08

Application of Derivatives

8.01 Introduction

In previous chapter we have studied derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithemic functions. In this chapter we will study applications of the derivative in various disciplines, e.g. in engineering, science, social science and many other fields. For instance we will learn how the derivative can be used to determine rate of change of quantities or to find the equations of tangent and normal to a curve at a point.

8.02 Rate of change of quantities

Let P be a variable quantity, that changes with respect to time. Let small change in time t is δt, then corresponding change in P is δP. Then $\frac{\delta P}{\delta t}$, is average rate of change in P, and the instantaneous rate of change in P is $\frac{dP}{dt}$ where $\frac{dP}{dt} = \lim_{\delta t \to 0} \frac{\delta P}{\delta t}$.

Where, $\frac{dP}{dt}$, rate of change in P with respect to time t. Further, if two variable v and r are functions of another variable t, then

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$$

Thus, the rate of change of any one of v and r can be calculate using the rate of change in other qantity with respect to time t.

Illustrative Examples

Example 1: Find the rate of change of volume of a sphere with respect to its surface area when radius of sphere is 2 cm.

Solution : $\because$ Volume of sphere $= V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

Surface area of sphere $s = 4\pi r^2 \Rightarrow \frac{ds}{dr} = 8\pi r$

$$\frac{dV}{ds} = \frac{dV/dr}{ds/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$$

$$\left(\frac{dV}{ds}\right)_{r=2} = \frac{2}{2} = 1 \text{ cm.}$$

Example 2. A ladder 10 m, long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 1.2 m / s. How fast is its height on the wall decreasing when the foot of the ladder is 6 m. away from the wall.

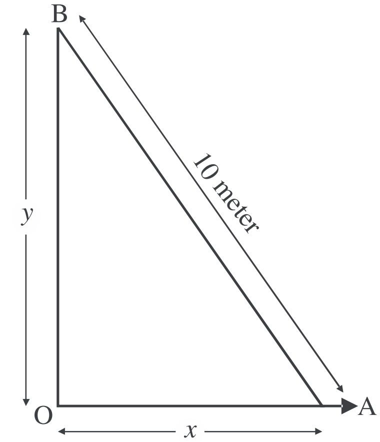

Fig. 8.01

Solution : Let AB be position of ladder at time t

Let $\quad OA = x,\ OB = y$ then $x^2 + y^2 = 10^2 \quad (1)$

It is given $\quad \dfrac{dx}{dt} = 1.2$ m/s

Diferentiating (1) with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad (2)$$

For $x = 6$, from (1) $6^2 + y^2 = 10^2 \Rightarrow y = 8$ m.

From (2) $\quad 2 \times 6 \times 1.2 + 2 \times 8\dfrac{dy}{dt} = 0$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{14.4}{16} = -0.9 \text{ m/s. (towards ground)}$$

Example 3. The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres?

Solution : Let x be the length of a side, V be the volume and s be the surface area of the cube. Then, $V = x^3,\ S = 6x^2$, where x is a function of time t.

Now, $\quad \dfrac{dV}{dt} = 9$ cm³ / s.

$$\Rightarrow \quad 9 = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\frac{dx}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3}{x^2} \qquad (1)$$

and $\quad \dfrac{dS}{dt} = \dfrac{d}{dt}(6x^2) = \dfrac{d}{dx}(6x^2)\dfrac{dx}{dt} = 12x\left(\dfrac{3}{x^2}\right) = \dfrac{36}{x} \qquad$ [From (1) से]

$\because \quad x = 10$ cm.

$$\Rightarrow \quad \frac{dS}{dt} = \frac{36}{10} = 3.6 \text{ cm}^2 / \text{s}.$$

Example 4. The surface area of a bubble is increasing at the rate of 2 cm² / s. At what rate is the volume of the bubble increasing when the radius is 6 cm.

Solution : Let surface area and volume of a bubble of radius r be S and V respectively.

Then $\quad S = 4\pi r^2 \Rightarrow \dfrac{dS}{dr} = 8\pi r$

and $$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

Given that $$\frac{dS}{dt} = 2 \text{ cm}^2 / \text{s}$$

$\because$ $$\frac{dS}{dt} = \frac{dS}{dr}.\frac{dr}{dt} \Rightarrow 2 = 8\pi r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r}$$

$\because$ $$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} = 4\pi r^2.\frac{1}{4\pi r} = r$$

Hence $$\left(\frac{dV}{dt}\right)_{r=6} = 6 \text{ cm}^3 / \text{s}$$

Example 5. The length x of a rectangle is decreasing at the rate of 3 cm / minute and the width y is increasing at the rate of 2 cm / minute. When $x = 12$ and $y = 6$. Find the rate of change of the perimeter and the area of the rectange.

