CIRCLES [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parallel to the coordinate axes. One vertex of the square is

   a. $(1 + \sqrt{2}, -2)$ **b.** $(1 - \sqrt{2}, -2)$

   c. $(1, -2 + \sqrt{2})$ **d.** none of these

   (IIT-JEE 1980)

2. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then the equation of the circle through their points of intersection and the point (1, 1) is

   a. $x^2 + y^2 - 6x + 4 = 0$

   b. $x^2 + y^2 - 3x + 1 = 0$

   c. $x^2 + y^2 - 4y + 2 = 0$

   d. none of these **(IIT-JEE 1980)**

3. The equation of the circle passing through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is

a. $4x^2 + 4y^2 - 30x - 10y - 25 = 0$
b. $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
c. $4x^2 + 4y^2 - 17x - 10y + 25 = 0$
d. none of these **(IIT-JEE 1983)**

4. The locus of the midpoint of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin is
a. $x + y = 2$ b. $x^2 + y^2 = 1$
c. $x^2 + y^2 = 2$ d. $x + y = 1$
(IIT-JEE 1984)

5. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, then the equation of the locus of its center is
a. $2ax + 2by - (a^2 + b^2 + k^2) = 0$
b. $2ax + 2by - (a^2 - b^2 + k^2) = 0$
c. $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
d. $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - k^2) = 0$
(IIT-JEE 1988)

6. If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect at two distinct points, then
a. $2 < r < 8$ b. $r < 2$
c. $r = 2$ d. $r > 2$ **(IIT-JEE 1989)**

7. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 sq. units. Then the equation of this circle is
a. $x^2 + y^2 + 2x - 2y = 62$
b. $x^2 + y^2 + 2x - 2y = 47$
c. $x^2 + y^2 - 2x + 2y = 47$
d. $x^2 + y^2 - 2x + 2y = 62$ **(IIT-JEE 1989)**

8. The center of the circle passing through the points (0, 0) and (1, 0) and touching the circle $x^2 + y^2 = 9$ is
a. (3/2, 1/2) b. (1/2, 3/2)
c. (1/2, 1/2) d. $(1/2, -\sqrt{2})$
(IIT-JEE 1992)

9. The locus of the center of the circle which touches the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ externally and also touches the y-axis is given by equation
a. $x^2 - 6x - 10y + 14 = 0$
b. $x^2 - 10x - 6y + 14 = 0$
c. $y^2 - 6x - 10y + 14 = 0$
d. $y^2 - 10x - 6y + 14 = 0$ **(IIT-JEE 1993)**

10. The angle between the pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2\alpha + 13 \cos^2\alpha = 0$ is 2α. Then the equation of the locus of the point P is
a. $x^2 + y^2 + 4x - 6y + 4 = 0$
b. $x^2 + y^2 + 4x - 6y - 9 = 0$
c. $x^2 + y^2 + 4x - 6y - 4 = 0$
d. $x^2 + y^2 + 4x - 6y + 9 = 0$ **(IIT-JEE 1996)**

11. If two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 - px - qy = 0$ (where $pq \neq 0$) are bisected by the x-axis, then
a. $p^2 = q^2$ b. $p^2 = 8q^2$
c. $p^2 < 8q^2$ d. $p^2 > 8q^2$ **(IIT-JEE 1999)**

12. Triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (−4, 3), respectively, then $\angle QPR$ is equal to
a. $\pi/2$ b. $\pi/3$ c. $\pi/4$ d. $\pi/6$
(IIT-JEE 2000)

13. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is
a. 2 or −3/2 b. −2 or −3/2
c. 2 or 3/2 d. −2 or 3/2
(IIT-JEE 2000)

14. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the center. Then, the locus of the centroid of triangle PAB as P moves on the circle is
a. a parabola b. a circle
c. an ellipse d. a pair of straight lines
(IIT-JEE 2001)

15. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals
a. $\sqrt{PQ \cdot RS}$ b. $\frac{(PQ + RS)}{2}$
c. $\frac{2PQ \times RS}{PQ + RS}$ d. $\frac{\sqrt{(PQ^2 + RS^2)}}{2}$
(IIT-JEE 2001)

16. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets a straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is
a. 4 b. $2\sqrt{5}$ c. 5 d. $3\sqrt{5}$
(IIT-JEE 2002)

17. The center of the circle inscribed in a square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is
a. (4, 7) b. (7, 4) c. (9, 4) d. (4, 9)
(IIT-JEE 2003)

18. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with center (2, 1), then the radius of the circle is
a. $\sqrt{3}$ b. $\sqrt{2}$ c. 3 d. 2
(IIT-JEE 2004)

19. A circle is given by $x^2 + (y - 1)^2 = 1$. Another circle C touches it externally and also the x-axis. Then the locus of its center is
a. $\{(x, y): x^2 = 4y\} \cup \{(x, y): y \leq 0\}$
b. $\{(x, y): x^2 + (y - 1)^2 = 4\} \cup \{(x, y): y \leq 0\}$
c. $\{(x, y): x^2 = y\} \cup \{(0, y): y \leq 0\}$
d. $\{(x, y): x^2 = 4y\} \cup \{(0, y): y \leq 0\}$ **(IIT-JEE 2005)**

20. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at points A and B. The equation of the circumcircle of triangle PAB is

a. $x^2 + y^2 + 4x - 6y + 19 = 0$

b. $x^2 + y^2 - 4x - 10y + 19 = 0$

c. $x^2 + y^2 - 2x + 6y - 20$

d. $x^2 + y^2 - 6x - 4y + 19 = 0$ **(IIT-JEE 2009)**

21. The circle passing through the point $(-1, 0)$ and touching the y-axis at $(0, 2)$ also passes through the point

a. $(-3/2, 0)$ b. $(-5/2, 2)$

c. $(-3/2, 5/2)$ d. $(-4, 0)$ **(IIT-JEE 2011)**

22. The locus of the midpoint of the chord of contact of tangents drawn from the points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is

a. $20(x^2 + y^2) - 36x + 45y = 0$

b. $20(x^2 + y^2) + 36x - 45y = 0$

c. $36(x^2 + y^2) - 20x + 45y = 0$

c. $36(x^2 + y^2) + 20x - 45y = 0$ **(IITJEE 2012)**

Multiple Correct Answers Type

1. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are

a. $x = 0$ b. $y = 0$

c. $(h^2 - r^2)x - 2rhy = 0$ d. $(h^2 - r^2)x + 2rhy = 0$

(IIT-JEE 1988)

2. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, and $S(x_4, y_4)$, then

a. $x_1 + x_2 + x_3 + x_4 = 0$ b. $y_1 + y_2 + y_3 + y_4 = 0$

c. $x_1x_2x_3x_4 = c^4$ d. $y_1y_2y_3y_4 = c^4$

(IIT-JEE 1998)

3. Circle(s) touching the x-axis at a distance of 3 units from the origin and having an intercept of length $2\sqrt{7}$ on the y-axis is (are)

a. $x^2 + y^2 - 6x + 8y + 9 = 0$

b. $x^2 + y^2 - 6x + 7y + 9 = 0$

c. $x^2 + y^2 - 6x - 8y + 9 = 0$

d. $x^2 + y^2 - 6x - 7y + 9 = 0$ **(JEE Advanced 2013)**

4. A circle S passes through the point $(0, 1)$ and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then

a. radius of S is 8 b. radius of S is 7

c. centre of S is $(-7, 1)$ d. centre of S is $(-8, 1)$

(JEE Advanced 2014)

Linked Comprehension Type

For Problems 1–3

ABCD is a square of side length 2. C_1 is a circle inscribed in the square and C_2 is a circle circumscribing the square. P and Q are any two points on C_1 and C_2, respectively. Also R, is fixed point and L is a fixed line in the same plane. A circle C touches line L and circle C_1 externally. S is a point which is equidistant from given point R and fixed line L. Point R coincides with B. **(IIT-JEE 2006)**

1. $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2} =$

a. 1 b. 1.75 c. 0.75 d. 1.5

2. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line. Then the locus of the center of the circle is a/an

a. ellipse b. hyperbola

c. parabola d. pair of straight lines

3. A line M through A is drawn parallel to BD. Point S moves such that its distance from the line BD and the vertex A is the same. If the locus of S cuts M at T_2 and T_3 and AC at T_1, then the area of $\Delta T_1T_2T_3$ is

a. 1/2 sq. units b. 2/3 sq. units

c. 1 sq. units d. 2 sq. units

For Problems 4–6

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, and RP are D, E, and F, respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $(3\sqrt{3}/2, 3/2)$. Further, it is given that the origin and the center of C are on the same side of the line PQ. **(IIT-JEE 2008)**

4. The equation of circle C is

a. $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$

b. $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$

c. $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

d. $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

5. Points E and F are given by

a. $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$ b. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

c. $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ d. $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

6. The equations of the sides QR and RP are, respectively,

a. $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$

b. $y = \frac{1}{\sqrt{3}}x, y = 0$

c. $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$

d. $y = \sqrt{3}x, y = 0$

For Problems 7–8

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT, is a tangent to the circle $(x - 3)^2 + y^2 = 1$. **(IIT-JEE 2012)**

7. A possible equation of L is
   a. $x - \sqrt{3}y = 1$ b. $x + \sqrt{3}y = 1$
   c. $x - \sqrt{3}y = -1$ d. $x + \sqrt{3}y = 5$
8. A common tangent of the two circles is
   a. $x = 4$ b. $y = 2$
   c. $x + \sqrt{3}y = 4$ d. $x + 2\sqrt{2}y = 6$

Matching Column Type

1. Match the statements in Column I with the statements in Column II.

	Column I		Column II
(a)	Two intersecting circles	(p)	have a common tangent
(b)	Two mutually external circles	(q)	have a common normal
(c)	Two circles, one strictly inside the other,	(r)	do not have a common tangent
(d)	Two branches of a hyperbola	(s)	do not have a common normal

(IIT-JEE 2007)

2. Match the conics in Column I with the statements/ expressions in Column II.

	Column I		Column II
(a)	Circle	(p)	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(b)	Parabola	(q)	Points z in the complex plane satisfying $\|z+2\| - \|z-2\| = \pm 3$
(c)	Ellipse	(r)	Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(d)	Hyperbola	(s)	The eccentricity of the conic lies in the interval $1 \le x < \infty$
		(t)	Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \|z\|^2 + 1$

(IIT-JEE 2009)

Integer Answer Type

1. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts.
   If $S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\}$
   then the number of point(s) in S lying inside the smaller part is **(IIT-JEE 2011)**
2. The centers of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centers of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then find the radius of circle C. **(IIT-JEE 2009)**

Assertion-Reasoning Type

1. Tangents are drawn from the point $(17, 7)$ to the circle $x^2 + y^2 = 169$.
   Statement 1: The tangents are mutually perpendicular.
   Statement 2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.
   a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.
   b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.
   c. Statement 1 is True and Statement 2 is False.
   d. Statement 1 is False and Statement 2 is True.
   (IIT-JEE 2007)
2. Consider
   L_1: $2x + 3y + p - 3 = 0$
   L_2: $2x + 3y + p + 3 = 0$
   where p is a real number
   and C: $x^2 + y^2 + 6x - 10y + 30 = 0$
   Statement 1: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.
   Statement 2: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.
   a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.
   b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.
   c. Statement 1 is True and Statement 2 is False.
   d. Statement 1 is False and Statement 2 is True.
   (IIT-JEE 2008)

Fill in the Blanks Type

1. If A and B are points in the plane such that $PA/PB = k$ (constant) for all P on a given circle, then the value of k cannot be equal to ________. **(IIT-JEE 1982)**
2. The points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are ________ and ________. **(IIT-JEE 1983)**
3. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle. The radius of this circle is________. **(IIT-JEE 1984)**
4. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from the point $(4, 5)$ with a pair of radii form a quadrilateral of area ________. **(IIT-JEE 1985)**
5. From the origin, chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the midpoints of these chords is ________. **(IIT-JEE 1985)**

6. The equation of the line passing through the points of intersection of the circles $3x^2 + 3y^2 - 2x + 12y - 9 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$ is ________. **(IIT-JEE 1986)**

7. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that $AM = 2AB$. The equation of the locus of M is ________. **(IIT-JEE 1986)**

8. The area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact is ________. **(IIT-JEE 1988)**

9. If the circle C_1: $x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to 3/4, then the coordinates of the center of C_2 are ________. **(IIT-JEE 1988)**

10. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is ________. **(IIT-JEE 1989)**

11. If a circle passes through the points of intersection of the coordinates axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is ________. **(IIT-JEE 1991)**

12. The equation of the locus of the midpoints of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $2\pi/3$ at its center is ________. **(IIT-JEE 1993)**

13. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. The equation of the circle with AB as a diameter is ________. **(IIT-JEE 1996)**

14. Two vertices of an equilateral triangle are (−1, 0) and (1, 0), and its third vertex lies above the x-axis. The equation of its circumcircle is ________. **(IIT-JEE 1997)**

15. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through the point ________. **(IIT-JEE 1997)**

True/False Type

1. No tangent can be drawn from the point (5/2, 1) to the circumcircle of the triangle with vertices $(1, \sqrt{3})$, $(1, -\sqrt{3})$, $(3, -\sqrt{3})$. **(IIT-JEE 1985)**

2. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$. **(IIT-JEE 1989)**

Subjective Type

1. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5). **(IIT-JEE 1978)**

2. Find the equation of the circle which passes through the point (2, 0) and whose center is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2 y = 1$ as c tends to 1. **(IIT-JEE 1979)**

3. Let A be the center of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point C. Find the area of quadrilateral $ABCD$. **(IIT-JEE 1981)**

4. Find the equations of the circle passing through (−4, 3) and touching the lines $x + y = 2$ and $x - y = 2$. **(IIT-JEE 1982)**

5. Through a fixed point (h, k), secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the midpoints of the secants by the circle is $x^2 + y^2 = hx + ky$. **(IIT-JEE 1983)**

6. The abscissa of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter. **(IIT-JEE 1984)**

7. Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a circle C_1 of diameter 6. If the center of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. **(IIT-JEE 1986)**

8. Let a given line L_1 intersects the x- and y-axes at P and Q, respectively. Let another line L_2, perpendicular to L_1, cuts the x- and the y-axis at R and S, respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin. **(IIT-JEE 1987)**

9. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcenter of the triangle is $x + y - xy + k(x^2 + y^2)^{1/2} = 0$. Find k. **(IIT-JEE 1987)**

10. Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends a right angle at the origin. **(IIT-JEE 1988)**

11. If $(m_i, 1/m_i)$, $m_i > 0$, $i = 1, 2, 3, 4$, are four distinct points on a circle, then show that $m_1m_2m_3m_4 = 1$. **(IIT-JEE 1989)**

12. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point (−10, 2) in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle. **(IIT-JEE 1990)**

13. Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangents is $4x + 3y = 10$, find the equations of the circles. **(IIT-JEE 1991)**

14. Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords, each

bisected by the x-axis, can be drawn to the circle from $(a, b/2)$. **(IIT-JEE 1992)**

15. Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point. **(IIT-JEE 1993)**

16. Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y = -4$ and $x^2 + y^2 - 12x - 8y = -36$ touch each other. Also find the equations of common tangents touching the circles at distinct points. **(IIT-JEE 1993)**

17. Find the intervals of the values of a for which the line $y + x = 0$ bisects two chords drawn from the point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$. **(IIT-JEE 1996)**

18. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line from point P intersects the curve at points Q and R. If the product $PQ \cdot PR$ is independent of the slope of the line, then show that the curve is a circle. **(IIT-JEE 1997)**

19. Let C be any circle with center $(0, \sqrt{2})$. Prove that at the most, two rational points can be there on C (a rational point is a point both of whose coordinates are irrational numbers). **(IIT-JEE 1997)**

20. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of triangle PAB lies on C_1. **(IIT-JEE 1998)**

21. Let T_1 and T_2 be two tangents drawn from $(-2, 0)$ onto the circle $C: x^2 + y^2 = 1$. Determine the circles touching C and having T_1 and T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time. **(IIT-JEE 1999)**

22. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3, with center in the first quadrant. If A is one of the points of contact, find the length of OA. **(IIT-JEE 2001)**

23. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum. **(IIT-JEE 2003)**

24. Find the equation of the circle touching the line $2x + 3y + 1 = 0$ at $(1, -1)$ and cutting orthogonally the circle having line segment joining $(0, 3)$ and $(-2, -1)$ as diameter. **(IIT-JEE 2004)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. d.	**2.** b.	**3.** b.	**4.** c.	**5.** a.
6. a.	**7.** c.	**8.** d.	**9.** d.	**10.** d.
11. d.	**12.** c.	**13.** a.	**14.** b.	**15.** a.
16. c.	**17.** a.	**18.** c.	**19.** d.	**20.** b.
21. d.	**22.** a.			

Multiple Correct Answers Type

1. a., c **2.** a., b., c., d **3.** a., c. **4.** b., c.

Linked Comprehension Type

1. a.	**2.** c.	**3.** c.	**4.** d.	**5.** a.
6. d.	**7.** a.	**8.** d.		

Matching Column Type

1. (a) – (p), (q); (b) – (p), (q); (c) – (q), (r)
2. (p) – (a)

Integer Answer Type

1. 2 **2.** 8 units

Assertion–Reasoning Type

1. a. **2.** c.

Fill in the Blanks Type

1. 1 **2.** (4, 2) and (−2, −6) **3.** 3/4
4. 8 sq. units **5.** $x^2 + y^2 - x = 0$
6. $10x - 3y - 18 = 0$ **7.** $x^2 + y^2 + 8x - 6y + 9 = 0$
8. $\frac{192}{25}$ sq. units **9.** $\left(\pm\frac{9}{5}, \mp\frac{12}{5}\right)$
10. $2\sqrt{3}$ sq. units **11.** 2
12. $(x - 3/2)^2 + (y + 1/2)^2 = 9/16$
13. $x(x - 1) + y(y - 1) = 0$
14. $x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$ **15.** (1/2, 1/4)

True/False Type

1. True
2. True

Subjective Type

1. $x^2 + y^2 - 18x - 16y + 120 = 0$
2. $25x^2 + 25y^2 - 20x + 2y - 60 = 0$
3. 75 sq. units
4. $(x-h)^2 + y^2 = \dfrac{(h-2)^2}{2}$, where $h = -10 \pm 3\sqrt{6}$
6. $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$
7. $(x-5)^2 + (y-2)^2 = 5^2$
9. ± 1
10. $x^2 + y^2 + gx + fy + \dfrac{c}{2} = 0$
12. $x^2 + y^2 + 18x - 2y + 32 = 0$
13. $x^2 + y^2 - 10x - 10y + 25 = 0$ and $x^2 + y^2 + 6x - 2y - 15 = 0$
14. $a^2 > 2b^2$
15. (2, 23/3)
16. $\left(\dfrac{14}{5}, \dfrac{8}{5}\right)$, $y = 0$, $7y - 24x + 16 = 0$
17. $(-\infty, -2) \cup (2, \infty)$
21. $(x-4)^2 + y^2 = 3^2$ and $\left(x+\dfrac{4}{3}\right)^2 + y^2 = \left(\dfrac{1}{3}\right)^2$

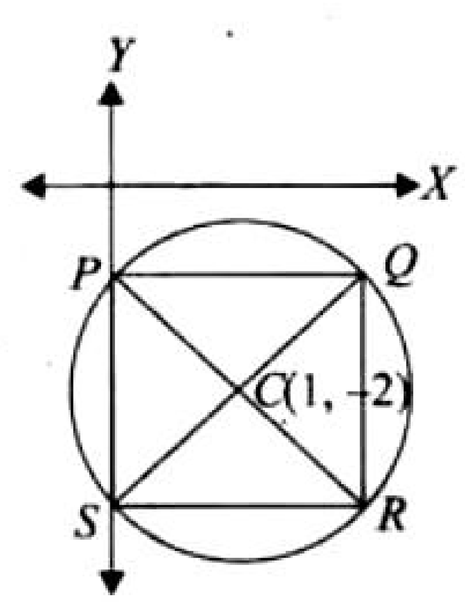
    $y = \pm \dfrac{5}{\sqrt{39}}\left(x + \dfrac{4}{5}\right)$
22. $3(3 + \sqrt{10})$
23. $r = 5$
24. $2x^2 + 2y^2 - 10x - 5y + 1 = 0$

Hints and Solutions

JEE Advnced

Single Correct Answer Type

1. **d.**



Radius of the circle,

$CQ = \sqrt{2}$

Since $\angle QSR = 45°$, the coordinates of Q and S are given by $(1 \pm \sqrt{2}\cos 45°, -2 \pm \sqrt{2}\sin 45°)$ or $Q(2, -1)$ and $S(0, -3)$.

