COORDINATE SYSTEM [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. The points $(-a, -b)$, $(0, 0)$, (a, b) and (a^2, ab) are

a. collinear

b. vertices of a parallelogram

c. vertices of a rectangle

d. none of these **(IIT-JEE 1979)**

2. The point (4, 1) undergoes the following three transformations successively.

i. Reflection about the line $y = x$.

ii. Translation through a distance 2 units along the positive direction of the x-axis.

iii. Rotation through an angle $\pi/4$ about the origin in the counterclockwise direction.

Then the final position of the point is given by the oordinates

a. $(1/\sqrt{2}, 7/\sqrt{2})$ **b.** $(-\sqrt{2}, 7\sqrt{2})$

c. $(-1/\sqrt{2}, 7/\sqrt{2})$ **d.** $(\sqrt{2}, 7\sqrt{2})$

(IIT-JEE 1980)

3. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in GP with the same common ratio, then the points (x_1, y_1), (x_2, y_2), and (x_3, y_3)

a. lie on a straight line

b. lie on an ellipse

c. lie on a circle

d. are the vertices of a triangle **(IIT-JEE 1999)**

4. The incenter of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$, and $(2, 0)$ is

a. $(1, \sqrt{3}/2)$ **b.** $(2/3, 1/\sqrt{3})$

c. $(2/3, \sqrt{3}/2)$ **d.** $(1, 1/\sqrt{3})$

(IIT-JEE 2000)

5. Let $0 < \alpha < \pi/2$ be a fixed angle. If $P \equiv (\cos\theta, \sin\theta)$ and $Q \equiv (\cos(\alpha - \theta), \sin(\alpha - \theta))$, then Q is obtained from P by the

a. clockwise rotation around the origin through an angle α

b. anticlockwise rotation around the origin through an angle α

c. reflection in the line through the origin with slope $\tan\alpha$

d. reflection in the line through the origin with slope $\tan(\alpha/2)$ **(IIT-JEE 2002)**

6. The number of integral points (integral point means both the coordinates should be integers) exactly in the interior of the triangle with vertices (0, 0), (0, 21), and (21, 0) is

a. 133 b. 190 c. 233 d. 105

(IIT-JEE 2003)

7. The orthocenter of the triangle with vertices (0, 0), (3, 4), and (4, 0) is

a. (3, 5/4) b. (3, 12) c. (3, 3/4) d. (3, 9)

(IIT-JEE 2003)

8. Let $O(0, 0)$, $P(3, 4)$, and $Q(6, 0)$ be the vertices of triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

a. (4/3, 3) b. (3, 2/3) c. (3, 4/3) d. (4/3, 2/3)

(IIT-JEE 2007)

9. Consider three points $P \equiv (-\sin(\beta - \alpha), -\cos\beta)$, $Q \equiv (\cos(\beta - \alpha), \sin\beta)$, and $R \equiv (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \pi/4$. Then

a. P lies on the line segment RQ

b. Q lies on the line segment PR

c. R lies on the line segment QP

d. P, Q, R are non-collinear **(IIT-JEE 2008)**

Multiple Correct Answers Type

1. If the vertices P, Q, and R of a triangle PQR are rational points, which of the following points of triangle PQR is (are) always rational points(s)? (A rational point is a point whose coordinates are rational numbers.)

a. Centroid b. Incenter

c. Circumcenter d. Orthocenter

(IIT-JEE 1982)

2. The points (0, 8/3), (1, 3), and (82, 30) are the vertices of

a. an obtuse-angled triangle

b. an acute-angled triangle

c. a right-angled triangle

d. none of these **(IIT-JEE 1986)**

3. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$, and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then

a. $a = 2, b = 4$ b. $a = 3, b = 4$

c. $a = 2, b = 3$ d. $a = 1$ or $b = -1$

(IIT-JEE 1998)

Subjective Type

1. The area of a triangle is 5. Two of its vertices are $A(2, 1)$ and $B(3, -2)$. The third vertex C is on $y = x + 3$. Find C. **(IIT-JEE 1978)**

2. A straight line segment of length l moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line segment in the ratio 1 : 2. **(IIT-JEE 1978)**

3. Two vertices of a triangle are (5, –1) and (–2, 3). If the orthocenter of the triangle is the origin, find the coordinates of the third point. **(IIT-JEE 1979)**

