7.9 Plane

Point coordinates: $x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, \ldots$
Real numbers: $A, B, C, D, A_1, A_2, a, b, c, a_1, a_2, \lambda, p, t, \ldots$
Normal vectors: $\vec{n}, \vec{n}_1, \vec{n}_2$
Direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$
Distance from point to plane: d

675. General Equation of a Plane

$$Ax + By + Cz + D = 0$$

676. Normal Vector to a Plane

The vector $\vec{n}(A, B, C)$ is normal to the plane

$$Ax + By + Cz + D = 0.$$

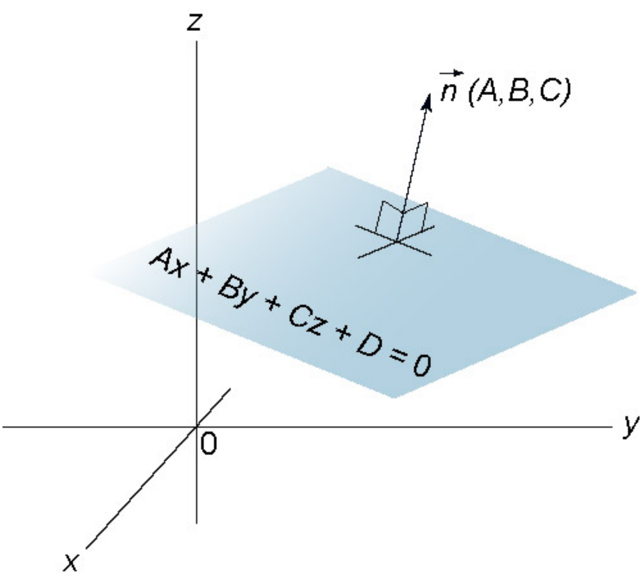

Figure 127.

677. Particular Cases of the Equation of a Plane
$Ax + By + Cz + D = 0$

If $A = 0$, the plane is parallel to the x-axis.
If $B = 0$, the plane is parallel to the y-axis.
If $C = 0$, the plane is parallel to the z-axis.
If $D = 0$, the plane lies on the origin.

If $A = B = 0$, the plane is parallel to the xy-plane.
If $B = C = 0$, the plane is parallel to the yz-plane.
If $A = C = 0$, the plane is parallel to the xz-plane.

678. Point Direction Form
$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$,
where the point $P(x_0, y_0, z_0)$ lies in the plane, and the vector (A, B, C) is normal to the plane.

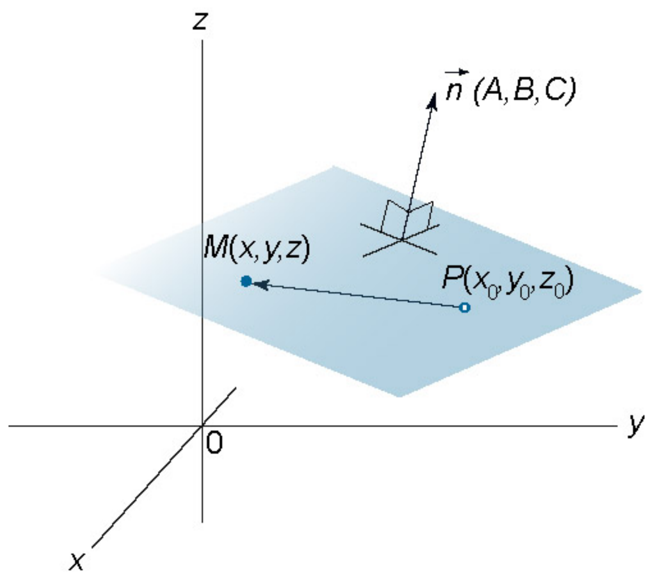

Figure 128.

679. Intercept Form

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

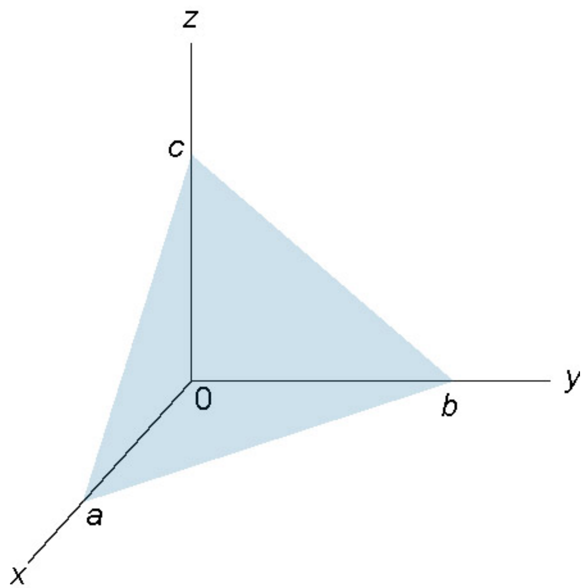

Figure 129.