The coordinates of P and R are given by $(1 \pm \sqrt{2}\cos 135°, -2 \pm \sqrt{2}\sin 135°)$ or $P(0, -1)$ and $S(2, -3)$.

2. **b.** The circle through the points of intersection of the two circles $x^2 + y^2 - 6 = 0$ and $x^2 + y^2 - 6x + 8 = 0$ is

$(x^2 + y^2 - 6) + \lambda(x^2 + y^2 - 6x + 8) = 0$

As it passes through (1, 1),

$(1 + 1 - 6) + \lambda(1 + 1 - 6 + 8) = 0$

or $\lambda = 1$

Therefore, the required circle is

$2x^2 + 2y^2 - 6x + 2 = 0$

or $x^2 + y^2 - 3x + 1 = 0$

3. **b.** Circle through points of intersection of given two circles is

$$(x^2 + y^2 + 13x - 3y) + \lambda(2x^2 + 2y^2 + 4x - 7y - 25) = 0$$

$$\Rightarrow \quad (1 + 2\lambda)x^2 + (1 + 2\lambda)y^2 + (13 + 4\lambda)x + (-3 - 7\lambda)y - 25\lambda = 0$$

As it passes through (1, 1),

$\therefore\quad 1 + 2\lambda + 1 + 2\lambda + 13 + 4\lambda - 3 - 7\lambda - 25\lambda = 0$

$\Rightarrow\quad -24\lambda + 12 = 0 \quad \Rightarrow \quad \lambda = 1/2$

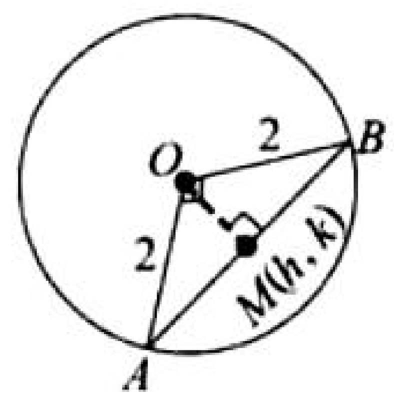
$\therefore$ Required circle is

$$2x^2 + 2y^2 + 15x - \frac{13y}{2} - \frac{25}{2} = 0$$

or $\quad 4x^2 + 4y^2 + 30x - 13y - 25 = 0$

4. c. Let AB be the chord with midpoint $M(h, k)$.

As $\angle AOB = 90°$,



$AB = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\therefore AM = \sqrt{2}$

By the property of right-angled triangles,

$AM = MB = OM$

$\therefore OM = \sqrt{2}$ or $h^2 + k^2 = 2$

Therefore, the locus of (h, k) is $x^2 + y^2 = 2$.

5. a. Let the equation of the circle through (a, b) be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

So, $a^2 + b^2 + 2ag + 2bf + c = 0$ (ii)

Since circle (i) cuts $x^2 + y^2 = k^2$ orthogonally, we have

$2g(0) + 2f(0) = c - k^2$ or $c = k^2$

Putting $c = k^2$ in (ii), we get

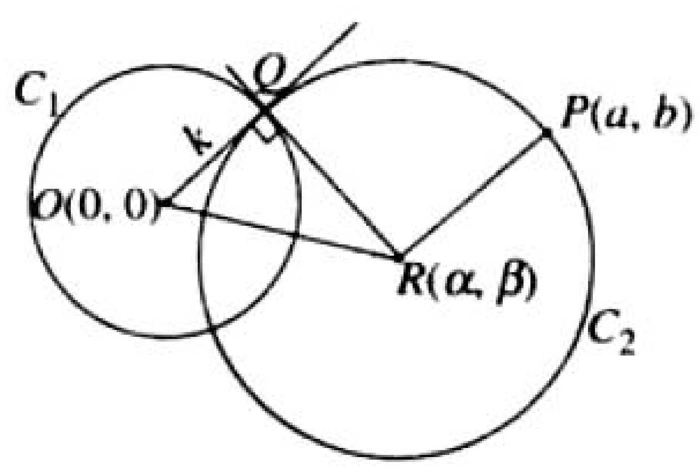
$2ag + 2bf + (a^2 + b^2 + k^2) = 0$

So, the locus of the center $(-g, -f)$ is

$-2ax - 2by + (a^2 + b^2 + k^2) = 0$

or $2ax + 2by - (a^2 + b^2 + k^2) = 0$

Alternative Method:



As shown in the figure, circle C_2 intersect the circle $C_1 : x^2 + y^2 = k^2$ orthogonally at point Q.

Circle C_2 passes through the point $P(a, b)$. We have to find the locus of center $R(\alpha, \beta)$ of circle C_2.

Now from the figure, $RQ = RP =$ radius of the circle C_2

$\therefore\quad RP^2 = RQ^2$

$\therefore\quad RP^2 = OR^2 - OQ^2$

$\therefore\quad (\alpha - a)^2 + (\beta - b)^2 = (\alpha^2 + \beta^2) - k^2$

$\therefore\quad 2a\alpha + 2b\beta - (a^2 + b^2 + k^2) = 0$

$\therefore\quad$ locus is $2ax + 2by - (a^2 + b^2 + k^2) = 0$

6. a. If d is the distance between the centers of two circles of radii r_1 and r_2, then they intersect at two distinct point if $|r_1 - r_2| < d < r_1 + r_2$

Here, the radii of two circles are r and 3 and the distance between the centers is 5. Thus,

$|r - 3| < 5 < r + 3$

$\therefore -5 < r - 3 < 5$ and $r > 2$

$\therefore 0 < r < 8$ and $r > 2$

$\therefore 2 < r < 8$

7. c. The given diameter are

$2x - 3y = 5$ (i)

and $3x - 4y = 7$ (ii)

Solving (i) and (ii), we get

$x = 1, y = -1$

Thus $(1, -1)$ is the center. Now,

Area of circle $= \pi r^2 = 154$

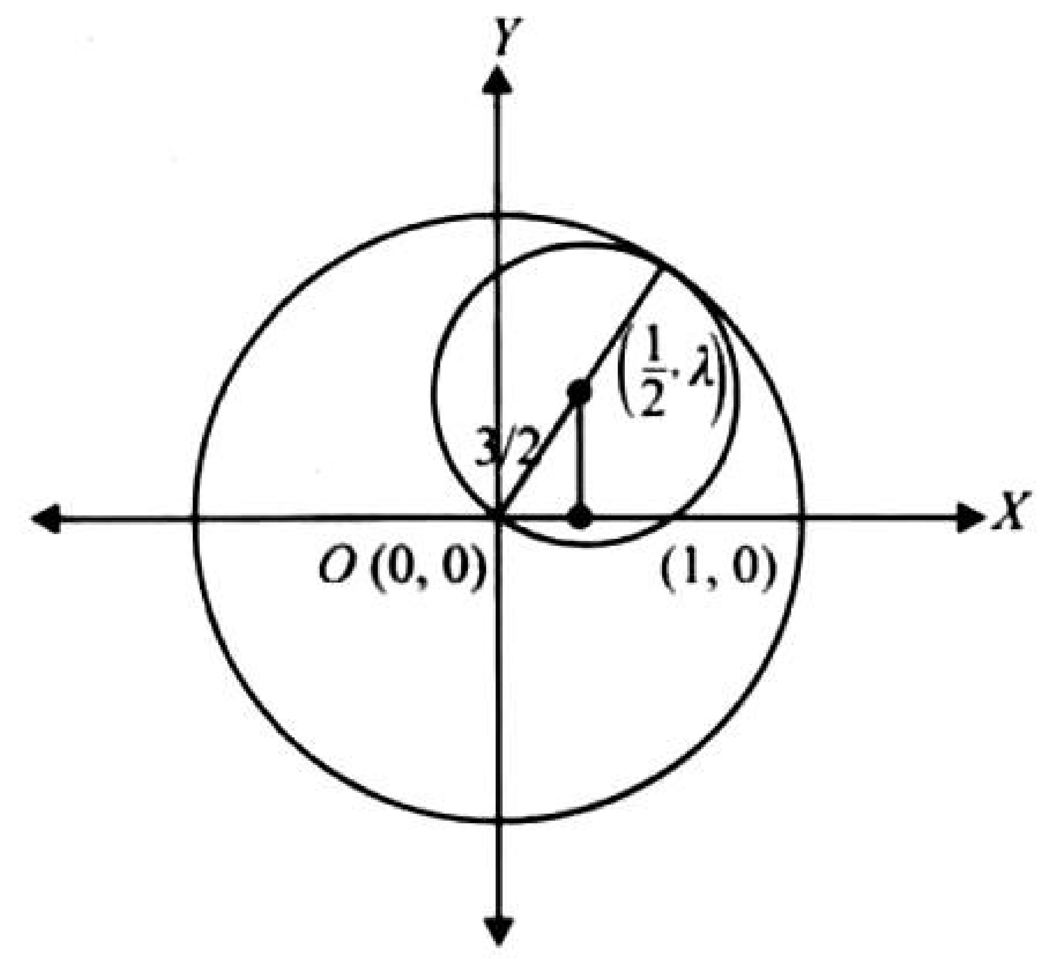
$\therefore r^2 = \frac{154}{22} \times 7 = 49$

Hence, the equation of the circle is

$(x - 1)^2 + (y + 1)^2 = 49$

or $x^2 + y^2 - 2x + 2y = 47$

8. d.



From the diagram,

$$\sqrt{\left(\frac{1}{2}\right)^2 + \lambda^2} = \frac{3}{2} \text{ or } \lambda = \pm\sqrt{2}$$

Hence, the centers of the circle are $(1/2, \pm\sqrt{2})$.

9. d. Let the center of the circle be (h, k).

Since the circle touches the axis of y, its radius is h.

The radius of the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ is 2 and it has its center at (3, 3).

Since the two circles touch each other externally,

Distance between the centers = Sum of the radii

or $\sqrt{(h-3)^2 + (k-3)^2} = |h + 2|$

or $k^2 - 10h - 6k + 14 = 0$

Hence, the locus of (h, k) is $y^2 - 10x - 6y + 14 = 0$.

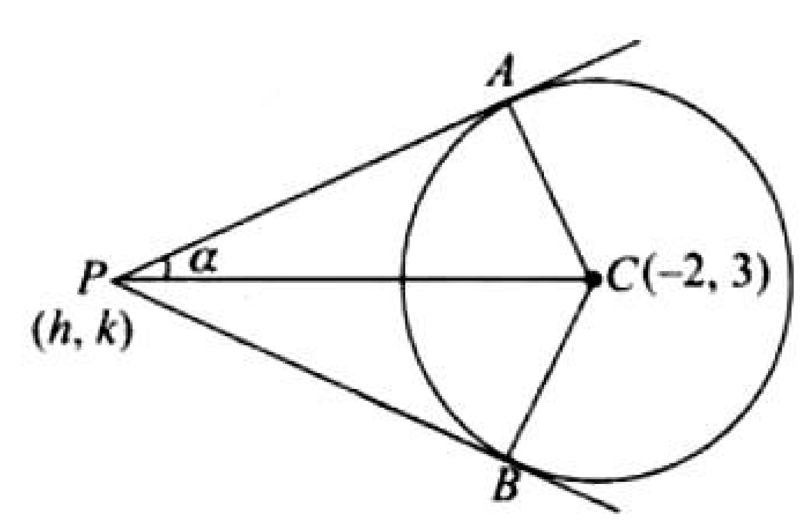
10. d. The center of the circle

$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$

is $C(-2, 3)$ and its radius is

$\sqrt{2^2 + (-3)^2 - 9\sin^2\alpha - 13\cos^2\alpha}$

$\sqrt{4 + 9 - 9\sin^2\alpha - 13\cos^2\alpha} = |2\sin\alpha|$



Let $P(h, k)$ be any point on the locus. Then $\angle APC = \alpha$.

From the diagram,

$$\sin \alpha = \frac{AC}{PC} = \frac{2 \sin \alpha}{\sqrt{(h+2)^2 + (k-3)^2}}$$

or $(h + 2)^2 + (k - 3)^2 = 4$

or $h^2 + k^2 + 4h - 6k + 9 = 0$

Thus, the required equation of the locus is

$x^2 + y^2 + 4x - 6y + 9 = 0$

11. **d.** Let $B(h, 0)$ be the midpoint of the chord drawn from point $A(p, q)$.

Also, the center is $C(p/2, q/2)$.

Then, we have $BC \perp AB$. Therefore,

$$\frac{(q/2) - 0}{(p/2) - h}\left(\frac{q - 0}{p - h}\right) = -1$$

$$\therefore \quad \left(\frac{q}{p - 2h}\right)\left(\frac{q - 0}{p - h}\right) = -1$$

$$\therefore \quad 2h^2 - 3ph + p^2 + q^2 = 0$$

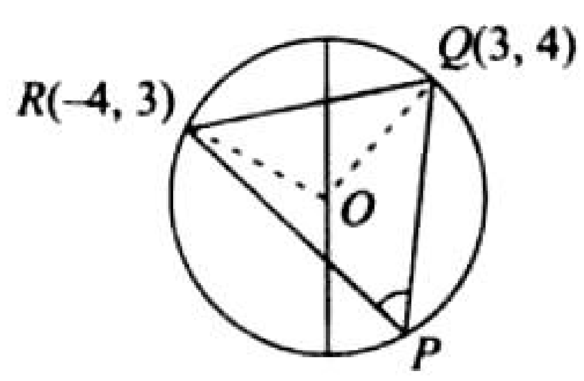
Since two such chords exist, the above equation must have two distinct real roots, i.e.,

Discriminant > 0

$\therefore 9p^2 - 8(p^2 + q^2) > 0$

or $p^2 > 8q^2$

12. **c.** We know that



$$\angle QPR = \frac{1}{2} \angle QOR$$

O being the center $(0, 0)$ of the given circle $x^2 + y^2 = 25$.

Let m_1 = Slope of $OQ = \frac{4}{3}$

and m_2 = Slope of $OR = -\frac{3}{4}$

As $m_1 m_2 = -1$, we have

$$\angle QOR = \frac{\pi}{2} \text{ or } \angle QPR = \frac{\pi}{4}$$

13. **a.** Given circles are

$x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$

We have $g = 1, f = k, c = 6$ and $g' = 0, f' = k$ and $c' = k$

Circles intersect orthogonally

$\therefore 2gg' + 2ff' = c + c'$

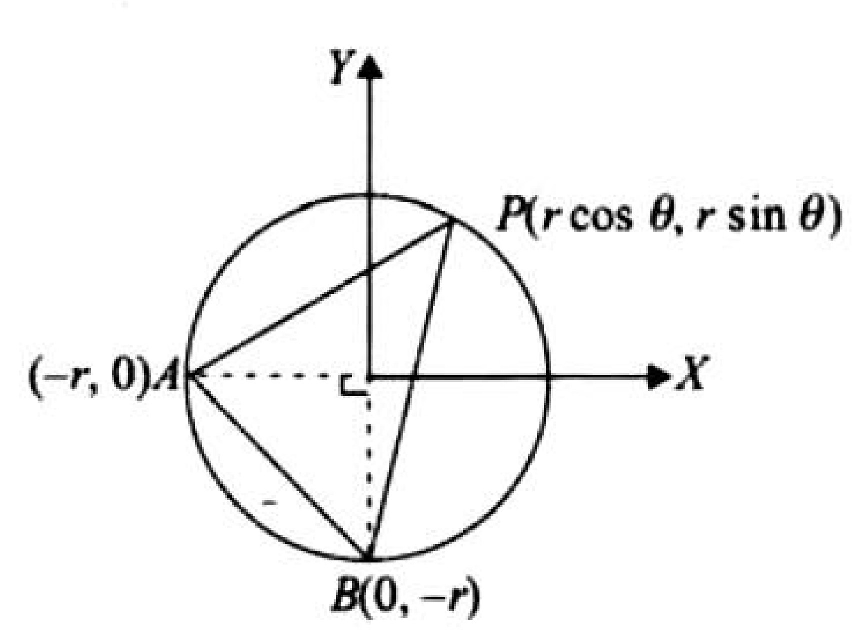
or $2 \times 1 \times 0 + 2 \cdot k \cdot k = 6 + k$

or $2k^2 - k - 6 = 0$

or $(2k + 3)(k - 2) = 0$

$\therefore k = 2, -\frac{3}{2}$

14. **b.** $x^2 + y^2 = r^2$ is a circle with center at $(0, 0)$ and radius r units.



Any arbitrary point P on it is $(r \cos \theta, r \sin \theta)$ choosing A and B as $(-r, 0)$ and $(0, -r)$, respectively.

For the locus of the centroid of ΔABP,

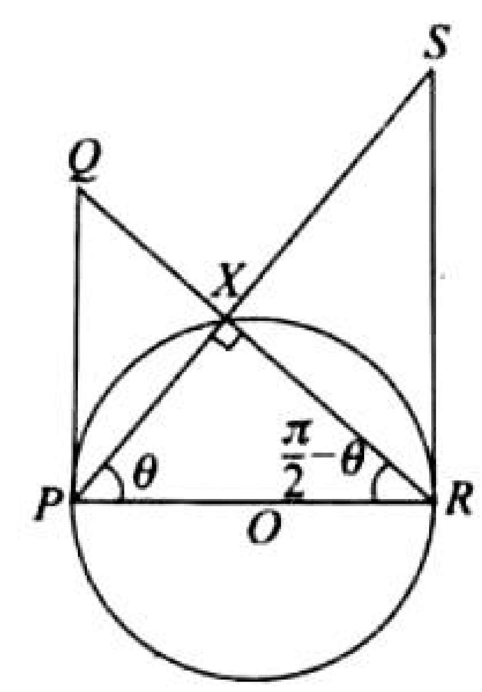
$$\left(\frac{r \cos \theta - r}{3}, \frac{r \sin \theta - r}{3}\right) \equiv (x, y)$$

$\therefore r \cos \theta - r = 3x$ and $r \sin \theta - r = 3y$

or $r \cos \theta = 3x + r$ and $r \sin \theta = 3y + r$

Squaring and adding, we get $(3x + r)^2 + (3y + r)^2 = r^2$, which is a circle.