4. The vertices of a triangle are $(at_1t_2, a(t_1 + t_2))$, $(at_2t_3, a(t_2 + t_3))$, and $(at_3t_1, a(t_3 + t_1))$. Find the orthocenter of the triangle. **(IIT-JEE 1983)**

5. The coordinates of A, B, C are (6, 3), (–3, 5), (4, –2), respectively, and P is any point (x, y). Show that the ratio of the area of ΔPBC to that of ΔABC is $|x + y - 2|/7$. **(IIT-JEE 1983)**

6. A line cuts the x-axis at $A(7, 0)$ and the y-axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the x-axis at P and the y-axis at Q. If AQ and BP intersect at R, then find the locus of R. **(IIT-JEE 1990)**

7. For points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ of the coordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O = (0, 0)$ and $A = (3, 2)$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram. **(IIT-JEE 2000)**

Answer Key

JEE Advanced

Single Correct Answer Type

1. a. 2. c. 3. a. 4. d. 5. d.
6. b. 7. c. 8. c. 9. d.

Multiple Correct Answers Type

1. a., c., d. 2. d. 3. c.

Subjective Type

1. (7/2, 13/2) or (–3/2, 3/2)
3. (–4, –7)
4. $(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$
6. $x^2 + y^2 - 7x + 5y = 0$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. a. We have points $A(-a, -b)$, $B(0, 0)$, $C(a, b)$ and $D(a^2, ab)$

Slope of AB, $m_1 = \frac{b}{a}$

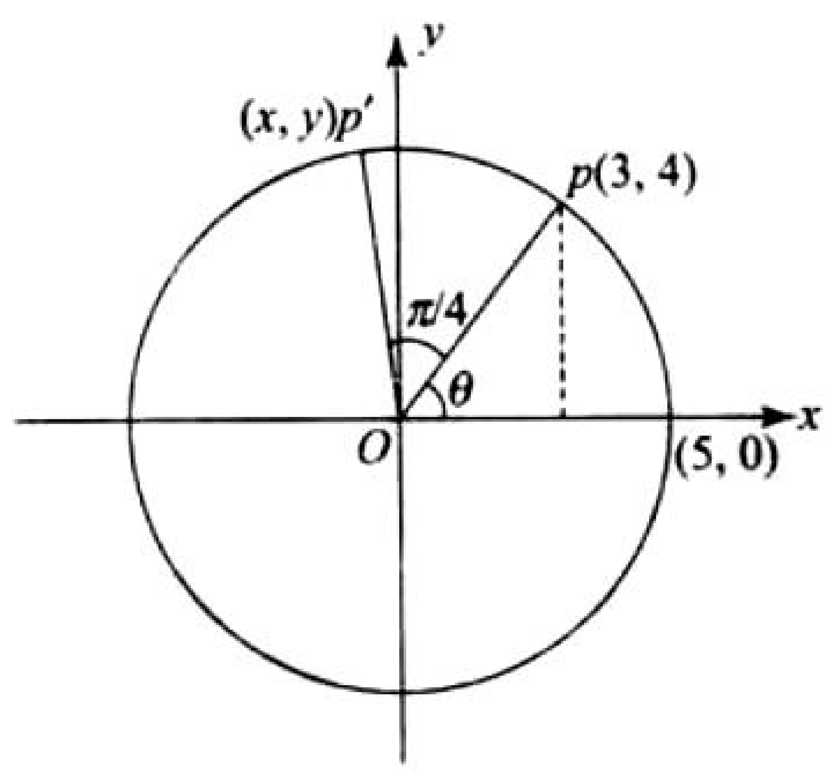
Slope of BC, $m_2 = \frac{b}{a}$

Slope of CD, $m_3 = \frac{ab-b}{a^2-a} = \frac{b(a-1)}{a(a-1)} = \frac{b}{a}$

Slope of AD, $m_3 = \frac{b-(-b)}{a-(-a)} = \frac{b}{a}$

Thus points A, B, C, D are collinear.

2. c. Reflection about the line $y = x$ changes the point (4, 1) to (1, 4). On the translation of (1, 4) through a distance of 2 units along the positive direction of the x-axis, the point becomes (1 + 2, 4), i.e., (3, 4).



