

Complex Numbers and Quadratic Equations

Short Answer Type Questions

Q. 1 For a positive integer n , find the value of $(1-i)^n \left(1 - \frac{1}{i}\right)^n$.

Sol. Given expression $= (1-i)^n \left(1 - \frac{1}{i}\right)^n$

$$\begin{aligned}
 &= (1-i)^n (i-1)^n \cdot i^{-n} = (1-i)^n (1-i)^n (-1)^n \cdot i^{-n} \\
 &= [(1-i)^2]^n (-1)^n \cdot i^{-n} = (1+i^2-2i)^n (-1)^n i^{-n} \quad [\because i^2 = -1] \\
 &= (1-1-2i)^n (-1)^n i^{-n} = (-2)^n \cdot i^n (-1)^n i^{-n} \\
 &= (-1)^{2n} \cdot 2^n = 2^n
 \end{aligned}$$

Q. 2 Evaluate $\sum_{n=1}^{13} (i^n + i^{n+1})$, where $n \in N$.

Thinking Process

Use $i^2 = -1, i^4 = (-1)^2 = 1, i^3 = -i$, and $i^5 = i$ to solve it

Sol. Given that, $\sum_{n=1}^{13} (i^n + i^{n+1}), n \in N$

$$\begin{aligned}
 &= (i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13}) \\
 &\quad + (i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13} + i^{14}) \\
 &= (i + 2i^2 + 2i^3 + 2i^4 + 2i^5 + 2i^6 + 2i^7 + 2i^8 + 2i^9 + 2i^{10} + 2i^{11} + 2i^{12} + 2i^{13} + i^{14}) \\
 &= i - 2 - 2i + 2 + 2i + 2(i^4)i^2 + 2(i^4)i^3 + 2(i^2)^4 + 2(i^2)^4 i + 2(i^2)^5 \\
 &\quad + 2(i^2)^5 \cdot i + 2(i^2)^6 + 2(i^2)^6 \cdot i + (i^2)^7 \\
 &= i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 1 - 1 + i
 \end{aligned}$$

Alternate Method

$$\begin{aligned}
 \sum_{n=1}^{13} (i^n + i^{n+1}), n \in N &= \sum_{h=1}^{13} i^h (1+i) \\
 &= (1+i)[i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13}] \\
 &= (1+i)[i^{13}] \quad [\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{12} i^n = 0] \\
 &= (1+i)i \\
 [\because (i^4)^3 \cdot i &= i] \\
 &= (i^2 + i) = i - 1
 \end{aligned}$$

Q. 3 If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y) .

Thinking Process

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal
i.e., $z_1 = z_2 \Rightarrow x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$.

Sol. Given that, $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$... (i)

$$\begin{aligned}
 \therefore \left(\frac{1+i}{1-i}\right)^3 &= \frac{1+i^3 + 3i(1+i)}{1-i^3 - 3i(1-i)} = \frac{1-i + 3i + 3i^2}{1+i - 3i + 3i^2} \\
 &= \frac{2i-2}{-2i-2} = \frac{i-1}{-i-1} = \frac{1-i}{1+i} \\
 &= \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1+i^2-2i}{1+1} = \frac{1-1-2i}{2}
 \end{aligned}$$

$$\Rightarrow \left(\frac{1+i}{1-i}\right)^3 = -i \quad \dots (ii)$$

$$\text{Similarly, } \left(\frac{1-i}{1+i}\right)^3 = \frac{-1}{i} = \frac{i^2}{i} = i \quad \dots (iii)$$

Using Eqs. (ii) and (iii) in Eq. (i), we get

$$\begin{aligned}
 -i - i &= x + iy \\
 -2i &= x + iy
 \end{aligned}$$

On comparing real and imaginary part of complex number, we get

$$\begin{aligned}
 x &= 0 \text{ and } y = -2 \\
 \text{So, } (x, y) &= (0, -2)
 \end{aligned}$$

Q. 4 If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

Sol. Given that, $\frac{(1+i)^2}{2-i} = x + iy$

$$\Rightarrow \frac{(1+i^2+2i)}{2-i} = x + iy \Rightarrow \frac{2i}{2-i} = x + iy$$

$$\Rightarrow \frac{2i(2+i)}{(2-i)(2+i)} = x + iy \Rightarrow \frac{4i+2i^2}{4-i^2} = x + iy$$

$$\Rightarrow \frac{4i-2}{4+1} = x + iy \Rightarrow \frac{-2}{5} + \frac{4i}{5} = x + iy$$

On comparing both sides, we get

$$x = -2/5 \Rightarrow y = 4/5$$

$$\Rightarrow x + y = \frac{-2}{5} + \frac{4}{5} = 2/5$$

Q. 5 If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b) .

Sol. Given that, $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$

$$\Rightarrow \left[\frac{(1-i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)}\right]^{100} = a + ib \Rightarrow \left(\frac{1+i^2-2i}{1-i^2}\right)^{100} = a + ib$$

$$\Rightarrow \left(\frac{-2i}{2}\right)^{100} = a + ib \quad [\because i^2 = -1]$$

$$\Rightarrow (i^4)^{25} = a + ib \Rightarrow 1 = a + ib$$

$$\text{Then, } a = 1 \text{ and } b = 0 \quad [\because i^4 = 1]$$

$$\therefore (a, b) = (1, 0)$$

Q. 6 If $a = \cos\theta + i\sin\theta$, then find the value of $\frac{1+a}{1-a}$.

Thinking Process

To solve the above problem use the trigonometric formula $\cos\theta = 2\cos^2\theta/2 - 1 = 1 - 2\sin^2\theta/2$ and $\sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$.

Sol. Given that, $a = \cos\theta + i\sin\theta$

$$\therefore \frac{1+a}{1-a} = \frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$$

$$= \frac{1+2\cos^2\theta/2-1+2i\sin\theta/2 \cdot \cos\theta/2}{1-1+2\sin^2\theta/2-2i\sin\theta/2 \cdot \cos\theta/2} = \frac{2\cos\theta/2(\cos\theta/2+i\sin\theta/2)}{2\sin\theta/2(\sin\theta/2-i\cos\theta/2)}$$

$$= -\frac{2\cos\theta/2(\cos\theta/2+i\sin\theta/2)}{2i\sin\theta/2(\cos\theta/2+i\sin\theta/2)} = -\frac{1}{i}\cot\theta/2$$

$$= \frac{+i^2}{i}\cot\theta/2 = i\cot\theta/2 \quad \left[\because \frac{-1}{i} = \frac{i^2}{i}\right]$$

Q. 7 If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Sol. We have, $(1+i)z = (1-i)\bar{z} \Rightarrow \frac{z}{\bar{z}} = \frac{(1-i)}{(1+i)}$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{(1-i)(1-i)}{(1+i)(1-i)} \Rightarrow \frac{z}{\bar{z}} = \frac{1+i^2-2i}{1-i^2} \quad [\because i^2 = -1]$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-1-2i}{2} \Rightarrow \frac{z}{\bar{z}} = -i$$

$\therefore$

$$z = -i\bar{z}$$

Hence proved.

Q. 8 If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in \mathbb{R}$, represents a circle.

Sol. Given that,

$$z = x + iy$$

Then,

$$\bar{z} = x - iy$$

Now,

$$z\bar{z} + 2(z + \bar{z}) + b = 0$$

$$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$$

$$\Rightarrow x^2 + y^2 + 4x + b = 0, \text{ which is the equation of a circle.}$$

Q. 9 If the real part of $\frac{\bar{z} + 2}{\bar{z} - 1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Sol. Let

$$z = x + iy$$

Now,

$$\frac{\bar{z} + 2}{\bar{z} - 1} = \frac{x - iy + 2}{x - iy - 1}$$

$$= \frac{[(x + 2) - iy][(x - 1) + iy]}{[(x - 1) - iy][(x - 1) + iy]}$$

$$= \frac{(x - 1)(x + 2) - iy(x - 1) + iy(x + 2) + y^2}{(x - 1)^2 + y^2}$$

$$= \frac{(x - 1)(x + 2) + y^2 + i[(x + 2)y - (x - 1)y]}{(x - 1)^2 + y^2}$$

$$[\because -iy^2 = 1]$$

$$\text{Taking real part, } \frac{(x - 1)(x + 2) + y^2}{(x - 1)^2 + y^2} = 4$$

$$\Rightarrow x^2 - x + 2x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$$

$$\Rightarrow 3x^2 + 3y^2 - 9x + 6 = 0, \text{ which represents a circle.}$$

Hence, z lies on the circle.