15. **a.** From the figure, we have



$$\frac{PQ}{PR} = \tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

and $\frac{RS}{PR} = \tan \theta$

$$\therefore \frac{PQ}{PR} \cdot \frac{RS}{PR} = 1$$

or $(PR)^2 = PQ \cdot RS$

or $(2r)^2 = PQ \cdot RS$

or $2r = \sqrt{PQ \cdot RS}$

16. **c.** The line $5x - 2y + 6 = 0$ is interested by a tangent at P to the circle $x^2 + y^2 + 6x + 6y - 2 = 0$ on the y-axis at $Q(0, 3)$.

In other words, the tangent passes through $(0, 3)$. Therefore,

PQ = Length of tangent to circle from $(0, 3)$

$= \sqrt{0 + 9 + 0 + 18 - 2}$

$= \sqrt{25} = 5$

17. **a.** $x^2 - 8x + 12 = 0$ or $(x - 6)(x - 2) = 0$

$y^2 - 14y + 45 = 0$ or $(y - 5)(y - 9) = 0$

Thus, the sides of square are

$x = 2, x = 6, y = 5, y = 9$

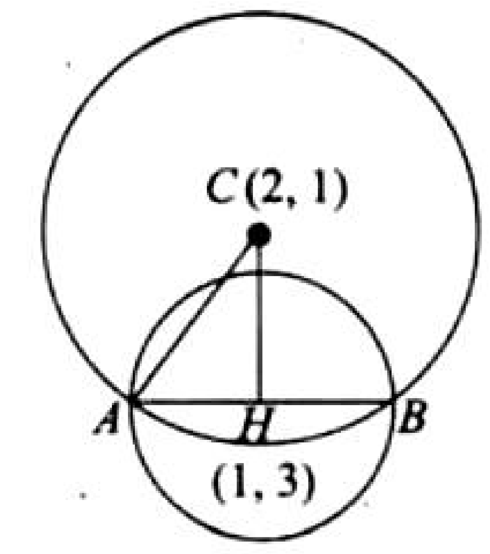
Then the center of the circle inscribed in the square will be

$$\left(\frac{2 + 6}{2}, \frac{5 + 9}{2}\right) \equiv (4, 7)$$

18. **c.** The given circle is $x^2 + y^2 - 2x - 6y + 6 = 0$.

Its center is $H(1, 3)$ and radius is 2.

$AH = 2$

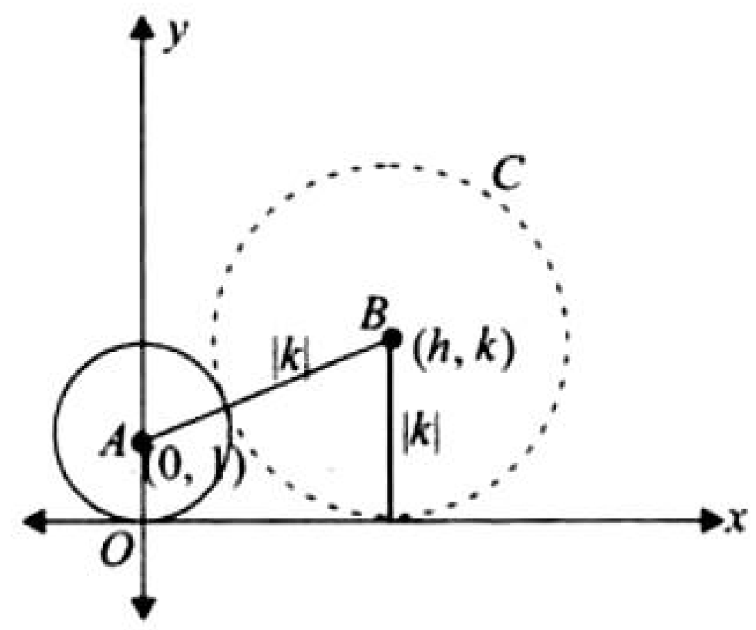


Radius of the required circle = AC

$$= \sqrt{AH^2 + CH^2} = \sqrt{2^2 + 5} = 3$$

19. **d.** Let the center of the circle C be (h, k).

As this circle touches the axis of x, its radius is $|k|$.



Also, it touches the given circle $x^2 + (y - 1)^2 = 1$, with center (0, 1) and radius 1, externally. Therefore,

Distance between centers = Sum of radii

or $\sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$

or $h^2 + k^2 - 2k + 1 = 1 + 2|k| + k^2$

or $h^2 = 2k + 2|k|$

Therefore, the locus of (h, k) is $x^2 = 2y + 2|y|$.

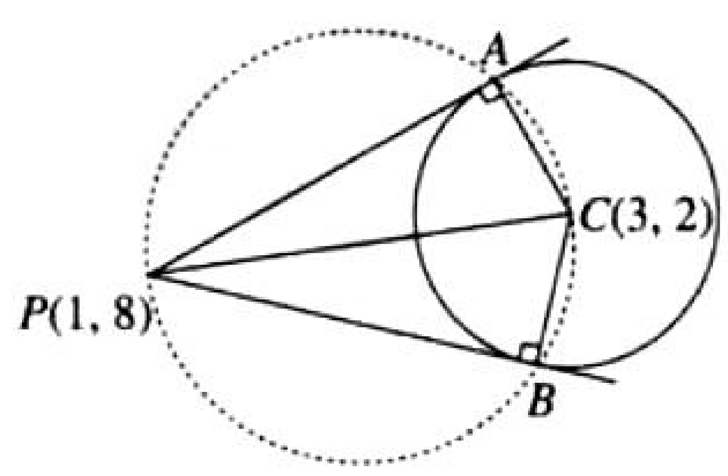
Now, if $y > 0$, it becomes $x^2 = 4y$.

Also, if $y \le 0$, it becomes $x = 0$.

Combining the two, the required locus is

$\{(x, y): x^2 = 4y\} \cup \{(0, y): y \le 0\}$

20. b.



The center of the circle is $C(3, 2)$.

Since CA and CB are perpendicular to PA and PB, CP is the diameter of the circumcircle of triangle PAB. Its equation is

$(x - 3)(x - 1) + (y - 2)(y - 8) = 0$

or $x^2 + y^2 - 4x - 10y + 19 = 0$

21. d. The circle touching the y-axis at (0, 2) is

$(x - 0)^2 + (y - 2)^2 + \lambda x = 0$

It passes through (–1, 0). Therefore,

$1 + 4 - \lambda = 0$ or $\lambda = 5$

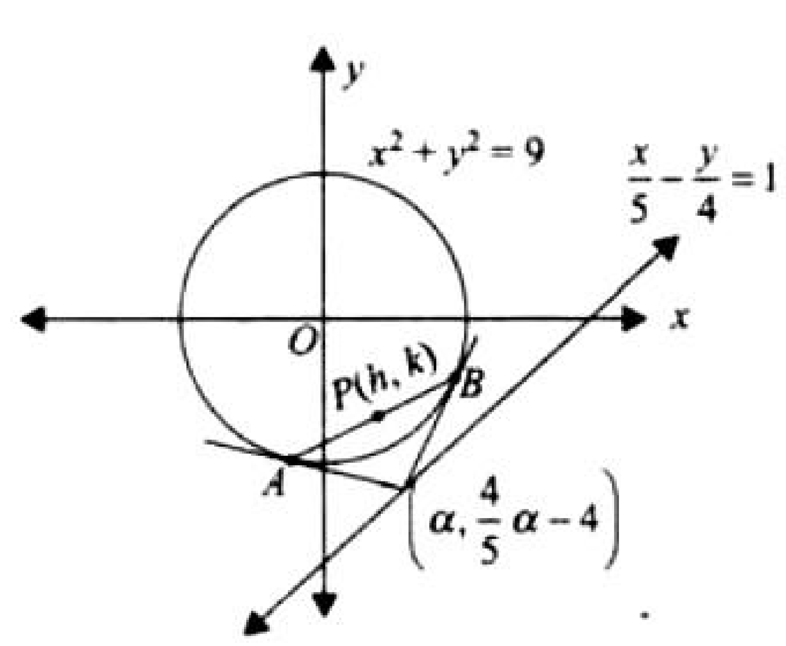
$\therefore$ Equation of circle is

$x^2 + y^2 + 5x - 4y + 4 = 0$

Putting $y = 0$, we get $x = -1, -4$.

Therefore, the circle passes through (–4, 0).

22. a.



The equation of the chord AB bisected at $P(h, k)$ is

$hx + ky = h^2 + k^2$ (i)

Let any point on the given line be

$\left(\alpha, \frac{4}{5}\alpha - 4\right)$

The equation of the chord of contact AB is

$\alpha x + \left(\frac{4}{5}\alpha - 4\right)y = 9$ (ii)

Comparing (i) and (ii), we get

$\frac{h}{\alpha} = \frac{k}{(4/5)\alpha - 4} = \frac{h^2 + k^2}{9}$

$\therefore \alpha = \frac{20h}{4h - 5k}$ (From 1st and 2nd ratio)

$\therefore \frac{h(4h - 5k)}{20h} = \frac{h^2 + k^2}{9}$

or $20(h^2 + k^2) = 9(4h - 5k)$

or $20(x^2 + y^2) - 36x + 45y = 0$

Multiple Correct Answers Type

1. a., c. The equation of any line through the origin (0, 0) is

$y = mx$ (i)

If line (i) is tangent to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, then the length of perpendicular from center (r, h) on (i) is equal to the radius of the circle, i.e.,

$\frac{|mr - h|}{\sqrt{m^2 + 1}} = \sqrt{r^2 + h^2 - h^2}$

$(mr - h)^2 = (m^2 + 1)r^2$

$0 \cdot m^2 + (2hr)m + (r^2 - h^2) = 0$

$m = \infty, \frac{h^2 - r^2}{2hr}$

Substituting these values in (i), we get the tangents as

$x = 0$ and $(h^2 - r^2)x - 2rhy = 0$

2. a., b., c., d. Putting $y = c^2/x$ in $x^2 + y^2 = a^2$, we get

$x^2 + \frac{c^4}{x^2} = a^2$

or $x^4 - a^2x^2 + c^4 = 0$ (i)

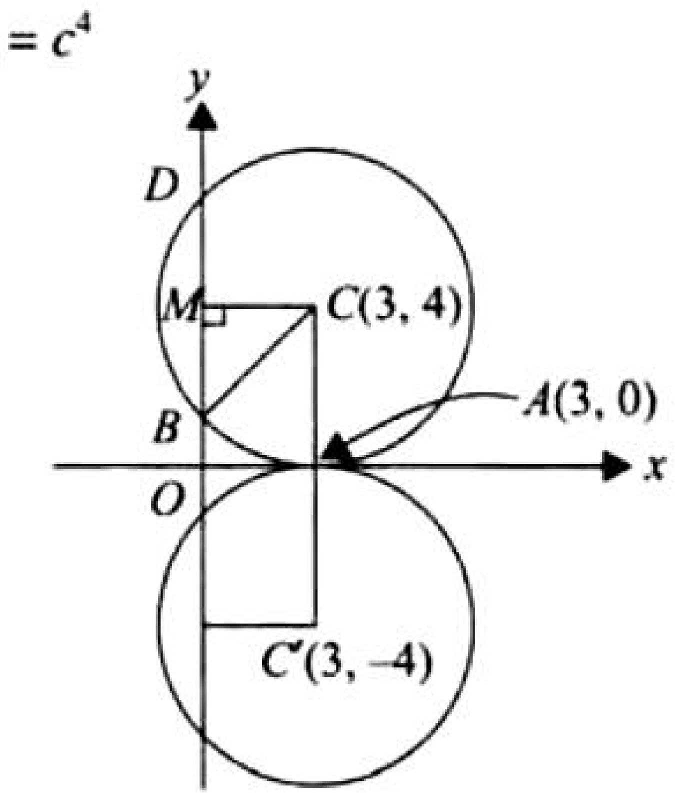
As x_1, x_2, x_3, and x_4 are the roots of (i), we have

$x_1 + x_2 + x_3 + x_4 = 0$ and $x_1x_2x_3x_4 = c^4$

Similarly, forming equation in y, we get

$y_1 + y_2 + y_3 + y_4 = 0$ and $y_1y_2y_3y_4 = c^4$

3. a., c. The figure shows circles touching the x-axis at $A(3, 0)$ and having intercept $BD = 2\sqrt{7}$ on the y-axis.



From the figure, the abscissa of center is 3.

Also, BC = Radius of the circle

$= \sqrt{CM^2 + BM^2} = \sqrt{9 + 7} = 4$.

Therefore, $AC = 4$

So, the centers of the circles are $C(3, 4)$ and $C'(3, -4)$.

Hence, the equations of circles are

$(x-3)^2 + (y \pm 4)^2 = 16$

or $x^2 + y^2 - 6x \pm 8y + 9 = 0$

4. b., c.

Given circles

$S_1 : x^2 + y^2 - 2x - 15 = 0$

and $S_2 : x^2 + y^2 - 1 = 0$

Center of the circle which intersects above two circles orthogonally lies on the radical axis of the circles which is $S_1 - S_2 = 0$ or $x + 7 = 0$

Let the centre of the required circle be $C(-7, k)$.

Circle passes through the point $A(0, 1)$.

$\therefore$ radius, $r = \sqrt{7^2 + (k-1)^2}$

Also, radius = length of the tangent from C to the any one of the given circles.

$\therefore \quad r = \sqrt{7^2 + k^2 - 1}$

On comparing, we get $7^2 + (k-1)^2 = 7^2 + k^2 - 1$

or $\quad -2k + 1 = -1$

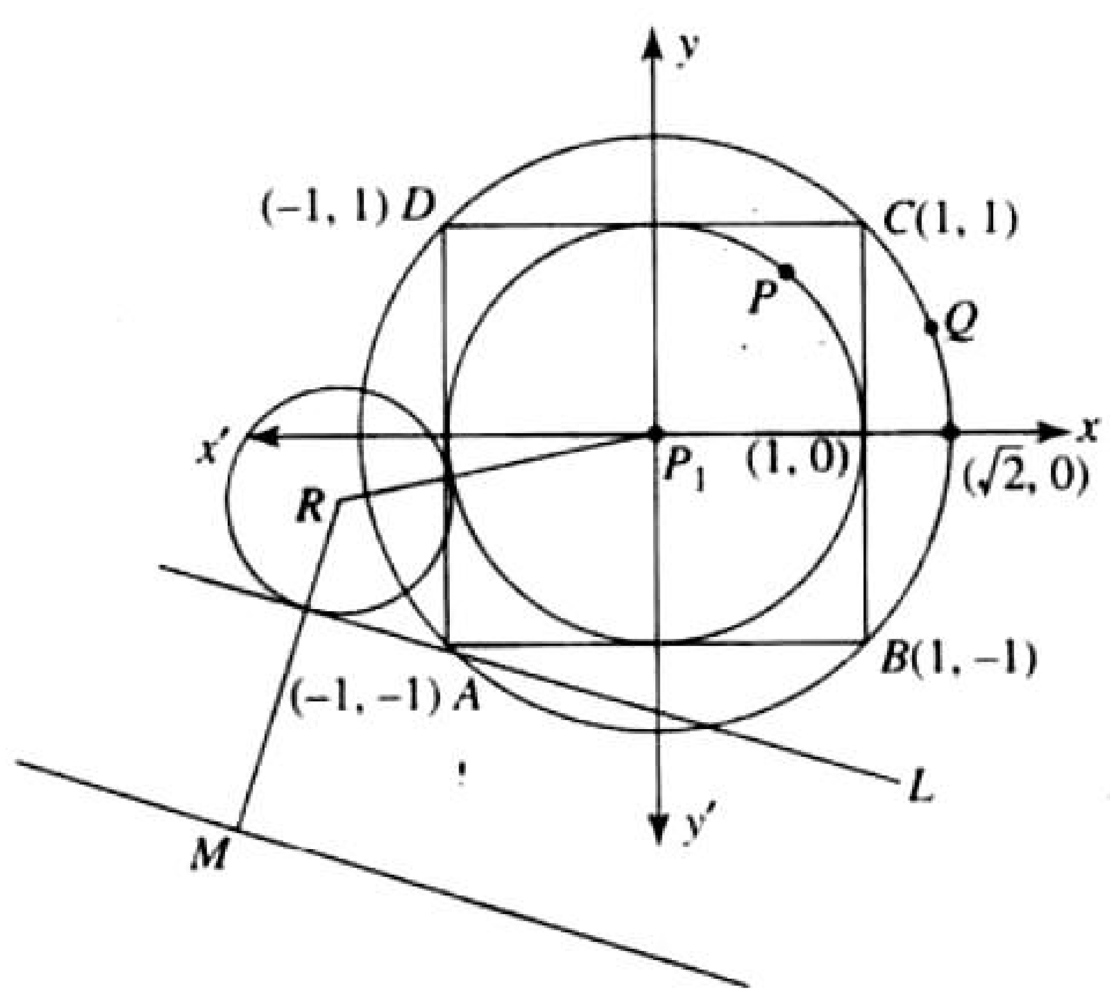
or $\quad k = 1$

$\therefore \quad r = 7$

Linked Comprehension Type

For Problems 1–3

1. c.



Consider square $ABCD$ with coordinates as shown in the figure.

Clearly, circle inscribed in square is $C_1: x^2 + y^2 = 1$

and circle circumscribing the square is $C_2 : x^2 + y^2 = 2$

Let $P(\cos\theta, \sin\theta)$, $Q(\sqrt{2}\cos\alpha, \sqrt{2}\sin\alpha)$

$\Rightarrow \quad PA^2 + PB^2 + PC^2 + PD^2$

$= 2(\cos\theta - 1)^2 + 2(\sin\theta - 1)^2 + 2(\cos\theta + 1)^2 + 2(\sin\theta + 1)^2$

$= 2[2(1 + \cos^2\theta + 1 + \sin^2\theta)] = 4 \times 3 = 12$ units

Similarly, $QA^2 + QB^2 + QC^2 + QD^2$

$= 2[(\sqrt{2}\cos\alpha - 1)^2 + (\sqrt{2}\cos\alpha + 1)^2 + (\sqrt{2}\sin\alpha - 1)^2 + (\sqrt{2}\sin\alpha + 1)^2]$

$= 2[2(2\cos^2\alpha + 1 + 2\sin^2\alpha + 1)] = 4 \times 4 = 16$

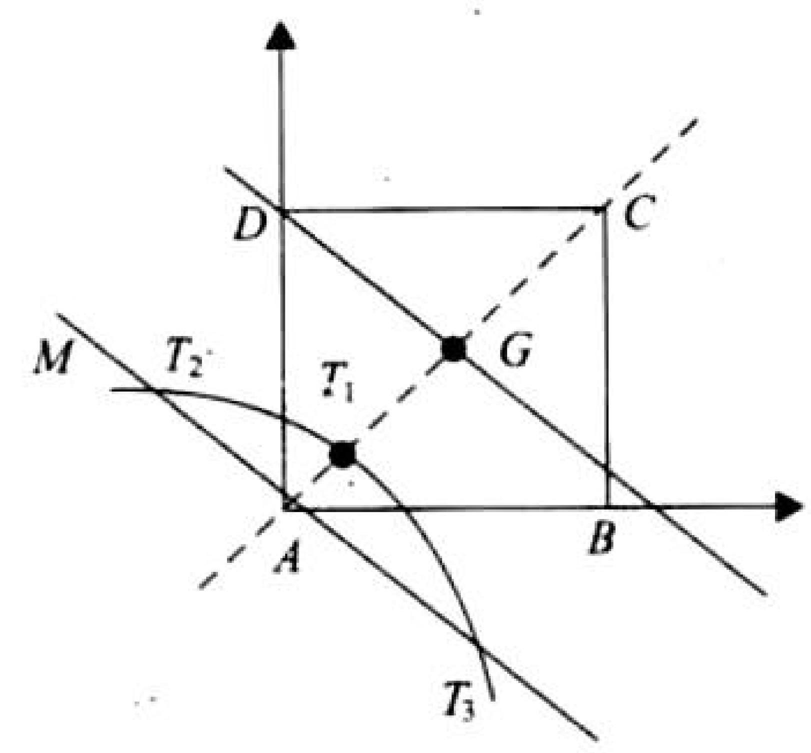
$$\therefore \quad \frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2} = \frac{12}{16} = \frac{3}{4} = 0.75$$

2. c. Let R be the center of the required circle.

