14

Three Dimensional Geometry

14.01 Introduction

The objects we come acrose around us are only three dimenstional. So, the study of such objects is of utmost importance for our better understanding of this world. In the previous chapter, we have studied vectors in the 3-dimensional space. Vectors are very useful tools to study the 3-dimensional analytic geometry, which is also called the solid geometry. Most of the results are obtained in vector form, which look very simple, and then translate these results to the cartesian form. In solving the problems, we may use either of these two forms.

14.02 Direction Cosines of a Line

Directions cosines of any line L are defined as direction cosines of any vector $\overrightarrow{AB}$ whose support is given line. Let $\overrightarrow{OP} \parallel \overrightarrow{AB}$. If $\overrightarrow{OP}$ makes angles α, β and γ with positive directions of axes OX, OY and OZ then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of $\overrightarrow{OP}$. Direction cosines of $\overrightarrow{OP}$ and $\overrightarrow{AB}$ are similar, because they are parallel and make same angles with axes. In general, direction cosines are represented by ℓ, m, n respectively

$$\therefore \qquad \ell = \cos\alpha, m = \cos\beta, n = \cos\gamma .$$

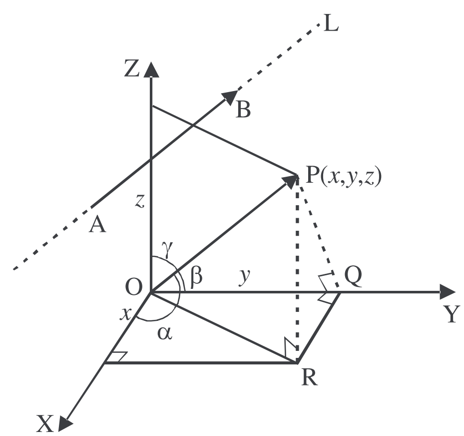

Fig. 14.01

Note:

1. Direction cosines never be written in bracket.
2. $\overrightarrow{BA}$ makes angle $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$ with co-ordinate axes OX, OY and OZ respectively. Therefore, directions cosines of $\overrightarrow{BA}$ will be $\cos(\pi-\alpha), \cos(\pi-\beta), \cos(\pi-\gamma)$ i.e. $-\ell, -m, -n$.
   So, if ℓ, m, n are direction cosines of any line, then $-\ell, -m, -n$ are also its direction cosines just because $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have a common support line L.
3. Direction cosines of X-axis : 1, 0, 0
   Direction cosines of Y-axis : 0, 1, 0
   Direction cosines fo Z-axis : 0, 0, 1

14.03 Relation among the Direction Cosines of a Line

Consider a vector $\overrightarrow{AB}$ with direction cosines ℓ, m, n with base line L. Through the origin, draw a line parallel to the given line and take a point P (x, y, z) on this line, such that $\overrightarrow{OP} \parallel \overrightarrow{AB}$. From P, draw a perpendicular PQ on the Y-axis (Fig. 14.01)

If $OP = r$, then $\cos\beta = \dfrac{y}{r}$

$\Rightarrow \qquad y = r\cos\beta = mr$. Similarly, $z = nr$ and $x = \ell r$

Again, $\qquad OP = r$

$$\Rightarrow \qquad (OP)^2 = r^2$$

$$\Rightarrow \qquad x^2 + y^2 + z^2 = r^2$$

$$\Rightarrow \qquad r^2\left(\ell^2 + m^2 + n^2\right) = r^2$$

$$\Rightarrow \qquad \ell^2 + m^2 + n^2 = 1$$

14.04 Direction ratios of a line

Definition : The direction ratios of a line are proportional to the direction cosines of the vector whose support is the line.

Let a, b, c be direction ratios of a line and let ℓ, m, n be the direction cosines of the vector whose support is give line. Then

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$$

$\therefore$ Direction ratios of any line be the direction ratio of that vector whose support is the given line.

Notes:

1. If a, b, c are direction ratios of a line, then $ka, kb, kc,$ where $k \neq 0$ are also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional. Also, for any line there are infinitely many sets of direction ratios.

2. For direction cosine ℓ, m, n, we have $\ell^2 + m^2 + n^2 = 1$ but for direction ratios a, b, c, we have $a^2 + b^2 + c^2 \neq 1$ till a, b, c become direction cosines.

3. $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (let)

$$\therefore \qquad \ell = ak, m = bk, n = ck$$

but $\qquad \ell^2 + m^2 + n^2 = 1$

$$\Rightarrow \qquad k^2\left(a^2 + b^2 + c^2\right) = 1$$

$$\Rightarrow \qquad k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \qquad \ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}};\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}};\ \ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

4. Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\therefore \qquad \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\right)i + \left(\frac{b}{\sqrt{a^2 + b^2 + c^2}}\right)j + \left(\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)k$$

$$= \ell\hat{i} + m\hat{j} + n\hat{k}$$

where $\ell = \frac{a}{\sqrt{a^2 + b^2 + c^2}};\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}};\ \ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Thus in vector $\vec{r}$, coefficient of $\hat{i},\ \hat{j},\ \hat{k}$ are the direction ratios of that vector.

14.05 Direction cosines of a line passing through Two Points

Let L be the line passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

$$\overrightarrow{PQ} = \text{(position vector of } Q) - \text{(position vector of } P)$$

$$= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$\therefore$ d.r's (direction ratios) of $\overrightarrow{PQ}$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and its d.c's (direction cosines) are

$$\frac{x_2 - x_1}{|\overrightarrow{PQ}|}, \frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|},$$

where, $\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Illustrative Examples

Example 1. A line makes an angle of 30^o and 60^o with the positive direction of X and Y-axis. Find the angle formed by the line with the positive direction of Z–axis.

Solution : Let the line makes an angle γ with the positive direction of Z-axis. Thus it makes angle 30^o, 60^o and γ with the three axes.

$\therefore$ the d.c's of line are $\cos 30^o$, $\cos 60^o$, $\cos\gamma$ i.e. $\frac{\sqrt{3}}{2}, \frac{1}{2}$, $\cos\gamma$

We know that, $\ell^2 + m^2 + n^2 = 1$

$\therefore$ $\left(\sqrt{3}/2\right)^2 + (1/2)^2 + (\cos\gamma)^2 = 1$

or $\cos^2\gamma = 1 - 1$

$\Rightarrow$ $\cos^2\gamma = 0$

$\Rightarrow$ $\cos\gamma = 0$

or, $\gamma = 90^o$

Thus the line makes an angle of 90^o with the Z-axis.

Example 2. If the vector makes an angle of α, β and γ with OX, OY and OZ axes respectively, then Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Solution : Let the d.c's of the given vector be ℓ, m, n

then, $\cos\alpha = \ell$, $\cos\beta = m$, $\cos\gamma = n$

we know that $\ell^2 + m^2 + n^2 = 1$

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow$ $\left(1 - \sin^2\alpha\right) + \left(1 - \sin^2\beta\right) + \left(1 - \sin^2\gamma\right) = 1$

$\Rightarrow$ $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Example 3. Find the direction cosines of a line joining the points (1, 0, 0) and (0, 1, 1).

Solution : The direction ratios of the line joining (1, 0, 0) and (0, 1, 1) are

$$0-1,\ 1-0,\ 1-0 = -1,\ 1,\ 1$$

Thus, the direction cosine will be

$$\mp\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}$$

Example 4. Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear.

Solution : The direction ratios of the line joining the points A and B thus it is clear that direction ratios of AB and BC are proportional therefore

AB || BC

But in AB and BC B is common

$\therefore$ A, B and C are colinear.

Example 5. If a line makes an angle 90°, 135° and 45° with the X, Y and Z-axes respectively then find the direction cosine of the line.

Solution : Direction angles are $90^\circ,\ 135^\circ,\ 45^\circ$

$\therefore$ direction cosines are

$$\ell = \cos 90^o = 0,\ m = \cos 135^\circ = -\frac{1}{\sqrt{2}},\ n = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

thus, the d.c's of the given line are

$$0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}.$$

Exercise 14.1

1. Find the direction cosines of a line whcih makes equal angles with the coordinate axes.
2. Find the direction cosines of the line passing through two points (4, 2, 3) and (4, 5, 7).
3. If the direction ratios of the line are 2, –1, –2, then find the direction cosines.
4. A vector $\vec{r}$, makes angle of 45°, 60°, 120° with the X, Y and Z-axes respectively and the magnitude of $\vec{r}$ is 2 units, then find $\vec{r}$.

14.6 Equation of a line in Space

We shall now study the vector and cartesian equations of a line in space. A line is uniquely determined if

(i) it passes through a given point and has given direction, or

(ii) it passes through two given points.

(i) Equation of a line through a given point A $(\vec{a})$ and parallel to a given vector $\vec{m}$

Let the line be L whose equation is to be determined. Let the line is parallel to the vector $\vec{m}$ and passes through the point A whose position vector is $\vec{a}$. Let O be the origin, therefore $\overrightarrow{OA} = \vec{a}$.

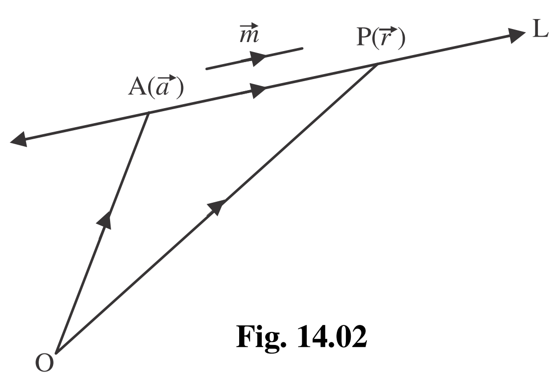

Fig. 14.02

Let P be any point on the line L whose position vector is $\vec{r}$,

then $\overrightarrow{OP} = \vec{r}$

clearly $\overrightarrow{AP} \parallel \overrightarrow{m}$

$\Rightarrow$ $\overrightarrow{AP} = \lambda\overrightarrow{m}$

$\Rightarrow$ (position vector P) $-$ (position vector of A) $= \lambda\vec{m}$

$\Rightarrow$ $\overrightarrow{OP} - \overrightarrow{OA} = \lambda\vec{m}$

$\Rightarrow$ $\vec{r} - \vec{a} = \lambda\vec{m}$

$\Rightarrow$ $\vec{r} = \vec{a} + \lambda\vec{m}$

for each value of the parameter λ , this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda\vec{m} \qquad (1)$$

Cartesian Form

Let $A(x_1, y_1, z_1)$ be the given point and the direction ratios of the line be a, b, c. Consider the coordinates of any point P be (x, y, z) then,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

Since, the direction ratios of the given line be a, b, c therefore, it is parallel to $\vec{m}$

$\therefore$ $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$

Now, vector equation of the line is

$$\vec{r} = \vec{a} + \lambda\vec{m}$$

$\Rightarrow$ $x\hat{i} + y\hat{j} + z\hat{k} = \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) + \lambda\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$

$\Rightarrow$ $x\hat{i} + y\hat{j} + z\hat{k} = (x_1 + \lambda a)i + (y_1 + \lambda b)\, j + (z_1 + \lambda c)$

$\Rightarrow$ $x = x_1 + \lambda a; \quad y = y_1 + \lambda b; \quad z = z_1 + \lambda c$

$\Rightarrow$ $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = \lambda$

$\therefore$ equation of line passing through $A(x_1, y_1, z_1)$ with direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

(ii) Equation of a line passing through two given points

Vector form

Let a line L passes through the two points A and B whose position vectors are $\vec{a}_1$ and $\vec{a}_2$. If O is the origin, then $\overrightarrow{OA} = \vec{a}_1$ and $\overrightarrow{OB} = \vec{a}_2$

$\therefore \qquad \overrightarrow{AB} =$ (position vector of B) $-$ (position vector of A)

$= \vec{a}_2 - \vec{a}_1$

Let there be point P on the line L whose position vector is $\vec{r}$, then $\overrightarrow{OP} = \vec{r}$