Q. 10 Show that the complex number z , satisfying the condition $\arg\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{4}$ lies on a circle.

Thinking Process

First use, $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. Also apply $\arg(z) = \theta = \tan^{-1}\frac{y}{x}$, where $z = x + iy$

and then use the property $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$

Sol. Let

$$z = x + iy$$

Given that,

$$\arg\left(\frac{z - 1}{z + 1}\right) = \pi/4$$

$$\Rightarrow \arg(z - 1) - \arg(z + 1) = \pi/4$$

$$\Rightarrow \arg(x + iy - 1) - \arg(x + iy + 1) = \pi/4$$

$$\Rightarrow \arg(x - 1 + iy) - \arg(x + 1 + iy) = \pi/4$$

$$\begin{aligned}
 &\Rightarrow \tan^{-1} \frac{y}{x-1} - \tan^{-1} \frac{y}{x+1} = \pi/4 \\
 &\Rightarrow \tan^{-1} \left[\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)} \right] = \pi/4 \\
 &\Rightarrow \frac{y \left[\frac{x+1-x+1}{x^2-1} \right]}{\frac{x^2-1+y^2}{x^2-1}} = \tan \pi/4 \\
 &\Rightarrow \frac{2y}{x^2+y^2-1} = 1 \\
 &\Rightarrow x^2+y^2-1=2y \\
 &\Rightarrow x^2+y^2-2y-1=0, \text{ which represents a circle.}
 \end{aligned}$$

Q. 11 Solve the equation $|z| = z + 1 + 2i$.

Sol. The given equation is $|z| = z + 1 + 2i$... (i)

Let $z = x + iy$

From Eq. (i), $|x + iy| = x + iy + 1 + 2i$

$$\Rightarrow \sqrt{x^2 + y^2} = x + iy + 1 + 2i \quad \left[\because |z| + \sqrt{x^2 + y^2} \right]$$

$$\Rightarrow \sqrt{x^2 + y^2} = (x+1) + i(y+2)$$

On squaring both sides, we get

$$x^2 + y^2 = (x+1)^2 + i^2(y+2)^2 + 2i(x+1)(y+2)$$

$$\Rightarrow x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4 + 2i(x+1)(y+2)$$

On comparing real and imaginary parts,

$$x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4y - 4$$

$$\text{i.e.,} \quad 2y^2 = 2x - 4y - 3 \quad \dots (ii)$$

$$\text{and} \quad 2(x+1)(y+2) = 0$$

$$(x+1) = 0 \text{ or } (y+2) = 0$$

$$\Rightarrow x = -1 \text{ or } y = -2$$

$$\text{For } x = -1, \quad 2y^2 = -2 - 4y - 3$$

$$2y^2 + 4y + 5 = 0 \quad [\text{using Eq. (ii)}]$$

$$\Rightarrow y = \frac{-4 \pm \sqrt{16 - 2 \times 4 \times 5}}{4}$$

$$\Rightarrow y = \frac{-4 \pm \sqrt{-24}}{4} \notin R$$

$$\text{For } y = -2, \quad 2(-2)^2 = 2x - 4(-2) - 3 \quad [\text{using Eq. (ii)}]$$

$$\Rightarrow 8 = 2x + 8 - 3$$

$$\Rightarrow 2x = 3 \Rightarrow x = 3/2$$

$$\therefore z = x + iy = 3/2 - 2i$$

Long Answer Type Questions

Q. 12 If $|z + 1| = z + 2(1 + i)$, then find the value of z .

Sol. Given that, $|z + 1| = z + 2(1 + i)$... (i)

$$z = x + iy$$

Then, $|x + iy + 1| = x + iy + 2(1 + i)$

$$\Rightarrow |x + 1 + iy| = (x + 2) + i(y + 2)$$

$$\Rightarrow \sqrt{(x + 1)^2 + y^2} = (x + 2) + i(y + 2)$$

On squaring both sides, we get

$$(x + 1)^2 + y^2 = (x + 2)^2 + i^2(y + 2)^2 + 2i(x + 2)(y + 2)$$

$$\Rightarrow x^2 + 2x + 1 + y^2 = x^2 + 4x + 4 - y^2 - 4y - 4 + 2i(x + 2)(y + 2)$$

$$\Rightarrow x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y + 2i(x + 2)(y + 2)$$

On comparing real and imaginary parts, we get

$$x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y$$

$$\Rightarrow 2y^2 - 2x + 4y + 1 = 0 \quad \dots (ii)$$

and $2(x + 2)(y + 2) = 0$

$$\Rightarrow x + 2 = 0 \text{ or } y + 2 = 0$$

$$x = -2 \text{ or } y = -2 \quad \dots (iii)$$

For $x = -2$, $2y^2 + 4 + 4y + 1 = 0$ [using Eq. (ii)]

$$\Rightarrow 2y^2 + 4y + 5 = 0$$

$$\Rightarrow 16 - 4 \times 2 \times 5 < 0$$

$$\therefore \text{Discriminant, } D = b^2 - 4ac < 0$$

$$\Rightarrow 2y^2 + 4y + 5 \text{ has no real roots.}$$

For $y = -2$, $2(-2)^2 - 2x + 4(-2) + 1 = 0$ [using Eq. (ii)]

$$\Rightarrow 8 - 2x - 8 + 1 = 0$$

$$\Rightarrow x = 1/2$$

$$\therefore z = x + iy = \frac{1}{2} - 2i$$

Q. 13 If $\arg(z - 1) = \arg(z + 3i)$, then find $x - 1 : y$, where $z = x + iy$.

Sol. Given that, $\arg(z - 1) = \arg(z + 3i)$

and let $z = x + iy$

Now, $\arg(z - 1) = \arg(z + 3i)$

$$\Rightarrow \arg(x + iy - 1) = \arg(x + iy + 3i)$$

$$\Rightarrow \arg(x - 1 + iy) = \arg[x + i(y + 3)]$$

$$\Rightarrow \tan^{-1} \frac{y}{x - 1} = \tan^{-1} \frac{y + 3}{x}$$

$$\Rightarrow \frac{y}{x - 1} = \frac{y + 3}{x} \Rightarrow xy = (x - 1)(y + 3)$$

$$\Rightarrow xy = xy - y + 3x - 3 \Rightarrow 3x - 3 = y$$

$$\Rightarrow \frac{3(x-1)}{y} = 1 \Rightarrow \frac{x-1}{y} = \frac{1}{3}$$

$$\therefore (x-1) : y = 1 : 3$$

Q. 14 Show that $\left| \frac{z-2}{z-3} \right| = 2$ represents a circle. Find its centre and radius.

Thinking Process

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $\left| \frac{z_1}{z_2} \right| = \left| \frac{z_1}{z_2} \right|$, ($z_2 \neq 0$), use this concept to solve the above problem. Also, we know that general equation of a circle is $xx^2 + y^2 + 2gx + 2fy + c = 0$, with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Sol. Let $z = x + iy$

Given, equation is $\left| \frac{z-2}{z-3} \right| = 2 \Rightarrow \left| \frac{z-2}{z-3} \right| = 2$

$$\Rightarrow \left| \frac{x+iy-2}{x+iy-3} \right| = 2$$

$$\Rightarrow |x-2+iy| = 2|x-3+iy|$$

$$\Rightarrow \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2} \quad \left[\because |x+iy| = \sqrt{x^2 + y^2} \right]$$