Now, draw a line parallel to L at a distance of r_1 (radius of C_1) from it.

Now, $RP_1 = RM$. Therefore, R lies on a parabola.

3. c.



Since, $AG = \sqrt{2}$, we have $AT_1 = T_1G$. (As A is the focus, T_1 is the vertex and BD is the directrix of parabola.)

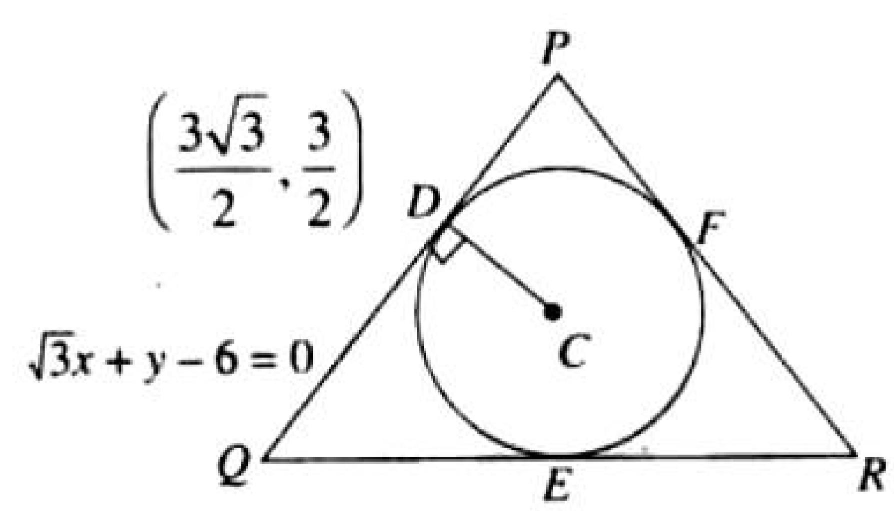
Also, T_2T_3 is a latus rectum. Therefore,

$T_2T_3 = 4 \times \frac{1}{\sqrt{2}}$

$\therefore$ Area of $\Delta T_1T_2T_3 = \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 1$

For Problems 4–6

4. d.



Equation of PQ is $L_1 : \sqrt{3}x + y - 6 = 0$ ...(1)

Slope of PQ is $-\sqrt{3}$

$\therefore \quad$ slope of CD is $\frac{1}{\sqrt{3}}$.

$\therefore \quad$ Equation of line CD in parametric from is

$$\frac{x - \frac{3\sqrt{3}}{2}}{\cos 30°} = \frac{y - \frac{3}{2}}{\sin 30°} = r$$

or $$\frac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{y - \frac{3}{2}}{\frac{1}{2}} = r,$$

For centre C, $r = \pm 1$

Two possible co-ordinates of centre are $(2\sqrt{3}, 2)$, $(\sqrt{3}, 1)$.

According to the question (center) lies on the same side where origin lies with respect to line PQ.

Now $L_1(0, 0) = -6 < 0$

And $L_1(\sqrt{3}, 1) = 3 + 1 - 6 < 0$

So Centre C must be $(\sqrt{3}, 1)$

Hence, equation of the circle is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

5. a. By simple geometry $PD = \sqrt{3}$ (ΔPQR is equilateral)

Equation of PQ in parametric form is

$$\frac{x - \frac{3\sqrt{3}}{2}}{\cos 120°} = \frac{y - \frac{3}{2}}{\sin 120°} = r$$

or $$\frac{x - \frac{3\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{y - \frac{3}{2}}{\frac{\sqrt{3}}{2}} = r$$

Now points at distance $\sqrt{3}$ from point D on line PQ are given by

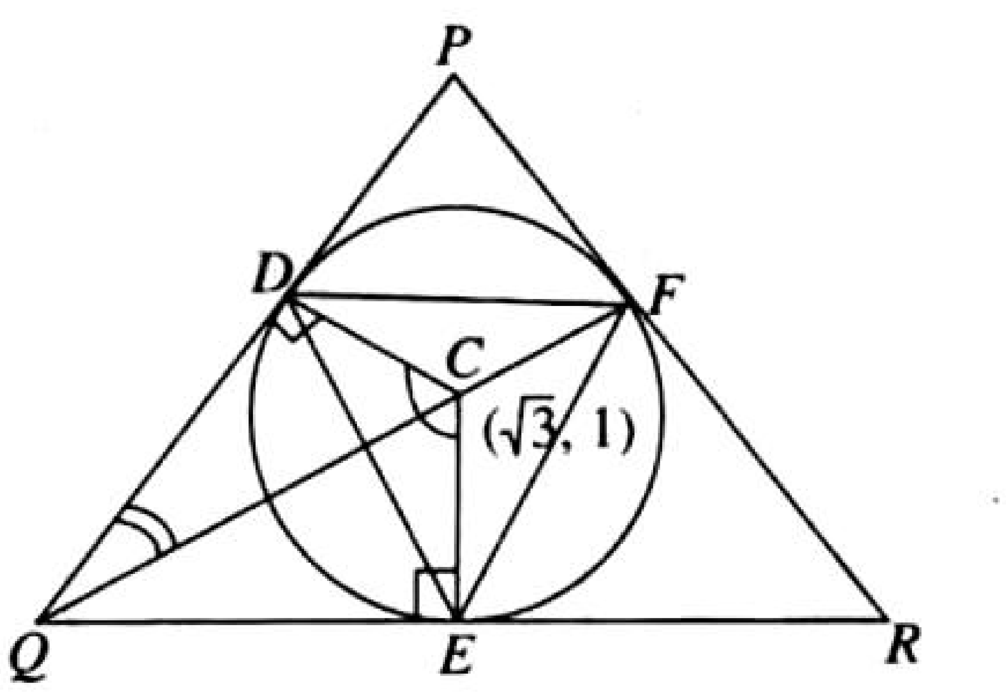
$$\left(\frac{3\sqrt{3}}{2} \pm \sqrt{3}\left(-\frac{1}{2}\right), \frac{3}{2} \pm \sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\right) \text{ or } (2\sqrt{3}, 0) \text{ and } (\sqrt{3}, 3)$$

Point C divides the join of P and E in the ratio 2 : 1.

Similarly, C divides join of Q and F in the ratio 2 : 1.

Thus, co-ordinates of E and F are $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ and $(\sqrt{3}, 0)$.

6. d.



Equation of line PR which is parallel to DE and passes through F is $(y - 0) = 0\,(x - \sqrt{3}) \Rightarrow y = 0$.

Similarly, equation of line QR which is parallel to DF and passes through the point E is

$$\left(y - \frac{3}{2}\right) = \left(\frac{\frac{3}{2} - 0}{\frac{3\sqrt{3}}{2} - \sqrt{3}}\right)\left(x - \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad y = \sqrt{3}\,x$$

For Problems 7–8

7. a. The equation of tangent at $P(\sqrt{3}, 1)$ is

$\sqrt{3}x + y = 4$

The slope of line perpendicular to the above tangent is $1/\sqrt{3}$.

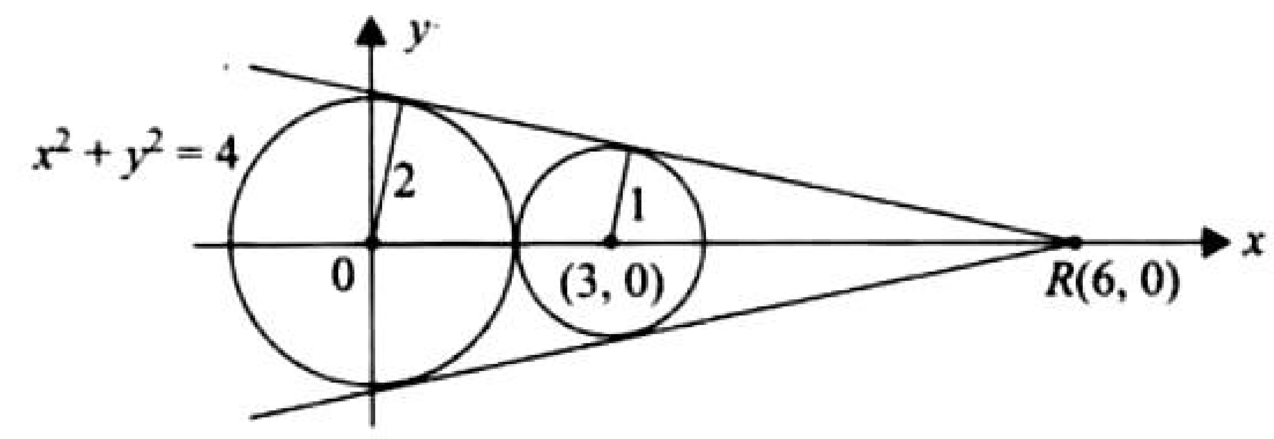
So, the equations of tangents with slope $1/\sqrt{3}$ to $(x - 3)^2 + y^2 = 1$ will be

$$y = \frac{1}{\sqrt{3}}(x - 3) \pm 1\sqrt{1 + \frac{1}{3}}$$

or $\sqrt{3}\,y = x - 3 \pm (2)$

i.e., $\sqrt{3}\,y = x - 1$ or $\sqrt{3}\,y = x - 5$

8. d. The point of intersection of direct common tangents is (6, 0).



So, let the equation of common tangent be

$y - 0 = m(x - 6)$

As it touches $x^2 + y^2 = 4$, we have

$$\left|\frac{0 - 0 - 6m}{\sqrt{1 + m^2}}\right| = 2$$

or $9m^2 = 1 + m^2$

or $m = \pm\frac{1}{2\sqrt{2}}$

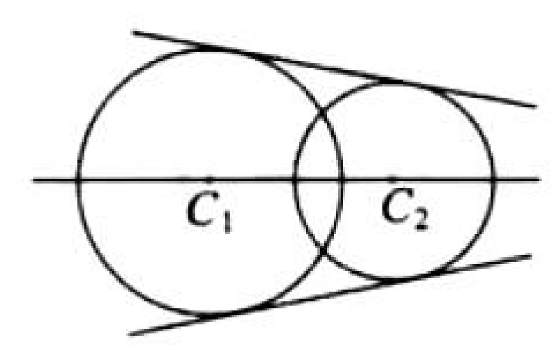
So, the equations of common tangents are

$$y = \frac{1}{2\sqrt{2}}(x - 6),\ y = -\frac{1}{2\sqrt{2}}(x - 6), \text{ and also } x = 2$$

Matching Column Type

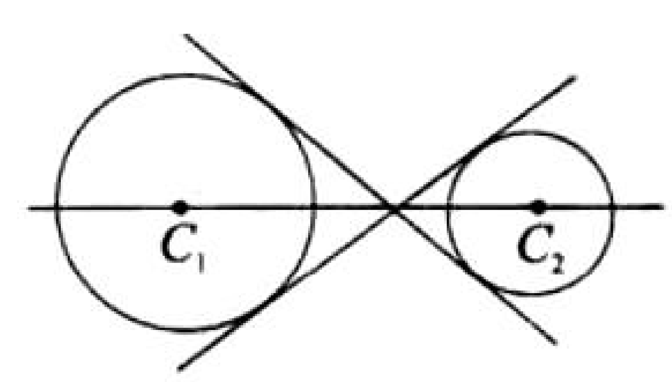
1. (a) – (p), (q); (b) – (p), (q); (c) – (q), (r)

a.



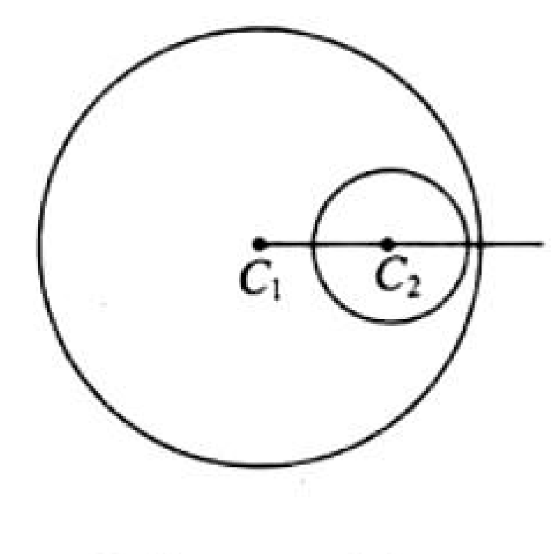
It is clear from the figure that two intersecting circles have a common tangents and a common normal joining the centers.

b.



It is clear from the figure that two intersecting circles have a common tangents and a common normal joining the centers.

c. Two circles, when one is completely inside the other, have a common normal C_1C_2, but no common tangent.



Note: Solutions of the remaining parts are given in their respective chapters.

2. **(p) – (a)**

$$\frac{1}{k^2} = 4\left(1+\frac{h^2}{k^2}\right)$$

$\Rightarrow \quad 1 = 4(k^2 + h^2)$

$\therefore \quad h^2 + k^2 = \left(\frac{1}{2}\right)^2$ which is a circle.

Note: Solutions of the remaining parts are given in their respective chapters.

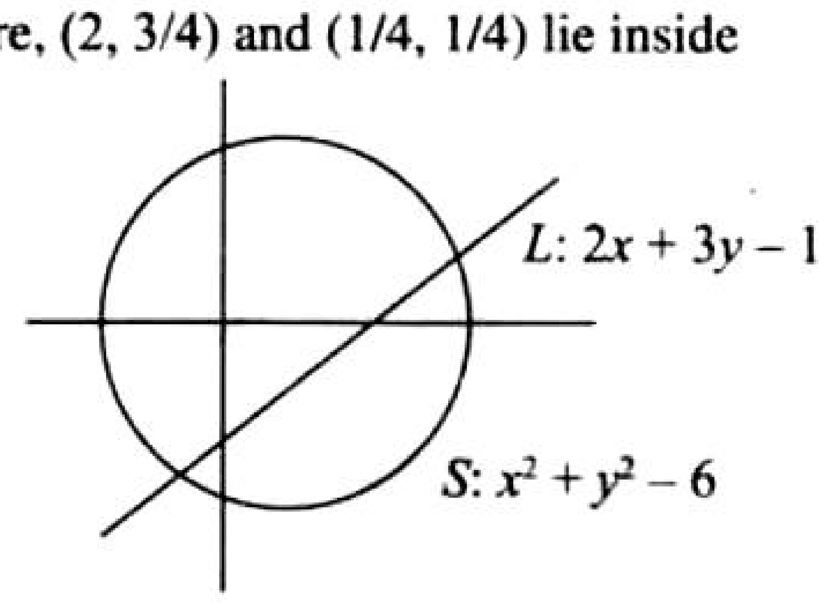
Integer Answer Type

1. **(2)** $L: 2x - 3y - 1$

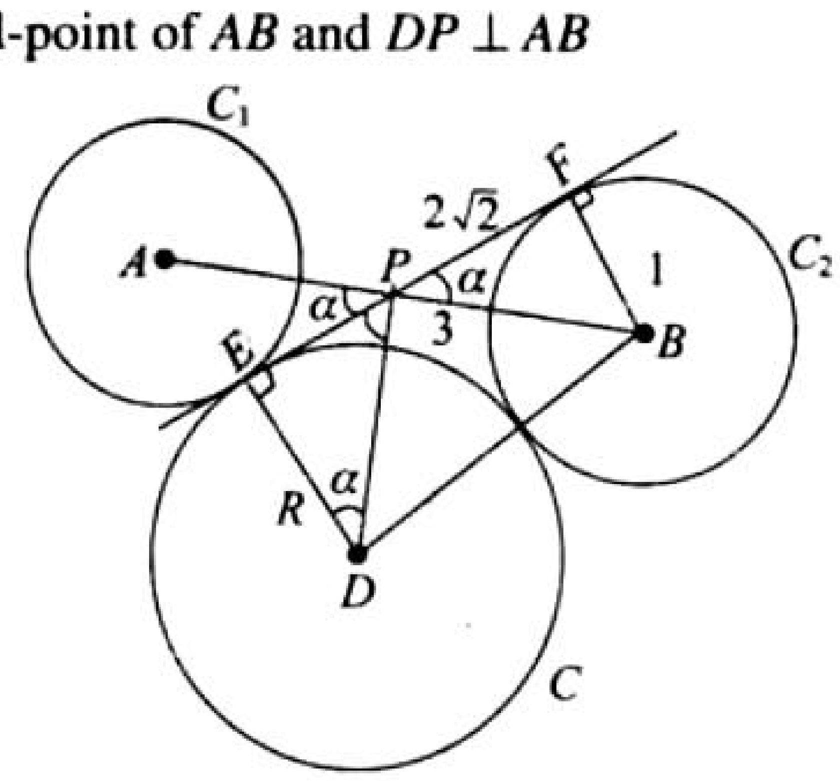
$S: x^2 + y^2 - 6$

If $L_1 > 0$ and $S_1 < 0$, then the point lies in the smaller part.

Therefore, (2, 3/4) and (1/4, 1/4) lie inside



2. P is mid-point of AB and $DP \perp AB$



In ΔPFB $\cos\alpha = \frac{2\sqrt{2}}{3}$

$\therefore \sin\alpha = \frac{1}{3}$

In ΔDEP

$\tan\alpha = \frac{2\sqrt{2}}{R}$

or $\quad R = \frac{2\sqrt{2}}{\tan\alpha} = 8$ units

Assertion-Reasoning Type

1. **a.** The equation of director circle of the given circle $x^2 + y^2 = 169$ is $x^2 + y^2 = 2 \times 169 = 338$.

We know that from every point on a director circle of given circle, the tangents drawn to the given circle are perpendicular to each other.

Here, (17, 7) lies on director circle.

Therefore, the tangents from (17, 7) to the given circle are mutually perpendicular.

2. **c.** Circle $\equiv (x + 3)^2 + (y - 5)^2 = 4$

Distance between L_1 and $L_2 = \frac{6}{\sqrt{13}} <$ Radius

Therefore, statement 2 is false.

But statement 1 is correct.

Fill in the Blanks Type

1. Let given points are $A(x_1, y_1)$ and $B(x_2, y_2)$.

Now, consider locus of point $P(x, y)$ such that $\frac{PA}{PB} = k$

or $\quad PA^2 = k^2PB^2$

or $\quad (x - x_1)^2 + (y - y_1)^2 = k^2[(x - x_2)^2 + (y - y_2)^2]$

Clearly, locus in above equation is circle for any real value k other than '1' as for $k = 1$, we have $PA = PB$. In this case locus of P is perpendicular bisector of line segment AB.