$\therefore \qquad \overrightarrow{AP} = \vec{r} - \vec{a}_1$

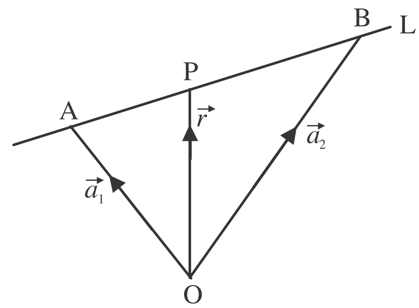

Fig. 14.03

since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear vectors, then

$\Rightarrow \qquad \overrightarrow{AP} = \lambda\left(\overrightarrow{AB}\right), \lambda \in R$

$\Rightarrow \qquad \vec{r} - \vec{a}_1 = \lambda\left(\vec{a}_2 - \vec{a}_1\right)$

$\Rightarrow \qquad \vec{r} = \vec{a}_1 + \lambda\left(\vec{a}_2 - \vec{a}_1\right)$

$\therefore$ The vector equation of line L passing through the points $A(\vec{a}_1)$ and $B(\vec{a}_2)$ is

$$\vec{r} = \vec{a}_1 + \lambda\left(\vec{a}_2 - \vec{a}_1\right) \qquad (2)$$

Cartesian Form:

Let the line L, passes through the points $A\left(x_1, y_1, z_1\right)$ and $B\left(x_2, y_2, z_2\right)$. Let the coordinates of any point P on the line be $\left(x, y, z\right)$.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear, then

$\Rightarrow \qquad \left(x\hat{i} + y\hat{j} + z\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) = \lambda\left\{\left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)\right\}$

$\Rightarrow \qquad \left(x - x_1\right)\hat{i} + \left(y - y_1\right)\hat{j} + \left(z - z_1\right)\hat{k} = \lambda\left(x_2 - x_1\right)\hat{i} + \lambda\left(y_2 - y_1\right)\hat{j} + \lambda\left(z_2 - z_1\right)\hat{k}$

$\Rightarrow \qquad x - x_1 = \lambda\left(x_2 - x_1\right);\ y - y_1 = \lambda\left(y_2 - y_1\right);\ z - z_1 = \lambda\left(z_2 - z_1\right)$

$\Rightarrow \qquad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

which is the required equation of line.

Illustrative Examples

Example 6. Find the vector and cartesian equation of the line passing through the point (5, 2, –4) and parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Solution :

Let $\quad \vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$

The vector equation of the line is $\qquad \vec{r} = \vec{a} + \lambda\left(\vec{m}\right)$

$\therefore \qquad x\hat{i} + y\hat{j} + z\hat{k} = 5\hat{i} + 2\hat{j} - 4k + \lambda\left(3\hat{i} + 2\hat{j} - 8\hat{k}\right)$

or, $\qquad x\hat{i} + y\hat{j} + z\hat{k} = (5 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 - 8\lambda)\hat{k}$

or, $\qquad x - 5 = 3x,\ y - 2 = 2x,\ z + 4 = -8x$

or, $$\frac{x-5}{3}=\frac{y-2}{2}=\frac{z+4}{-8}=\lambda$$

Thus, equation in cartesian form will be $\frac{x-5}{3}=\frac{y-2}{2}=\frac{z+4}{-8}$

Example 7. Find the vector equation of the line passign through the points (–1, 0, 2) and (3, 4, 6).

Solution : Let the position vector of points A(–1, 0, 2) and B (3, 4, 6) be $\vec{a}$ and $\vec{b}$ respectively.

then, $\vec{a}=-\hat{i}+2\hat{k}$

and $\vec{b}=3\hat{i}+4\hat{j}+6\hat{k}$

$\therefore$ $\vec{b}-\vec{a}=4\hat{i}+4\hat{j}+4\hat{k}$

Let the position vector of any point P be $\vec{r}$, then the vector equation of the line is-

$$\vec{r}=-\hat{i}+2\hat{k}+\lambda\left(4\hat{i}+4\hat{j}+4\hat{k}\right)$$

Example 8. Find the vector equation of a line passing through point A (2, –1, 1) and parallel to the line joining the points B (–1, 4, 1) and C (1, 2, 2). Also find its the cartesian equation.

Solution : For the vector equation

position vector of $B=-\hat{i}+4\hat{j}+\hat{k}$

and position vector of C $=\hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ $\overrightarrow{BC}=$ position vector of C– position vector of B

$$=\left(\hat{i}+2\hat{j}+2\hat{k}\right)-\left(-\hat{i}+4\hat{j}+\hat{k}\right)=2\hat{i}-2\hat{j}+\hat{k}$$

position vector of A, $\vec{r}_1=2\hat{i}-\hat{j}+\hat{k}$

$\therefore$ Vector equation of the line

$$\vec{r}=\vec{r}_1+\lambda(\overrightarrow{BC})$$

$$\Rightarrow \quad \vec{r}=\left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right) \qquad (1)$$

Certesian equation of the line,

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right), \text{ when } \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\Rightarrow \quad \left(x\hat{i}+y\hat{j}+z\hat{k}\right)=(2+2\lambda)\hat{i}+(-1-2\lambda)\hat{j}+(1+\lambda)\hat{k}$$

On comparing,

$$\Rightarrow \quad \frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{1}=\lambda$$

Thus, the cartesian equation of line is $\frac{x-2}{2}=\frac{y+1}{-2}=\frac{z-1}{1}$

Example 9. The cartesian equation of a line $6x-2=3y+1=2z-2$. Find

(a) direction ratios of the line.

(b) the vector and cartesian equation of a line passing through (2, –1, –1) and parallel to the given line.

Solution : Equation of a line

$$6x-2=3y+1=2z-2$$

$$\Leftrightarrow \quad \frac{x-(1/3)}{1/6}=\frac{y+(1/3)}{1/3}=\frac{z-1}{1/2}$$

$$\Leftrightarrow \quad \frac{x-(1/3)}{1}=\frac{y+(1/3)}{2}=\frac{z-1}{3}$$

(a) Therefore, the d.r's of the given line are 1, 2, 3.

(b) Equation of a line passing through (2, –1, –1) and parallel to the given line.

$$\frac{x-2}{1}=\frac{y+1}{2}=\frac{z+1}{3}$$

New vector equation of a line passing through A (2, –1, –1) and i.e. $\vec{a}=2\hat{i}-\hat{j}-\hat{k}$ parallel to $\vec{m}=\hat{i}+2\hat{j}+3\hat{k}$ is

$$\vec{r}=\vec{a}+\lambda\vec{m}$$

or, $$\vec{r}=\left(2\hat{i}-\hat{j}-\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$

Exercise 14.2

1. Find the equation of the line passing through the point (5, 7, 9) and parallel to the following given axis:
   (i) *X*–axis (ii) *Y*–axis (iii) *Z*–axis
2. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and is parallel to the vector $3\hat{i}+4\hat{j}-5\hat{k}$.
3. Find the equation of the line which passes through the point (5, –2, 4) and is parallel to the vector $2\hat{i}-\hat{j}+3\hat{k}$
4. Find the eqation of the line whcih passes through the point (2, –1, 1) and is parallel to the line $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$.
5. Find the vector equation of the line whose cartesian equation is
   $$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$
6. Find the cartesian equation of the line whcih passes through the point (1, 2, 3) and is parallel to the line $\frac{x-2}{1}=\frac{y+3}{7}=\frac{2z-6}{3}$.

7. The coordinates of the three vertices of a parallelogram ABCD are A (4, 5, 10), B (2, 3, 4) and C (1, 2, –1). Find the vector and cartesian equation of AB and BC. Also find the coordinates of D.

8. The cartesian equation of a line is $3x+1=6y-2=1-z$. Find the point through which it passes and also find the direction ratios and vector equation.

9. Find the equation of the line whcih passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$.

10. Find the vector and cartesian equation of a line passing through the point whose position vector is $2\hat{i}-\hat{j}+4\hat{k}$ and in the direction of the vector $\hat{i}+2\hat{j}-\hat{k}$.

11. Find the cartesian equation of the line which passes through the point (–2, 4, –5) and is parallel to the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$.

12. The cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Find its vector equation.

13. Find the vector and cartesian equation of a line passing through the origin and the point (5, –2, 3).

14. Find the vector and cartesian equation of a line passing through the point (3, –2, –5) and (3, –2, 6).

14.07 Angle between Two Lines

Vector form:

Let the vector equation of two lines be

$$\vec{r}=\vec{a}_1+\lambda\vec{m}_1,\ \lambda\in R \quad \text{and} \quad \vec{r}=\vec{a}_2+\mu\vec{m}_2,\ \mu\in R$$

If the angle between them is θ, then it is clear from figure 14.04 that, the angle between vector $\vec{m}_1$ and vector $\vec{m}_2$ is also θ. Thus $\cos\theta=\frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$.

Cartesian form:

Let the cartesian equation of two lines be

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \quad \text{and} \quad \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

$\therefore \quad \vec{m}_1=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{m}_2=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$

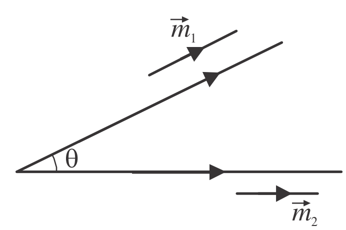

Fig. 14.04

but $\quad \cos\theta=\frac{\vec{m}_1\cdot\vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$

$$\Rightarrow \quad \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Note:

1. If the direction cosines of two lines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and the angle between them is θ, then $\cos\theta=\ell_1\ell_2+m_1m_2+n_1n_2$.

2. If the two lines are perpendicualr, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

3. If the two lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Illustrative Examples

Example 10. Find the angle between the lines $\frac{5-x}{3} = \frac{y+3}{-4} = \frac{z-7}{0}$ and $\frac{x}{1} = \frac{1-y}{2} = \frac{z-6}{2}$.

Solution : Given lines are

$$\frac{x-5}{-3} = \frac{y+3}{-4} = \frac{z-7}{0} \qquad (1)$$

$$\frac{x}{1} = \frac{y-1}{-2} = \frac{z-6}{2} \qquad (2)$$

Let the vectors parallel to line (1) and (2) be $\vec{m}_1$ and $\vec{m}_2$ respectively, then $\vec{m}_1 = -3\hat{i} - 4\hat{j} + 0k$ and $\vec{m}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$. Let the angle between $\vec{m}_1$ and $\vec{m}_2$ be θ, then

$$\cos\theta = \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$$

$$\Rightarrow \qquad \cos\theta = \frac{\{(-3)\times 1 + (-4)\times(-2) + 0\times 2\}}{\left\{\sqrt{(-3)^2 + (-4)^2 + 0^2}\right\}\left\{\sqrt{1^2 + (-2)^2 + 2^2}\right\}} = \frac{1}{3}$$

$$\Rightarrow \qquad \theta = \cos^{-1}(1/3).$$

Example 11. Find the angle betwene the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \text{ and } \vec{r} = 5\hat{i} - 2\hat{j} + \mu\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right).$$

Solution : Let the angle between the lines which are parallel to $\vec{b_1} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b_2} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ respectively be θ, therefore

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)\cdot\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)}{\sqrt{1+4+4}\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

$$\therefore \qquad \theta = \cos^{-1}(19/21)$$

Example 12. Find the equation of line passing through (–1, 3, –2) and perpendicular to the line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$$

Solution : Let < *a, b, c* > be the d.r's of the required line. Since this required line is are perpendicular to the given lines, then

$$a + 2b + 3c = 0 \quad (1)$$

and $$-3a + 2b + 5c = 0 \quad (2)$$

By cross-multiplication method in (1) and (2), we get

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

or $$\frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = k \text{ (Let)}$$

$\therefore$ The line passes through (–1, 3, –2), with d.r's < 2, –7, 4 > be given by

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

Exercise 14.3

1. Find the angle between the lines-

$$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right) \text{ and } \vec{r} = 7\hat{i} - 6\hat{j} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

2. Find the angle between the lines-

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

3. Show that the line passing through the points (1, –1, 2) and (3, 4, –2) is perpendicular to the line passing through the points (0, 3, 2) and (3, 5, 6).

4. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are mutually perpendicular, then find the value of *k*.

5. Find the vector equation of the line passing through the point (1, 2, –4) and perpendicular to the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

6. Find the cartesian equation of the line passing through (–2, 4, –5) and parallel to the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

14.08 Intersection of Two Lines

If two lines intersect in a plane, then there is one common point between them so that the distance between them is zero. The following methods are used to find the point of intersection of two lines.

(1) Equation of lines in vector form:

Let two lines be $$\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \lambda(m_1\hat{i} + m_2\hat{j} + m_3\hat{k}) \quad (1)$$

and $$\vec{r} = (a'_1\hat{i} + a'_2\hat{j} + a'_3\hat{k}) + \mu(m'_1\hat{i} + m'_2\hat{j} + m'_3\hat{k}) \quad (2)$$

(i) $\because$ Lines intersect, therefore

$$(a_1 i + a_2 j + a_3 k) + \lambda(m_1\hat{i} + m_2\hat{j} + m_3\hat{k}) = \vec{r} = (a'_1\hat{i} + a'_2\hat{j} + a'_3\hat{k}) + \mu(m'_1\hat{i} + m'_2\hat{j} + m'_3\hat{k})$$

On comparing, we get

$$a_1 + \lambda m_1 = a'_1 + \mu m'_1; \quad a_2 + \lambda m_2 = a'_2 + \mu m'_2; \quad a_3 + \lambda m_3 = a'_3 + \mu m'_3$$

(ii) On solving the two equations, we get the value of λ and μ. If these values satisfy the third equation, then the lines are intersecting otherwise not.

(iii) To get the position vector of intersecting point, put the value of λ, μ in (1) and (2).

(2) Equation of lines in cartesian form

lines $$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} = r_1 \text{ (let)} \quad (1)$$

and $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} = r_2 \text{ (let)} \quad (2)$$

(i) Point on line (1) and (2) are

$$(a_1 r_1 + x_1,\ b_1 r_1 + y_1,\ c_1 r_1 + z_1) \text{ and } (a_2 r_2 + x_2,\ b_2 r_2 + y_2,\ c_2 r_2 + z_2)$$

$\because$ Lines intersect, therefore

$$a_1 r_1 + x_1 = a_2 r_2 + x_2; \quad b_1 r_1 + y_1 = b_2 r_2 + y_2 \text{ and } c_1 r_1 + z_1 = c_2 r_2 + z_2$$

(ii) Find the value of r_1 and r_2 by solving any two of the equations. If the values of r_1 and r_2 satisfy the third equation, then the lines intersect otherwise not.

(iii) Substituting the values of r_1 and r_2 in the general point, we get the point of intersection.

Illustrative Examples

Example 13. Prove that the lines

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} \text{ and } \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$$

intersect each other. Find the coordinates of their intersecting points.

Solution : Let the coordinates of any point on $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = r_1$ (let)

be $(r_1 + 4, -4r_1 - 3, 7r_1 - 1)$. Similarly,

$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = r_2$ the coordinates of the point be $(2r_2 + 1, -3r_2 - 1, 8r_2 - 10)$ on the line.

These lines will intersect each other, if they have a common point between them i.e.

$$r_1 + 4 = 2r_2 + 1 \quad (1)$$

$$-4r_1 - 3 = -3r_2 - 1 \quad (2)$$

$$7r_1 - 1 = 8r_2 - 10. \quad (3)$$

Solving equation (1) and (2), we have $r_1 = 1,\ r_2 = 2,$ which satisfies equation (3) also. Thus, the two lines intersect each other at the point (5, –7, 6).

Example 14. Prove that the lines

$$\vec{r} = (i + j - k) + \lambda(3i - j) \text{ and } \vec{r} = (4i - k) + \mu(2i + 3k)$$

intersect each other and find the point of intrsection.

Solution : Let the position vector of the points of intersection be $\vec{r}$.

$$\therefore \qquad (i + j - k) + \lambda(3i - j) = (4i - k) + \mu(2i + 3k)$$

$$1 + 3\lambda = 4 + 2\mu \qquad \Rightarrow \qquad 3\lambda - 2\mu = 3 \qquad (1)$$

$$1 - \lambda = 0 \qquad \Rightarrow \qquad \lambda = 1 \qquad (2)$$

$$-1 = -1 + 3\mu \qquad \Rightarrow \qquad \mu = 0 \qquad (3)$$

(on comparing the coefficients of i, j, k)

From (2) and (3), $\lambda = 1, \mu = 0$, which satisfie (1). Also putting $\lambda = 1$ in the equation $\vec{r} = (i + j - k) + \lambda(3i - j)$, we have

$$\vec{r} = 4i + 0j - k$$

thus, the coordinates of point of intersection are (4, 0, –1).

Example 15. Show that the lines,

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} \text{ and } \frac{x-2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$$

do not intersect each other.

Solution : Given lines are

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} = \lambda \qquad (1)$$

$$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+1}{-2} = \mu \qquad (2)$$

Let $P(3\lambda + 1, 2\lambda - 1, 5\lambda + 1)$ be any point on (1) and $Q(4\mu + 2, 3\mu + 1, -2\mu - 1)$ be any point on (2). If the lines (1) and (2) intersect, then

$$3\lambda + 1 = 4\mu + 2 \quad ; \quad 2\lambda - 1 = 3\mu + 1 \quad ; \quad 5\lambda + 1 = -2\mu - 1$$

$$\Leftrightarrow \qquad 3\lambda - 4\mu = 1 \qquad (3)$$

$$2\lambda - 3\mu = 2 \qquad (4)$$

$$5\lambda + 2\mu = -2 \qquad (5)$$

Solving (3) and (4), we have $\lambda = -5$ and $\mu = -4$.

But the value of λ and μ , do not satisfy (5) Therefore, these two lines do not intersect each other.

14.09 Perpendicular distance of a point from a line

Vector form:

Let the foot of perpendicular drawn from point $P(\vec{\alpha})$ on the line be L

$\because$ $\vec{r}$ is any arbitrary point on the line. Therefore, the position vector of point L will be $\vec{a} + \lambda\vec{b}$

$\therefore \quad \overrightarrow{PL} =$ Position vector of L– position vector of P

$$= \vec{a} + \lambda\vec{b} - \vec{\alpha}$$

$$= (\vec{a} - \vec{\alpha}) + \lambda\vec{b}$$

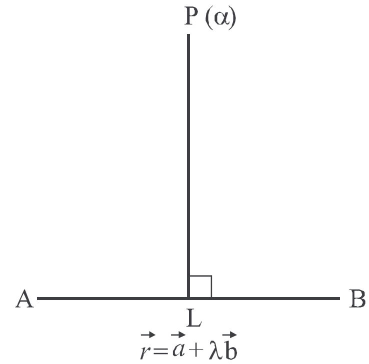

Fig. 14.05

$\because \quad$ vector $\overrightarrow{PL}$ is perpendicular to the line parallel to $\vec{b}$ therefore

$$\overrightarrow{PL} \cdot \vec{b} = 0$$

$$\left\{(\vec{a} - \vec{\alpha}) + \lambda\vec{b}\right\} \cdot \vec{b} = 0$$

$$(\vec{a} - \vec{\alpha}) \cdot \vec{b} + \lambda |\vec{b}|^2 = 0$$

$$\lambda = -\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}$$

Now position vector of L $\quad = \vec{a} + \lambda\vec{b}$

$$= \vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

$\therefore \quad$ Equation of $\overrightarrow{PL}$ $\quad \vec{r} = \vec{\alpha} + \mu\left[\left\{\vec{a} - \left(\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}\right\} - \alpha\right]$

$$= \vec{\alpha} + \mu\left[(a - \vec{\alpha}) - \left\{\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}\right\}\vec{b}\right]$$

$\therefore \quad$ Magnitude of $\overrightarrow{PL}$ is length of PL

Cartesian Form : To find the length of perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

Let the foot of perpendicular drawn from point $P(\alpha, \beta, \gamma)$ to the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$

Let the coordinates of L be $(x_1 + a\lambda,\ y_1 + b\lambda,\ z_1 + c\lambda)$

$\therefore \quad$ direction ratios of PL be $x_1 + a\lambda - \alpha,\ y_1 + b\lambda - \beta$ and $z_1 + c\lambda - \gamma$

d.r's of line AB be a, b, c

$\because \quad$ PL and AB are mutually perpendicular. Therefore,

$$(x_1 + a\lambda - \alpha)a + (y_1 + b\lambda - \beta)b + (z_1 + c\lambda - \gamma)c = 0$$

$$\Rightarrow \quad \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

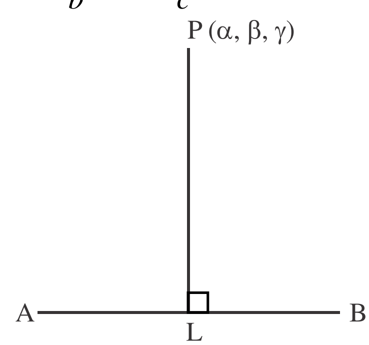

Fig. 14.06

By putting the value of λ in the coordinates of L, we get the actual coordinates of L. We can find the distance of PL by using distance formula.

Illustrative Examples

Example 16. Find the length of perpendicular drawn from point the (1, 2, 3) on the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$.

Solution : Let the foot of perpendicular from point P (1, 2, 3) on the line be L.

$\therefore$ Coordinates of L are $(3\lambda+6,\ 2\lambda+7,\ -2\lambda+7)$ (1)

$\therefore$ d.r's of PL

$3\lambda+6-1,\ 2\lambda+7-2,\ -2\lambda+7-3$

i.e. $3\lambda+5,\ 2\lambda+5,\ -2\lambda+4$

d.r's of line are 3, 2, –2. Since PL is perpendicular to the given line. Therefore,

$$3(3\lambda+5)+2(2\lambda+5)+(-2)(-2\lambda+4)=0$$

$$\Rightarrow \qquad \lambda=-1$$

Putting the value of $\lambda=-1$ in (1), the coordiantes of L are (3, 5, 9)

$$PL=\sqrt{(3-1)^2+(5-2)^2-(9-3)^2}$$

$$= 7 \text{ units}$$

Required length of perpendicular is 7 units.

Exercise 14.4

1. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ are mutually intersecting. Find the point of intersection.
2. Examine that the lines $\vec{r}=(\hat{i}-\hat{j})+\lambda(2\hat{i}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j})+\mu(\hat{i}+\hat{j}-\hat{k})$ are intersecting or not.
3. Find the foot of perpendicular from the point (2, 3, 4) to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also find the perpendicular distance of the line from the point.
4. Find the vector equation of the line passing through the point (2, 3, 2) and parallel to the line $\vec{r}=(-2\hat{i}+3\hat{j})+\mu(2\hat{i}-3\hat{j}+6\hat{k})$. Also find the distance between them.

14.10 Skew Lines and Shortest Distance between Two Skew Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line. Further, in a space, there are lines whcih are neither intersecting nor parallel. In fact, such pair of lines are *non coplanar* and are called *skew lines.*

By the shortest distance between two lines, we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest. For skew lines, the line of the shortest distance will be perpendicular to both the lines.

Note: If two lines intersect at a point, then the shortest distance between them is zero.