On squaring both sides, we get

$$x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$$

$$\Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \quad \dots(i)$$

On comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$\Rightarrow 2g = \frac{-20}{3} \Rightarrow g = \frac{-10}{3}$$

and $2f = 0 \Rightarrow f = 0$ and $c = \frac{32}{3}$

$$\therefore \text{Centre} = (-g, -f) = (10/3, 0)$$

Also, radius $(r) = \sqrt{(10/3)^2 + 0 - 32/3}$ $[\because r = \sqrt{g^2 + f^2 - c}]$

$$= \frac{1}{3} \sqrt{(100 - 96)} = 2/3$$

Q. 15 If $\frac{z-1}{z+1}$ is a purely imaginary number ($z \neq -1$), then find the value of $|z|$.

Thinking Process

If $z = x + iy$ is a purely imaginary number, then its real part must be equal to zero i.e., $x=0$,

Sol. Let

$$z = x + iy$$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}, z \neq -1$$

$$= \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$\begin{aligned}
 &= \frac{(x^2 - 1) + iy(x + 1) - iy(x - 1) - i^2 y^2}{(x + 1)^2 - (iy)^2} \\
 \Rightarrow \quad \frac{z - 1}{z + 1} &= \frac{(x^2 - 1) + y^2 + i[y(x + 1) - y(x - 1)]}{(x + 1)^2 + y^2} \\
 \text{Given that, } \frac{z - 1}{z + 1} &\text{ is a purely imaginary numbers.} \\
 \text{Then, } \quad \frac{(x^2 - 1) + y^2}{(x + 1)^2 + y^2} &= 0 \\
 \Rightarrow \quad x^2 - 1 + y^2 = 0 &\Rightarrow x^2 + y^2 = 1 \\
 \Rightarrow \quad \sqrt{x^2 + y^2} = \sqrt{1} &\Rightarrow |z| = 1 \quad [\because |z| = \sqrt{x^2 + y^2}]
 \end{aligned}$$

Q. 16 z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Sol. Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ are two complex numbers.

Given that, $|z_1| = |z_2|$

and $\arg(z_1) + \arg(z_2) = \pi$

If $|z_1| = |z_2|$

$\Rightarrow r_1 = r_2$... (i)

and if $\arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \theta_1 + \theta_2 = \pi$

$\Rightarrow \theta_1 = \pi - \theta_2$

Now, $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$\Rightarrow z_1 = r_2 [\cos(\pi - \theta_2) + i \sin(\pi - \theta_2)]$ [$\because r_1 = r_2$ and $\theta_1 = (\pi - \theta_2)$]

$\Rightarrow z_1 = r_2 (-\cos \theta_2 + i \sin \theta_2)$

$\Rightarrow z_1 = -r_2 (\cos \theta_2 - i \sin \theta_2)$

$\Rightarrow z_1 = -[r_2 (\cos \theta_2 - i \sin \theta_2)]$

$\Rightarrow z_1 = -\bar{z}_2$ [$\because \bar{z}_2 = r_2 (\cos \theta_2 - i \sin \theta_2)$]

Q. 17 If $|z_1| = 1$ ($z_1 \neq -1$) and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

Sol. Let $z_1 = x + iy$

$\Rightarrow |z_1| = \sqrt{x^2 + y^2} = 1$ [$\because |z_1| = 1$, given] ... (i)

Now, $z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{x + iy - 1}{x + iy + 1}$

$$\begin{aligned}
 &= \frac{x - 1 + iy}{x + 1 + iy} = \frac{(x - 1 + iy)(x + 1 - iy)}{(x + 1 + iy)(x + 1 - iy)} \\
 &= \frac{x^2 - 1 + iy(x + 1) - iy(x - 1) - i^2 y^2}{(x + 1)^2 - i^2 y^2} \\
 &= \frac{x^2 - 1 + ixy + iy - ixy + iy + y^2}{(x + 1)^2 + y^2}
 \end{aligned}$$

$$= \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2} = \frac{1-1+2iy}{(x+1)^2 + y^2} \quad [\because x^2 + y^2 = 1]$$

$$= 0 + \frac{2yi}{(x+1)^2 + y^2}$$

Hence, the real part of z_2 is zero.

Q. 18 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$.

Thinking Process

First let, $z = r(\cos \theta + i \sin \theta)$, then conjugate of z i.e., $\bar{z} = r(\cos \theta - i \sin \theta)$. Use the property $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Sol. Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$,

Then, $z_2 = \bar{z}_1 = r_1(\cos \theta_1 - i \sin \theta_1) = r_1[\cos(-\theta_1) + i \sin(-\theta_1)]$

Also, let $z_3 = r_2(\cos \theta_2 + i \sin \theta_2)$,

Then, $z_4 = \bar{z}_3 = r_2(\cos \theta_2 - i \sin \theta_2)$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$$

$$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2 \quad [\because \arg(z) = \theta]$$

$$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$$

Q. 19 If $|z_1| = |z_2| = \dots = |z_n| = 1$, then show that

$$|z_1 + z_2 + z_3 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \dots + \frac{1}{z_n} \right|$$

Sol. Given that,

$$|z_1| = |z_2| = \dots = |z_n| = 1$$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \dots = |z_n|^2 = 1$$

$$\Rightarrow z_1 \bar{z}_1 = z_2 \bar{z}_2 = z_3 \bar{z}_3 = \dots = z_n \bar{z}_n = 1$$

$$\Rightarrow z_1 = \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2} = \dots = z_n = \frac{1}{\bar{z}_n}$$

Now, $|z_1 + z_2 + z_3 + z_4 + \dots + z_n|$

$$= \left| \frac{z_1 \bar{z}_1}{\bar{z}_1} + \frac{z_2 \bar{z}_2}{\bar{z}_2} + \frac{z_3 \bar{z}_3}{\bar{z}_3} + \dots + \frac{z_n \bar{z}_n}{\bar{z}_n} \right| \quad \left[\because z_1 \bar{z}_1 = 1, \text{ where } z_1 = \frac{1}{\bar{z}_1}, z_1 = \frac{\bar{z}}{\bar{z} - \bar{z}}, z_1 = \bar{z} \right]$$

$$= \left| \frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + \dots + \frac{|z_n|^2}{\bar{z}_n} \right|$$

$$= \left| \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \dots + \frac{1}{\bar{z}_n} \right| = \sqrt{\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}}$$

$$= \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|$$

Hence proved.

Q. 20 If the complex numbers z_1 and z_2 , $\arg(z_1) - \arg(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Sol. Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$
 and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$
 $\Rightarrow \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$
 Given that, $\arg(z_1) - \arg(z_2) = 0$
 $\theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$
 $z_2 = r_2 (\cos \theta_1 + i \sin \theta_1)$ [$\because \theta_1 = \theta_2$]
 $z_1 - z_2 = (r_1 \cos \theta_1 - r_2 \cos \theta_1) + i (r_1 \sin \theta_1 - r_2 \sin \theta_1)$
 $|z_1 - z_2| = \sqrt{(r_1 \cos \theta_1 - r_2 \cos \theta_1)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_1)^2}$
 $= \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos^2 \theta_1 - 2 r_1 r_2 \sin^2 \theta_1}$
 $= \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 (\sin^2 \theta_1 + \cos^2 \theta_1)}$
 $= \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2} = \sqrt{(r_1 - r_2)^2}$
 $\Rightarrow |z_1 - z_2| = r_1 - r_2$ [$\because r = |z|$]
 $= |z_1| - |z_2|$ Hence proved.