2. Line: $4x - 3y - 10 = 0$ (i)

Circle: $x^2 + y^2 - 2x + 4y - 20 = 0$ (ii)

Solving (i) and (ii), we get

$$\left(\frac{3y+10}{4}\right)^2 + y^2 - 2\left(\frac{3y+10}{4}\right) + 4y - 20 = 0$$

or $9y^2 + 60y + 100 + 16y^2 - 24y - 80 + 64y - 320 = 0$

or $25y^2 + 100y - 300 = 0$

or $y^2 + 4y - 12 = 0$

or $y = 2, -6$

$\therefore x = 4, -2$

Hence, the points are (4, 2) and (–2, –6).

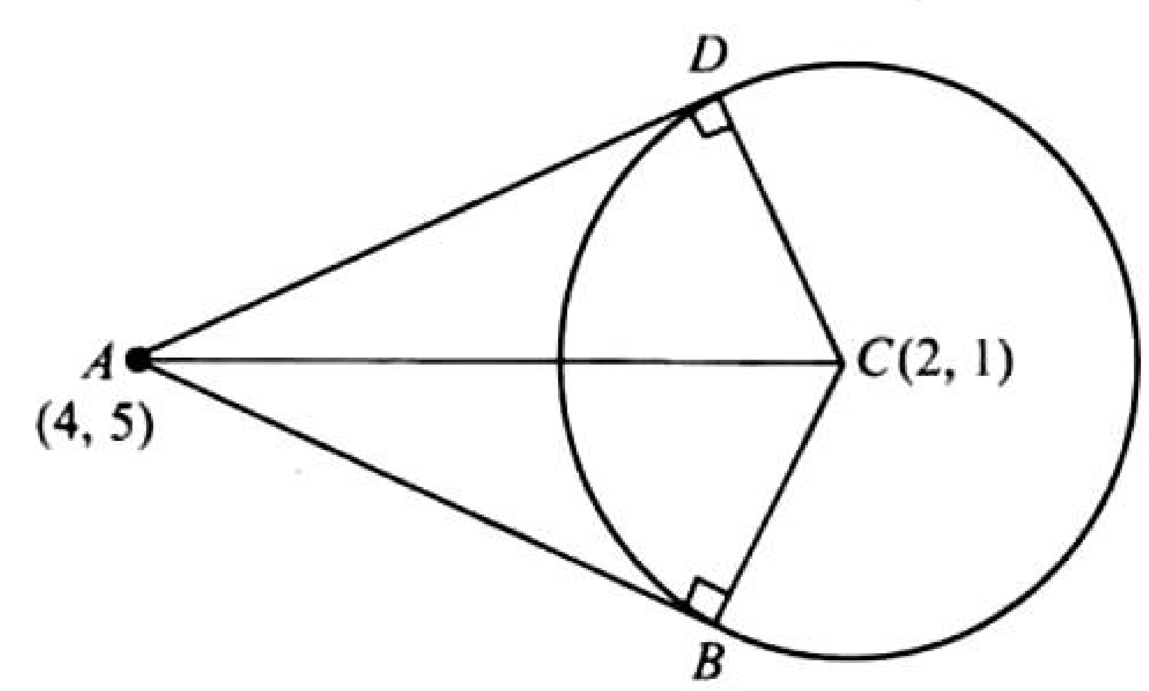
3. Since the given lines $6x - 8y + 8 = 0$ and $6x - 8y - 7 = 0$ are parallel and touching the circle, the diameter of the circle is the distance between parallel lines, i.e.,

$\left|\frac{8+7}{\sqrt{36+64}}\right| = \frac{15}{10} = \frac{3}{2}$

$\therefore$ Radius of circle $= \frac{1}{2}(AB) = \frac{3}{4}$

4. The equation of the circle is $x^2 + y^2 - 4x - 2y - 11 = 0$.

Its center is (2, 1) and radius is $\sqrt{4+1+11} = 4 = BC$.



$AC = 2\sqrt{5}$

$\therefore AB = 2$

$\therefore$ Area of quadrilateral $ABCD = 2 \times$ (Area of ΔABC)

$= 2 \times \frac{1}{2} \times AB \times BC$

$= 2 \times \frac{1}{2} \times 2 \times 4$

$= 8$ sq. units

5. The origin $O(0, 0)$ satisfies the circle.

If the midpoint of the chord from the origin is $P(h, k)$, then P is the midpoint of OR, where $R(2h, 2k)$ lies on the circle. Hence,

$(2h - 1)^2 + (2k)^2 = 1$

or $x^2 + y^2 - x = 0$

6. The equations of the two circles are

$S_1 \equiv x^2 + y^2 - \frac{2}{3}x + 4y - 3 = 0$ (i)

and $S_2 \equiv x^2 + y^2 + 6x + 2y - 15 = 0$ (ii)

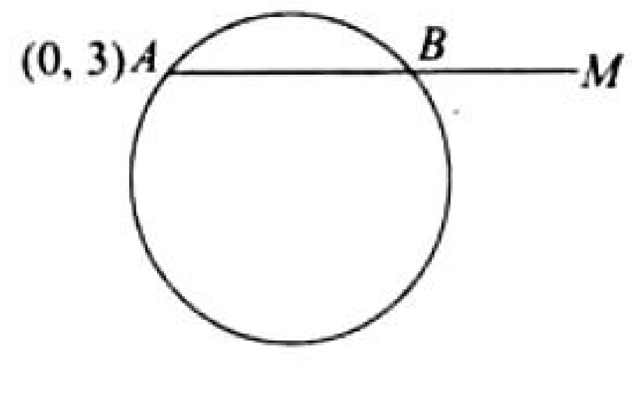
Now, we know that the equation of common chord of two circles $S_1 = 0$ and $S_2 = 0$ is

$S_1 - S_2 = 0$

or $\frac{20x}{3} - 2y - 12 = 0$

or $10x - 3y - 18 = 0$

7. The equation of the circle is



$x^2 + y^2 + 4x - 6y + 9 = 0$ (i)

Given that $AM = 2AB$

or $AB = BM$

Let the coordinates of M be (h, k).

Then B is the midpoint of AM. Therefore,

$$B \equiv \left(\frac{0+h}{2}, \frac{3+k}{2}\right) \equiv \left(\frac{h}{2}, \frac{k+3}{2}\right)$$

As B lies on the circle, we have (i)

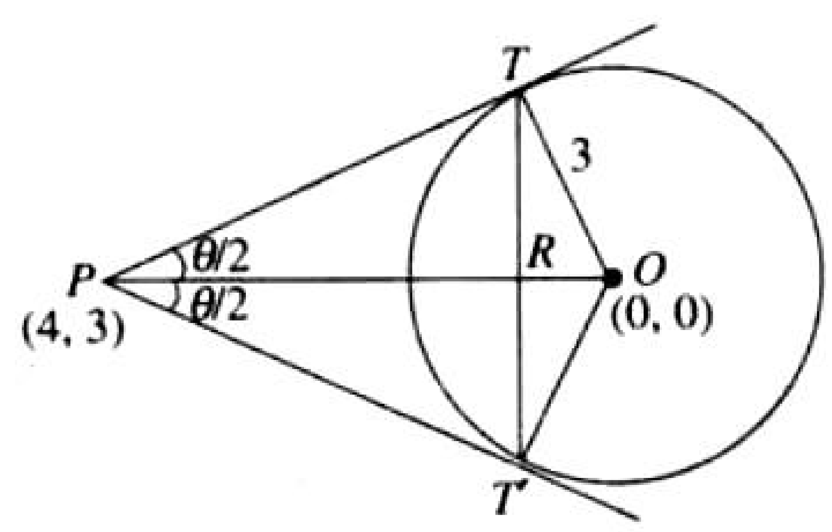
$$\left(\frac{h}{2}\right)^2 + \left(\frac{k+3}{2}\right)^2 + 4\left(\frac{h}{2}\right) - 6\left(\frac{k+3}{2}\right) + 9 = 0$$

or $h^2 + k^2 + 6k + 9 + 8h - 12k - 36 + 36 = 0$

or $h^2 + k^2 + 8h - 6k + 9 = 0$

Therefore, the locus of (h, k) is $x^2 + y^2 + 8x - 6y + 9 = 0$.

8.



Let R be the point of intersection of OP and TT'.

The equation of the chord of contact TT' is

$4x + 3y = 9$ (i)

Now, OR = Length of perpendicular from O to TT'

$$= \left|\frac{4 \times 0 + 3 \times 0 - 9}{\sqrt{4^2 + 3^2}}\right| = \frac{9}{5}$$

OT = Radius of circle = 3

$$\therefore TR = \sqrt{OT^2 - OR^2} = \sqrt{9 - \frac{81}{25}} = \frac{12}{5}$$

Also, $OP = 5$

$$\therefore PR = OP - OR = 5 - \frac{9}{5} = \frac{16}{5}$$

$$\text{Area of } \Delta PTT' = \frac{1}{2} PR \times TT' = \frac{1}{2}\frac{16}{5} \times \frac{24}{5} = \frac{192}{25}$$

Alternative Method:

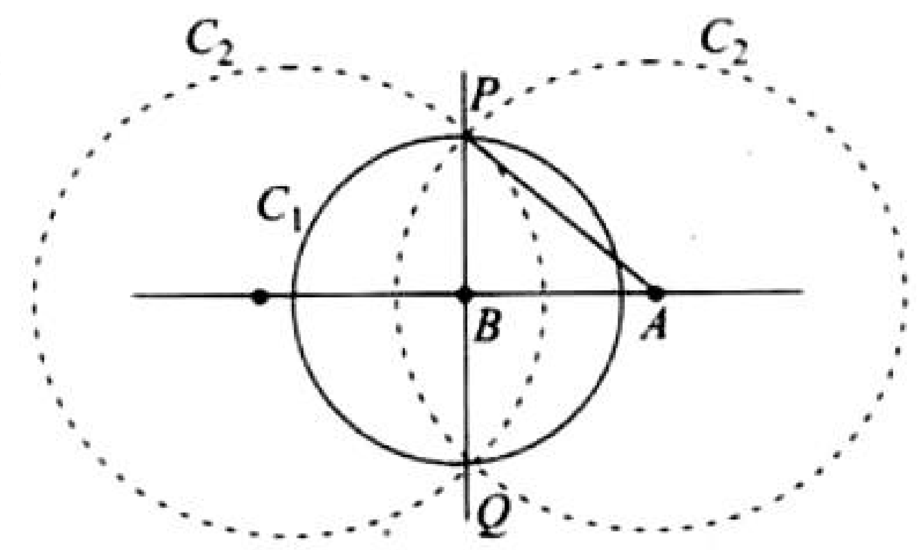
From the figure in ΔPTO, $\sin\frac{\theta}{2} = \frac{OT}{OP} = \frac{3}{5}$

$$\therefore \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$$

$\therefore$ Area of $\Delta PTT'$

$$= \frac{1}{2} PT \times PT' \sin\theta = \frac{1}{2} \times 4 \times 4 \times \frac{24}{25} = \frac{192}{25} \text{ sq. units}$$

9.



Clearly, the diameter of C_1 will be the common chord.

Let the common chord be PQ and the center of C_2 be $A(h, k)$.

We have $AP = 5$, $PB = 4$. Therefore, $AB = 3$ units, where $B \equiv (1, 2)$.

Using the parametric equation of line, we get

$$\frac{h-0}{-3/5} = \frac{k-0}{4/5} = \pm 3$$

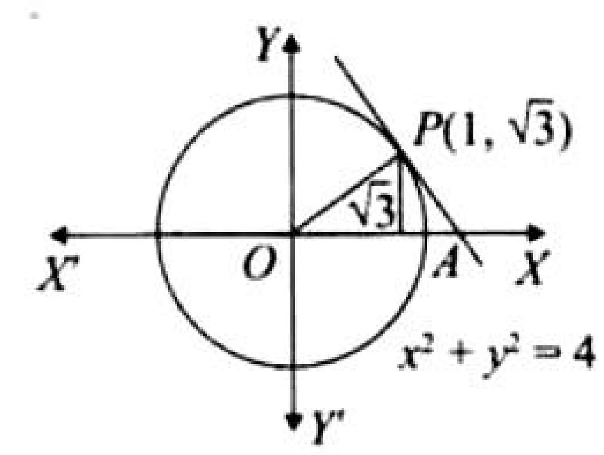
$$\therefore h = \frac{-9}{5}, k = \frac{12}{5}$$

or $h = \frac{9}{5}, k = \frac{-12}{5}$

$\therefore$ Centers are $\left(\pm\frac{9}{5}, \mp\frac{12}{5}\right)$

or $h = -\frac{7}{5}, k = \frac{26}{5}$ or $h = \frac{17}{5}, k = -\frac{6}{5}$

10. The equations of the tangent and the normal to $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ are, respectively, $x + \sqrt{3}y = 4$ and $y = \sqrt{3}x$.

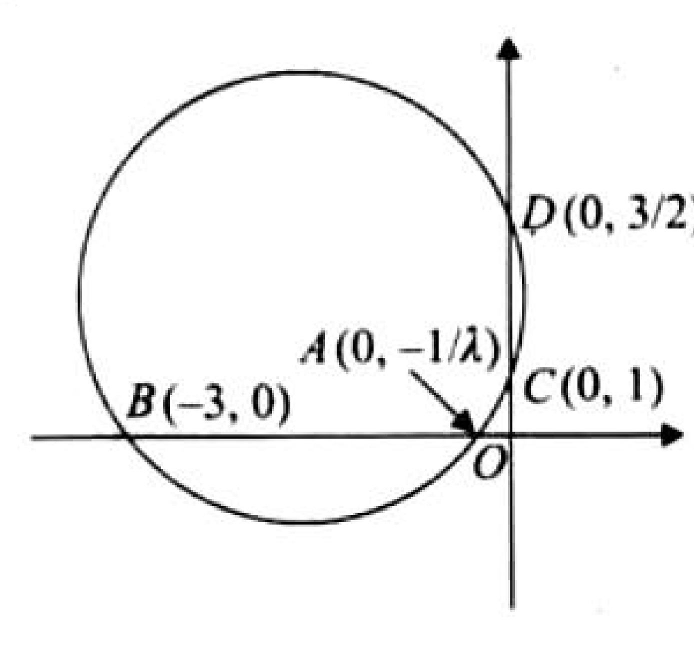


The tangent meets the x-axis at $(4, 0)$. Therefore,

$$\text{Area of } \Delta OAP = \frac{1}{2}(4)\sqrt{3}$$

$$= 2\sqrt{3} \text{ sq. units}$$

11. The given lines are $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$ which meet the x-axis at $A(-1/\lambda, 0)$ and $B(-3, 0)$ and the y-axis at $C(0, 1)$ and $D(0, 3/2)$, respectively.



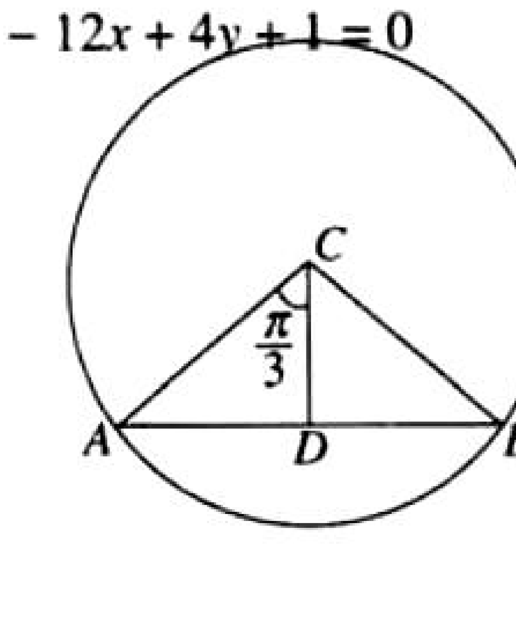
Then, we must have

$OA \times OB = OC \times OD$

or $\left(\frac{1}{\lambda}\right)(3) = 1 \times \frac{3}{2}$

or $\lambda = 2$

12. Given equation of circle is $4x^2 + 4y^2 - 12x + 4y + 1 = 0$



or $\quad x^2 + y^2 - 3x + y + \frac{1}{4} = 0$

Center of the circle is $C\left(\frac{3}{2}, -\frac{1}{2}\right)$.

$$\text{Radius } AC = \sqrt{\frac{9}{4} + \frac{1}{4} - \frac{1}{4}} = \frac{3}{2}$$

Given $\angle ACB = \frac{2\pi}{3}$

$\angle ACD = \frac{\pi}{3}$

Now in ΔACD, $\cos\frac{\pi}{3} = \frac{CD}{AC} = \frac{\sqrt{\left(h-\frac{3}{2}\right)^2 + \left(k+\frac{1}{2}\right)^2}}{\frac{3}{2}}$

$\therefore$ Locus is $\left(x-\frac{3}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{9}{16}$.

13. The points of intersection of the line $y = x$ and the circle $x^2 + y^2 - 2x = 0$ can be obtained by solving these two equations. Solving, we get

$x^2 + x^2 - 2x = 0$

$\therefore x = 0, 1$

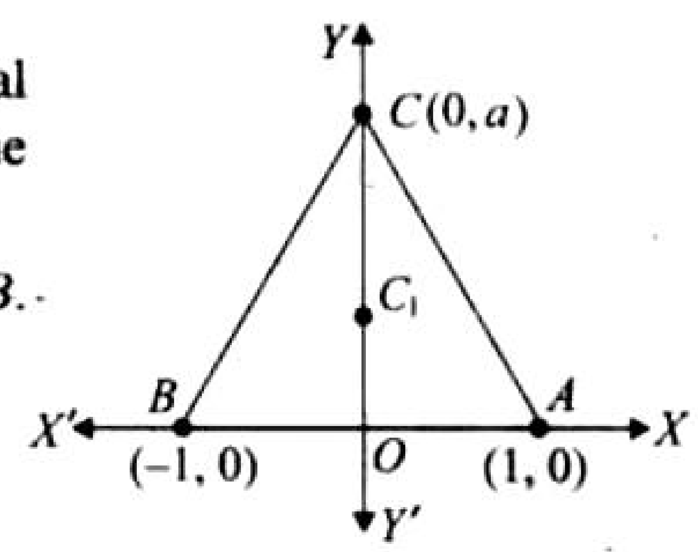
$\therefore y = 0, 1$

$\therefore A \equiv (0, 0), B \equiv (1, 1)$

Now, the equation of circle with AB as diameter is

$x(x - 1) + y(y - 1) = 0$

14. Let ABC be the given equilateral triangle. Then C must lie on the y-axis.



Let $C \equiv (0, a)$. Also, $AC = AB$.

Therefore,

$\sqrt{1 + a^2} = 2$

or $1 + a^2 = 4$

or $a = \sqrt{3}$

$\therefore C \equiv (0, \sqrt{3})$

Then, the centroid of ΔABC is $(0, 1/\sqrt{3})$.

But in an equilateral triangle, the circumcenter coincides with the centroid. Therefore, the circumcenter is $(0, 1/\sqrt{3})$. Also,

Radius of circumcircle $= C_1B$

$$= \sqrt{(1-0)^2 + \left(0 - \frac{1}{\sqrt{3}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$$

Therefore, the equation of circumcircle is

$$(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$$

or $x^2 + y^2 - \frac{2y}{\sqrt{3}} + \frac{1}{3} = \frac{4}{3}$

or $x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$

15. Let (h, k) be any point on the given line.

Then, $2h + k = 4$. Chord of contact to circle from (h, k) is

$hx + ky = 1$

or $hx + (4 - 2h)y = 1$

or $(4y - 1) + h(x - 2y) = 0$

It passes through the point of intersection of $4y - 1 = 0$ and $x - 2y = 0$ or $(1/2, 1/4)$.