Solution : Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt} = -3 \text{ cm / minute}, \quad \frac{dy}{dt} = 2 \text{ cm / minute}$$

$\because$ Perimeter of rectnage $$p = 2(x+y)$$

$\Rightarrow$ $$\frac{dp}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3+2) = -2 \text{ cm / minute}$$

and area of rectange $$A = x.y$$

$\Rightarrow$ $$\frac{dA}{dt} = x\frac{dy}{dt} + \frac{dx}{dt}.y$$

$$= (12)(2) + (-3).6$$

$$= 24 - 18$$

$$= 6 \text{ cm}^2 / \text{ minute}$$

Example 6. Water is dripping out from a conical funnel at uniform rate 4 cm^3 / s through a tiny hole at the vertex in the bottom. When the slant height of the water is 4 cm. Find the rate of the decrease of the slant height of the water, given that the semi vertical angle of the funnel is 60º.

Solution : Let volume of water at time t is V.

$\therefore$ The volume of cone of water PEF is V and slant height $PE = \ell$

$\therefore$ $$O'E = \ell \sin 60^\circ = \ell.\frac{\sqrt{3}}{2}$$

and $$O'P = \ell \cos 60^\circ = \ell.\frac{1}{2}$$

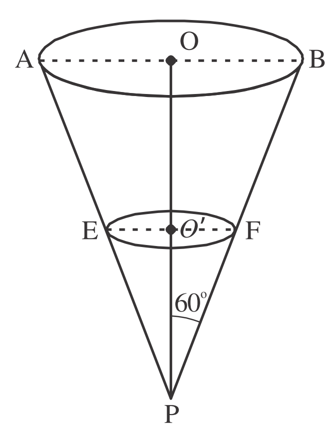

Fig. 8.02

$\because \quad V = \frac{1}{3}\pi (O'E)^2 . O'P$

$$= \frac{1}{3}\pi \left(\frac{\ell\sqrt{3}}{2}\right)^2 . \left(\frac{\ell}{2}\right)$$

$\Rightarrow \quad V = \frac{\pi \ell^3}{8}$

$\Rightarrow \quad \frac{dV}{dt} = \frac{3\pi \ell^2}{8}\frac{d\ell}{dt}$

It is given that $\quad \frac{dV}{dt} = -4$

So $\quad -4 = \frac{3\pi\ell^2}{8}\frac{d\ell}{dt}$

$\Rightarrow \quad \frac{d\ell}{dt} = -\frac{32}{3\pi\ell^2}$

So, at $\ell = 4 \quad \frac{d\ell}{dt} = \frac{-32}{3\pi(4)^2} = -\frac{2}{3\pi}$ cm / s.

Exercise 8.1

1. Find the rate of change of the area of a circle with respect to radius *r,* when $r = 3$ cm and $r = 4$ cm.
2. A particle is moving along the curve $y = \frac{2}{3}x^3 + 1$. Find the points on the curve at which the *y*-coordinate is changing twice as fast as the *x* coordinate.
3. A ladder 13 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, from the wall, at the rate of 1.5 m / s. How fast is its height on the wall decreasing when the foot of the ladder is 12 m away from the wall?
4. An edge of a variable cube is increasing at the rate of 3 cm / s. Fnd the rate at which the volume of the cube increasing when the edge is 10 cm long?
5. A ballon which always remains spherical on inflation, is being inflated by pumping at the rate of 900 cm^3 /s. of gas. Find the rate at whcih the radius of ballon increases when the radius is 15 cm.
6. A ballon, which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate at which its volume is increasing with respect to *x*.
7. The total cost *C*(*x*) rupees, associated with the production of *x* units of an item is given by

$$C(x) = 0.005\, x^3 - 0.02\, x^2 + 30\, x + 5000$$

   Find the marginal cost when 3 units are produced, here by marginal cost we mean the instantaneous rate of change of total cost at any level of output.
8. The radius of a soap bubble is increasing at the rate of 0.2 cm / s. Find the rate of increase in surface area when the radius is 7 cm. Also find the rate of change in volume when the radius is 5 cm.