Q. 21 Solve the system of equations $\operatorname{Re}(z^2) = 0$, $|z| = 2$.

Sol. Given that, $\operatorname{Re}(z^2) = 0$, $|z| = 2$
 Let $z = x + iy$
 $|z| = \sqrt{x^2 + y^2}$
 $\therefore \sqrt{x^2 + y^2} = 2$
 $\Rightarrow x^2 + y^2 = 4$... (i)
 and $\operatorname{Re}(z) = x$
 Also, $z = x + iy$
 $\Rightarrow z^2 = x^2 + 2ixy - y^2$
 $\Rightarrow z^2 = (x^2 - y^2) + 2ixy$
 $\Rightarrow \operatorname{Re}(z^2) = x^2 - y^2$ [$\because \operatorname{Re}(z^2) = 0$]
 $\Rightarrow x^2 - y^2 = 0$... (ii)

From Eqs. (i) and (ii),

$$\begin{aligned} x^2 + x^2 &= 4 \\ \Rightarrow 2x^2 &= 4 \Rightarrow x^2 = 2 \\ \Rightarrow x &= \pm \sqrt{2} \\ \therefore y &= \pm \sqrt{2} \\ \therefore z &= x + iy \\ \Rightarrow z &= \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2} \end{aligned}$$

Q. 22 Find the complex number satisfying the equation $z + \sqrt{2} |(z+1)| + i = 0$.

Sol. Given equation is $z + \sqrt{2} |(z+1)| + i = 0$... (i)

Let $z = x + iy$

$$\Rightarrow x + iy + \sqrt{2} |x + iy + 1| + i = 0$$

$$\Rightarrow x + i(1+y) + \sqrt{2} [\sqrt{(x+1)^2 + y^2}] = 0$$

$$\Rightarrow x + i(1+y) + \sqrt{2} \sqrt{x^2 + 2x + 1 + y^2} = 0$$

$$\Rightarrow x + \sqrt{2} \sqrt{x^2 + 2x + 1 + y^2} = 0$$

$$\Rightarrow x^2 = 2(x^2 + 2x + 1 + y^2)$$

$$\Rightarrow x^2 + 4x + 2y^2 + 2 = 0 \quad \dots (ii)$$

$$1 + y = 0$$

$$\Rightarrow y = -1$$

For $y = -1$, $x^2 + 4x + 2 + 2 = 0$ [using Eq. (ii)]

$$\Rightarrow x^2 + 4x + 4 = 0 \Rightarrow (x+2)^2 = 0$$

$$\Rightarrow x + 2 = 0 \Rightarrow x = -2$$

$$\therefore z = x + iy = -2 - i$$

Q. 23 Write the complex number $z = \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$ in polar form.

Sol. Given that,
$$z = \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{-\sqrt{2} \left[\frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right]}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$$

$$= \frac{-\sqrt{2} [\cos(\pi - \pi/4) + i \sin(\pi - \pi/4)]}{\cos \pi/3 + i \sin \pi/3}$$

$$= \frac{-\sqrt{2} [\cos 3\pi/4 + i \sin 3\pi/4]}{\cos \pi/3 + i \sin \pi/3}$$

$$= -\sqrt{2} \left[\cos \left(\frac{3\pi}{4} - \frac{\pi}{3} \right) + i \sin \left(\frac{3\pi}{4} - \frac{\pi}{3} \right) \right]$$

$$= -\sqrt{2} \left[\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right]$$

Q. 24 If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

Sol. Let $z = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $w = r_2 (\cos \theta_2 + i \sin \theta_2)$

Also, $|zw| = |z| |w| = r_1 r_2 = 1$

$$\therefore r_1 r_2 = 1$$

Further, $\arg(z) = \theta_1$ and $\arg(w) = \theta_2$

$$\text{But} \quad \arg(z) - \arg(w) = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\Rightarrow \quad \arg\left(\frac{z}{w}\right) = \frac{\pi}{2}$$

Now, to prove $\bar{z}w = -i$

$$\text{LHS} = \bar{z}w$$

$$= r_1(\cos \theta_1 - i \sin \theta_1) r_2(\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 r_2 [\cos(\theta_2 - \theta_1) + i \sin(\theta_2 - \theta_1)]$$

$$= r_1 r_2 [\cos(-\pi/2) + i \sin(-\pi/2)]$$

$$= 1[0 - i]$$

$$= -i = \text{RHS}$$

Hence proved.

Q. 25 Fill in the blanks of the following.

- (i) For any two complex numbers z_1, z_2 and any real numbers a, b ,
 $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \dots$
- (ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ...
- (iii) The number $\frac{(1-i)^3}{1-i^3}$ is equal to ...
- (iv) The sum of the series $i + i^2 + i^3 + \dots$ upto 1000 terms is ...
- (v) Multiplicative inverse of $1 + i$ is ...
- (vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_1 = \dots$
- (vii) $\arg(z) + \arg \bar{z}$ where, ($\bar{z} \neq 0$) is ...
- (viii) If $|z + 4| \leq 3$, then the greatest and least values of $|z + 1|$ are ... and ...
- (ix) If $\left| \frac{z-2}{z+2} \right| = \frac{\pi}{6}$, then the locus of z is ...
- (x) If $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then $z = \dots$

Sol. (i) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$$= |az_1|^2 + |bz_2|^2 - 2\operatorname{Re}(az_1 \cdot b\bar{z}_2) + |bz_1|^2 + |az_2|^2 + 2\operatorname{Re}(az_1 \cdot b\bar{z}_2)$$

$$= (a^2 + b^2)|z_1|^2 + (a^2 + b^2)|z_2|^2$$

$$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

(ii) $\sqrt{-25} \times \sqrt{-9} = i\sqrt{25} \times i\sqrt{9} = i^2 (5 \times 3) = -15$

(iii) $\frac{(1-i)^3}{1-i^3} = \frac{(1-i)^3}{(1-i)(1+i+i^2)}$

$$= \frac{(1-i)^2}{i} = \frac{1+i^2-2i}{i} = \frac{-2i}{i} = -2$$

(iv) $i + i^2 + i^3 + \dots$ upto 1000 terms $= i + i^2 + i^3 + i^4 + \dots i^{1000} = 0$

$$\left[\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in \text{Ni.e.}, \sum_{n=1}^{1000} i^n = 0 \right]$$

(v) Multiplicative inverse of $1 + i = \frac{1}{1+i} = \frac{1-i}{1-i^2} = \frac{1}{2} (1-i)$

(vi) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, which is real.

If $z_1 + z_2$ is real, then $y_1 + y_2 = 0$

$\Rightarrow y_1 = -y_2$

$\therefore z_2 = x_2 - iy_1$

$\Rightarrow z_2 = \bar{z}_1$ [when $x_1 = x_2$]

(vii) $\arg(z) + \arg(\bar{z})$, ($\bar{z} \neq 0$)

$\Rightarrow \theta + (-\theta) = 0$

(viii) Given that, $|z + 4| \leq 3$

For the greatest value of $|z + 1|$.

$$\begin{aligned} \Rightarrow |z + 1| &= |z + 4 - 3| \leq |z + 4| + |-3| \\ &= |z + 4 - 3| \leq 3 + 3 \\ &= |z + 4 - 3| \leq 6 \end{aligned}$$

So, greatest value of $|z + 1| = 6$

For, now, least value of $|z + 1|$, we know that the least value of the modulus of a complex number is zero. So, the least value of $|z + 1|$ is zero.

(ix) Given that, $\left| \frac{z-2}{z+2} \right| = \frac{\pi}{6}$

$$\Rightarrow \frac{|x + iy - 2|}{|x + iy + 2|} = \frac{\pi}{6} \Rightarrow \frac{|x - 2 + iy|}{|x + 2 + iy|} = \frac{\pi}{6}$$

$$\Rightarrow 6|x - 2 + iy| = \pi|x + 2 + iy|$$

$$\Rightarrow 6\sqrt{(x-2)^2 + y^2} = \pi\sqrt{(x+2)^2 + y^2}$$

$$\Rightarrow 36[x^2 + 4 - 4x + y^2] = \pi^2[x^2 + 4x + 4 + y^2]$$

$$\Rightarrow (36 - \pi^2)x^2 + (36 - \pi^2)y^2 - (144 + 4\pi^2)x + 144 + 4\pi^2 = 0, \text{ which is a circle.}$$

(x) Given that, $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$

Let $z = x + iy = r(\cos \theta + i \sin \theta)$

$\Rightarrow |z| = r = 4$ and $\arg(z) = \theta$

$\therefore \tan \theta = \frac{5\pi}{6}$

$$\Rightarrow z = 4\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) = 4[\cos(\pi - \pi/6) + i \sin(\pi - \pi/6)]$$

$$= 4\left[-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right] = 4\left[\frac{-\sqrt{3}}{2} + i \frac{1}{2}\right] = -2\sqrt{3} + 2i$$

True/False

Q. 26 State true or false for the following.

- (i) The order relation is defined on the set of complex numbers.
- (ii) Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.
- (iii) For any complex number z , the minimum value of $|z| + |z - 1|$ is 1.
- (iv) The locus represented by $|z - 1| = |z - i|$ is a line perpendicular to the join of the points (1, 0) and (0, 1).
- (v) If z is a complex number such that $z \neq 0$ and $\operatorname{Re}(z) = 0$, then, $\operatorname{Im}(z^2) = 0$.
- (vi) The inequality $|z - 4| < |z - 2|$ represents the region given by $x > 3$.
- (vii) Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1 - z_2) = 0$.
- (viii) 2 is not a complex number.