True/False Type

1. The circle passes through the points $A(1, \sqrt{3})$, $B(1, -\sqrt{3})$, and $C(3, -\sqrt{3})$.

Here, line AB is parallel to the y-axis and line BC is parallel to the x-axis.

Hence, $\angle ABC = 90°$.

So, AC is a diameter of the circle.

Therefore, the equation of the circle is

$(x - 1)(x - 3) + (y - \sqrt{3})(y + \sqrt{3}) = 0$

or $x^2 + y^2 - 4x = 0$ (i)

Let us check the position of point (5/2, 1) with respect to the circle (i).

$S_1 = \frac{25}{4} + 1 - 10 < 0$

Therefore, the point lies inside the circle.

So, no tangent can be drawn to the given circle from point (5/2, 1).

Therefore, the given statement is true.

2. The center of the circle $x^2 + y^2 - 6x + 2y = 0$ is $(3, -1)$ which lies on $x + 3y = 0$.

Therefore, the statement is true.

Subjective Type

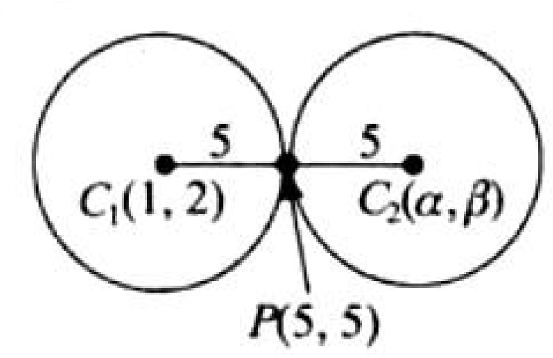
1. The given circle is

$x^2 + y^2 - 2x - 4y - 20 = 0$

whose center is (1, 2) and radius is 5.

The radius of the required circle is also 5.

Let its center be $C_2(\alpha, \beta)$.



Both the circles touch each other at $P(5, 5)$.

It is clear from the figure that $P(5, 5)$ is the midpoint of C_1C_2.

Therefore,

$\frac{1+\alpha}{2} = 5$ and $\frac{2+\beta}{2} = 5$

or $\alpha = 9$ and $\beta = 8$

Therefore, the center of the required circle is (9, 8) and its equation is

$(x - 9)^2 + (y - 8)^2 = 25$

or $x^2 + y^2 - 18x - 16y + 120 = 0$

2. The point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2 y = 1$ is

$$\left(\frac{1-c^2}{2+c-3c^2}, \frac{c-1}{2+c-3c^2}\right)$$

If (h, k) is the center of the required circle, then

$$h = \lim_{c \to 1} \frac{1-c^2}{2+c-3c^2}$$

$$= \lim_{c \to 1} \frac{(1-c)(1+c)}{(1-c)(2+3c)}$$

$$= \lim_{c \to 1} \frac{1+c}{2+3c} = \frac{2}{5}$$

and $k = \lim_{c \to 1} \dfrac{c-1}{5(2+c-3c^2)}$

$= \lim_{c \to 1} \dfrac{-1}{5(2+3c)} = \dfrac{-1}{25}$

Therefore, the center is (2/5, –1/25).

Also, the circle passes through (2, 0). Therefore,

$$\text{Radius} = \sqrt{\left(2-\frac{2}{5}\right)^2 + \left(\frac{1}{25}\right)^2} = \frac{\sqrt{1601}}{25}$$

Thus, the equation of the required circle is

$$\left(x-\frac{2}{5}\right)^2 + \left(y+\frac{1}{25}\right)^2 = \frac{1601}{625}$$

or $25x^2 + 25y^2 - 20x + 2y - 60 = 0$

3. The equation of the circle is

$x^2 + y^2 - 2x - 4y - 20 = 0$

Center $\equiv (1, 2)$ and radius $= \sqrt{1+4+20} = 5$

The equation of tangent at (1, 7) is

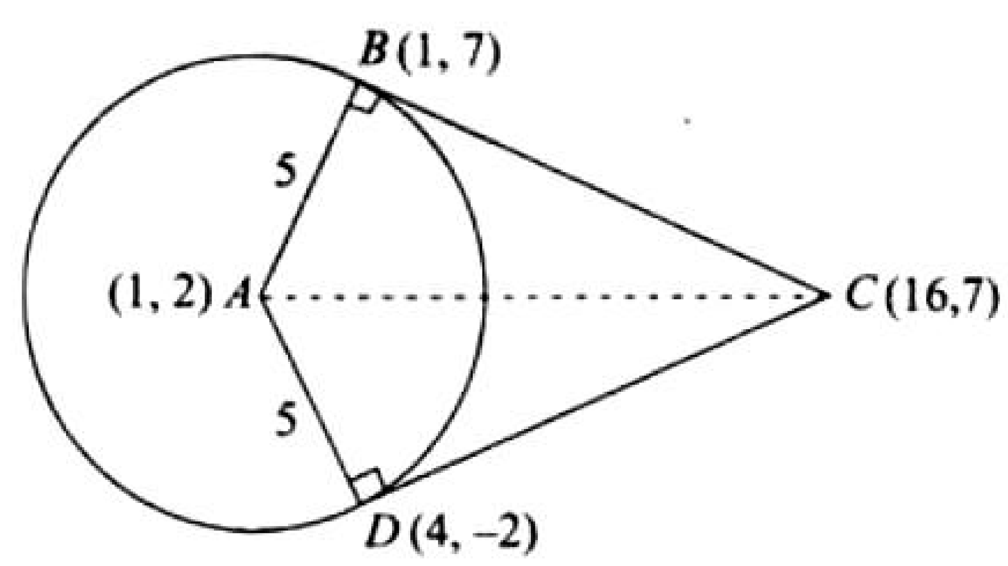
$x \cdot 1 + y \cdot 7 - (x+1) - 2(y+7) - 20 = 0$

or $y - 7 = 0$ (i)

Similarly, the equation of tangent at (4, –2) is

$4x - 2y - (x+4) - 2(y-2) - 20 = 0$

or $3x - 4y - 20 = 0$ (ii)



For point C, solving (i) and (ii), we get $x = 16$ and $y = 7$. Therefore,

$C \equiv (16, 7)$

Now, clearly,

ar(quad. $BCDA$) = 2 × ar(ΔABC)

$= 2 \times \frac{1}{2} \times AB \times BC$

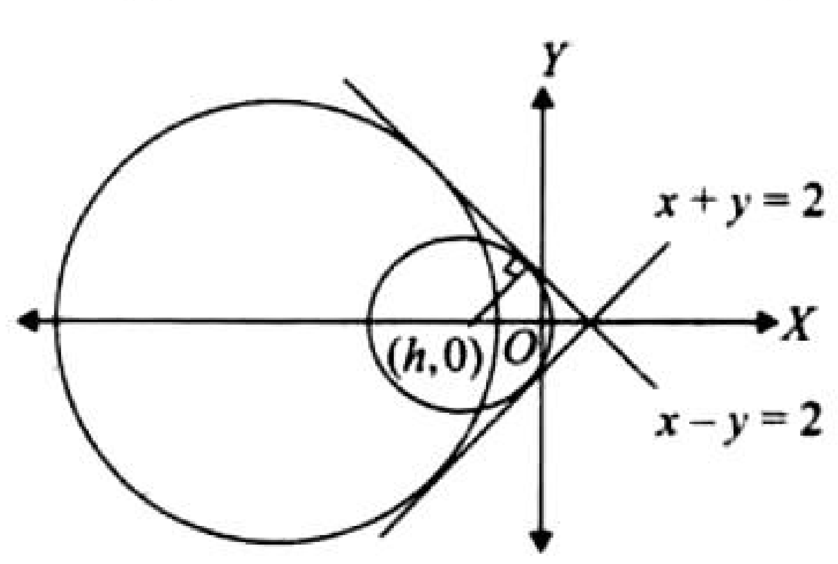
$= AB \times BC$

where AB = Radius of circle = 5

and $BC = 15$

∴ ar(quad. $ABCD$) = 5 × 15 = 75 sq. units

4.



Since the circle touches both the lines $x + y = 2$ and $x - y = 2$, its center must lie on the x-axis. Let the center of the circle be $(h, 0)$.

Now, the radius of the circle is equal to the perpendicular distance of the point $(h, 0)$ from the line $x + y - 2 = 0$, i.e.,

$$\text{Radius} = \frac{|h+0-2|}{\sqrt{2}} = \frac{|h-2|}{\sqrt{2}}$$

Then, the equation of the circle is

$$(x-h)^2 + (y-0)^2 = \frac{(h-2)^2}{2}$$

Since the circle passes through the point (–4, 3), we have

$$(-4-h)^2 + (3-0)^2 = \frac{(h-2)^2}{2}$$

or $h^2 + 20h + 46 = 0$

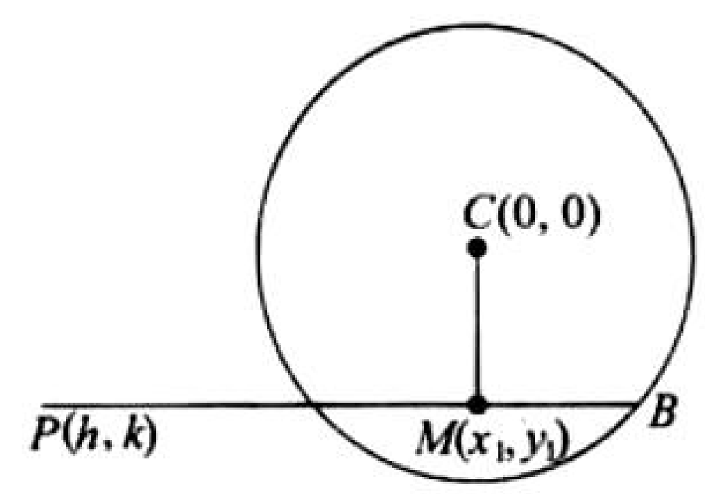
or $h = \dfrac{-20 \pm \sqrt{400-184}}{2}$

$= -10 \pm 3\sqrt{6}$

5. From the diagram,

$CM \perp BP$

$\therefore \left(\frac{y_1}{x_1}\right)\left(\frac{y_1-k}{x_1-h}\right) = -1$



So, the locus of M is

$x^2 + y^2 = hx + ky$

6. Let x_1, x_2 and y_1, y_2 be the roots of $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$, respectively. Then,

$x_1 + x_2 = -2a$, $x_1x_2 = -b^2$

and $y_1 + y_2 = -2p$, $y_1y_2 = -q^2$

The equation of the circle with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as the endpoints of diameter is

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

or $x^2 + y^2 - x(x_1+x_2) - y(y_1+y_2) + x_1x_2 + y_1y_2 = 0$

or $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

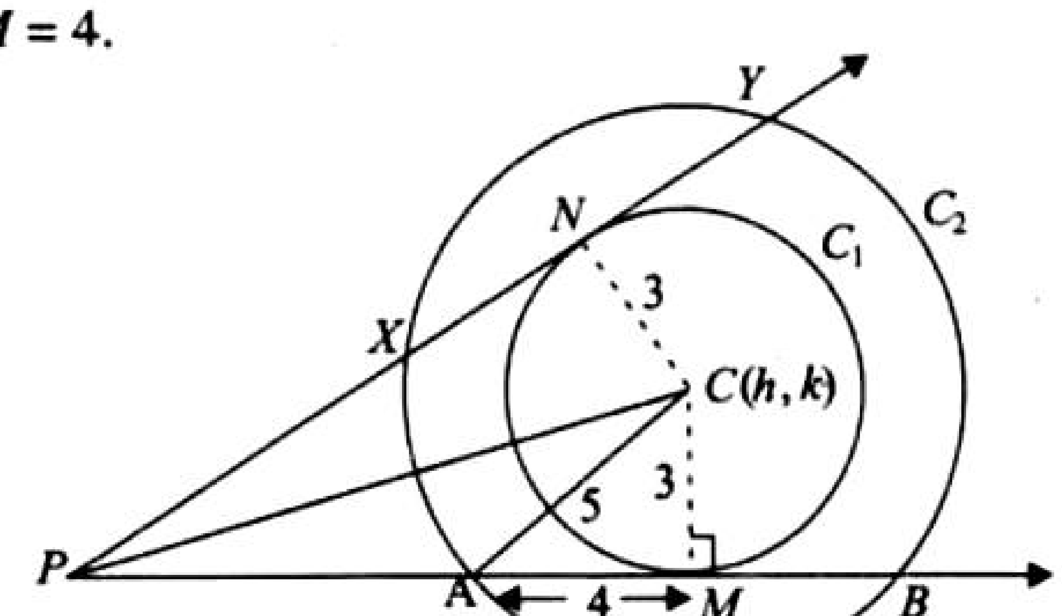
7. Let the equation of tangent PAB be $5x + 12y - 10 = 0$ and that of PXY be $5x - 12y - 40 = 0$.

Now, let the center of circles C_1 and C_2 be $C(h, k)$.

Let $CM \perp PAB$. Then

CM = Radius of $C_1 = 3$

Also, C_2 makes an intercept of length 8 units on PAB. So, $AM = 4$.



Then in ΔAMC, we get $AC = \sqrt{4^2 + 3^2} = 5$.

Therefore, the radius of C_2 is 5 units.

Also, as

$5x + 12y - 10 = 0$ (i)

and $5x - 12y - 40 = 0$ (ii)

are tangents to C_1, the length of perpendicular from C to AB is 3 units. Therefore,

$$\frac{5h + 12k - 10}{13} = \pm 3$$

i.e., $5h + 12k - 49 = 0$ (i)

or $5h + 12k + 29 = 0$ (ii)

Similarly,

$$\frac{5h - 12k - 40}{13} = \pm 3$$

i.e., $5h - 12k - 79 = 0$ (iii)

or $5h - 12k - 1 = 0$ (iv)

As C lies in the first quadrant, h and k are positive.

Therefore, (ii) is not possible.

Solving (i) and (iii), we get $h = 64/5$ and $k = -5/4$ which is also not possible.

Now, solving (i) and (iv), we get $h = 5$ and $k = 2$.

Thus, center for C_2 is (5, 2) and radius is 5.

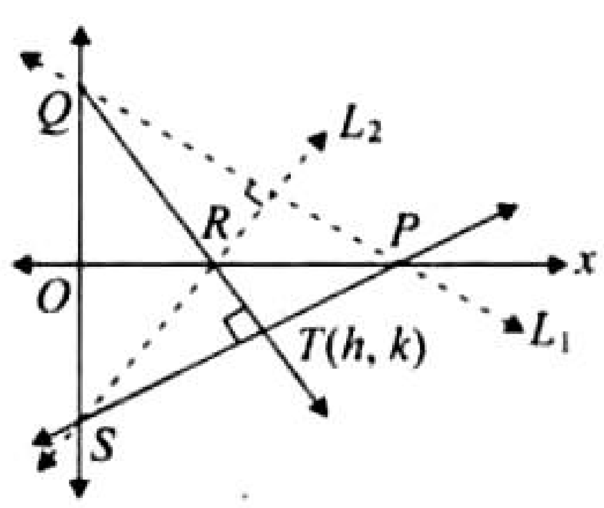
Hence, the equation of C_2 is $(x - 5)^2 + (y - 2)^2 = 5^2$.

8. From the figure in ΔPQS,
$PQ \perp SR$, $PR \perp QS$

Then we have $QT \perp PS$ (as the altitudes of a triangle are concurrent).

Therefore, points O, R, T, and S are concyclic.

Hence, the locus of T is a circle which passes through the origin.

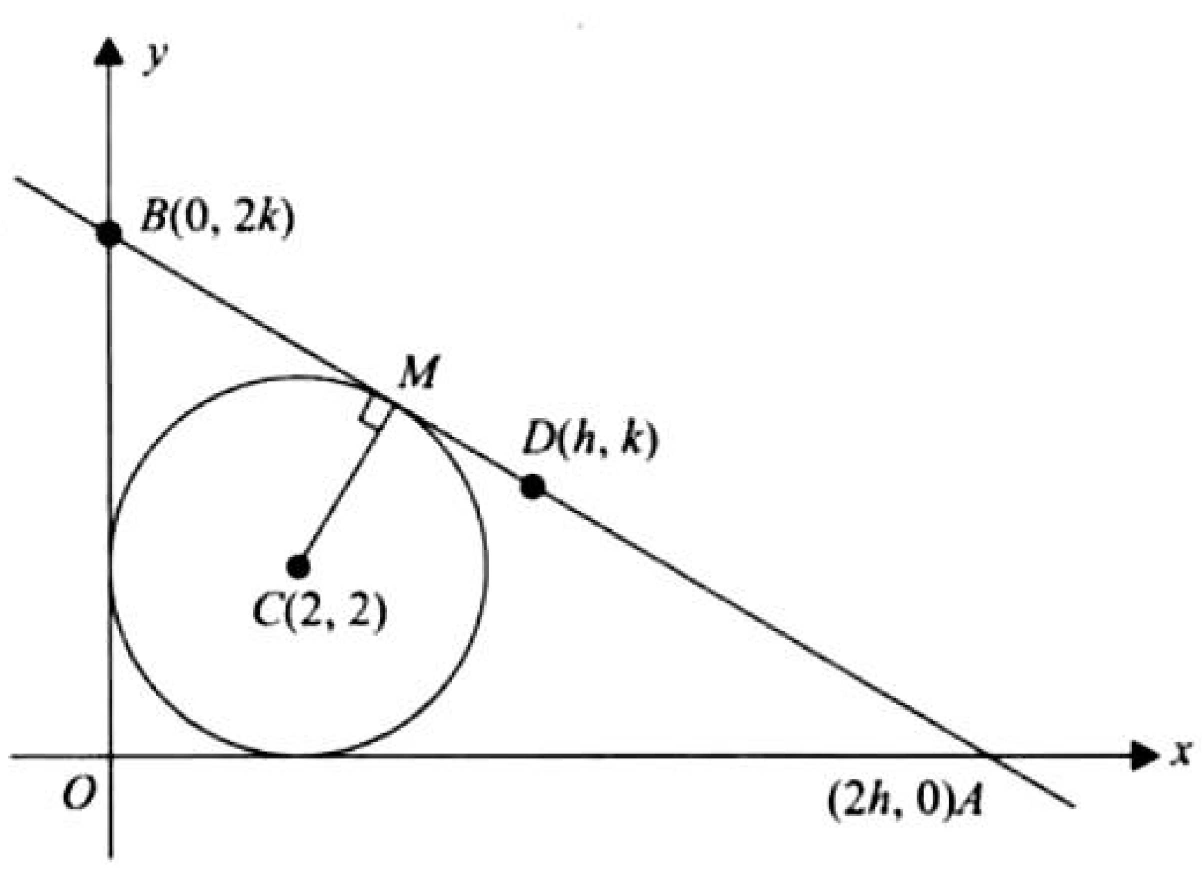


9. The equation of the given circle is

$x^2 + y^2 - 4x - 4y + 4 = 0$

or $(x - 2)^2 + (y - 2)^2 = 4$

This circle touches both the axes.



Variable line AB touches the circle.

Now, the circumcircle of triangle OAB is the midpoint of hypotenuse AB.

Let the midpoint of AB be $M(h, k)$.

Therefore, the coordinates of A and B are $(2h, 0)$ and $(0, 2k)$, respectively.