9. Sand is pouring from a pipe at the rate of 12 cm^3 / s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of base. How fast is the height of the sand cone increasing when the height is 4 cm?

10. The total revenue in rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26\,x + 15$$

Find the marginal revenue when $x = 15$.

8.03 Increasing and Decreasing Functions

In this section, we will use differentiation to find out wheather a function is increasing or decreasing.

Increasing Function : A function $f(x)$ is called an increasing function in open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

Strictly Increasing Function : A function $f(x)$ is called a strictly increasing function is open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

i.e. if x increases in open interval (a, b) then $f(x)$ will also increase.

Decreasing Function : A function $f(x)$ is called a decreasing function in open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

Strictly Decreasing Function : A function $f(x)$ is called a strictly decreasing function in open interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \quad \forall \quad x_1, x_2 \in (a, b)$$

i.e. in open interval (a, b) when x increases $f(x)$ decreases.

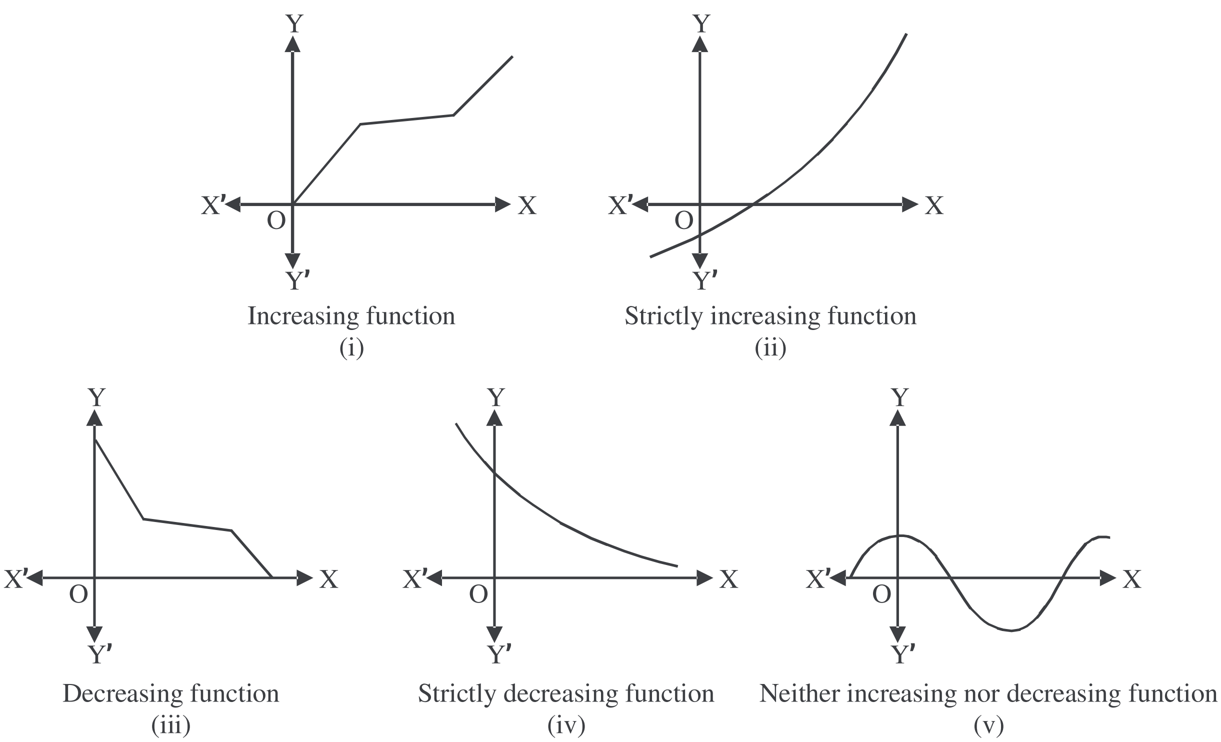

Fig. 8.03

8.04 Theorem

Let f be continuous on $[a, b]$ and differentiable in the open interval (a, b). Then

(i) f is increasing in [a, b] if $f'(x) > 0$ for each $x \in [a,b]$

(ii) f is decreasing in [a, b] if $f'(x) < 0$ for each $x \in [a,b]$

(iii) f is constant function in [a, b] if $f'(x) = 0$ for each $x \in [a,b]$

Proof : (i) Let $x_1, x_2 \in [a,b]$ be such that $x_1 < x_2$

Then by Lagrange's mean value theorem there exist a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$\Rightarrow$ $f(x_2) - f(x_1) > 0$ $\quad (\because f'(c) > 0)$

$\Rightarrow$ $f(x_2) > f(x_1)$

So, $\forall \quad x_1, x_2 \in [a,b]$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

So, $f(x)$ is increasing function in [a, b]

Similary parts (ii) and (iii) can be proved.