Sol. (i) **False**

We can compare two complex numbers when they are purely real. Otherwise comparison of complex number is not possible.

(ii) **False**

$$(x, y) \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \text{ which is false.}$$

(iii) **True**

Let

$$z = x + iy$$

$$|z| + |z - 1| = \sqrt{x^2 + y^2} + \sqrt{(x - 1)^2 + y^2}$$

If $x = 0, y = 0$, then the value of $|z| + |z - 1| = 1$.

(iv) **True**

Let

$$z = x + iy$$

$$|z - 1| = |z - i|$$

Then,

$$|x - 1 + iy| = |x - i(1 - y)|$$

$$(x - 1)^2 + y^2 = x^2 + (1 - y)^2$$

$$x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$$

$$-2x + 1 = 1 - 2y$$

$$-2x + 2y = 0$$

$$x - y = 0$$

...(i)

Equation of a line through the points (1, 0) and (0, 1),

$$y - 0 = \frac{1 - 0}{0 - 1}(x - 1)$$

$\Rightarrow$

$$y = -(x - 1) \Rightarrow x + y = 1$$

...(ii)

which is perpendicular to the line $x - y = 0$.

(v) False

Let $z = x + iy$, $z \neq 0$ and $\operatorname{Re}(z) = 0$

i.e., $x = 0$

$\therefore z = iy$

$\operatorname{Im}(z^2) = i^2 y^2 = -y^2$ which is real.

(vi) True

Given inequality,

$$|z - 4| < |z - 2|$$

Let

$$z = x + iy$$

$\therefore$

$$\frac{|x - 4 + iy|}{\sqrt{(x - 4)^2 + y^2}} < \frac{|x - 2 + iy|}{\sqrt{(x - 2)^2 + y^2}}$$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

(vii) False

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$$\Rightarrow \frac{|x_1 + iy_1 + x_2 + iy_2|}{\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}} = \frac{|x_1 + iy_1| + |x_2 + iy_2|}{\sqrt{(x_1^2 + y_1^2)} + \sqrt{(x_2^2 + y_2^2)}}$$

On squaring both sides, we get

$$\begin{aligned} (x_1 + x_2)^2 + (y_1 + y_2)^2 &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ \Rightarrow x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 &= x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ \Rightarrow 2x_1x_2 + 2y_1y_2 &= 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ \Rightarrow x_1x_2 + y_1y_2 &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \end{aligned}$$

On squaring both sides, we get

$$\begin{aligned} x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 &= x_1^2x_2^2 + y_1^2x_2^2 + x_1^2y_2^2 + y_1^2y_2^2 \\ \Rightarrow (x_1y_2 - x_2y_1)^2 &= 0 \\ \Rightarrow x_1y_2 &= x_2y_1 \\ \Rightarrow \frac{y_1}{x_1} &= \frac{y_2}{x_2} \\ \Rightarrow \left(\frac{y_1}{x_1}\right) - \left(\frac{y_2}{x_2}\right) &= 0 \\ \Rightarrow \arg(z_1) - \arg(z_2) &= 0 \end{aligned}$$

(viii) True

We know that, 2 is a real number.

Since, 2 is not a complex number.

Q. 27 Match the statements of Column A and Column B.

Column A	Column B
(i) The polar form of $i + \sqrt{3}$ is	(a) Perpendicular bisector of segment joining $(-2, 0)$ and $(2, 0)$.
(ii) The amplitude of $-1 + \sqrt{-3}$ is	(b) On or outside the circle having centre at $(0, -4)$ and radius 3.
(iii) It $ z + 2 = z - 2 $, then locus of z is	(c) $\frac{2\pi}{3}$
(iv) It $ z + 2i = z - 2i $, then locus of z is	(d) Perpendicular bisector of segment joining $(0, -2)$ and $(0, 2)$.
(v) Region represented by	(e) $2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$
(vi) Region represented by $ z + 4 \leq 3$ is	(f) On or inside the circle having centre $(-4, 0)$ and radius 3 units.
(vii) Conjugate of $\frac{1+2i}{1-i}$ lies in	(g) First quadrant
(viii) Reciprocal of $1 - i$ lies in	(h) Third quadrant

Sol. (i) Given,

$$z = i + \sqrt{3} = r(\cos \theta + i \sin \theta)$$

$$\therefore r \cos \theta = \sqrt{3}, r \sin \theta = 1$$

$$\Rightarrow r^2 = 1 + 3 = 4 \Rightarrow r = 2 \quad [\because r > 0]$$

$$\Rightarrow \tan \alpha = \frac{r \sin \theta}{r \cos \theta} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$$

$$\therefore x > 0, y > 0$$

$$\text{and } \arg(z) = \theta = \frac{\pi}{6}$$

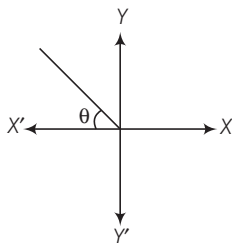
$$\text{So the polar form of } z \text{ is } 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right).$$

(ii) Given that,

$$z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$$

$$\therefore \tan \alpha = \frac{|\sqrt{3}|}{|-1|} = \sqrt{3}$$

$$\Rightarrow \tan \alpha = \tan \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}$$



$$\therefore x < 0, y > 0$$

$$\Rightarrow \theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

(iii) Given that,

$$\begin{aligned}
 &|z + 2| = |z - 2| \\
 \Rightarrow &|x + 2 + iy| = |x - 2 + iy| \\
 \Rightarrow &(x + 2)^2 + y^2 = (x - 2)^2 + y^2 \\
 \Rightarrow &x^2 + 4x + 4 = x^2 - 4x + 4 \Rightarrow 8x = 0 \\
 \therefore &x = 0
 \end{aligned}$$

It is a straight line which is a perpendicular bisector of segment joining the points $(-2, 0)$ and $(2, 0)$.

(iv) Given that,

$$\begin{aligned}
 &|z + 2i| = |z - 2i| \\
 \Rightarrow &|x + i(y + 2)| = |x + i(y - 2)| \\
 \Rightarrow &x^2 + (y + 2)^2 = x^2 + (y - 2)^2 \\
 \Rightarrow &4y = 0 \Rightarrow y = 0
 \end{aligned}$$

It is a straight line, which is a perpendicular bisector of segment joining $(0, -2)$ and $(0, 2)$.

(v) Given that,

$$\begin{aligned}
 &|z + 4i| \geq 3 = |x + iy + 4i| \geq 3 \\
 \Rightarrow &= |x + i(y + 4)| \geq 3 \\
 \Rightarrow &= \sqrt{x^2 + (y + 4)^2} \geq 3 \\
 \Rightarrow &= x^2 + (y + 4)^2 \geq 9 \\
 \Rightarrow &= x^2 + y^2 + 8y + 16 \geq 9 \\
 \Rightarrow &= x^2 + y^2 + 8y + 7 \geq 0
 \end{aligned}$$

Which represent a circle. On or outside having centre $(0, -4)$ and radius 3.