680. Three Point Form

$$\begin{vmatrix} x-x_3 & y-y_3 & z-z_3 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \\ x_2-x_3 & y_2-y_3 & z_2-z_3 \end{vmatrix}=0,$$

or

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

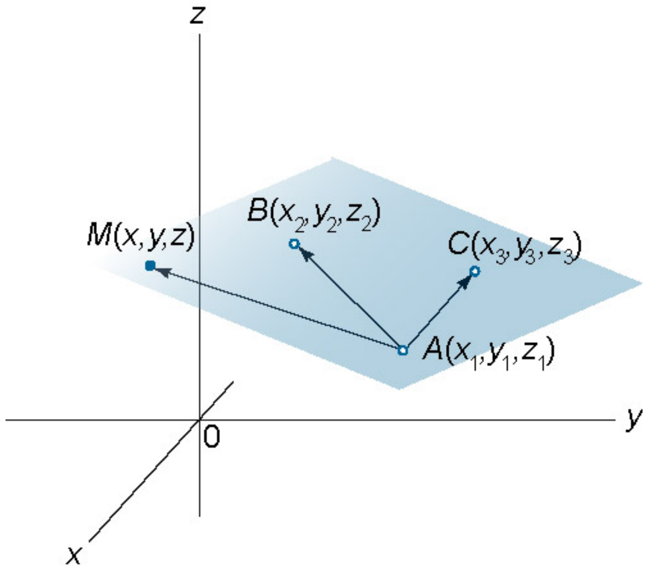

Figure 130.

681. Normal Form

$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$,

where p is the perpendicular distance from the origin to the plane , and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of any line normal to the plane.

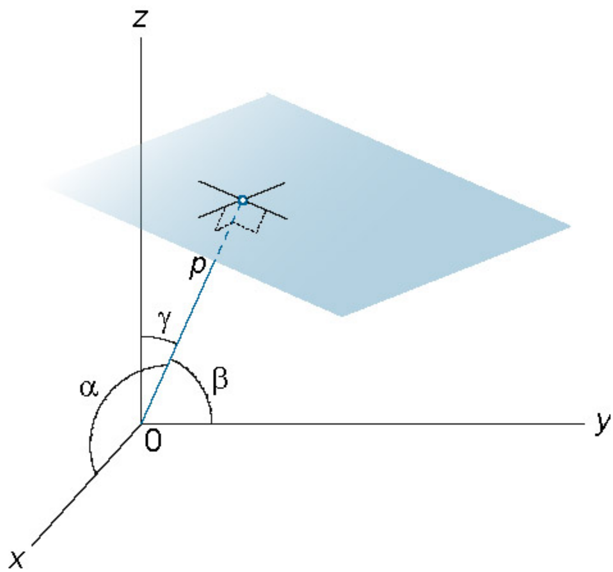

Figure 131.

682. Parametric Form

$$\begin{cases} x = x_1 + a_1 s + a_2 t \\ y = y_1 + b_1 s + b_2 t \, , \\ z = z_1 + c_1 s + c_2 t \end{cases}$$

where (x, y, z) are the coordinates of any unknown point on the line , the point $P(x_1, y_1, z_1)$ lies in the plane, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel to the plane.

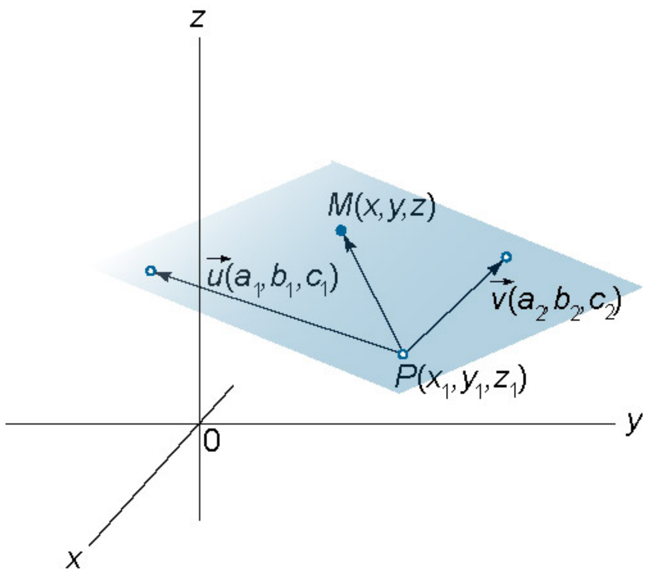

Figure 132.