Illustrative Examples

Example 7. Find the intervals in whcih the function $f(x) = 2x^3 - 9x^2 + 12x + 3$,

(a) increasing (b) Decreasing

Solution : $f(x) = 2x^3 - 9x^2 + 12x + 3$

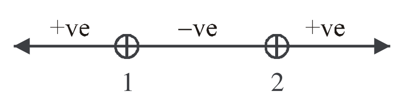

Fig. 8.04

$\Rightarrow$ $f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

Now $f'(x) = 0 \Rightarrow 6(x^2 - 3x + 2) = 0$

$\Rightarrow$ $(x-2)(x-1) = 0$

$\Rightarrow$ $x = 1, 2$ are critical points.

(a) $f(x)$ is increasing and $f'(x) > 0$

$\Rightarrow$ $6(x^2 - 3x + 2) > 0$

$\Rightarrow$ $(x-1)(x-2) > 0$

$\Rightarrow$ $x < 1$ or $x > 2$

$\Rightarrow$ $x \in (-\infty, 1) \cup (2, \infty)$

Hence, $f(x)$ is increasing in $(-\infty, 1) \cup (2, \infty)$

(b) $f(x)$ is decreasing then $f'(x) < 0$

$\Rightarrow$ $6(x^2 - 3x + 2) < 0$

$\Rightarrow$ $(x-1)(x-2) < 0$

$\Rightarrow \quad x > 1$ or $x < 2$

$\Rightarrow \quad x \in (1, 2)$

Hence, $f(x)$ is decreasing in interval (1, 2)

Example 8. Show that the function *f* given by $f(x) = x^3 - 3x^2 + 4x,$ is strictly increasing on R.

Solution : $\because \quad f(x) = 3x^3 - 3x^2 + 4x$

$\Rightarrow \quad f'(x) = 3x^2 - 6x - 4$

$= 3(x^2 - 2x + 1) + 1$

$= 3(x-1)^2 + 1 > 0, \quad \forall \quad x \in R$

Therefore, the function *f* is strictly increasing on R

Example 9. Find the intervals is which the function $f(x) = -2x^3 + 3x^2 + 12x + 25$

(a) Increasing (b) decreasing

Solution : $\because \quad f(x) = -2x^3 + 3x^2 + 12x + 25$

$\Rightarrow \quad f'(x) = -6x^2 + 6x + 12$

$= -6(x^2 - x - 2)$

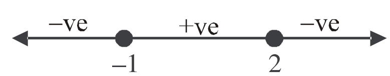

Fig. 8.05

So, $f'(x) = 0 \Rightarrow -6(x^2 - x - 2) = 0$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x+1)(x-2) = 0$

$\Rightarrow \quad x = -1, 2$ are critical points.

(a) If $f(x)$ is increasing then $f'(x) > 0$

$\Rightarrow \quad -6(x^2 - x - 2) > 0$

$\Rightarrow \quad x^2 - x - 2 < 0$

$\Rightarrow \quad (x+1)(x-2) < 0$

$\Rightarrow \quad x > -1$ or $x < 2$

$\Rightarrow \quad x \in (-1, 2)$

Hence $f(x)$, is increasing in $(-1, 2)$

(b) If $f(x)$ is decreasing $f'(x) < 0$

$\Rightarrow \quad -6(x^2 - x - 2) < 0$

$\Rightarrow \quad x^2 - x - 2 > 0$

$\Rightarrow \quad (x+1)(x-2) > 0$

$\Rightarrow \quad x < -1$ or $x > 2$

$\Rightarrow \quad x \in (-\infty, -1) \cup (2, \infty)$

Hence, $f(x)$ is decreasing in $(-\infty, -1) \cup (2, \infty)$

Example 10. Find the interval in which function $f(x) = \sin x - \cos x$ is increasing or decreasing.