(vi) Given that,

$$\begin{aligned}
 &|z + 4| \leq 3 \\
 \Rightarrow &|x + iy + 4| \leq 3 \\
 \Rightarrow &|x + 4 + iy| \leq 3 \\
 \Rightarrow &\sqrt{(x + 4)^2 + y^2} \leq 3 \\
 \Rightarrow &(x + 4)^2 + y^2 \leq 9 \\
 \Rightarrow &x^2 + 8x + 16 + y^2 \leq 9 \\
 \Rightarrow &x^2 + 8x + y^2 + 7 \leq 0
 \end{aligned}$$

It represent the region which is on or inside the circle having the centre $(-4, 0)$ and radius 3.

(vii) Given that,

$$\begin{aligned}
 z &= \frac{1 + 2i}{1 - i} = \frac{(1 + 2i)(1 + i)}{(1 - i)(1 + i)} \\
 &= \frac{1 + 2i + i + 2i^2}{1 - i^2} = \frac{1 - 2 + 3i}{1 + 1} = \frac{-1 + 3i}{2} \\
 \therefore \bar{z} &= \frac{-1}{2} - \frac{3i}{2}
 \end{aligned}$$

Hence, $\left(\frac{-1}{2}, \frac{-3}{2}\right)$ lies in third quadrant.

(viii) Given that, $z = 1 - i$

$$\therefore \frac{1}{z} = \frac{1}{1 - i} = \frac{1 + i}{(1 - i)(1 + i)} = \frac{1 + i}{1 - i^2} = \frac{1}{2}(1 + i)$$

Hence, $\left(\frac{1}{2}, \frac{1}{2}\right)$ lies in first quadrant.

Hence, the correct matches are **(a)** $\rightarrow$ (v), **(b)** $\rightarrow$ (iii), **(c)** $\rightarrow$ (i), **(d)** $\rightarrow$ (iv), **(e)** $\rightarrow$ (ii), **(f)** $\rightarrow$ (vi), **(g)** $\rightarrow$ (viii), **(h)** $\rightarrow$ (vii)

Q. 28 What is the conjugate of $\frac{2-i}{(1-2i)^2}$?

Sol. Given that,

$$\begin{aligned} z &= \frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i} \\ &= \frac{2-i}{1-4-4i} = \frac{2-i}{-3-4i} \\ &= \frac{(2-i)}{-(3+4i)} = - \left[\frac{(2-i)(3-4i)}{(3+4i)(3-4i)} \right] \\ &= - \left(\frac{6-8i-3i+4i^2}{9+16} \right) = - \frac{(-11i+2)}{25} \\ &= \frac{-1}{25}(2-11i) \Rightarrow z = \frac{1}{25}(-2+11i) \\ \therefore \bar{z} &= \frac{1}{25}(-2-11i) = \frac{-2}{25} - \frac{11}{25}i \end{aligned}$$

Q. 29 If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$.

Sol. Given that,

$$\begin{aligned} \text{Let } |z_1| &= |z_2| \\ z_1 &= x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2 \\ \Rightarrow |x_1 + iy_1| &= |x_2 + iy_2| \\ \Rightarrow x_1^2 + y_1^2 &= x_2^2 + y_2^2 \\ \Rightarrow x_1^2 = x_2^2, y_1^2 &= y_2^2 \\ \Rightarrow x_1 = \pm x_2, y_1 &= \pm y_2 \\ \Rightarrow z_1 = x_1 + iy_1 \text{ or } z_1 &= \pm x_2 \pm iy_2 \end{aligned}$$

Hence, it is not necessary that $z_1 = z_2$.

Q. 30 If $\frac{(a^2+1)^2}{2a-i} = x + iy$, then what is the value of $x^2 + y^2$?

Sol. Given that,

$$\begin{aligned} \frac{(a^2+1)^2}{2a-i} &= x + iy \Rightarrow \frac{(a^2+1)^2}{(2a-i)} = x + iy \\ \Rightarrow \frac{(a^2+1)^2(2a+i)}{(2a-i)(2a+i)} &= x + iy \\ \Rightarrow \frac{(a^2+1)^2(2a+i)}{4a^2+1} &= x + iy \\ \Rightarrow x = \frac{2a(a^2+1)^2}{4a^2+1} \text{ and } y &= \frac{(a^2+1)^2}{4a^2+1} \\ \therefore x^2 + y^2 &= 4a^2 \left[\frac{(a^2+1)^2}{4a^2+1} \right]^2 + \left[\frac{(a^2+1)^2}{4a^2+1} \right]^2 \\ &= \frac{(4a^2+1)(a^2+1)^4}{(4a^2+1)^2} = \frac{(a^2+1)^4}{(4a^2+1)} \end{aligned}$$

Q. 31 Find the value of z , if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

Sol. Let $z = r(\cos \theta + i \sin \theta)$
 Also, $|z| = r = 4$ and $\theta = \frac{5\pi}{6}$ [$\because \arg(z) = \theta$]
 $\therefore z = 4 \left[\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right]$ [$\because z = r(\cos \theta + i \sin \theta)$]
 $= 4 \left[\cos \left(\pi - \frac{\pi}{6} \right) + i \sin \left(\pi - \frac{\pi}{6} \right) \right]$
 $= 4 \left[-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right]$
 $= 4 \left[-\frac{\sqrt{3}}{2} + i \frac{1}{2} \right] = -2\sqrt{3} + 2i$

Q. 32 Find the value of $\left| (1+i) \frac{(2+i)}{(3+i)} \right|$.

Thinking Process

First, convert the given expression in the form $a + ib$, then use $|a + ib| = \sqrt{a^2 + b^2}$.

Sol. Given that, $\left| (1+i) \frac{(2+i)}{(3+i)} \right| = \left| \frac{(2+i+2i+i^2)}{(3+i)} \right| = \left| \frac{2+3i-1}{3+i} \right|$
 $= \left| \frac{1+3i}{3+i} \right| = \left| \frac{(1+3i)(3-i)}{(3+i)(3-i)} \right|$
 $= \left| \frac{3+9i-i-3i^2}{9-i^2} \right| = \left| \frac{3+8i+3}{9+1} \right| = \left| \frac{6+8i}{10} \right|$
 $= \sqrt{\frac{6^2}{100} + \frac{8^2}{100}} = \sqrt{\frac{36+64}{100}} = \sqrt{\frac{100}{100}} = 1$

Q. 33 Find the principal argument of $(1+i\sqrt{3})^2$.

Thinking Process

Let $z = a + ib$, then the polar form of z is $r(\cos \theta + i \sin \theta)$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan \theta = \frac{b}{a}$. Here, θ is argument or amplitude of z i.e., $\arg(z) = \theta$. The principal argument is a unique value of θ such that $-\pi \leq \theta \leq \pi$.

Sol. Given that, $z = (1+i\sqrt{3})^2$
 $\Rightarrow z = 1 - 3 + 2i\sqrt{3} \Rightarrow z = -2 + i2\sqrt{3}$
 $\Rightarrow \tan \alpha = \left| \frac{2\sqrt{3}}{-2} \right| = |-\sqrt{3}| = \sqrt{3}$ [$\because \tan \alpha = \left| \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right|$]
 $\Rightarrow \tan \alpha = \tan \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}$
 $\because \operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) > 0$
 $\Rightarrow \arg(z) = \pi - \frac{\pi}{3} \Rightarrow = \frac{2\pi}{3}$

Q. 34 Where does z lie, if $\left| \frac{z - 5i}{z + 5i} \right| = 1$?

Thinking Process

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$.

Also, use the modulus property i.e., $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$,

Sol. Let

$$z = x + iy$$

Given that, $\left| \frac{z - 5i}{z + 5i} \right| = \left| \frac{x + iy - 5i}{x + iy + 5i} \right|$

$$\Rightarrow \left| \frac{z - 5i}{z + 5i} \right| = \left| \frac{x + i(y - 5)}{x + i(y + 5)} \right| \quad \left[\because \left| \frac{z - 5i}{z + 5i} \right| = 1 \right]$$

$$\Rightarrow \left| \frac{z - 5i}{z + 5i} \right| = \frac{\sqrt{x^2 + (y - 5)^2}}{\sqrt{x^2 + (y + 5)^2}}$$

On squaring both sides, we get

$$x^2 + (y - 5)^2 = x^2 + (y + 5)^2$$

$$\Rightarrow -10y = +10y$$

$$\Rightarrow 20y = 0$$

$$\therefore y = 0$$

So, z lies on real axis.