On rotation about the origin through an angle $\pi/4$, point P takes the position P' such that $OP = OP'$. Also, $OP = 5 = OP'$ and $\cos\theta = 3/5$, $\sin\theta = 4/5$. Now,

$$x = OP'\cos\left(\frac{\pi}{4}+\theta\right)$$
$$= 5\left(\cos\frac{\pi}{4}\cos\theta - \sin\theta\right)$$
$$= 5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right)$$
$$= -\frac{1}{\sqrt{2}}$$
$$y = OP'\sin\left(\frac{\pi}{4}+\theta\right)$$
$$= 5\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)$$
$$= 5\,\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}} = \frac{7}{\sqrt{2}}$$
$$\therefore P' \equiv \left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

3. a. Let $x_2 = x_1 r$, $x_3 = x_1 r^2$ and so is $y_2 = y_1 r$, $y_3 = y_1 r^2$, where r is the common ratio of G.P. Therefore,

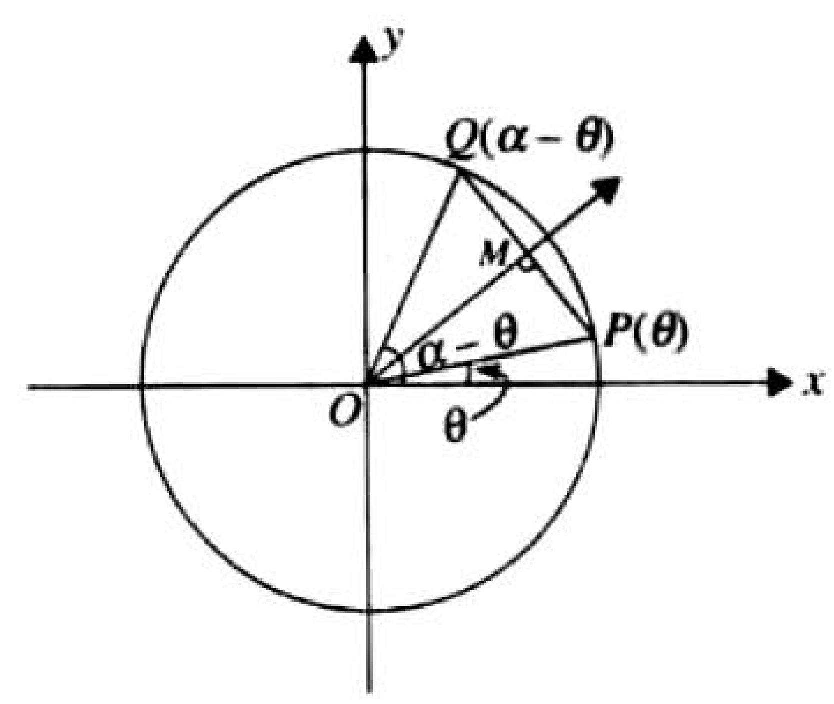
$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} x_1 & y_1 & 1 \\ rx_2 & ry_1 & 1 \\ r^2x_3 & r^2y_1 & 1 \end{vmatrix}$$
$$= r \times r^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0$$

Hence, the points are collinear.

4. d. Vertices of Δ are $A(1, \sqrt{3})$, $B(0, 0)$ and $C(2, 0)$. Here, $AB = BC = CA = 2$. So, it is an equilateral triangle and the incenter coincides with the centroid. Therefore, the centroid is

$$\left(\frac{0+1+2}{3}, \frac{0+0+\sqrt{3}}{3}\right) \equiv \left(1, \frac{1}{\sqrt{3}}\right)$$

5. d. Clearly points $P(\cos\theta, \sin\theta)$ and $Q(\cos(\alpha-\theta), \sin(\alpha-\theta))$ lie on circle of unit radius.



In the fig., $\angle POX = \theta$

$\angle QOX = \alpha - \theta$

$\therefore \quad \angle QOP = \alpha - 2\theta$

Now ΔQOP is isosceles.

$\therefore$ Altitude or angle bisector OM is perpendicular bisector of PQ.