14.11 To find the Shortest Distance between Two Skew Lines

Vector form

We now determine the shortest distance between two skew lines in the following way:

Let L_1 and L_2 be two skew lines with equations

$$L_1 : \vec{r} = \vec{a}_1 + \lambda \vec{b}_1$$

$$L_2 : \vec{r} = \vec{a}_2 + \lambda \vec{b}_2$$

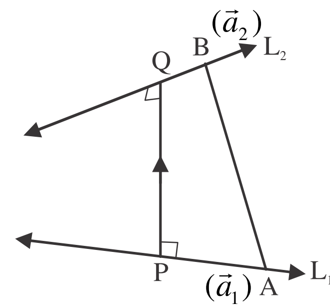

Fig. 14.07

Take any point A on L_1 with position vector $A(\vec{a}_1)$ and B on L_2, with position vector $B(\vec{a}_2)$. Then, the magnitude of the shortest distance vector will be equal to that of the projection of AB along the direction of the line of shortest distance. If $\overrightarrow{PQ}$ is the shortest distance vector between L_1 and L_2, then it is being perpendicular to both $\vec{b}_1$ and $\vec{b}_2$. Thus, unit vector $\hat{n}$ along $\overrightarrow{PQ}$ will be

$$\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{\left|\vec{b}_1 \times \vec{b}_2\right|}$$

$\therefore$ $\qquad \overrightarrow{PQ} = (PQ)\hat{n} = d\hat{n},$ where $PQ = d$ (Shortest Distance)

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{PQ}$, then

$$PQ = AB\cos\theta \qquad (1)$$

But, $$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{PQ}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{PQ}\right|} \qquad (2)$$

$\therefore$ $$\cos\theta = \frac{(\vec{a}_2 - \vec{a}_1)\cdot(d\hat{n})}{(AB)(d)}, \qquad \overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$$

$$= \frac{(\vec{a}_2 - \vec{a}_1)\cdot\hat{n}}{(AB)}$$

From (1), $$PQ = (AB)\frac{(\vec{a}_2 - \vec{a}_1)\cdot\hat{n}}{(AB)}$$

$$= (\vec{a}_2 - \vec{a}_1)\cdot\hat{n}$$

$$= \frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}$$

$\therefore$ Required shortest distance $= d = PQ = \left|\dfrac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

Note: If two lines mutually intersect, then the shortest distance distance between them is zero.

i.e. $$\left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 0$$

$$\Rightarrow \quad (\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$$

$$\Rightarrow \quad \left[(\vec{a}_2 - \vec{a}_1)\ \vec{b}_1\ \ \vec{b}_2\right] = 0$$

Cartesian Form:

The shortest distance between lines $L_1 : \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $L_2 : \dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

14.12 Distance between two parallel lines

If two lines L_1 and L_2 are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = a_1 + \lambda\vec{b} \quad \text{and} \quad \vec{r} = \vec{a}_2 + \mu\vec{b}$$

where, $\vec{a}_1$ is the position vector of a point A to L_1 and $\vec{a}_2$ is the position vector of a point B to L_2.

As L_1, L_2 are coplanar. Therefore, according to fig. 14.08 the foot of the perpendicualr from B on the line L_1 is C, then the distance between the lines L_1 and $L_2 = BC$

Let θ be the angle between $\overrightarrow{AB}$ and $\vec{b}$

$$\therefore \quad \vec{b} \times \overrightarrow{AB} = \left(|\vec{b}||\overrightarrow{AB}|\sin\theta\right)\hat{n}$$

where $\hat{n}$, is the unit vector perpendicular to the plane of the lines L_1 and L_2

$$\Rightarrow \quad \vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|(BC)\hat{n}, \text{ where } BC = (AB)\sin\theta$$

$$\Rightarrow \quad |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|(BC), \text{ where } |\hat{n}| = 1$$

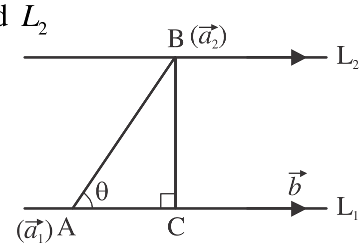

Fig. 14.08

$$\Rightarrow \qquad BC = \frac{\left|\vec{b} \times \left(\vec{a}_2 - \vec{a}_1\right)\right|}{|\vec{b}|}$$

Thus, the distance between the two given parallel lines,

$$d = BC = \left|\frac{\vec{b} \times \left(\vec{a}_2 - \vec{a}_1\right)}{|\vec{b}|}\right|$$

Illustrative Examples

Example 17. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and } \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Solution : $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$

We see that $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$$\therefore \qquad \left(\vec{a}_2 - \vec{a}_1\right) = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right)$$

and
$$\left(\vec{b}_1 \times \vec{b}_2\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}\left(-3-6\right) + \hat{j}\left(4-1\right) + \hat{k}\left(3+6\right) = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\therefore \qquad |\vec{b}_1 \times \vec{b}_2| = \sqrt{81+9+81} = \sqrt{171}$$

$$\text{S.D.} = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

$$\therefore \qquad \text{S.D.} = \frac{|(3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (-9\hat{i} + 3\hat{j} + 9\hat{k})|}{\sqrt{171}}$$

$$= \frac{|-27+9+27|}{\sqrt{171}} = \frac{9}{\sqrt{171}} = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}}$$

Example 18. Find the shortest distance between two lines whose equations are

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-3} \text{ and } \frac{x-1}{-1} = \frac{y-3}{3} = \frac{z-1}{2}.$$

Solution : The given equations are

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-3} \qquad (1)$$

$$\frac{x-1}{-1}=\frac{y-3}{3}=\frac{z-1}{2} \qquad (2)$$

From (1) line passes through (3, 4, –1) and its d.r's are 2, 1, –3.

$\therefore$ Its the vector equation is $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$, where $\vec{a}_1=3\hat{i}+4\hat{j}-\hat{k},\ \vec{b}_1=2\hat{i}+\hat{j}-3\hat{k}$

Similarly from line (2),

$$\vec{a}_2=\hat{i}+3\hat{j}+\hat{k},\ \vec{b}_2=-\hat{i}+3\hat{j}+2\hat{k}$$

Now, $$\vec{a}_2-\vec{a}_1=\left(\hat{i}+3\hat{j}+\hat{k}\right)-\left(3\hat{i}+4\hat{j}-\hat{k}\right)=-2\hat{i}-\hat{j}+2\hat{k}$$

and $$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -3\\ -1 & 3 & 2\end{vmatrix}=11\hat{i}-\hat{j}+7\hat{k}$$

$\therefore$ $$|\vec{b}_1\times\vec{b}_2|=|11\hat{i}-\hat{j}+7\hat{k}|=\sqrt{121+1+49}=\sqrt{171}=3\sqrt{19}$$

Shortest distance $$=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

$$=\left|\frac{(-2\hat{i}-\hat{j}+2\hat{k})\cdot(11\hat{i}-\hat{j}+7\hat{k})}{3\sqrt{19}}\right|=\left|\frac{-22+1+14}{3\sqrt{19}}\right|=\frac{7}{3\sqrt{19}}$$

Example 19. Find the shortest distance between the lines L_1 and L_2 whose vector equations are

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right) \text{ and } \vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

Solution : The given lines are parallel. Comparing with $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$, we have

$$\vec{a}_1=\hat{i}+2\hat{j}-4\hat{k},\ \vec{a}_2=3\hat{i}+3\hat{j}-5\hat{k}$$

and $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

Hence, the distance between the lines

$$d=\left|\frac{\vec{b}\times\left(\vec{a}_2-\vec{a}_1\right)}{|\vec{b}|}\right|=\left|\frac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 6\\ 2 & 1 & -1\end{vmatrix}}{\sqrt{4+9+36}}\right|=\frac{\left|-9\hat{i}+14\hat{j}-4\hat{k}\right|}{\sqrt{49}}=\frac{\sqrt{293}}{\sqrt{49}}=\frac{\sqrt{293}}{7}$$

Exercise 14.5

1. Find the shortest distance between the lines whose vector equations are $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

2. Find the shortest distance between the lines whose equation are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

3. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \quad \text{and} \quad \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

4. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \quad \text{and} \quad \vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

5. Find the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y+1}{3} = z \quad \text{and} \quad \frac{x+1}{3} = \frac{y-2}{1}, z = 2$$

Also find the equations of line of shortest distance.

14.13 Plane

Definition : A plane is a surface such that if any two points are taken on it, the line segment joining them, lies completely on the surface.

A plane is determined uniquely if any one of the following be known:

(i) the normal to the plane and its distance from the origin be given, i.e., equation of a plane in normal form.

(ii) it passes through a point and it is perpendicualr to a given direction.

(iii) it passes through three given non collinear points.

Now we shall find vector and Cartesian equations of the planes.

14.14 General Equation of a Plane

To prove that, every first degree equation in x, y and z represents a plane.

Let the equation be

$$ax + by + cz + d = 0, \tag{1}$$

where a, b, c and d are the constants and a, b, c are non-zero.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ satisfy the equation (1)

$$ax_1 + by_1 + cz_1 + d = 0 \tag{2}$$

and
$$ax_2 + by_2 + cz_2 + d = 0 \tag{3}$$

multiplying (2) with m_2 and (3) with m_1 (where $m_1 + m_2 \neq 0$) and adding

$$a(m_2x_1 + m_1x_2) + b(m_2y_1 + m_1y_2) + c(m_2z_1 + m_1z_2) + d(m_1 + m_2) = 0$$

or
$$a\left(\frac{m_2x_1 + m_1x_2}{m_1 + m_2}\right) + b\left(\frac{m_2y_1 + m_1y_2}{m_1 + m_2}\right) + c\left(\frac{m_2z_1 + m_1z_2}{m_1 + m_2}\right) + d = 0$$

The point dividing the points P and Q in the ratio $m_1 : m_2$ is given by

$$R\left(\frac{m_2x_1+m_1x_2}{m_1+m_2},\ \frac{m_2y_1+m_1y_2}{m_1+m_2},\ \frac{m_2z_1+m_1z_2}{m_1+m_2}\right)$$

for every value of m_1, m_2 (except $m_1 = -m_2$), point R satisfies equation (1)

Here, we have shown that $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on (1) and the point R joining the point P and Q also lies on (1) i.e. line lies in the plane given by (1).

Thus, equation (1) denotes a plane in General form. Therefore, a linear equation with variables x, y, z always denotes an equation of plane.

Corollary : One Point Form:

To prove that, the equation of a plane passing through the point (x_1, y_1, z_1) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0,$$

Let equation of plane be

$$ax+by+cz+d=0, \tag{1}$$

since, it passes through (x_1, y_1, z_1)

$$\therefore \qquad ax_1+by_1+cz_1+d=0. \tag{2}$$

subtracting (2) from (1),

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0, \tag{3}$$

whcih is the required equation of plane.

Special cases: In the general equation of plane $ax+by+cz+d=0$,

	If		Form of Plane		Conclusion
1.	$d=0$	$\Rightarrow$	$ax+by+cz=0$	$\Rightarrow$	Plane passes through the origin
2. (i)	$a=0$	$\Rightarrow$	$by+cz+d=0$	$\Rightarrow$	Plane parallel to X - axis
(ii)	$b=0$	$\Rightarrow$	$ax+cz+d=0$	$\Rightarrow$	Plane parallel to Y - axis
(iii)	$c=0$	$\Rightarrow$	$ax+by+d=0$	$\Rightarrow$	Plane parallel to Z - axis
3. (i)	$a=0, d=0$	$\Rightarrow$	$by+cz=0$	$\Rightarrow$	Plane passes through X - axis
(ii)	$b=0, d=0$	$\Rightarrow$	$ax+cz=0$	$\Rightarrow$	Plane passes through Y - axis
(iii)	$c=0, d=0$	$\Rightarrow$	$ax+by=0$	$\Rightarrow$	Plane passes through Z - axis
4. (i)	$b=0, c=0$	$\Rightarrow$	$ax+d=0$	$\Rightarrow$	Plane perpendicular to X - aixs
(ii)	$a=0, c=0$	$\Rightarrow$	$by+d=0$	$\Rightarrow$	Plane perpendicular to Y - axis
(iii)	$a=0, b=0$	$\Rightarrow$	$cz+d=0$	$\Rightarrow$	Plane perpendicular to Z - axis
5- (i)	$a=b=d=0$	$\Rightarrow$	$cz=0$	$\Rightarrow$	Plane coincides to with XY - plane
(ii)	$b=c=d=0$	$\Rightarrow$	$ax=0$	$\Rightarrow$	Plane coincides to with YZ - plane
(iii)	$a=c=d=0$	$\Rightarrow$	$by=0$	$\Rightarrow$	Plane coincides to with ZX - plane

Note: Since there are three independent constants in the equation of plane, hence to get the complete equation of plane, we must find the vlaues of the three constants.