So, the equation of AB is

$$\frac{x}{2h} + \frac{y}{2k} = 1$$

Since this line touches the given circle,

$CM = 2$

$$\therefore \frac{\left|\frac{2}{2h} + \frac{2}{2k} - 1\right|}{\sqrt{\left(\frac{1}{2h}\right)^2 + \left(\frac{1}{2k}\right)^2}} = 2$$

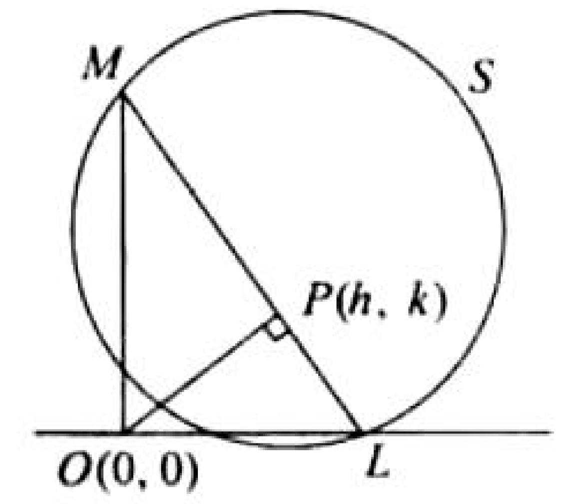
or $|h + k - hk| = \sqrt{h^2 + k^2}$

or the locus is $(x + y - xy) \pm \sqrt{x^2 + y^2} = 0$. Therefore, $k = \pm 1$.

10. The equation of the given circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

Let $P(h, k)$ be the foot of perpendicular drawn from the origin to the chord LM of circle S.



Slope of $OP = \frac{k}{h}$

$\therefore$ Slope of $LM = -\frac{h}{k}$

Therefore, the equation of LM is

$y - k = -\frac{h}{k}(x - h)$

or $ky - k^2 = -kx + h^2$

or $hx + ky = h^2 + k^2$ (ii)

Now, the combined equations of lines joining the points of intersection of (i) and (ii) to the origin can be obtained by making (i) homogeneous with the help of (ii) as follows:

$$x^2 + y^2 + (2gx + 2fy)\left(\frac{hx + ky}{h^2 + k^2}\right) + c\left(\frac{hx + ky}{h^2 + k^2}\right)^2 = 0$$

or $(h^2 + k^2)^2(x^2 + y^2) + (h^2 + k^2)(2gx + 2fy)(hx + ky)$
$+ c(hx + ky)^2 = 0$ (iii)

Equation (iii) represents the combined equation of OL and OM.

But $\angle LOM = 90°$.

Therefore, from (iii), we must have

Coeff. of x^2 + Coeff. of $y^2 = 0$

or $(h^2 + k^2)^2 + 2gh(h^2 + k^2) + ch^2 + (h^2 + k^2)^2 + 2fk(h^2 + k^2)$
$+ ck^2 = 0$

or $h^2 + k^2 + gh + fk + \frac{c}{2} = 0$

Therefore, the locus of (h, k) is $x^2 + y^2 + gx + fy + \frac{c}{2} = 0$.

11. Given that $(m_i, 1/m_i)$, $m_i > 0$, $i = 1, 2, 3, 4$, are four distinct points on a circle.

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

As the point $(m_i, 1/m_i)$ lies on it, we have

$$m^2 + \frac{1}{m^2} + 2gm + \frac{2f}{m} + c = 0$$

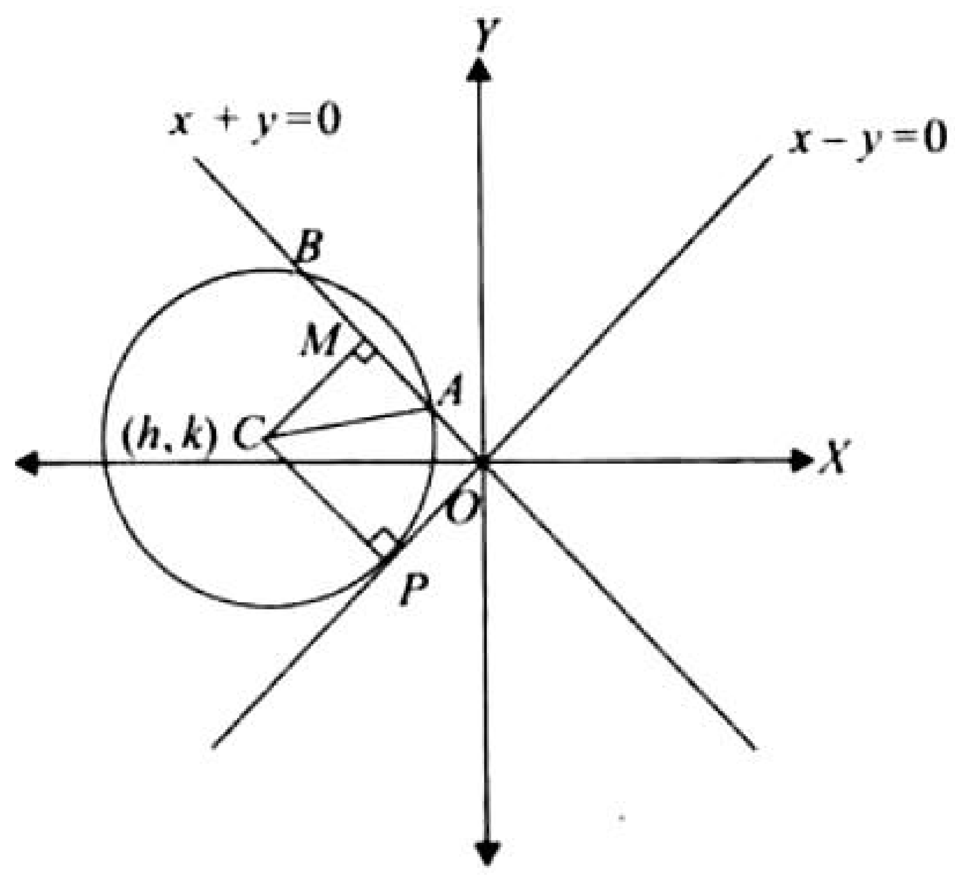
or $m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$

Since m_1, m_2, m_3, and m_4 are the roots of this equation, the product of roots is 1, i.e.,

$$m_1m_2m_3m_4 = 1$$

12. Let AB be the length of chord intercepted by the circle on $y + x = 0$.

Let CM be perpendicular to BA from center $C(h, k)$.



Also, $y - x = 0$ and $y + x = 0$ are perpendicular to each other.

Therefore, $OPCM$ is a rectangle. So,

$$CM = OP = 4\sqrt{2}$$

Let r be the radius of the circle.

Also, $AM = 3\sqrt{2}$.

Therefore, in ΔCAM,

$$AC^2 = AM^2 + MC^2$$

or $\quad r^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2$

or $\quad r^2 = (5\sqrt{2})^2$

or $\quad r = 5\sqrt{2}$

Also, the coordinates of P are

$$(0 - 4\sqrt{2}\cos 45°, 0 - 4\sqrt{2}\sin 45°) \text{ or } (-4, -4)$$

The slope of PC is -1 and $CP = 5\sqrt{2}$.

Therefore, the coordinates of C are

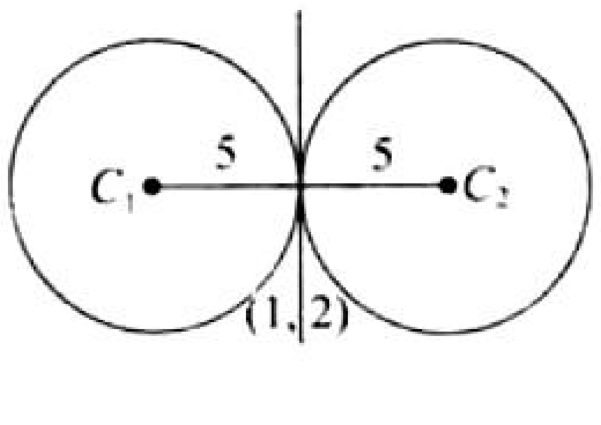
$$(-4 + 5\sqrt{2}\cos 135°, -4 + 5\sqrt{2}\sin 135°) \text{ or } (-9, 1)$$

Hence, the equation of circle is

$$(x + 9)^2 + (y - 1)^2 = 50$$

or $\quad x^2 + y^2 + 18x - 2y + 32 = 0$

13. Since the circles touch at (1, 2), the line joining the centers and the point (1, 2) are collinear and perpendicular to the common tangent $4x + 3y = 10$. Also, the centers are at a distance of 5 units from (1, 2). Now, the slope of the line perpendicular to the common tangent is $3/4 = \tan\alpha$.



Therefore, the equation of the line perpendicular to the common tangent through (1, 2) [in symmetrical form] is given by

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = r$$

For C_1 and C_2

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} = \pm 5$$

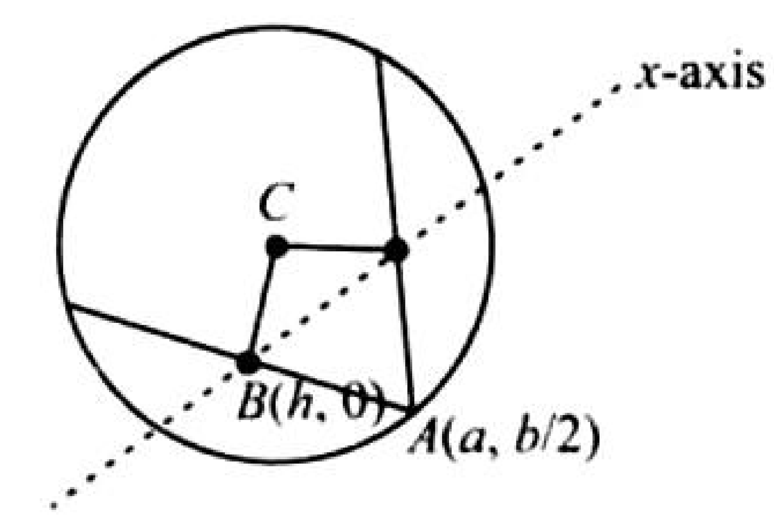
So, the centers are (5, 5) and (−3, −1).

Hence, the equations of the circles are

$(x - 5)^2 + (y - 5)^2 = 25$ and $(x + 3)^2 + (y + 1)^2 = 25$

i.e., $x^2 + y^2 - 10x - 10y + 25 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$

14.



The given circle is

$$2x(x - a) + y(2y - b) = 0 \; (a, b \neq 0)$$

or $\quad x^2 + y^2 - ax - \frac{b}{2}y = 0 \qquad$ (i)

Its center is $C(a/2, b/4)$.

Also, point $A(a, b/2)$ lies on the circle.

Since chord is bisected at point $B(h, 0)$ on the x-axis, we have

(Slope of BC) × (Slope of AB) = −1

or $\quad \left\{\frac{(b/4) - 0}{(a/2) - h}\right\}\left\{\frac{(b/2) - 0}{a - h}\right\} = -1$

$$h^2 - 3\frac{a}{2}h + \frac{a^2}{2} + \frac{b^2}{8} = 0$$

Since two such chords exist, the above equation must have two distinct real roots, for which its discriminant must be zero, i.e.,

or $\quad \frac{9a^2}{4} - 4\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$

or $\quad a^2 > 2b^2$

15. The equation of the line passing through the points $A(3, 7)$ and $B(6, 5)$ is

$$y - 7 = -\frac{2}{3}(x - 3)$$

or $2x + 3y - 27 = 0$

Also, the equation of the circle with A and B as the endpoints of diameter is

$(x - 3)(x - 6) + (y - 7)(y - 5) = 0$

Now, the equation of the family of circles through A and B is

$(x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27) = 0 \qquad$ (i)

The equation of the common chord of (i) and $x^2 + y^2 - 4x - 6y - 3 = 0$ is the radical axis, which is

$[(x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27)] - [x^2 + y^2 - 4x - 6y - 3] = 0$

or $(2\lambda - 5)x + (3\lambda - 6)y + (-27\lambda + 56) = 0$

or $(-5x - 6y + 56) + \lambda(2x + 3y - 27) = 0$

This is the family of lines which passes through the point of intersection of $-5x - 6y + 56 = 0$ and $2x + 3y - 27 = 0$, i.e., (2, 23/3).

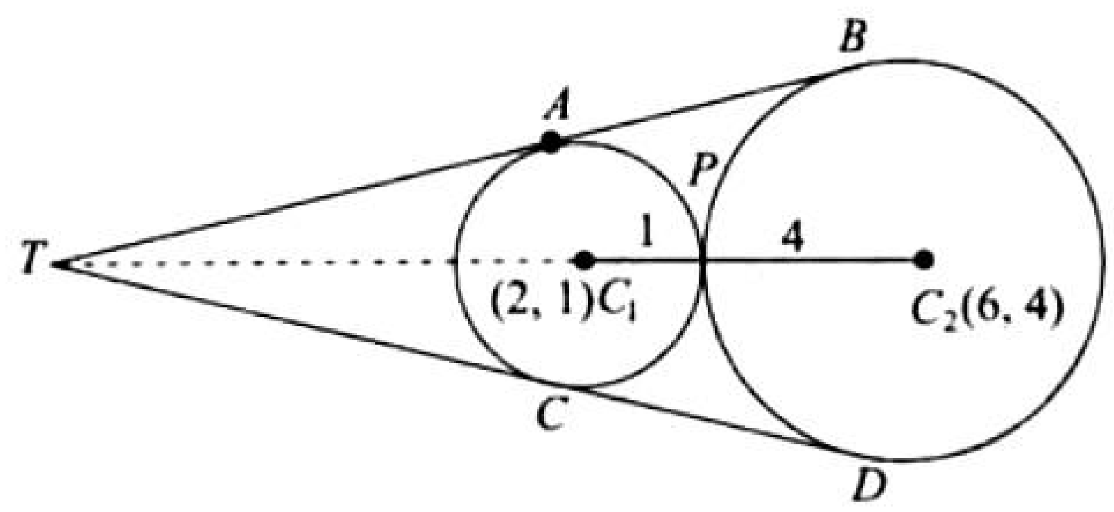
16. The given circles are

$$x^2 + y^2 - 4x - 2y + 4 = 0 \quad \text{(i)}$$

and $$x^2 + y^2 - 12x - 8y + 36 = 0 \quad \text{(ii)}$$

with centers $C_1(2, 1)$ and $C_2(6, 4)$ and radii 1 and 4, respectively. Also, $C_1C_2 = 5$.

As $r_1 + r_2 = C_1C_2$, the two circles touch each other externally at P.



Clearly, P divides C_1C_2 in the ratio 1 : 4.

Therefore, the coordinates of P are

$$\left(\frac{1 \times 6 + 4 \times 2}{1+4}, \frac{1 \times 4 + 4 \times 1}{1+4}\right) \equiv \left(\frac{14}{5}, \frac{8}{5}\right)$$

Let AB and CD be two common tangents of the given circles, meeting each other at T.

Then T divides C_1C_2 externally in the ratio 1 : 4. Hence,

$$T \equiv \left(\frac{1 \times 6 - 4 \times 2}{1-4}, \frac{1 \times 4 - 4 \times 1}{1-4}\right) \equiv \left(\frac{2}{3}, 0\right)$$

Let m be the slope of the tangent. Then the equation of tangent through (2/3, 0) is

$$y - 0 = m\left(x - \frac{2}{3}\right)$$

or $$y - mx + \frac{2}{3}m = 0$$

Now, the length of perpendicular from (2, 1) to the above tangent is the radius of the circle. Therefore,

$$\left|\frac{1 - 2m + \frac{2}{3}m}{\sqrt{m^2+1}}\right| = 1$$

or $(3 - 4m)^2 = 9(m^2 + 1)$

or $9 - 24m + 16m^2 = 9m^2 + 9$

or $7m^2 - 24m = 0$

or $m = 0, \frac{24}{7}$

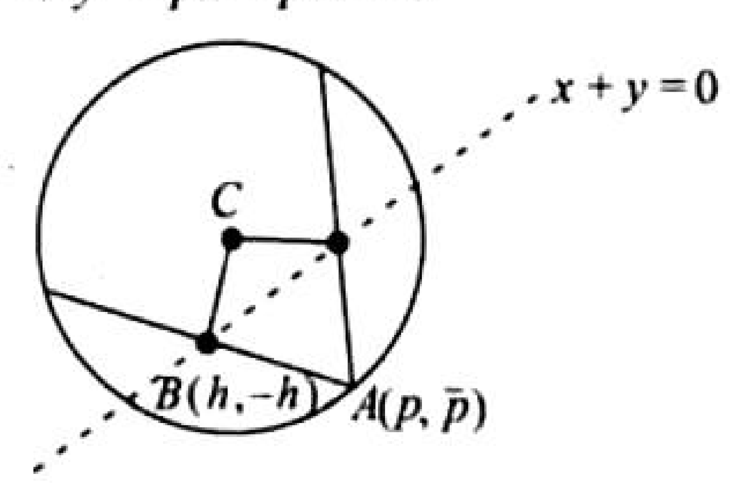
Thus, the equations of the tangents are $y = 0$ and $7y - 24x + 16 = 0$.

17. Let the given point be

$$(p, \bar{p}) \equiv \left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$$

Therefore, the equation of the circle becomes

$$x^2 + y^2 - px - \bar{p}x = 0$$



Since the chord is bisected by the line $x + y = 0$, its midpoint can be chosen as $(h, -h)$. From the figure, $BC \perp AB$. Therefore,

$$\left\{\frac{(\bar{p}/2) + h}{(p/2) - h}\right\}\left\{\frac{\bar{p} + h}{p - h}\right\} = -1$$

or $$\left(\frac{\bar{p} + 2h}{p - 2h}\right)\left(\frac{\bar{p} + h}{p - h}\right) = -1$$

or $\bar{p}^2 + 3\bar{p}h + 2h^2 = -p^2 + 3ph - 2h^2$

or $4h^2 - 3(\bar{p} - p)h + \bar{p}^2 + p^2 = 0$

Since two such chords exist, the above equation must have two distinct real roots, for which its discriminant must be zero. Therefore,

$9(\bar{p} - p)^2 - 16(\bar{p}^2 + p^2) > 0$

or $7(\bar{p}^2 + p^2) + 18\,\bar{p}p < 0$

or $\left(\frac{7}{4}\right)(2 + 4a^2) + \left(\frac{18}{4}\right)(1 - 2a^2) < 0$

or $32 - 8a^2 < 0\ a^2 > 4$

or $a \in (-\infty, -2) \cup (2, \infty)$

18. The given curve is

$$ax^2 + 2hxy + by^2 = 1 \quad \text{(i)}$$

Let the point P not lying on curve (i) be (x_1, y_1).

Let θ be the inclination of line through P which intersects the given curve at Q and R.

The equation of line through P is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

or $x = \cos\theta + x_1$

and $y = r\sin\theta + y_1$

For point Q or R, the above point must lie on curve (i). So,

$$a[x_1 + r\cos\theta]^2 + 2h[x_1 + r\cos\theta][y_1 + r\sin\theta] + b[y_1 + r\sin\theta]^2 = 1$$

or $(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)r^2 + 2(ax_1\cos\theta + hx_1\sin\theta + hy_1\cos\theta + by_1\sin\theta)\,r + \left(ax_1^2 + 2hx_1y_1 + by_1^2 - 1\right) = 0$

It is a quadratic in r giving two values of r for PQ and PR. Therefore,

Product of roots $= PQ \cdot PR$

$$= \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}$$

Here, $ax_1^2 + 2hx_1y_1 + by_1^2 - 1 \neq 0$ as (x_1, y_1) does not lie on curve (i). Also,

Denominator $= a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$

$= a + 2h\sin\theta\cos\theta + (b - a)\sin^2\theta$

$= a + h\sin 2\theta + \frac{(b-a)}{2}.(1 - \cos 2\theta)$

$= \left(\frac{a+b}{2}\right) + h\sin 2\theta + \left(\frac{a-b}{2}\right)\cos 2\theta$

which is independent of θ if $h = 0$, and $(a - b)/2 = 0$, i.e., $h = 0$ and $a = b$.

