Question: 8

Let Q be the set

Solution:

(i) * is an operation as a*b = a+ b+ ab where a, b ∈ Q- {-1}. Let $a = 1$ and $b = \frac{-3}{2}$ two rational numbers.

$a*b = 1+\frac{-3}{2}+1.\frac{-3}{2} \Rightarrow \frac{2-3}{2}-\frac{3}{2} = \frac{-1-3}{2} \Rightarrow \frac{-4}{2} = -2 \in Q-\{-1\}$

So, * is a binary operation from $Q-\{-1\} \times Q-\{-1\} \to Q-$ {-1}.

(ii) For commutative binary operation, a*b = b*a.

$b*a = \frac{-3}{2}+1+\frac{-3}{2}.1 \Rightarrow \frac{-3+2}{2}-\frac{3}{2} = \frac{-1-3}{2} \Rightarrow \frac{-4}{2} = -2 \in Q-\{-1\}$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c

a+(b*c) = a*(b+ c+ bc) = a+ (b+ c+ bc) +a(b+ c+ bc)

= a+ b+ c+ bc+ ab+ ac+ abc

(a*b) *c = (a+ b+ ab) *c = a+ b+ ab+ c+ (a+ b+ ab) c

= a+ b+ c+ ab+ ac+ bc+ abc

Now as a*(b*c) = (a*b) *c, hence an associative binary operation.

(iv) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

a*e = a+ e+ ae (1)

e*a = e+ a +e a (2)

using a*e = a

a+ e+ ae = a

e+ ae = 0

e(1+a) = 0

either e = 0 or a = -1 as operation is on Q excluding -1 so a≠-1, hence e = 0.

So identity element e = 0.

(v) for a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 0

a+ b+ ab = 0

b(1+a) = -a

$$b = \frac{-a}{(1+a)}$$

$$a^{-1} = \frac{-a}{(a+1)}$$

Question: 9

Let A = N × N. De

Solution:

(i) A is said to be closed on * if all the elements of (a, b) *(c, d) = (a+ c, b+ d) belongs to N×N for A = N×N.

Let a = 1, b = 3, c = 8, d = 2

(1, 3) * (8, 2) = (1+8, 3+2)

= (9, 5) ∈N×N

Hence A is closed for *.

(ii) For commutative,

(c, d) *(a, b) = (c+ a, d+ b)

As addition is commutative a+ c = c+ a and b+ d = d+ b, hence * is commutative binary operation.

(iii) For associative,

(a, b) *((c, d) *(e, f)) = (a, b) *(c+ e, d+ f)

= (a+ c+ e, b+ d+ f)

((a, b) *(c, d)) *(e, f) = (a+ c, b+ d) *(e, f)

= (a+ c+ e, b+ d+ f)

As (a, b) *((c, d) *(e, f)) = ((a, b) *(c, d)) *(e, f), hence * is an associative binary operation.

(iv) For identity element (e_1, e_2), (a, b) $*(e_1, e_2) = (e_1, e_2) *$(a, b) = (a, b) in a binary operation.

(a, b) $*(e_1, e_2)$ = (a, b)

$(a+e_1, b+e_2)$ = (a, b)

(e_1, e_2) = (0, 0)

As (0,0) ∉N×N, hence identity element does not exist for *.

Question: 10

Let A = (1, -1, i

Solution:

(i) A is said to be closed on * if all the elements of a*b ∈A. composition table is

×	**1**	**-1**	**i**	**-i**
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(as $i^2 = -1$)

As table contains all elements from set A, A is close for multiplication operation.

(ii) For associative, a× (b× c) = (a× b) ×c

1× (-i× i) = 1×1 = 1

(1× -i) ×i = -i× i = 1

a× (b× c) = (a× b) ×c, so A is associative for multiplication.

(iii) For commutative, a× b = b× a

1× -1 = -1

-1× 1 = -1

a× b = b× a, so A is commutative for multiplication.

(iv) For multiplicative identity element e, a× e = e× a = a where a ∈A.

a× e = a

a(e-1) = 0

either a = 0 or e = 1 as a≠0 hence e = 1.

So, multiplicative identity element e = 1.

(v) For multiplicative inverse of every element of A, a*b = e where a, b∈A.

$1 \times b_1 = 1$

$b_1 = 1$

$-1 \times b_2 = 1$

$b_2 = -1$

$i \times b_3 = 1$

$$b_3 = \frac{1}{i} \Rightarrow \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} \Rightarrow \frac{i}{-1} = -i$$

$-i \times b_4 = 1$

$$b_4 = \frac{1}{-i} \Rightarrow \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} \Rightarrow \frac{i}{-(-1)} = i$$

So, multiplicative inverse of A = {1, -1, -i, i}