Illustrative Examples

Example 20. Find the ratio in which the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by the plane $ax+by+cz+d=0$.

Solution : Let the line joining the points P and Q is divided by the plane $ax+by+cz+d=0$ in the ratio $\lambda:1$.

Let the intersecting point of the line and Plane be R. Thus, R lies on PQ whcih divides PQ in the ratio $\lambda:1$. Therefore, the coordinates of R will be $\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right)$.

Since the point R lies on the plane, therefore it will satisfy the equation of the plane

$$\therefore \qquad a\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)+b\left(\frac{\lambda y_2+y_1}{\lambda+1}\right)+c\left(\frac{\lambda z_2+z_1}{\lambda+1}\right)+d=0$$

or, $$a(\lambda x_2+x_1)+b(\lambda y_2+y_1)+c(\lambda z_2+z_1)+d(\lambda+1)=0$$

or, $$\lambda(ax_2+by_2+cz_2+d)=-(ax_1+by_1+cz_1+d)$$

or, $$\lambda=-\frac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$$

This is the required ratio.

Example 21. Find the ratio in whcih the line joining the points P (–2, 4, 7) and Q (3, –5, 8) is cut by the co-ordinate planes.

Solution : The coordinates of point R on the line joining the points P (–2, 4, 7) and Q (3, –5, 8) and dividing in the ratio $\lambda:1$ be $\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right)$.

(i) If R, lies on YZ plane i.e. at x = 0, then $\frac{3\lambda-2}{\lambda+1}=0$ or $\lambda=\frac{2}{3}$ i..e the required ratio is 2 : 3.

(ii) If R, lies on ZX plane i.e. at y = 0, then $\frac{-5\lambda+4}{\lambda+1}=0$ or $\lambda=\frac{4}{5}$ i.e. the required ratio is 4 : 5.

(iii) If R, lies on XY plane i.e. at z = 0, then $\frac{8\lambda+7}{\lambda+1}=0$ or $\lambda=-\frac{7}{8}$ i.e. the required ratio is –7 : 5.

14.15 Intercept Form of a Plane

In this section, we shall deduce the equation of a plane in terms of the intercepts a, b and made by the plane on the coordinate axes. i.e. on X, Y and Z-axes resectively as -

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Let the equation of the plane be

$$Ax+By+Cz+D=0 \qquad (1)$$

Let the plane makes intercepts a, b, c on X Y and Z axes respectively, such that

$OP = a$, $OQ = b$ and $OR = c$

Thus, the coordinates P, Q and R be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

Since, point P $(a, 0, 0)$ lies on plane (1),

$$\therefore \quad A \cdot a + B \cdot 0 + C \cdot 0 + D = 0 \quad \Rightarrow \quad A = -\frac{D}{a}$$

Similarly, plane (1) passes through point Q and R. Therefore,

$B = -D/b$ and $C = -D/c$

Substituting the vlaues of A, B, C in (1), we have

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

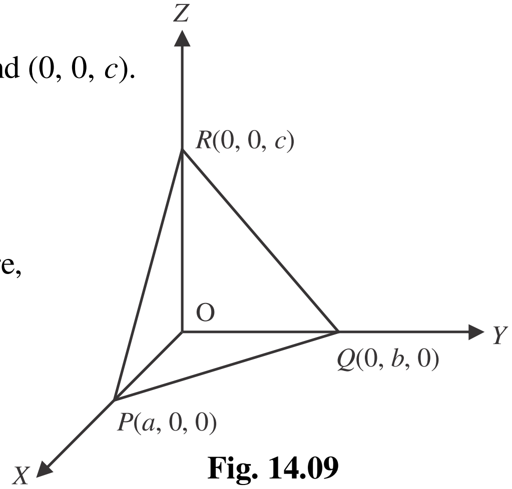

Fig. 14.09

This is the required equation of plane in intercept form.

Note: By converting general equation of plane in intercept from, we obtain the intercepts made by plane on axes.

Illustrative Examples

Example 22. Convert the equation of plane $3x - 4y + 2z = 12$ in the intercept form and find the intercepts made on the coordinate axes.

Solution : Given equation is $3x - 4y + 2z = 12$,

$$\Rightarrow \quad \frac{3x}{12} - \frac{4y}{12} + \frac{2z}{12} = 1$$

$$\Rightarrow \quad \frac{x}{4} + \frac{y}{(-3)} + \frac{z}{6} = 1$$

On comparing with the intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have the intercepts made on X, Y and Z axes are 4, –3 and 6 respectively.

Example 23. A plane meets the coordinate axes at points A, B and C such that the coordinates of the centroid of the triangle ABC so formed is K (p, q, r). Show that the required equation of the plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$.

Solution : Let the equation of plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Thus, the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Therefore, the coordinates of the centroid will be K $(a/3, b/3, c/3)$. But it is given that the centroid is K (p, q, r),

$$\therefore \quad \frac{a}{3} = p, \qquad \frac{b}{3} = q, \qquad \frac{c}{3} = r$$

$$\Rightarrow \quad a = 3p, \qquad b = 3q, \qquad c = 3r.$$

Substituting the values of a, b and c in, we get the required equation

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1, \quad \text{i.e.} \quad \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3,$$

Example 24. A variable plane moves in a space in such a way that the sum of reciprocals of the intercepts made by it on the coordinate axes is a constant. Prove that the plane passes through the fixed point.

Solution : Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ (1)

$\therefore$ The intercpets made by the plane on the coordinate axes are a, b and c respectively.

According to the question, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\text{constant}=\frac{1}{\lambda}$ (let)

or, $$\frac{\lambda}{a}+\frac{\lambda}{b}+\frac{\lambda}{c}=1 \qquad (2)$$

$\therefore$ Equation (2) shows that, $(\lambda, \lambda, \lambda)$ satisfies equation (1). Thats means, plane (1) passes through the fixed point $(\lambda, \lambda, \lambda)$

14.16 Equation of a Plane in Normal Form

Vector form : Consider a plane whose perpendicualr distance from the origin is p and $\hat{n}$ is the unit normal vector. Now, we have to find the equation of this plane.

Let O be the origin.

Let $ON = p$, length of perpendicular from the origin to the plane

Let $\hat{n}$ is the unit normal vector along ON whose direction from O to N is positive

$\therefore$ $$\overrightarrow{ON} = p\hat{n} \qquad (1)$$

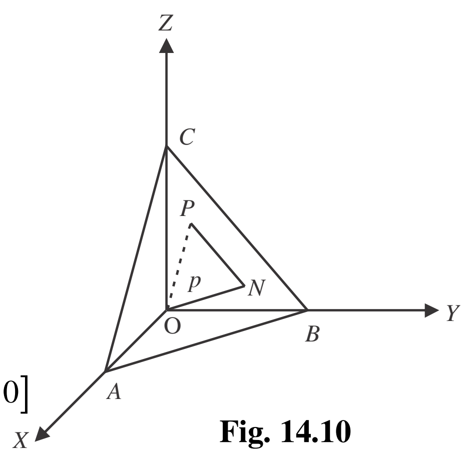

Fig. 14.10

Let the position vector of any point P be $\vec{r}$ then $\overrightarrow{NP}\perp\overrightarrow{ON}$.

$\therefore$ $$\overrightarrow{NP}\cdot\overrightarrow{ON}=0 \qquad (2)$$

but $$\overrightarrow{NP}=\vec{r}-p\hat{n} \qquad (3)$$

From (1), (2) and (3),

$$\left(\vec{r}-p\hat{n}\right)\cdot p\hat{n}=0$$

or $$\left(\vec{r}-p\hat{n}\right)\cdot\hat{n}=0 \qquad [\because\ p\neq 0]$$

or $$\vec{r}\cdot\hat{n}-p\hat{n}\cdot\hat{n}=0$$

or $$\vec{r}\cdot\hat{n}=p \quad [\because \hat{n}\cdot\hat{n}=1]$$

Cartesian Form : Let ABC is any plane and ON is perpendicular from the origin, where N is the foot of perpendicular. If the length of perpendicular from the origin to ON is p and direction cosines are l, m, n, then equation of plane will be in terms of l, m, n and p.

Clearly the coordinates of point N are (lp, mp, np). Let $P(x, y, z)$ be any point on the line lying in a plane. Then, the d.c's of PN are $\frac{x-lp}{PN}, \frac{y-mp}{PN}, \frac{z-np}{PN}$. Now since ON is perpendicualr on the plane, thus it is perpendicular to every line lying in the plane. Therefore, ON and PN are mutually perpendicular.

$$l\left(\frac{x-lp}{PN}\right)+m\left(\frac{y-mp}{PN}\right)+\left(\frac{z-np}{PN}\right)=0$$

$$\Rightarrow \qquad lx + my + nz = p\left(l^2 + m^2 + n^2\right)$$

$$\Rightarrow \qquad lx + my + nz = p \qquad \left[\because\ l^2 + m^2 + n^2 = 1\right]$$

This is the required eqation of a plane in Normal form.

Note: Let $\vec{n}$ be any vector in the direction of n, then $\vec{n} = n\hat{n}$.

From (4), $\qquad \vec{r}\cdot(\vec{n}/n) = p \Rightarrow \qquad \vec{r}\cdot\vec{n} = np$

or, $\qquad \vec{r}\cdot\vec{n} = q$, where $\qquad (5)$

$$q = np \qquad (6)$$

$\therefore$ This is the vector equation of the plane.

2. When origin lies on the plane, then $p = 0$, therefore, equation of plane passing through the origin and perpendicular to the vector $\vec{n}$ is $\vec{r}\cdot\vec{n} = 0$.

3. In the normal form of a plane, the direction of vector $\vec{n}$ is from origine to the plane and p is positive.

4. If the intercepts made by the plane $\vec{r}\cdot\vec{n} = q$ are $x_1,\ y_1,\ z_1$, respectively such that OA=x_1, OB=y_1 and OC=z_1 then the position vector of these points are $x_1 i,\ y_1 j$ and $z_1 k$. Since the point lies on the plane, therefore.

$$x_1 i\cdot\vec{n} = q \qquad x_1 j\cdot\vec{n} = q, \qquad z_1 k\cdot\vec{n} = q$$

$$\Rightarrow \qquad x_1 = \frac{q}{i\cdot\vec{n}}, \qquad y_1 = \frac{q}{j\cdot\vec{n}}, \qquad z_1 = \frac{q}{k\cdot\vec{n}}.$$

5. Vector equation of a plane is an equation which have the position vector of any arbitrary point lying in the plane.

Illustrative Examples

Example 25. Find the equation of plane whcih is at a distance of 4 units from the origin and perpendicular to the vector $i - 2j + 2k$.

Solution : Vector form : Here $p = 4$ and $\vec{n} = i - 2j + 2k$

$$\therefore \qquad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{i - 2j + 2k}{\sqrt{(1+4+4)}} = \frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k$$

thus, the required equation of plane is $\vec{r}\cdot\left(\frac{1}{3}i - \frac{2}{3} + \frac{2}{3}k\right) = 4$

or, $$\vec{r}\cdot(i - 2j + 2k) = 12$$

This is the required equation of plane.

Cartesian form : Substituting $\vec{r} = xi + yj + zk$ in the above equation, we have

equation $$(xi + yj + zk)\cdot(i - 2j + 2k) = 12$$

i.e. $$x - 2y + 2z = 12,$$

Example 26. Reduce the equation of plane $\vec{r}\cdot(i-2j+2k)=12$ into the Normal form and find the perpendicular distance from the origin.

Solution : Vector form : Given equation of plane is $\vec{r}\cdot(i-2j+2k)=12$

i.e. $\vec{r}\cdot\vec{n}=12,$

where, $\vec{n}=i-2j+2k.\quad \therefore\quad |\vec{n}|=\sqrt{(1+4+4)}=3\neq 1$

Thus, the given equation is not in the nromal form.