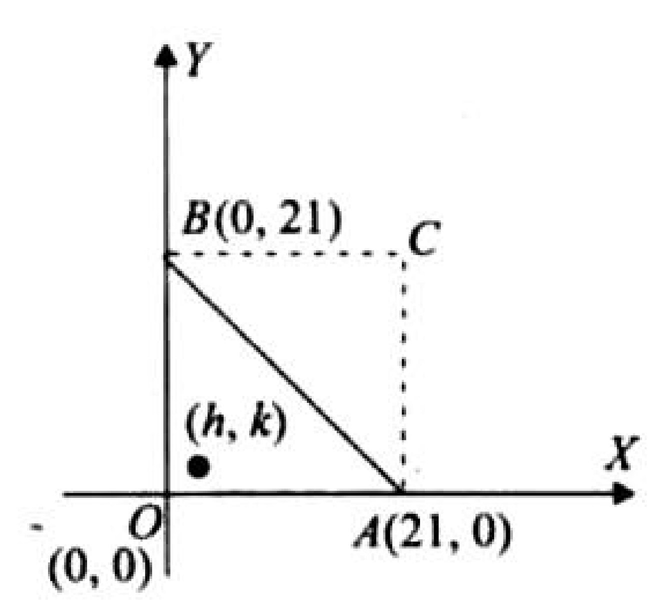
$$\therefore \quad \angle MOP = \frac{\alpha-2\theta}{2} = \frac{\alpha}{2} - \theta$$

$$\therefore \quad \angle MOX = \theta + \left(\frac{\alpha}{2} - \theta\right) = \frac{\alpha}{2}$$

Thus point P is reflection of point Q in line OM having slope $\tan\frac{\alpha}{2}$.

6. b. The total number of points within the square $OABC$ is $20 \times 20 = 400$.

The total number of points on line AB, $x + y = 21$ is 20, the points being (1, 20), (2, 19), ..., (20, 1). Therefore, the total number of points within ΔOBC and ΔABC is $400 - 20 = 380$. By symmetry, the number of points within ΔOAB is $380°/2 = 190°$.

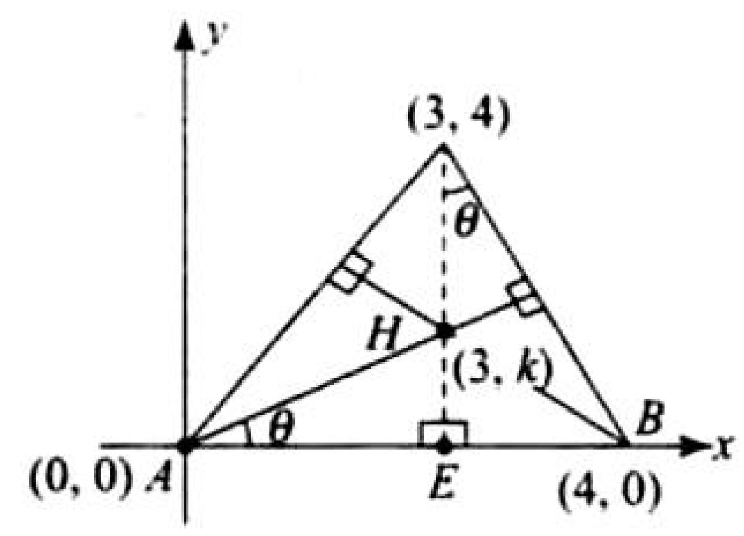


7. c. In ΔAEH, $\tan\theta = \frac{k}{3}$ (i)

In ΔCEB $\tan\theta = \frac{1}{4}$ (ii)

By (i) and (ii), $\frac{k}{3} = \frac{1}{4}$

$\therefore k = \frac{3}{4}$



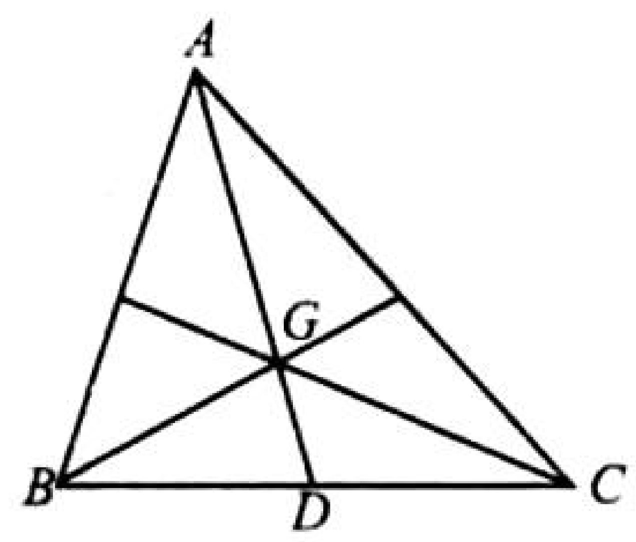
$\therefore$ Orthocenter $\equiv \left(3, \frac{3}{4}\right)$