Objective Type Questions

Q. 35 $\sin x + i \cos 2x$ and $\cos x - i \sin 2x$ are conjugate to each other for

(a) $x = n\pi$

(b) $x = \left(n + \frac{1}{2}\right) \frac{\pi}{2}$

(c) $x = 0$

(d) No value of x

Sol. (d)

Let

$$z = \sin x + i \cos 2x$$

and

$$\bar{z} = \sin x - i \cos 2x$$

...(i)

Given that,

$$\bar{z} = \cos x - i \sin 2x$$

...(ii)

$\therefore$

$$\sin x - i \cos 2x = \cos x - i \sin 2x$$

$\Rightarrow$

$$\sin x = \cos x \text{ and } \cos 2x = \sin 2x$$

$\Rightarrow$

$$\tan x = 1 \text{ and } \tan 2x = 1$$

$\Rightarrow$

$$\tan x = \tan \frac{\pi}{4} \text{ and } \tan 2x = \tan \frac{\pi}{4}$$

$\Rightarrow$

$$x = n\pi + \frac{\pi}{4} \text{ and } 2x = n\pi + \frac{\pi}{4}$$

$\Rightarrow$

$$2x - x = 0 \Rightarrow x = 0$$

Q. 36 The real value of α for which the expression $\frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$ is purely real is

- (a) $(n+1) \frac{\pi}{2}$ (b) $(2n+1) \frac{\pi}{2}$ (c) $n\pi$ (d) None of these

where, $n \in \mathbb{N}$

Thinking Process

First, convert the given expansion into $a + ib$ form and then check whether the complex number $a + ib$ is purely real.

Sol. (c) Given expression, $z = \frac{1 - i \sin \alpha}{1 + 2i \sin \alpha}$ [let]

$$\begin{aligned}
 &= \frac{(1 - i \sin \alpha)(1 - 2i \sin \alpha)}{(1 + 2i \sin \alpha)(1 - 2i \sin \alpha)} \\
 &= \frac{1 - i \sin \alpha - 2i \sin \alpha + 2i^2 \sin^2 \alpha}{1 - 4i^2 \sin^2 \alpha} \\
 &= \frac{1 - 3i \sin \alpha - 2 \sin^2 \alpha}{1 + 4 \sin^2 \alpha} \\
 &= \frac{1 - 2 \sin^2 \alpha}{1 + 4 \sin^2 \alpha} - \frac{3i \sin \alpha}{1 + 4 \sin^2 \alpha}
 \end{aligned}$$

It is given that z is a purely real.

$$\begin{aligned}
 \therefore \quad &\frac{-3 \sin \alpha}{1 + 4 \sin^2 \alpha} = 0 \\
 \Rightarrow \quad &-3 \sin \alpha = 0 \Rightarrow \sin \alpha = 0 \\
 &\alpha = n\pi
 \end{aligned}$$

Q. 37 If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant, if

- (a) $x > y > 0$ (b) $x < y < 0$ (c) $y < x < 0$ (d) $y > x > 0$

Sol. (b) Given that, $z = x + iy$ lies in third quadrant.

$$x < 0 \text{ and } y < 0.$$

Now,

$$\begin{aligned}
 \frac{\bar{z}}{z} &= \frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2} \\
 \frac{\bar{z}}{z} &= \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}
 \end{aligned}$$

Since, $\frac{\bar{z}}{z}$ also lies in third quadrant.

$$\begin{aligned}
 \therefore \quad &\frac{x^2 - y^2}{x^2 + y^2} < 0 \text{ and } \frac{-2xy}{x^2 + y^2} < 0 \\
 &x^2 - y^2 < 0 \text{ and } -2xy < 0 \\
 \Rightarrow \quad &x^2 < y^2 \text{ and } xy > 0 \\
 \text{So,} \quad &x < y < 0
 \end{aligned}$$

Q. 38 The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

- (a) $|z + 3|^2$ (b) $|z - 3|$ (c) $z^2 + 3$ (d) None of these

Sol. (a) Given that, $(z + 3)(\bar{z} + 3)$

$$\begin{aligned} \text{Let } z &= x + iy \\ \Rightarrow (z + 3)(\bar{z} + 3) &= (x + iy + 3)(x + 3 - iy) \\ &= (x + 3)^2 - (iy)^2 = (x + 3)^2 + y^2 \\ &= |x + 3 + iy|^2 = |z + 3|^2 \end{aligned}$$

Q. 39 If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

- (a) $x = 2n + 1$ (b) $x = 4n$ (c) $x = 2n$ (d) $x = 4n + 1$

where, $n \in N$

Sol. (b) Given that, $\left(\frac{1+i}{1-i}\right)^x = 1$

$$\Rightarrow \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x = 1 \Rightarrow \left[\frac{1+2i+i^2}{1-i^2}\right]^x = 1$$

$$\Rightarrow \left[\frac{2i}{1+1}\right]^x = 1 \Rightarrow \left[\frac{2i}{2}\right]^x = 1$$

$$\Rightarrow i^x = 1 \Rightarrow i^x = (i^{4n}) \quad [\therefore i^{4n} = 1, n \in N]$$

$$\Rightarrow x = 4n$$

Q. 40 A real value of x satisfies the equation $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta$ ($\alpha, \beta \in R$), if

$\alpha^2 + \beta^2$ is equal to

- (a) 1 (b) -1 (c) 2 (d) -2

Sol. (a) Given equation, $\left(\frac{3-4ix}{3+4ix}\right) = \alpha - i\beta$ ($\alpha, \beta \in R$)

$$\Rightarrow \left[\frac{3-4ix}{3+4ix}\right] = \alpha - i\beta$$

$$\text{Now, } (\alpha - i\beta) = \frac{(3-4ix)(3-4ix)}{(3+4ix)(3-4ix)} = \frac{9 + 16i^2x^2 - 24ix}{9 - 16i^2x^2}$$

$$\Rightarrow \alpha - i\beta = \frac{9 - 16x^2 - 24ix}{9 + 16x^2}$$

$$\Rightarrow \alpha - i\beta = \frac{9 - 16x^2}{9 + 16x^2} - \frac{i24x}{9 + 16x^2} \quad \dots(i)$$

$$\therefore \alpha + i\beta = \frac{9 - 16x^2}{9 + 16x^2} + \frac{i24x}{9 + 16x^2} \quad \dots(ii)$$

$$\begin{aligned}
 \text{So, } (\alpha - i\beta)(\alpha + i\beta) &= \left(\frac{9 - 16x^2}{9 + 16x^2}\right)^2 - \left(\frac{i24x}{9 + 16x^2}\right)^2 \\
 \therefore \alpha^2 + \beta^2 &= \frac{81 + 256x^4 - 288x^2 + 576x^2}{(9 + 16x^2)^2} \\
 &= \frac{81 + 256x^4 + 288x^2}{(9 + 16x^2)^2} \\
 &= \frac{(9 + 16x^2)^2}{(9 + 16x^2)^2} = 1
 \end{aligned}$$

Q. 41 Which of the following is correct for any two complex numbers z_1 and z_2 ?

- (a) $|z_1 z_2| = |z_1| |z_2|$ (b) $\arg(z_1 z_2) = \arg(z_1) \cdot \arg(z_2)$
 (c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 + z_2| \geq |z_1| - |z_2|$

Sol. (a) Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$
 $\Rightarrow |z_1| = r_1$... (i)
 and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$
 $\Rightarrow |z_2| = r_2$... (ii)
 Now, $z_1 z_2 = r_1 r_2 [\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2]$
 $= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$
 $\Rightarrow |z_1 z_2| = r_1 r_2$
 $\therefore |z_1 z_2| = |z_1| |z_2|$ [using Eqs. (i) and (ii)]

Q. 42 The point represented by the complex number $(2 - i)$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

- (a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

Thinking Process

Here, $z = 2 - i$ is a complex number, where modulus is r and argument $(\theta + \alpha)$. If $P(z)$ rotates in clockwise sense through an angle α , then its new position will be $z(\theta - i\alpha)$.