$\therefore$ Dividing both the sides by $|\vec{n}|=3$

$$(\vec{r}\cdot\vec{n})/3=12/3 \qquad \Rightarrow \qquad \vec{r}\cdot\left(\frac{1}{3}i-\frac{2}{3}j+\frac{2}{3}k\right)=4$$

This equation represents the equation of plane in normal form and the distance from the origin is 4 units.

Cartesian form: Cartesian form of the equation is

$$x-2y+2z=12$$

Here R.H.S. is positive, now dividing the equation by $\sqrt{(1+4+4)}=3\neq 1$ we have

$$\frac{1}{3}x-\frac{2}{3}y+\frac{2}{3}z=4,$$

The given equation represents the normal form, with d.r's $\frac{1}{3},-\frac{2}{3},\frac{2}{3}$

Example 27. Find the equation of plane whose distance from the origin is 2 units and the d.r's of its normal be 12, –3, 4.

Solution : Given $p=2$ and the d.r's of its normal be $12, -3, 4$

Thus, the direction cosines of the normal be $12/13, -3/13, 4/13$ $\{\because \sqrt{(12)^2+(-3)^2+(4)^2}=13\}$

$\therefore$ Thus, the equation of plane is,

$$\frac{12}{13}x-\frac{3}{13}y+\frac{4}{13}z=2, \qquad [\text{From } \ell x+my+nz=p]$$

or, $$12x-3y+4z=26,$$

which is the required equation of the plane.

Vector form : Let $\vec{n}$ be the vector perpendicular to the plane and the d.r's of $\vec{n}$ are 12, –3, 4

$$\therefore \quad \vec{n}=12i-3j+4k \qquad \Rightarrow \qquad |\vec{n}|=\sqrt{\{(12)^2+(-3)^2+(4)^2\}}=13\neq 1$$

$$\therefore \quad \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{12}{13}i-\frac{3}{13}j+\frac{4}{13}k$$

$\because$ The required plane is at a distance of 2 units from the origin. Therefore, the equation will be

$$\vec{r}\cdot\hat{n}=2$$

or, $$\vec{r}\cdot\left(\frac{12}{13}i-\frac{3}{13}j+\frac{4}{13}k\right)=2$$

This is the required equation of the plane in vector form.

Example 28. Find the direction cosines of the perpendicular dropped from the origin to the plane $\vec{r}\cdot(6i+2j-3k)+7=0$.

Solution : Cartesian form : The given equation of the plane can be written as

$$(xi+yj+zk)\cdot(6i+2j-3k)+7=0$$

or, $$6x+2y-3z+7=0$$

or, $$-6x-2y+3z=7 \qquad (1)$$

On dividing by 7, we have

$$-\frac{6}{7}x-\frac{2}{7}y+\frac{3}{7}z=1, \qquad (2)$$

Comparing the equation (2) with $\ell x+my+nz=p$, we get the required direction cosines as $-6/7,\ -2/7,\ 3/7$

Vector form : To find the direction cosines of the perpendicular we need to convert the given plane into normal form

Equation of plane $$\vec{r}\cdot(6i+2j-3k)+7=0,$$

i.e. $$\vec{r}\cdot(6i+2j-3k)=-7$$

$$\Rightarrow \quad \vec{r}\cdot(-6i-2j+3k)=7$$

$$\Rightarrow \quad \vec{r}\cdot\vec{n}=7, \qquad \text{where } \vec{n}=(-6i-2j+3k)$$

now $$|\vec{n}|=\sqrt{\left\{(-6)^2+(-2)^2+(3)^2\right\}}=7\neq 1.$$

On dividing by $|\vec{n}|=7$ in (1), we have

$$\frac{\vec{r}\cdot\vec{n}}{7}=\frac{7}{7}$$

or $$\vec{r}\cdot\left(-\frac{6}{7}i+\frac{2}{7}j+\frac{3}{7}k\right)=1$$

Therefore the d.c's of the perpendicular dropped from origin to the plaen are $-\frac{6}{7},-\frac{2}{7},\frac{3}{7}$.

Exercise 14.6

1. Find the equation of plane passing through the point (2, –1, 3) and perpendicular to the X-axis.
2. Find the equation of plane passing through the point (3, 2, 4) and X-axis.
3. A variable plane passes through the point (p, q, r) and meets the coordinate axes in point A, B and C respectively. Show that the locus of the common points of the planes parallel to the coordinate axes and passing through A, B and C is

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=1$$

4. Find the vector equation of the plane which is at a distance of 7 units from the origin and $\hat{i}$ is the unit normal vector to it.
5. Find the vector equation of the plane whcih is at a distance of 7 units from the origin and normal to the vector $6i+3j-2k$.
6. Write the equation of plane $\vec{r}\cdot\left(3i-4j+12k\right)=5$ in normal form and find the perpendicular distance from the origin, Also find the d.c's of the normal so obtained.

or

Write the equation of plane $3x-4y+12z=5$ in normal form and find the perpendicular distance from the origin. Also find the d.c's of the normal so obtained.

7. Find the vector equation of the plane whcih is at a distance fo 4 units from the origin and the direction ratios of the normal are 2, –1, 2.
8. Find the normal form of the equation of the plane $2x-3y+6z+14=0$.
9. Find the equation of plane, if the length of perpendicular drawn from origin is 13 units and the direction ratios of the perpendicular are 4, –3, 12.
10. Find the unit normal vector of the plane $x+y+z-3=0$.

14.17 Angle Between Two Planes

The angle between two planes is defined as the angle between their normals

Vector form: Let the equation of the plane be

$$\vec{r}\cdot\vec{n}_1=d_1 \qquad \text{and} \qquad \vec{r}\cdot\vec{n}_2=d_2$$

where $\vec{n}_1$ and $\vec{n}_2$ are the perpendicular vectors. Observe that if θ is an angle between the two planes, then angle between their normals is also θ

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \qquad \text{or} \qquad \theta=\cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

Note: (i) Two planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2=0$.

(ii) Two planes are parallel if $\vec{n}_1=\lambda\vec{n}_2$, where λ is a constant.

Cartesian form: Let the angle between the two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be θ . Let $\vec{n}_1$ and $\vec{n}_2$ are normal vectors to the plane.

$\therefore \qquad \vec{n}_1 = a_1 i + b_1 j + c_1 k$

and $\qquad \vec{n}_2 = a_2 i + b_2 j + c_2 k$

$$\therefore \qquad \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

Note: (i) Two planes are mutually perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

(ii) Two planes are parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

14.18 Angle Between a Plane and a Line

The angle between a plane and a line is the complement of the angle between the line and normal to the plane

Vector form: Let the equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and the equation of the plane is $\vec{r} \cdot \vec{n} = d$, where $\vec{n}$ is normal vector of plane. If θ is angle between plane and lien, then angle between line and normal to the plane will be $\left(\frac{\pi}{2} - \theta\right)$.

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \qquad \text{or} \qquad \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$$

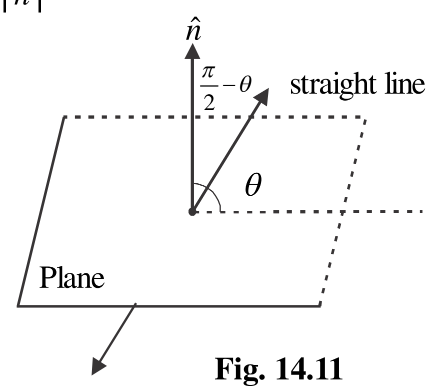

Fig. 14.11

Note: (i) line is perpendicualr to the plane, if $\vec{b} \times \vec{n} = \vec{O}$ or $\vec{b} = \lambda \vec{n}$.

(ii) line is parallel to the plane, if $\vec{b} \cdot \vec{n} = 0$.

Cartesian form: Let the equation of the plane be

$$ax + by + cz + d = 0 \qquad (1)$$

and equation of line be

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (2)$$

The d.c's of (1) are *a*, *b*, *c* and the d.r's of the line (2) are *l*, *m*, *n*. If the angle between the line and the plane is θ , then the angle between the normal and the line will be $\left(\frac{\pi}{2} - \theta\right)$.

$$\therefore \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$$

or

$$\sin\theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$$

Note: (i) line is perpendicualr to the plane, if $\frac{a}{\ell}=\frac{b}{m}=\frac{c}{n}$.

(ii) line is parallel to the plane, if $a\ell+bm+cn=0$.

Illustrative Examples

Example 29. Find the angle between the planes $\vec{r}\cdot(2i-3j+4k)=1$ and $\vec{r}\cdot(-i+j)=4$.

Solution : We know that the angle between the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is

$$\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here, $\vec{n}_1=2i-3j+4k$ and $\vec{n}_2=-i+j+0k$

$$\therefore \qquad \cos\theta=\frac{-2-3+0}{\sqrt{4+9+16}\sqrt{1+1}}=\frac{-5}{\sqrt{29}\sqrt{2}}$$

$$\Rightarrow \qquad \theta=\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$$

Example 30. Prove that the planes $2x+6y+6z=7$ and $3x+4y-5z=8$ are mutually perpendicular.

Solution : We know that the planes

$$2x+6y+6z=7$$

and $$3x+4y-5z=8$$

are mutually perpendicular, if their normals are mutually perpendicular

i.e., $2(3)+6(4)+6(-5)=0$

or, $6+24-30=0$, which is true.

Hence, the planes are mutually perpendicular.

Example 31. If the planes $\vec{r}\cdot(i+2j+3k)=7$ and $\vec{r}\cdot(\lambda i+2j-7k)=26$ are mutually perpendicular then find the vlaue of λ.

Solution : The planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ are mutually perpendicualr if

$$\vec{n}_1\cdot\vec{n}_2=0$$

Here, $\vec{n}_1=(i+2j+3k)$ and $\vec{n}_2=(\lambda i+2j-7k)$, therefore

$$(i+2j+3k)\cdot(\lambda i+2j-7k)=0$$

$$\Rightarrow \qquad \lambda+4-21=0 \quad \text{or} \quad \lambda=17$$

Example 32. Find the angle between the line $\vec{r}=(2i+2j+9k)+\lambda(2i+3j+4k)$ and plane $\vec{r}\cdot(i+j+k)=5$.

Solution : If the angle between the line $\vec{r}=\vec{a}+\lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n}=d$ is θ, then

$$\sin\theta=\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

On comparing with the standard equation, we have

$$\vec{b} = 2i + 3j + 4k \qquad \text{and} \qquad \vec{n} = i + j + k$$

$$\therefore \qquad \sin\theta = \frac{(2i+3j+4k)\cdot(i+j+k)}{\sqrt{4+9+16}\sqrt{1+1+1}} = \frac{9}{\sqrt{87}}$$

$$\Rightarrow \qquad \theta = \sin^{-1}\left(\frac{9}{\sqrt{87}}\right) \qquad \text{or} \qquad \theta = \sin^{-1}\left(\frac{3\sqrt{3}}{\sqrt{29}}\right)$$

Example 33. Find the angle between the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$ and the plane $2x + y - z = 4$.

Solution : The perpendicular vector to the plane $\qquad 2x + y - z = 4 \qquad$ (1)

is $\vec{n} = 2i + j - k$ and the parallel vector to the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$ is $\vec{b} = i - j + k$

If the angle between the line adn the plane is θ then

$$\sin\theta = \frac{(i-j+k)\cdot(2i+j-k)}{\sqrt{1+1+1}\sqrt{4+1+1}} = \frac{2-1-1}{\sqrt{3}\sqrt{6}} = 0 \qquad \Rightarrow \qquad \theta = 0$$

Example 34. If the line $\vec{r} = (i - 2j + k) + \lambda(2i + j + 2k),$ is parallel to the plane $\vec{r}\cdot(3i - 2j + mk) = 4$, then find the vlaue of m.

Solution : Given line is parallel to the vector $\vec{b} = 2i + j + 2k$ and normal vector to the plane is $\vec{n} = 3i - 2j + mk$. Since the given line is parallel to the plane,

$$\therefore \qquad \vec{b} \perp \vec{n}.$$

$$\Rightarrow \qquad \vec{b}\cdot\vec{n} = 0$$

$$\Rightarrow \qquad (2i + j + 2k)\cdot(3i - 2j + mk) = 0$$

$$\Rightarrow \qquad 6 - 2 + 2m = 0 \qquad \Rightarrow \qquad m = -2$$

14.19 Distance of a Point From a Plane

Consider a point P with position vector $\vec{a}$ and a plane whose equation is $\vec{r}\cdot\vec{n} = q$ We have to find the length of perpendicular from a point to the given plane.