8. c. Centroid of triangle divides triangle into three equal area triangles

Proof:

Let G be centroid of triangle ABC.

Let AG meets opposite side BC in point D.



$\therefore$ Area of triangle ABD = Area of triangle ACD

(as $BD = CD$ and length of altitude from A on BC is same for both triangles)

For the same reason,

area of triangle GBD = area of triangle GCD

$\therefore$ Area of triangle ABD – Area of triangle GBD = Area of triangle ACD – area of triangle GCD

$\therefore$ Area of triangle AGB = Area of triangle AGC

Similarly we can prove that, area of triangle AGC = Area of triangle BGC

Thus area of triangle AGB = Area of triangle AGC = Area of triangle BGC

Hence, centroid of triangle divides triangle into three equal area triangles.

So, in given question R is centroid of triangle OPQ, which is given by

$$R\left(\frac{3+6+0}{3}, \frac{4+0+0}{3}\right) \text{ or } R\left(3, \frac{4}{3}\right)$$

9. d. $P \equiv (-\sin(\beta-\alpha), -\cos\beta) \equiv (x_1, y_1)$

$Q \equiv (\cos(\beta-\alpha), \sin\beta) \equiv (x_2, y_2)$

$R \equiv (\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$.

or $R \equiv (x_2\cos\theta + x_1\sin\theta, y_2\cos\theta + y_1\sin\theta)$

If $T \equiv \left(\frac{x_2\cos\theta + x_1\sin\theta}{\cos\theta+\sin\theta}, \frac{y_2\cos\theta + y_1\sin\theta}{\cos\theta+\sin\theta}\right)$, for some '$\theta$'

then P, Q, and T are collinear.

$\therefore P, Q, R$ are collinear if $\cos\theta + \sin\theta = 1$, which is not possible as $0 < \theta < \pi/4$.

Hence, P, Q, R are non-collinear.

Multiple Correct Answers Type

1. a., c., d. If $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_2, y_3)$, where x_1, y_1, etc., are rational numbers, then Σx_1, Σy_1 are also rational.

So, the coordinates of the centroid $(\Sigma x_1/3, \Sigma y_1/3)$ will be rational.

As $AB = c = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$, c may or may not be rational and it may be an irrational number of the from $\sqrt{p}$. Hence, the coordinates of the incenter $(\Sigma ax_1/\Sigma a, \Sigma ay_1/\Sigma a)$ may or may not be rational. If $P(\alpha, \beta)$ is the circumcenter, then using $AP = BP = CP$, we get two equations in α and β with rational coefficients, solving which we get the rational values of α and β.

If $P(\alpha, \beta)$ is the orthocenter, then using [(slope of AP) $\times$ (slope of BC) = –1] and [(slope of BP) $\times$ (slope of AC) = –1], we get two equations in α and β with rational coefficients, solving which we get the rational values of α and β.

2. d. Let $A \equiv (0, 8/3)$, $B \equiv (1, 3)$, and $C \equiv (82, 30)$.

Now, the slope of line AB is $(3 - 8/3)/(1 - 0) = 1/3$. The slope of line BC is $(30 - 3)/(82 - 1) = 27/81 = 1/3$. Therefore, $AB \parallel BC$ and B is the common point. Hence, A, B, and C are collinear.

3. c. $PQRS$ will represent a parallelogram if and only if the midpoint of PR is the same as that of the midpoint of QS. Hence,

$$\frac{1+5}{2} = \frac{4+a}{2} \text{ and } \frac{2+7}{2} = \frac{6+b}{2}$$

or $a = 2$ and $b = 3$.