Hence, the given equation is a circle.

19. There cannot be three points on the circle with rational coordinates for then the centre of the circle, being the circumcentre of a triangle whose vertices have rational coordinates, must have rational coordinates ($\because$ the coordinates will be obtained by solving two linear equations in x, y having rational coefficients). But the point $(0, \sqrt{2})$ does not have rational coordinates.

Also the equation of the circle is

$(x-0)^2 + (y-\sqrt{2})^2 = r^2$

$\Rightarrow \quad y = \sqrt{2} \pm \sqrt{r^2 - x^2}$

For suitable r, x, where x is rational, y may have two rational values.

For example, $r = 2, x = 0, y = 1, -1$ satisfy $y = \sqrt{2} \pm \sqrt{r^2 - x^2}$

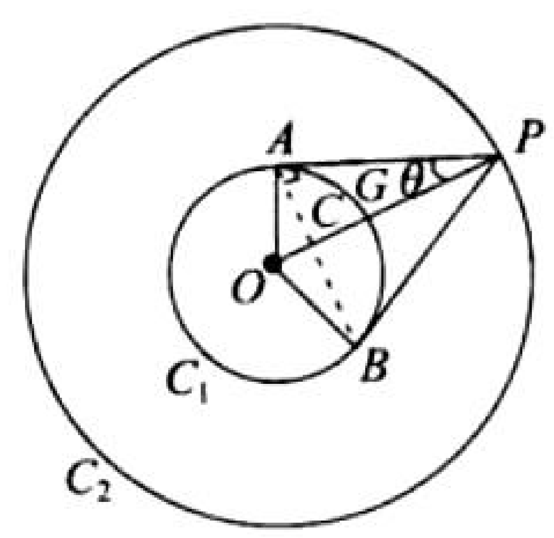
So we get two points $(0, 1), (0, -1)$ which have rational coordinates.

20. Let r be the radius of C_1 and $2r$ be the radius of C_2. Then,

$OA = OB = r$

and $OP = 2r$

Since PA and PB are tangents to C_1,



$\angle OAP = \angle OBP = 90°$

Let OP meets C_1 at G.

Let $\angle OPA = \theta$. Then,

$\sin\theta = \frac{OA}{OP} = \frac{r}{2r} = \frac{1}{2}$

or $\theta = 30°$

or $\angle AOP = 60°$

In ΔOAC,

$\cos 60° = \frac{OC}{OA}$

or $\frac{1}{2} = \frac{OC}{r}$ or $OC = \frac{r}{2}$

$\therefore \quad PC = 2r - \frac{r}{2} = \frac{3r}{2}$

Also, $CG = r - OC = r - \frac{r}{2} = \frac{r}{2}$

Clearly, $CG = \frac{1}{3}PC$

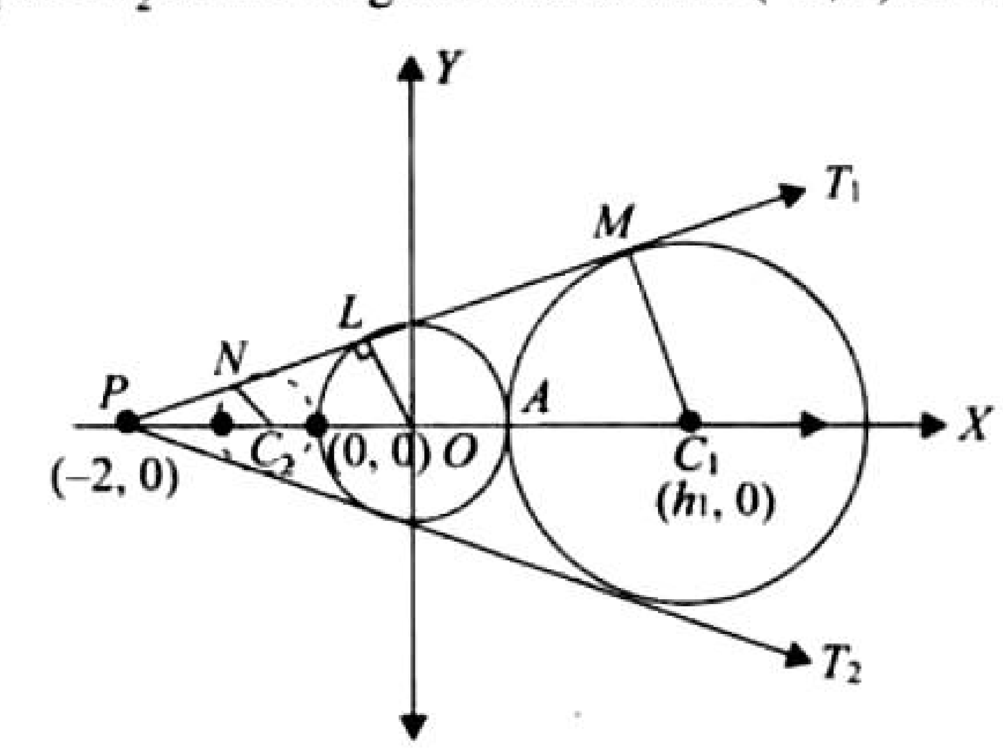
Therefore, G is the centroid of ΔABP.

21. The given circle is

$x^2 + y^2 = 1$ (i)

Center $\equiv O(0, 0)$ and radius $= 1$

Let T_1 and T_2 be the tangents drawn from $(-2, 0)$ to the circle (i).



From the figure, ΔPLO and ΔPMC_1 are similar. Therefore,

$\frac{OL}{OP} = \frac{C_1M}{C_1P}$

or $\frac{1}{2} = \frac{r_1}{h_1 + 2}$

or $2r_1 = h_1 + 2$ (ii)

Also, the circles are touching externally. Therefore,

$h_1 = r_1 + 1$ (iii)

From (ii) and (iii),

$r_1 = 3$ and $h_1 = 4$

Hence, the equation of the circle is

$(x-4)^2 + y^2 = 9$ (iv)

Also, ΔPLO and ΔPNC_2 are similar. Therefore,

$\frac{OL}{OP} = \frac{C_2N}{C_2P}$

or $\frac{1}{2} = \frac{r_2}{2 + h_2}$

or $2r_2 = 2 + h_2$ (v)

Also, the circles are touching externally. Therefore,

$-h_2 = 1 + r_2$ (vi)

From (v) and (vi),

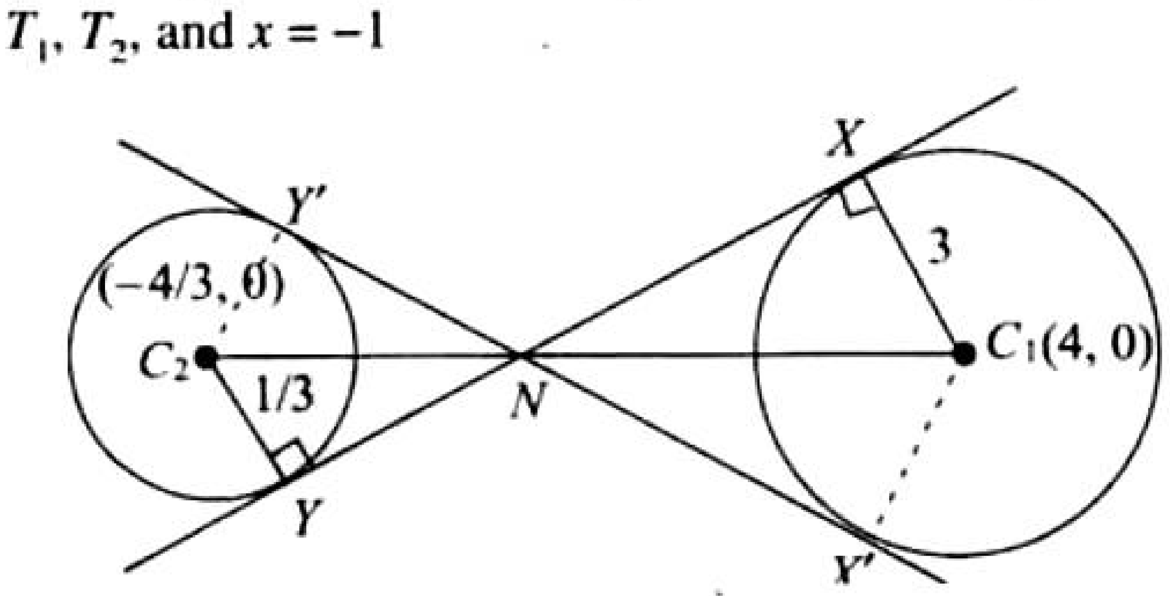
$r_2 = \frac{1}{3}$ and $h_2 = -\frac{4}{3}$

Hence, the equation of the circle is

$\left(x + \frac{4}{3}\right)^2 + y^2 = \left(\frac{1}{3}\right)^2$ (vii)

Since circles (i) and (iv) are two touching circles, they have three common tangents: T_1, T_2, and $x - 1 = 0$.

Similarly, the common tangents of circles (i) and (vii) are T_1, T_2, and $x = -1$



For the circles (iv) and (vii), there will be four common tangents: two direct and two indirect.

Two common direct tangents are T_1 and T_2.

Let us find two common indirect tangents.

$\frac{C_1N}{C_2N} = \frac{3}{1/3} = 9$

i.e., N divides C_1C_2 in the ratio $9:1$. Therefore,

$N \equiv \left(\frac{9 \times (-4/3) + 1 \times 4}{10}, 0\right) \equiv \left(\frac{-4}{5}, 0\right)$

Any line through N is

$y = m\left(x + \frac{4}{5}\right)$

or $5mx - 5y + 4m = 0$

If it is tangent to circle (iv), then

$\left|\frac{20m + 4m}{\sqrt{25m^2 + 25}}\right| = 3$

or $24m = 15\sqrt{m^2 + 1}$

or $64m^2 = 25m^2 + 25$

or $39m^2 = 25$

or $m = \pm \frac{5}{\sqrt{39}}$

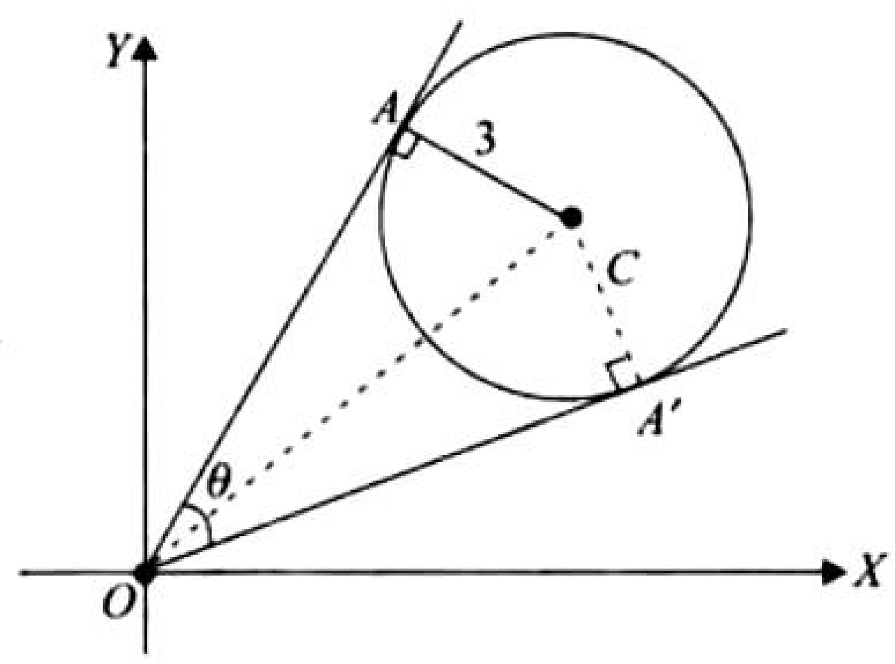
Therefore, the required tangents are

$$y = \pm \frac{5}{\sqrt{39}}\left(x + \frac{4}{5}\right)$$

22. The equation $2x^2 - 3xy + y^2 = 0$ represents a pair of tangents OA and OA'.

Let the angle between these two tangents be 2θ. Then,

$$\tan 2\theta = \frac{2\sqrt{(-3/2)^2 - 2 \times 1}}{2 + 1}$$



or $\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{3}$

or $\tan^2\theta + 6 \tan \theta - 1 = 0$

or $\tan \theta = \frac{-6 \pm \sqrt{36 + 4}}{2} = -3 \pm \sqrt{10}$

As θ is acute, $\tan \theta = \sqrt{10} - 3$.

Now, we know that the line joining the point through which tangents are drawn to the center bisects the angle between the tangents. Therefore,

$\angle AOC = \theta$

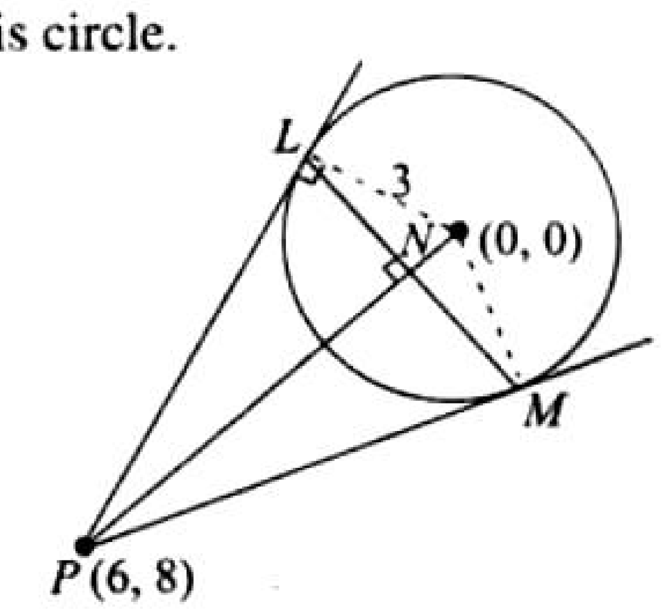
In ΔOAC,

$\tan \theta = \frac{3}{OA}$

or $OA = \frac{3}{\sqrt{10} - 3} \times \frac{\sqrt{10} + 3}{\sqrt{10} + 3}$

$\therefore$ $OA = 3(3 + \sqrt{10})$

23. The given circle is $x^2 + y^2 = r^2$. From point (6, 8), tangents are drawn to this circle.



Then, the length of tangent is

$$PL = \sqrt{6^2 + 8^2 - r^2} = \sqrt{100 - r^2}$$

Also, the equation of chord of contact LM is $6x + 8y - r^2 = 0$.

PN = Length of $\perp$ from P to LM

$$= \frac{36 + 64 - r^2}{\sqrt{36 + 64}} = \frac{100 - r^2}{10}$$

Now, in right-angled ΔPLN,

$LN^2 = PL^2 - PN^2$

$$= (100 - r^2) - \frac{(100 - r^2)^2}{100} = \frac{(100 - r^2)r^2}{100}$$

or $LN = \frac{r\sqrt{100 - r^2}}{10}$

$\therefore$ $LM = \frac{r\sqrt{100 - r^2}}{5}$ $\quad (\because LM = 2LN)$

$\therefore$ Area of $\Delta PLM = A = \frac{1}{2} \times LM \times PN$

$$= \frac{1}{2} \times \frac{r\sqrt{100 - r^2}}{5} \times \frac{100 - r^2}{10}$$

$$= \frac{1}{100}\left[r(100 - r^2)^{3/2}\right]$$

For the maximum value of A, we should have

$\frac{dA}{dr} = 0$

or $\frac{1}{100}\left[(100 - r^2)^{3/2} + r\frac{3}{2}(100 - r^2)^{1/2}(-2r)\right] = 0$

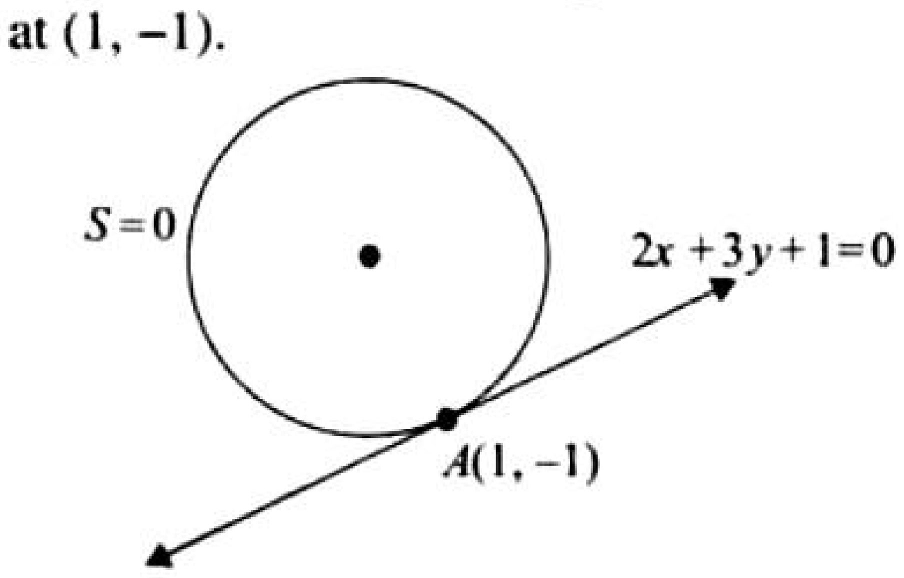
or $(100 - r^2)^{1/2}[100 - r^2 - 3r^2] = 0$

i.e., $r = 10$ or $r = 5$

But $r = 10$ gives the length of tangent PL. Therefore, $r \neq 10$.

Hence, $r = 5$.

24. We are given that line $2x + 3y + 1 = 0$ touches the circle $S = 0$ at $(1, -1)$.



So, the equation of this circle can be given by

$(x - 1)^2 + (y + 1)^2 + \lambda(2x + 3y + 1) = 0, \lambda \in R$

[Here, $(x - 1)^2 + (y + 1)^2 = 0$ represents a point circle at $(1, -1)$.]

or $x^2 + y^2 + 2x(\lambda - 1) + y(3\lambda + 2) + (\lambda + 2) = 0$ (i)

But this circle is orthogonal to the circle the extremities of whose diameter are (0, 3) and (−2, −1), i.e.,

$x(x + 2) + (y - 3)(y + 1) = 0$

or $x^2 + y^2 + 2x - 2y - 3 = 0$ (ii)

Applying the condition of orthogonality for (i) and (ii), we get

$2(\lambda - 1) \times 1 + 2\left(\frac{3\lambda + 2}{2}\right) \times (-1) = \lambda + 2 + (-3)$

$[2g_1g_2 + 2f_1f_2 = c_1 + c_2]$

or $2\lambda - 2 - 3\lambda - 2 = \lambda - 1$

or $2\lambda = -3$

or $\lambda = -\frac{3}{2}$

Substituting this value of λ in (i), we get the required circle as

$x^2 + y^2 - 5x - \frac{5}{2}y + \frac{1}{2} = 0$

or $2x^2 + 2y^2 - 10x - 5y + 1 = 0$