Let π be the given plane and the position vector of point P is $\vec{a}$. Let the length of perpendicualr drawn from point P on plane π be PM.

$\because$ line PM, passes through $P(\vec{a})$ and the unit normal vector $\vec{n}$ is parallel to the plane π

$\therefore$ the vector equation of the line PM is $\vec{r} = \vec{a} + \lambda\vec{n}$, where λ is a sacalr. (1)

Again point M, is the intersecting point of line PM and plane π , therefore point M will satisfy the equation of plane

$$\therefore \qquad (\vec{a} + \lambda\vec{n})\cdot\vec{n} = q$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{n} + \lambda\vec{n}\cdot\vec{n} = q$$

$$\Rightarrow \qquad \vec{a}\cdot\vec{n}+\lambda|\vec{n}|^2=q$$

$$\Rightarrow \qquad \lambda=\frac{q-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}$$

substituting the value of λ in (1), the position vector of M will be

$$\vec{r}=\vec{a}+\frac{q-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}$$

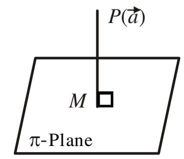

Fig. 14.12

$$\therefore \qquad \overrightarrow{PM} = \text{(position vector M)} - \text{(position vector of P)}$$

$$=\vec{a}+\frac{q-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}-\vec{a}=\frac{(q-\vec{a}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$$

$$\therefore \qquad PM=|\overrightarrow{PM}|=\frac{|(q-\vec{a}\cdot\vec{n})\vec{n}|}{|\vec{n}|^2}=\frac{|(q-\vec{a}\cdot\vec{n})||\vec{n}|}{|\vec{n}|^2}=\frac{|(q-\vec{a}\cdot\vec{n})|}{|\vec{n}|}$$

Thus, the required length is $\dfrac{|q-\vec{a}\cdot\vec{n}|}{|\vec{n}|}$ or $\dfrac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}$

Note: (i) $$\overrightarrow{PM}=(PM)\hat{n}=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}\hat{n}$$

$$=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|}\times\frac{\vec{n}}{|\vec{n}|}=\frac{|\vec{a}\cdot\vec{n}-q|\vec{n}}{|\vec{n}|^2}$$

(ii) length of perpendicular drawn from origin to the plane $\vec{r}\cdot\vec{n}=q$ is

$$=\frac{q}{|\vec{n}|} \qquad \text{[here } \vec{a}=\vec{0}\text{]}$$

Cartesian form: To find the length of perpendicualr drawn from point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$.

Let the foot of perpendicualr drawn from point $P(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ is M.

Therefore, the equation of the line *PM* is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \qquad (1)$$

($\because$ The direction ratios *a*, *b*, c of normal to the plane will also be the direction ratios of the line *PM)*

Now, the coordinates of any point on the line are $(x_1+ar,\ y_1+br,\ z_1+cr)$, where *r* is real number. If these are the coordinates of point *M,* then they will satisfy the equation of plane

$$\therefore \qquad a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$$

or, $$r=-\frac{ax_1+by_1+cz_1+d}{a^2+b^2+c^2} \qquad (2)$$

Now, $PM = \sqrt{\left\{(x_1 + ar - x_1)^2 + (y_1 + br - y_1)^2 + (z_1 + cr - z_1)^2\right\}}$

$= |r|\sqrt{(a^2 + b^2 + c^2)}$

Now, $PM = \left| -\dfrac{(ax_1 + by_1 + cz_1 + d)}{(a^2 + b^2 + c^2)} \right| \sqrt{\left(a^2 + b^2 + c^2\right)}$ [using (2)]

Therefore, the required length is $\left| \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$

Illustrative Examples

Example 35. Find the perpendicualr distance of the point with position vector $2i - j - 4k$ from the plane $\vec{r} \cdot (3i - 4j + 12k) - 9 = 0$.

Solution : We know that the perpendicular distance of the point, whose position vector is $\vec{a}$, from the plane $\vec{r} \cdot \vec{n} = q$ is $\left| \dfrac{\vec{a} \cdot \vec{n} - q}{|\vec{n}|} \right|$.

Here $\vec{a} = 2i - j - 4k$, $\vec{n} = 3i - 4j + 12k$ and $q = 9$.

$\therefore$ Required distance $= \dfrac{|(2i - j - 4k) \cdot (3i - 4j + 12k) - 9|}{\sqrt{(9 + 16 + 144)}} = \dfrac{47}{13}$

Example 36. Show that the points $A(1, -1, 3)$ and $B(3, 3, 3)$ are at equal distance from the plane $\vec{r} \cdot (5i + 2j - 7k) + 9 = 0$.

Solution : Position vector of point A is $i - j + 3k$.

$\therefore$ perpendicular distance of point A from the plane is

$$= \frac{|(i - j + 3k) \cdot (5i + 2j - 7k) + 9|}{\sqrt{(25 + 4 + 49)}} = \frac{9}{\sqrt{78}} \qquad (1)$$

Position vector of point B is $3i + 3j + 3k$.

$\therefore$ Perpendicular distance of point B from the plane is

$$= \frac{|(3i + 3j + 3k) \cdot (5i + 2j - 7k) + 9|}{\sqrt{(25 + 4 + 49)}} = \frac{9}{\sqrt{78}} \qquad (2)$$

Therefore, from (1) and (2), we conclude that the point is at equal distance from the given plane.

Exercise 14.7

1. Find the angle between the planes:

   (i) $\vec{r} \cdot (2i - j + 2k) = 6$ and $\vec{r} \cdot (3i + 6j - 2k) = 9$

   (ii) $\vec{r} \cdot (2i + 3j - 6k) = 5$ and $\vec{r} \cdot (i - 2j + 2k) = 9$

   (iii) $\vec{r} \cdot (i + j + 2k) = 5$ and $\vec{r} \cdot (2i - j + 2k) = 6$

2. Find the angle between the planes:

(i) $x+y+2z=9$ and $2x-y+z=15$

(ii) $2x-y+z=4$ and $x+y+2z=3$

(iii) $x+y-2z=3$ and $2x-2y+z=5$

3. Prove that the following planes are mutually perpendicular:

(i) $x-2y+4z=10$ and $18x+17y+4z=49$

(ii) $\vec{r}\cdot(2i-j+k)=4$ and $\vec{r}\cdot(-i-j+k)=3$

4. If the following planes are mutually perpendicualr, then find the vlaue of λ:

(i) $\vec{r}\cdot(2i-j+\lambda k)=5$ and $\vec{r}\cdot(3i+2j+2k)=4$

(ii) $2x-4y+3z=5$ and $x+2y+\lambda z=5$

5. Find the angle between the line $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z-2}{4}$ and plane $2x+y-3z+4=0$.

6. Find the angle between the line $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-3}{2}$ and plane $3x+4y+z+5=0$.

7. Find the angle between the line $\vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$.

8. Find the angle between the line $\vec{r}=(2i+3j+k)+\lambda(i+2j-k)$ and plane $\vec{r}\cdot(2i-j+k)=4$.

9. If the line $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ is parallel to the plane $\vec{r}\cdot(3\hat{i}-2\hat{j}+m\hat{k})=3$, then find the vlaue of m.

10. If the line $\vec{r}=i+\lambda(2i-mj-3k)$ is parallel to the plane $\vec{r}\cdot(mi+3j+k)=4$, then find the vlaue of m.

Miscellaneous Exercise 14

1. Whcih of the following group is not the direction cosines of a line:

(A) 1, 1, 1 (B) 0, 0, –1 (C) –1, 0, 0 (D) 0, –1, 0

2. Point P is such that OP = 6 and vector $\overrightarrow{OP}$ makes an angle 45° and 60° with OX-axis and OY-axis respectively, then the position vector of P will be

(A) $3i+3j\pm3\sqrt{2}k$ (B) $6i+6\sqrt{2}j\pm6k$ (C) $3\sqrt{2}i+3j\pm3k$ (D) $3i+3\sqrt{2}j\pm3k$

3. The angle between the two diagonals of the cube will be

(A) 30° (B) 45° (C) $\cos^{-1}(1/\sqrt{3})$ (D) $\cos^{-1}(1/3)$

4. The direction cosines of vector $3i$ are:

(A) 3, 0, 0 (B) 1, 0, 0 (C) –1, 0, 0 (D) –3, 0, 0

5. The vector form of the line $\frac{x-3}{-2}=\frac{y-4}{-5}=\frac{x+7}{13}$ is

(A) $(3i+4j-7k)+\lambda(-2i-5j+13k)$ (B) $(-2i-5j+13k)+\lambda(3i+4j-7k)$

(C) $(-3i-4j+7k)+\lambda(-2i-5j+13k)$ (D) none of these

6. If lines $\frac{x+1}{1}=\frac{y+2}{\lambda}=\frac{z-1}{-1}$ and $\frac{x-1}{-\lambda}=\frac{y+1}{-2}=\frac{z+1}{1}$ are mutually perpendicular, then the value of λ is

(A) 0 (B) 1 (C) –1 (d) 2

7. The shortest distance between the lines $\vec{r}=(5i+7j+3k)+\lambda(5i-16j+7k)$ and $\vec{r}=(9i+13j+15k)+\mu(3i+8j-5k)$ is

(A) 10 units (B) 12 units (C) 14 units (D) 7 units

8. The angle between the line $\vec{r}=(2i-j+k)+\lambda(-i+j+k)$ and the plane $\vec{r}\cdot(3i+2j-k)=4$ is

(A) $\sin^{-1}(-2/\sqrt{42})$ (B) $\sin^{-1}(2/\sqrt{42})$ (C) $\cos^{-1}(-2/\sqrt{42})$ (D) $\cos^{-1}(2/\sqrt{42})$

9. If the equation $lx+my+nz=p$ is the normal form of the plane then which of the following is true or false

(A) l, m, n are the d.c's of the normal to the plane

(B) p is the perpendicualr distance from the origin to the plane

(C) for every vlaue of p, the plane passes through the origin

(D) $\ell^2+m^2+n^2=1$

10. A plane meets the coordinate axes at the points A, B and C respectively such that the centroid of the triangle ABC is (1, 2, 3), then the equation of the plane is

(A) $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$ (B) $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=\frac{1}{6}$ (C) $\frac{x-1}{1}+\frac{y-2}{2}+\frac{z-3}{3}=1$ (D) $\frac{x}{3}+\frac{y}{6}+\frac{z}{9}=1$

11. If two points are $P(2i+j+3k)$ and $Q(-4i-2j+k)$, then the equation of the plane passing through point Q and perpendicualr to PQ is

(A) $\vec{r}\cdot(6i+3j+2k)=28$ (B) $\vec{r}\cdot(6i+3j+2k)=32$

(C) $\vec{r}\cdot(6i+3j+2k)+28=0$ (D) $\vec{r}\cdot(6i+3j+2k)+32=0$

12. The direction cosines of two lines are expressed with the following given relations, find them

$$l-5m+3n=0 \quad \text{and} \quad 7l^2+5m^2-3n^2=0$$

13. The projection of the line segment on the axes are –3, 4, –12 respectively. Find the length and direction cosines of the line segment.

14. Prove that the line joining the points (a, b, c) and (a', b', c') passes through the origin, If $aa'+bb'+cc'=pp'$, p and p' are the distances from the origin.

15. Find the equation of plane passing through $P(-2, 1, 2)$ and parallel to the vectors $\vec{a}=-i+2j-3k$ and $\vec{b}=5i-j+k$.

IMPORTANT POINTS

1. Any line OP (Vector $\overrightarrow{OP}$) makes angle α, β, γ with positive direction of co-ordinate axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of line OP (Vector $\overrightarrow{OP}$), whcih are generally denoted by l, m, n . Hence, $0 \le \alpha, \beta, \gamma \le \pi$.