Subjective Type

1. As C lies on the line $y = x + 3$, let the coordinates of C be $(\lambda, \lambda + 3)$. Also, $A \equiv (2, 1)$, $B \equiv (3, -2)$. Then the area of ΔABC is given by

$$\frac{1}{2}\left\|\begin{matrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ \lambda & \lambda+3 & 1 \end{matrix}\right\| = 5$$

or $|4\lambda - 4| = 10$

i.e., $4\lambda - 4 = 10$ or $4\lambda - 4 = -10$

i.e., $4\lambda = 14$ or $4\lambda = -6$

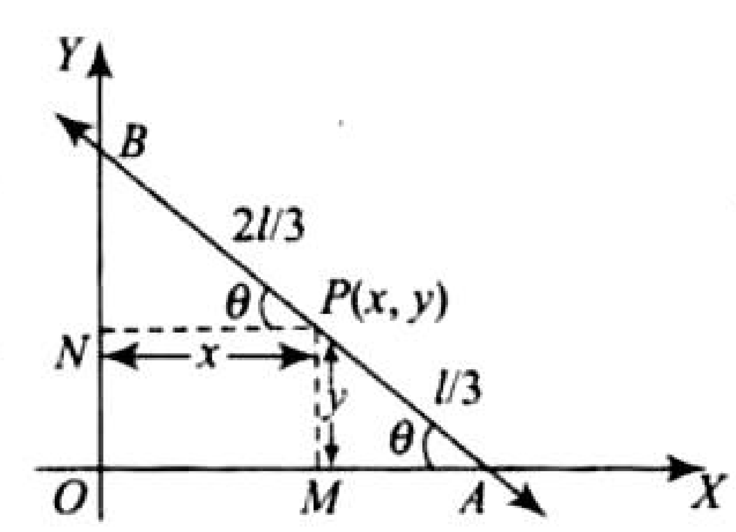
i.e., $\lambda = \frac{7}{2}$ or $\lambda = -\frac{3}{2}$

Hence, the coordinates of C are $(7/2, 13/2)$ or $(-3/2, 3/2)$.

2. Let $P(x, y)$ divide line segment AB in the ratio 1 : 2, so that $AP = l/3$ and $BP = 2l/3$ where $AB = l$

Then $PN = x$ and $PM = y$

Let $\angle PAM = \theta = \angle BPN$



In, ΔPMA, $\sin\theta = \frac{y}{l/3} = \frac{3y}{l}$

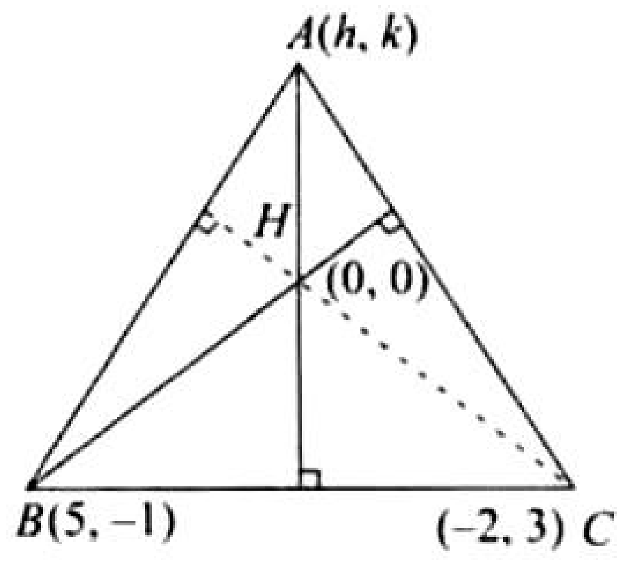
In ΔPNB, $\cos\theta = \frac{x}{2l/3} = \frac{3x}{2l}$

Now, $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \frac{9y^2}{l^2} + \frac{9x^2}{4l^2} = 1$$

$\Rightarrow$ $9x^2 + 36y^2 = 4l^2$ which is the required locus.