Solution : $\because$ $f(x) = \sin x - \cos x$

$\Rightarrow$ $f'(x) = \cos x + \sin x$

$\Rightarrow$ $f'(x) = 0$

$\Rightarrow$ $\cos x + \sin x = 0$

$\Rightarrow$ $\sin(\pi/2 + x) + \sin x = 0$

$\Rightarrow$ $2\sin(\pi/4 + x)\cdot\cos\pi/4 = 0$

$\Rightarrow$ $\sin(\pi/4 + x) = 0 = \sin\pi$

$\Rightarrow$ $\pi/4 + x = \pi$

$\Rightarrow$ $x = 3\pi/4$, which is a critical point.

when $f(x)$ is increasing then $f'(x) > 0$

$\Rightarrow$ $\cos x + \sin x > 0$

$\Rightarrow$ $2\sin(\pi/4 + x)\cos\pi/4 > 0$

$\Rightarrow$ $\sin(\pi/4 + x) > 0$

$\Rightarrow$ $\sin\{\pi - (\pi/4 + x)\} > 0$

$\Rightarrow$ $\sin(3\pi/4 - x) > 0$

$\Rightarrow$ $3\pi/4 - x > 0$

$\Rightarrow$ $x < 3\pi/4$

$\Rightarrow$ $x \in (0,\ 3\pi/4)$

Hence $f(x)$ is increasing if $x \in (0,\ 3\pi/4)$

If $f(x)$ is decreasing then $f'(x) < 0$

$\Rightarrow$ $\cos x + \sin x < 0$

$\Rightarrow$ $\sin(\pi/2 + x) + \sin x < 0$

$\Rightarrow$ $2\sin(\pi/4 + x)\cos\pi/4 < 0$

$\Rightarrow$ $\sin(\pi/4 + x) < 0$

$\Rightarrow$ $\sin\{\pi - (\pi/4 + x)\} < 0$

$\Rightarrow$ $\sin(3\pi/4 - x) < 0$

$\Rightarrow$ $3\pi/4 - x < 0$

$\Rightarrow$ $x > 3\pi/4 \Rightarrow x \in (3\pi/4, \pi)$

Hence $f(x)$ is decreasing if $x \in (3\pi/4, \pi)$

Example 11. Find the values of x for which $f(x)=\dfrac{x}{1+x^2}$ is increasing or decreasing?

Solution : Given $\quad f(x)=\dfrac{x}{1+x^2} \Rightarrow f'(x)=\dfrac{1-x^2}{(1+x^2)^2}$

$\therefore \quad f'(x)=0 \Rightarrow \dfrac{1-x^2}{(1+x^2)^2}=0$

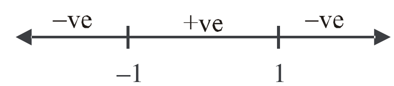

Fig. 8.06

$\Rightarrow \quad x^2-1=0$

$\Rightarrow \quad (x-1)(x+1)=0$

$\Rightarrow \quad x=-1, 1$ are critical points.

If $f(x)$ is increasing then $f'(x)>0$

$\Rightarrow \quad \dfrac{1-x^2}{(1+x^2)^2}>0$

$\Rightarrow \quad 1-x^2>0$

$\Rightarrow \quad -(x^2-1)>0$

$\Rightarrow \quad x^2-1<0$

$\Rightarrow \quad (x-1)(x+1)<0$

$\Rightarrow \quad x\in(-1, 1)$

Hence $f(x)$ is increasign for $x\in(-1, 1)$

If $f(x)$ is decreasign then $f'(x)<0$

$\Rightarrow \quad \dfrac{1-x^2}{(1+x^2)^2}<0$

$\Rightarrow \quad 1-x^2<0$

$\Rightarrow \quad x^2-1>0$

$\Rightarrow \quad (x-1)(x+1)>0$

$\Rightarrow \quad x\in(-\infty, -1)\cup(1, \infty)$

Hence $f(x)$ is decreasing for $x\in(-\infty, -1)\cup(1, \infty)$

Example 12. Find the intervals in whcih the following functions are increasing or decreasing