   (i) Vector $\overrightarrow{PO}$ makes angle $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$ with axes OX, OY, OZ respectively, then direction cosines of $\overrightarrow{PO}$ are $\cos(\pi-\alpha), \cos(\pi-\beta)$, $\cos(\pi-\gamma)$ i.e. $-l, -m, -n$.

   Therefore, if l, m, n are direction cosines of any line, then $-l, -m, -n$ are also direction cosines of the same line.

   (ii) Direction cosines of X, Y and Z axes are respectively 1, 0, 0; 0, 1, 0 and 0, 0, 1.

2. **Projection of any vector on co-ordinate axes :** If $\vec{r}$ is given position vector and l, m, n are its direction cosines, then its projection on X, Y, Z axes are lr, mr, nr respectively.

3. **Co-ordinates of a point in the form of direction cosines:** If $P(x, y, z)$ is a point, then its co-ordinates will be (lr, mr, nr), where l, m, n are direction cosines of $\overrightarrow{OP}$ and OP = r.

4. **To represent a unit vector $\hat{r}$ in the form of direction cosines:**

   $\hat{r}$ (unit vector in direction of $\vec{r}$) $= l\hat{i} + m\hat{j} + n\hat{k}$,

   where l, m, n are direction cosines of $\vec{r}$.

5. $l^2 + m^2 + n^2 = 1$, where l, m, n are direction cosines.

6. **Direction ratios of a line :** A set of three numbers for $\vec{r}$, which are proportional to the direction cosines l, m, n are called direction ratios.

7. **Conversion of direction ratios into direction cosines :** Let $\vec{r} = ai + bj + ck$ is a vector having direction ratios a, b, c, then its direction cosines l, m, n are given as follows:

$$l = \frac{a}{\sqrt{(a^2+b^2+c^2)}}, \quad m = \frac{b}{\sqrt{(a^2+b^2+c^2)}}, \quad n = \frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

8. **Direction ratio and direction cosines of a line joining two points:** Let two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are direction ratio of line PQ and direction cosines are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$,

   where $$PQ = \sqrt{\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\}}$$

9. Equation of a line whcih passes through point $P(x_1, y_1, z_1)$ and parallel to line having direction cosines l, m, n is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

10. Co-ordinates of a point lying on line, which is at a distance *r* from a point $P(x_1, y_1, z_1)$ on same line are $(lr+x_1, mr+y_1, nr+z_1)$, where *r* is a parameter.
11. If direction ratios *a*, *b*, *c* are given then the equation of line is

$$\frac{x-x_1}{a/\sqrt{a^2+b^2+c^2}}=\frac{y-y_1}{b/\sqrt{a^2+b^2+c^2}}=\frac{z-z_1}{c/\sqrt{a^2+b^2+c^2}}=k(\text{let})$$

or, $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=R$, where $R=\frac{k}{\sqrt{a^2+b^2+c^2}}$

12. A point has co-ordinates $(ar+x_1, br+y_1, cr+z_1)$, then at this position, it is not at a distance *r* from point $P(x_1, y_1, z_1)$.
13. Equation of line passes through a point having position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$, where λ is a real number.
14. If above line passes through origin, then $\vec{r}=\lambda\vec{b}$.
15. **Non coplanar lines (skew lines) :** Non parallel and non-intersecting lines which doesn't lie on same plane are called 'Non coplanar or Skew lines'.
16. **Shortest distance :** Distance between two skew lines, which is perpendicular to both, is called "Shortest Distance".
17. **Shortest distance :** Shortest distance between two skew lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \text{ is}$$

$$=\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\left\{\sum\left(m_1n_2-m_2n_1\right)^2\right\}}$$

18. If shortest distance becomes zero, then lines are coplanar with the following condition.

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

19. **Shortet Distance :** Shortest distance between two skew lines

$$\vec{r}=\vec{a}_1+\lambda\vec{b}_1 \quad \text{and} \quad \vec{r}=\vec{a}_2+\lambda\vec{b}_2$$

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

20. If θ is an angle between two planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, Then

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

(i) If planes are mutually perpendicular, then $\vec{n}_1\cdot\vec{n}_2 = 0$.

(ii) If planes are parallel, then $\vec{n}_1 = \lambda\vec{n}_2$, where λ is constant

21. If θ is an angle between two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) Planes are mutually perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) Planes are parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

22. If θ is an angle between two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then

$$\cos\theta = \frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right)$$

(i) Lines are perpendicular, if $\vec{b}_1\cdot\vec{b}_2 = 0$.

(ii) Lines are parallel, if $\vec{b}_1 = \lambda\vec{b}_2$, where λ is cosntant.

23. If θ is an angle between two lines

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) Lines are perpendicualr, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) Lines are parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

24. Angle between a line and a plane is complement of angle between normal of plane and given line. Let equation of plane is $\vec{r}\cdot\vec{n} = d$ and equation of line is $\vec{r} = \vec{a} + \lambda\vec{b}$ and θ is angle between them, then

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

(i) Line is perpendicular to plane, if $\vec{b}\times\vec{n}=\vec{0}$, or $\vec{b}=\lambda\vec{n}$.

(ii) Line is parallel to plane, if $\vec{b}\cdot\vec{n}=0$.

25. **General equation of plane :**

$$ax+by+cz+d=0,$$

where a, b, c, d are scalar quantity or constant and all a, b, c are not zero.

(a) Every first degree equation in x, y, z represetns a plane.

(b) There is only three independent constant in plane.

26. Equation of plane passing through a point $(x_1,\ y_1,\ z_1)$

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0,$$

where a, b, c are constant.

27. Equation of plane in intercpet form:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

where a, b, c are respectively intercepts on X, Y, Z axes respectively.

28. **Equation of plane in normal form:**

$$\vec{r}\cdot\hat{n}=p,$$

Here p is perpendicualr distance from origin to the plane and $\hat{n}$ is unti vector of normal of plane.

Note : Equation of plane in normal form may also be written as

$$\vec{r}\cdot\vec{n}=q$$

Here $q=|\vec{n}|\,p$.

29. Distance of a point from plane :

$$d=\frac{|\vec{a}\cdot\vec{n}-q|}{|\vec{n}|},$$

where $\vec{a}$ is position vector of point and $\vec{r}\cdot\vec{n}=q$ is equation of plane.

Answers

Exercise 14.1

1. $\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}$ 2. $0,\ \frac{1}{\sqrt{5}},\ \frac{4}{5}$ 3. $\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}$ 4. $\sqrt{2}\hat{i}+\hat{j}-\hat{k}$

Exercise 14.2

1. (i) $\frac{x-5}{0}=\frac{y-7}{0}=\frac{z-9}{0}$; (ii) $\frac{x-5}{0}=\frac{y-7}{1}=\frac{z-9}{0}$; (iii) $\frac{x-5}{0}=\frac{y-7}{0}=\frac{z-9}{1}$

2. $\vec{r}=\left(2\hat{i}-3\hat{j}+4\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{j}-5\hat{k}\right)$; $\frac{x-2}{3}=\frac{y+3}{4}=\frac{z-4}{-5}$ 3. $\vec{r}=5\hat{i}-2\hat{j}+4\hat{k}+\lambda\left(2\hat{i}-\hat{j}+3\hat{k}\right)$

4. $\vec{r}=\left(2\hat{i}-\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+7\hat{j}-3\hat{k}\right)$ 5. $\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$

6. $\frac{x-1}{-2}=\frac{y-2}{14}=\frac{z-3}{3}$

7. (i) Equation of AB : $\vec{r}=\left(4\hat{i}+5\hat{j}+10\hat{k}\right)+\mu\left(\hat{i}+\hat{j}+3\hat{k}\right)$; $\frac{x-4}{1}=\frac{z-5}{1}=\frac{z-10}{3}$

(ii) Equation of BC : $\vec{r}=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\lambda\left(\hat{i}+\hat{j}+5\hat{k}\right)$; $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{5}$;

(iii) Co-ordinates of D are $(3,\ 4,\ 5)$

8. $\left(-\frac{1}{3},\frac{1}{3},1\right)$; 2, 1, -6 ; $\vec{r}=-\frac{1}{3}\hat{i}+\frac{1}{3}\hat{j}+\hat{k}+\lambda\left(2\hat{i}+\hat{j}-6\hat{k}\right)$

9. $\vec{r}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$; $\frac{x-1}{3}=\frac{y-2}{2}=\frac{z-3}{-2}$

10. $\vec{r}=\left(2\hat{i}-\hat{j}+4\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-\hat{k}\right)$; $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$ 11. $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

12. $\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$ 13. $\frac{x}{5}=\frac{y}{-2}=\frac{z}{3}$; $\vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$

14. $\vec{r}=\left(3\hat{i}-2\hat{j}-5\hat{k}\right)+\lambda\left(11\hat{k}\right)$; $\frac{x-3}{0}=\frac{z+2}{0}=\frac{z+5}{11}$

Exercise 14.3

1. $\theta=\cos^{-1}\left(19/21\right)$ 2. $\theta=\cos^{-1}\left(2/3\right)$ 4. $k=-10/7$

5. $\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$; $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$ 6. $\frac{x+}{3}=\frac{y-4}{5}=\frac{z+5}{6}$

Exercise 14.4

1. (–1, –1, –1) 2. No 3. $\left(\frac{170}{49}, \frac{78}{49}, \frac{10}{49}\right); \frac{3}{7}\sqrt{101}$

4. $\vec{r} = \left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) + \lambda\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right); \ \frac{\sqrt{580}}{7}$

Exercise 14.5

1. $\frac{3\sqrt{2}}{2}$ 2. $2\sqrt{29}$ 3. $\frac{3}{\sqrt{19}}$ 4. $\frac{8}{\sqrt{29}}$ 5. $\frac{3}{\sqrt{59}}; \ \frac{59x-253}{1} = \frac{59y-232}{-3} = \frac{592-97}{7}$

Exercise 14.6

1. $x - 2 = 0$ 2. $2y - z = 0$ 4. $\vec{r} \cdot i = 7$

5. $\vec{r} \cdot \left(\frac{6}{7}i + \frac{3}{7}j - \frac{2}{7}k\right) = 7$;k $\vec{r} \cdot (6i + 3j - 2k) = 49$

6. $\vec{r} \cdot \left(\frac{3}{13}i - \frac{4}{13}j + \frac{12}{13}k\right) = \frac{5}{13}; \ \frac{5}{13}; \ \frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$ or $\frac{3}{13}x - \frac{4}{13}y + \frac{12}{13}z = \frac{5}{13}; \ \frac{5}{13}; \ \frac{3}{13}, -\frac{4}{13}, \frac{12}{13}$

7. $\vec{r} \cdot \left(\frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k\right) = 4$ 8. $-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$ 9. $\frac{4}{13}x - \frac{3}{13}y + \frac{12}{13}z = 13$ 10. $\frac{1}{\sqrt{3}}(i + j + k)$

Exercise 14.7

1. (i) $\cos^{-1}\left(-\frac{4}{21}\right)$; (ii) $\cos^{-1}\left(-\frac{16}{21}\right)$; (iii) $\cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)$

2. (i) $\theta = \frac{\pi}{3}$; (ii) $\theta = \frac{\pi}{3}$; (iii) $\cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$

4. (i) $\lambda = -2$; (ii) $\lambda = 2$ 5. $\sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$ 6. $\sin^{-1}\left(\sqrt{\frac{7}{52}}\right)$

7. $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ 8. $\sin^{-1}\left(-\frac{1}{6}\right)$ 9. $m = -2$ 10. $m = -3$

Miscellaneous Exercise 14

1. (A) 2. (C) 3. (D) 4. (B) 5. (C) 6. (B) 7. (A)
8. (A) 9. (C) 10. (D) 11. (C)

12. $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}; \ \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

13. 13; $-\frac{3}{13}, \frac{4}{13}, -\frac{12}{13}$

14. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right); \ x - 2y + z = 0$

15. $x + 14y + 9z = 30$