3. We have



$AH \perp BC$

$\therefore m_{AH} \times m_{BC} = -1$

or $\frac{k}{h} \times \frac{3+1}{-2-5} = -1$

or $4k - 7h = 0$ (i)

Also, $BH \perp AC$

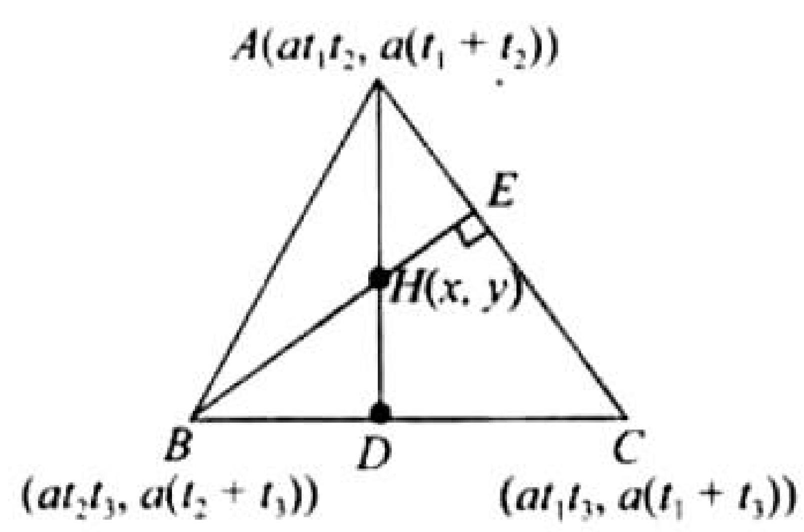
$\therefore \frac{-1}{5} \times \frac{3-k}{-2-h} = -1$

or $3 - k = -10 - 5h$

or $5h - k + 13 = 0$ (ii)

Solving (i) and (ii), we get $h = -4$, $k = -7$. Hence, the third vertex is $(-4, -7)$.

4.



We know that orthocenter of Δ is point of concurrency of altitudes.

Let orthocenter be $H(x, y)$.

Now, slope of $BC = \frac{a(t_1+t_3) - a(t_2+t_3)}{at_1t_3 - at_2t_3}$

$= \frac{a(t_1 + t_3 - t_2 - t_3)}{at_3(t_1 - t_2)} = \frac{1}{t_3}$

$\therefore$ Slope of $AD = -t_3$

Also, slope of AD = Slope of $AH = \frac{y - a(t_1+t_2)}{x - at_1t_2} = -t_3$

$\therefore \quad y - a(t_1 + t_2) = -t_3(x - at_1t_2)$

or $\quad xt_3 + y = at_1t_2t_3 + a(t_1 + t_2)$ (1)

Similarly, by symmetry from $AC \perp BE$, we get

$xt_1 + y = at_1t_2t_3 + a(t_2 + t_3)$ (2)

Solving (1) and (2) we get

$x = -a, y = a(t_1 + t_2 + t_3) + at_1t_2t_3$

$\therefore$ Orthocenter is $H(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$

5. Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 6 & 3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$

$= \frac{1}{2} \times [6(7) + 3(5) + 4(-2)] = \frac{49}{2}$

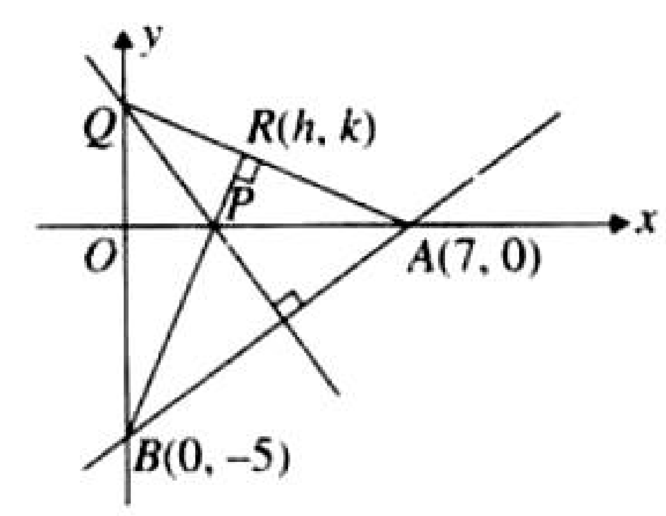
Area of $\Delta PBC = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$

$= \frac{1}{2}(7x + 7y - 14) = \frac{7}{2}|x + y - 2|$

Hence, the ratio of the area of triangles is given by

$\frac{\text{Ar}(\Delta PBC)}{\text{Ar}(\Delta ABC)} = \frac{(7/2)|x+y-2|}{49/2} = \left|\frac{x+y-2}{7}\right|$