683. Dihedral Angle Between Two Planes

If the planes are given by

$A_1x + B_1y + C_1z + D_1 = 0$,

$A_2x + B_2y + C_2z + D_2 = 0$,

then the dihedral angle between them is

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

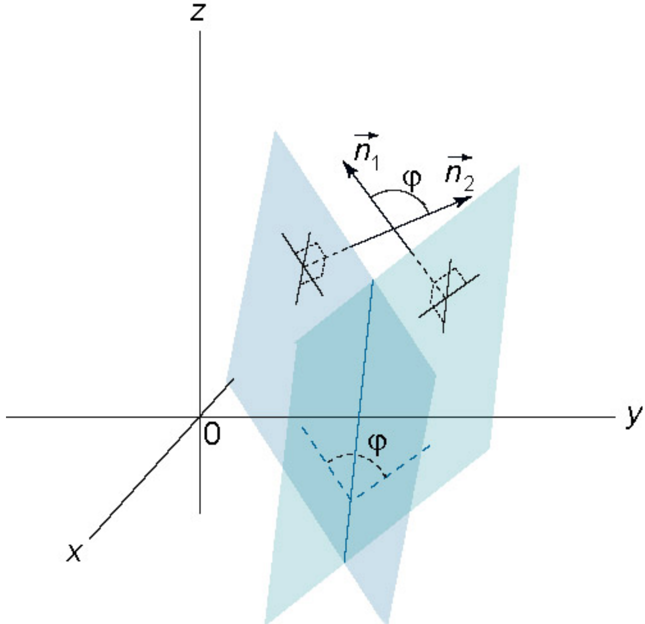

Figure 133.

684. Parallel Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

685. Perpendicular Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

686. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel To the Vectors $(a\ , b\ , c\ )$ and $(a\ , b\ , c\ )$ (Fig.132)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

687. **Equation of a Plane Through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and Parallel To the Vector (a, b, c)**

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

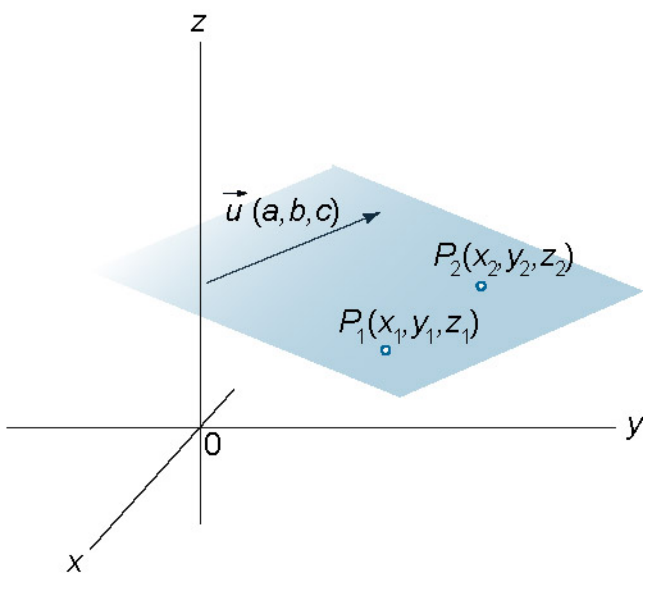

Figure 134.

688. **Distance From a Point To a Plane**
The distance from the point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$

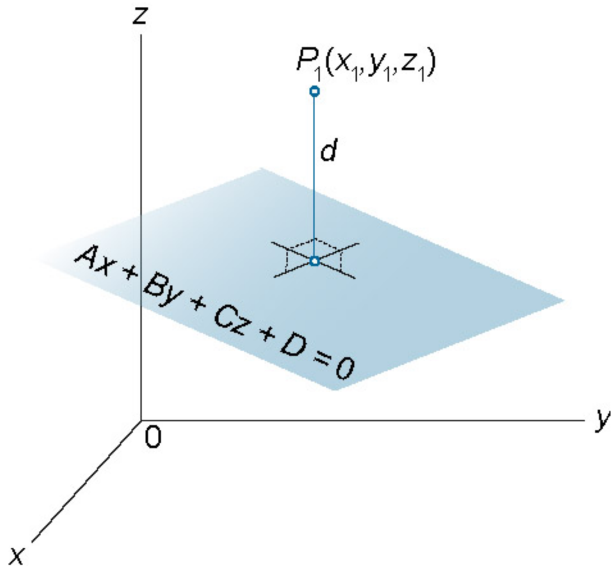

Figure 135.

689. Intersection of Two Planes

If two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ intersect, the intersection straight line is given by

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt, \\ z = z_1 + ct \end{cases}$$

or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where

$$a = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\ b = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},\ c = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

$$x_1 = \frac{b\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$y_1 = \frac{c\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - a\begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$z_1 = \frac{a\begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b\begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2 + b^2 + c^2}.$$