Sol. (b) Given that, $z = 2 - i$
 It is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction
 $\therefore$ New position $= ze^{-i\pi/2} = (2 - i)e^{-i\pi/2}$
 $= (2 - i) \left[\cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \right] = (2 - i)[0 - i]$
 $= -2i - 1 = -1 - 2i$

Q. 43 If $x, y \in R$, then $x + iy$ is a non-real complex number, if

- (a) $x = 0$ (b) $y = 0$ (c) $x \neq 0$ (d) $y \neq 0$

Sol. (d) Given that, $x, y \in R$
 Then, $x + iy$ is non-real complex number if and only if $y \neq 0$.

Q. 44 If $a + ib = c + id$, then

(a) $a^2 + c^2 = 0$

(b) $b^2 + c^2 = 0$

(c) $b^2 + d^2 = 0$

(d) $a^2 + b^2 = c^2 + d^2$

Thinking Process

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, then

$$|z_1| = |z_2| \Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

Sol. (d) Given that,

$$a + ib = c + id$$

$\Rightarrow$

$$|a + ib| = |c + id|$$

$\Rightarrow$

$$\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$$

On squaring both sides, we get

$$a^2 + b^2 = c^2 + d^2$$

Q. 45 The complex number z which satisfies the condition $\left| \frac{i + z}{i - z} \right| = 1$ lies on

(a) circle $x^2 + y^2 = 1$

(b) the X-axis

(c) the Y-axis

(d) the line $x + y = 1$

Sol. (b) Given that,

$$\left| \frac{i + z}{i - z} \right| = 1$$

Let

$$z = x + iy$$

$$\therefore \left| \frac{x + i(y + 1)}{-x - i(y - 1)} \right| = 1 \Rightarrow \frac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2} = 1$$

$\Rightarrow$

$$x^2 + (y + 1)^2 = x^2 + (y - 1)^2$$

$\Rightarrow$

$$4y = 0 \Rightarrow y = 0$$

So, z lies on X-axis (real axis).

Q. 46 If z is a complex number, then

(a) $|z^2| > |z|$

(b) $|z^2| = |z|^2$

(c) $|z^2| < |z|^2$

(d) $|z^2| \geq |z|^2$

Sol. (b) If z is a complex number, then $z = x + iy$

$$|z| = |x + iy| \text{ and } |z|^2 = |x + iy|^2$$

$\Rightarrow$

$$|z|^2 = x^2 + y^2$$

...(i)

and

$$z^2 = (x + iy)^2 = x^2 + i^2 y^2 + i2xy$$

$$z^2 = x^2 - y^2 + i2xy$$

$\Rightarrow$

$$|z^2| = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$$

$\Rightarrow$

$$|z^2| = \sqrt{x^4 + y^4 - 2x^2 y^2 + 4x^2 y^2}$$

$\Rightarrow$

$$|z^2| = \sqrt{x^4 + y^4 + 2x^2 y^2} = \sqrt{(x^2 + y^2)^2}$$

$\Rightarrow$

$$|z^2| = x^2 + y^2$$

...(ii)

From Eqs. (i) and (ii),

$$|z|^2 = |z^2|$$

Q. 47 $|z_1 + z_2| = |z_1| + |z_2|$ is possible, if

(a) $z_2 = \bar{z}_1$

(b) $z_2 = \frac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$

(d) $|z_1| = |z_2|$

Sol. (c) Given that, $|z_1 + z_2| = |z_1| + |z_2|$
 $\Rightarrow |r_1(\cos \theta_1 + i \sin \theta_1) + r_2(\cos \theta_2 + i \sin \theta_2)| = |r_1(\cos \theta_1 + i \sin \theta_1)| + |r_2(\cos \theta_2 + i \sin \theta_2)|$
 $\Rightarrow |(r_1 \cos \theta_1 + r_2 \cos \theta) + i(r_1 \sin \theta_1 + r_2 \sin \theta_2)| = r_1 + r_2$
 $\Rightarrow \sqrt{r_1^2 \cos^2 \theta_1 + r_2^2 \cos^2 \theta_2 + 2r_1r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \sin^2 \theta_1 + r_2^2 \sin^2 \theta_2} = r_1 + r_2$
 $\Rightarrow \sqrt{r_1^2 + r_2^2 + 2r_1r_2[\cos(\theta_1 - \theta_2)]} = r_1 + r_2$

On squaring both sides, we get

$$\begin{aligned} r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2) &= r_1^2 + r_2^2 + 2r_1r_2 \\ \Rightarrow 2r_1r_2[1 - \cos(\theta_1 - \theta_2)] &= 0 \\ \Rightarrow 1 - \cos(\theta_1 - \theta_2) &= 0 \\ \Rightarrow \cos(\theta_1 - \theta_2) &= 1 \\ \Rightarrow \cos(\theta_1 - \theta_2) &= \cos 0^\circ \\ \Rightarrow \theta_1 - \theta_2 &= 0^\circ \\ \Rightarrow \theta_1 &= \theta_2 \\ \therefore \arg(z_1) &= \arg(z_2) \end{aligned}$$

Q. 48 The real value of θ for which the expression $\frac{1 + i \cos \theta}{1 - 2i \cos \theta}$ is a real number is

(a) $n\pi + \frac{\pi}{4}$

(b) $n\pi + (-1)^n \frac{\pi}{4}$

(c) $2n\pi \pm \frac{\pi}{2}$

(d) None of these

Sol. (c) Given expression = $\frac{1 + i \cos \theta}{1 - 2i \cos \theta} = \frac{(1 + i \cos \theta)(1 + 2i \cos \theta)}{(1 - 2i \cos \theta)(1 + 2i \cos \theta)}$
 $= \frac{1 + i \cos \theta + 2i \cos \theta + 2i^2 \cos^2 \theta}{1 - 4i^2 \cos^2 \theta}$
 $= \frac{1 + 3i \cos \theta - 2 \cos^2 \theta}{1 + 4 \cos^2 \theta}$

For real value of θ , $\frac{3 \cos \theta}{1 + 4 \cos^2 \theta} = 0$

$\Rightarrow 3 \cos \theta = 0$

$\Rightarrow \cos \theta = \cos \frac{\pi}{2}$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{2}$

Q. 49 The value of $\arg(x)$, when $x < 0$ is

(a) 0

(b) $\frac{\pi}{2}$

(c) π

(d) None of these

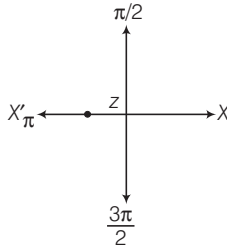
Sol. (c) Let

$$z = x + 0i \text{ and } x < 0$$

$$|z| = \sqrt{(-1)^2 + (0^2)} = 1$$

Since, the point $(x, 0)$ represent $z = x + 0i$ lies on the negative side of real axis.

$\therefore$ Principal $\arg(z) = \pi$



Q. 50 If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is equal to

(a) $\frac{|z|}{2}$

(b) $|z|$

(c) $2|z|$

(d) None of these

Sol. (a) Let

$$z = 1 + 2i$$

$\Rightarrow$

$$|z| = \sqrt{1+4} = \sqrt{5}$$

Now,

$$\begin{aligned} f(z) &= \frac{7-z}{1-z^2} = \frac{7-1-2i}{1-(1+2i)^2} \\ &= \frac{6-2i}{1-1-4i^2-4i} = \frac{6-2i}{4-4i} \\ &= \frac{(3-i)(2+2i)}{(2-2i)(2+2i)} \\ &= \frac{6-2i+6i-2i^2}{4-4i^2} = \frac{6+4i+2}{4+4} \\ &= \frac{8+4i}{8} = 1 + \frac{1}{2}i \end{aligned}$$

$$f(z) = 1 + \frac{1}{2}i$$

$\therefore$

$$|f(z)| = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{4+1}{4}} = \frac{\sqrt{5}}{2} = \frac{|z|}{2}$$