6. From the figure in ΔABQ,



$AP \perp BQ, PQ \perp AB$

Then, we have $BP \perp AQ$ (as the altitudes of triangle are concurrent). Hence,

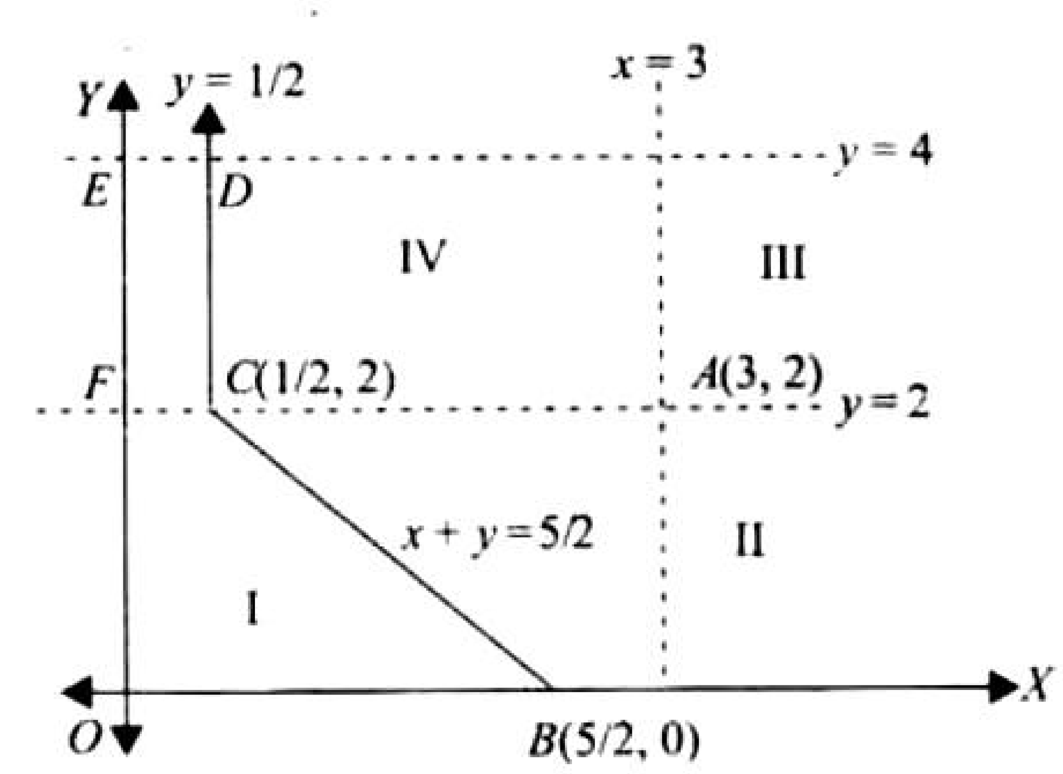
$BR \perp AR$

or $\frac{k-0}{h-7} \times \frac{k+5}{h-0} = -1$

Therefore, the locus of R is

$x^2 + y^2 - 7x + 5y = 0$

7.



Let $P = (h, k)$ be a general point in the first quadrant such that $d(P, A) = d(P, O)$.

$\Rightarrow \quad |h - 3| + |k - 2| = |h| + |k| = h + k$ (1)

[h and k are +ve, point $P(h, k)$ being in I quadrant]

If $h < 3$; $k < 2$ then (h, k) lies in region I.

$\Rightarrow \quad 3 - h + 2 - k = h + k \Rightarrow h + k = 5/2$

If $h > 3$, $k < 2$, (h, k) lies in region II.

$\Rightarrow \quad h - 3 + 2 - k = h + k \Rightarrow k = -1/2$, which is not possible

If $h > 3$, $k > 2$ (h, k) lies in region III.

$\Rightarrow \quad h - 3 + k - 2 = h + k \Rightarrow -5 = 0$, which is not possible

If $h < 3$, $k > 2$ (h, k) lies in region IV.

$\Rightarrow \quad 3 - h + k - 2 = h + k \Rightarrow h = 1/2$

Hence required set consists of line segment $x + y = 5/2$ of finite length as shown in the first region and the ray $x = 1/2$ in the fourth region.