(a) x^2+2x+5 (b) $10-6x-2x^2$ (c) $(x+1)^3(x-3)^3$

Solution : (a) Let $\quad f(x)=x^2+2x+5$

$\Rightarrow \quad f'(x)=2x+2=2(x+1)$

$\therefore \quad f'(x)=0 \Rightarrow 2(x+1)=0$

$\Rightarrow \quad x=-1$

Case-I: When $x < -1$

$\Rightarrow$ $x + 1 < 0$

$\therefore$ $f'(x) = 2(-ve) = \text{Negative} < 0$

Hence $f(x)$ is decreasing in $(-\infty, -1)$

Case-II: When $x > -1$

$\Rightarrow$ $x + 1 > 0$

$\therefore$ $f'(x) = \text{Positive} > 0$

Hence $f(x)$ is increasing in $(-1, \infty)$

(b) Let $f(x) = 10 - 6x - 2x^2$

$\Rightarrow$ $f'(x) = -6 - 4x = -2(3 + 2x)$

$\therefore$ $f'(x) = 0 \Rightarrow -2(3 + 2x) = 0$

$\Rightarrow$ $x = -3/2$

Case-I: When $x < -3/2$

$\Rightarrow$ $3 + 2x < 0$

$\Rightarrow$ $f'(x) = -2(-ve) = \text{Positive} > 0$

Hence $f(x)$ is increasing in $(-\infty, -3/2)$

Case-II: When $x > -3/2$

$\Rightarrow$ $3 + 2x > 0$

$\Rightarrow$ $f'(x) = -2(+ve) = \text{Negative} < 0$

$\Rightarrow$ Hence $f(x)$ is decreasing in $(-3/2, \infty)$

(c) Let $f(x) = (x+1)^3 (x-3)^3$

$\Rightarrow$ $f'(x) = 3(x+1)^2 (x-3)^3 + 3(x+1)^3 (x-3)^2$

$= 3(x+1)^2 (x-3)^2 \{x - 3 + x + 1\}$

$= 6(x+1)^2 (x-3)^2 (x-1)$

If $f(x)$ is increasing then $f'(x) > 0$

$\Rightarrow$ $6(x+1)^2 (x-3)^2 (x-1) > 0$

$\Rightarrow$ $x - 1 > 0$ $[\because 6(x+1)^2 (x-3)^2 > 0]$

$\Rightarrow$ $x > 1$

Hence $f(x)$ is increasing in $(1, \infty)$

$f(x)$ is decreasing function then $f'(x) < 0$

$\Rightarrow$ $6(x+1)^2 (x-3)^2 (x-1) < 0$

$\Rightarrow$ $x - 1 < 0$

$\Rightarrow$ $x < 1$

Hence $f(x)$, is decreasing in $(-\infty, 1)$

Example 13. Show that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $[0, \pi/2]$

Solution : Let $$f(\theta) = y = \frac{4\sin\theta}{2+\cos\theta} - \theta$$

$\Rightarrow$ $$f'(\theta) = \frac{(2+\cos\theta).4\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

$\therefore$ $$f'(\theta) = 0 \Rightarrow \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} = 0$$

$\Rightarrow$ $\cos\theta = 0$

$\Rightarrow$ $\theta = \pi/2$

When $0 < \theta < \pi/2$ then $f'(\theta) > 0$

Hence $y = f(\theta)$ is increasing in $(0, \pi/2)$

Example 14: Prove that the function f given by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$

Solution : Here $f(x) = x^2 - x + 1$

$\Rightarrow$ $f'(x) = 2x - 1$

$\therefore$ $f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = 1/2$

Case-I: When $-1 < x < 1/2$ then $f'(x) < 0$

Hence $f(x)$ is decreasing in $(-1, 1/2)$

Case-II: When $1/2 < x < 1$ then $f'(x) > 0$

Hence $f(x)$ is increasing in $(1/2, 1)$

Hence $f(x)$ is neither increasing nor decreasing in $(-1, 1)$

Example 15: Find the value of a for which the function $f(x) = x^2 + ax + 1$ is increasing on [1, 2].

Solution : Given $f(x) = x^2 + ax + 1$

$\Rightarrow$ $f'(x) = 2x + a$

If $f(x)$ is increasing in $[1, 2]$ then $f'(x) > 0 \quad \forall\ x \in R$

Now $f'(x) = 2x + a$

$\Rightarrow$ $f''(x) = 2 > 0, \quad \forall\ x \in R$

$\Rightarrow$ $f(x)$ is increasing at $x \in R$

$\Rightarrow$ $f'(x)$ is increasing at $[1, 2]$

$\Rightarrow$ The least value of $f'(x)$ is $f'(1)$ at $[1, 2]$

$\therefore$ $f'(x) > 0 \quad \forall \ x \in [1, 2]$

$f'(1) > 0 \Rightarrow 2 + a > 0$

$\Rightarrow$ $a > -2$

$\Rightarrow$ $a \in (-2, \infty)$

Exercise 8.2

1. Show that $f(x) = x^2$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$
2. Show that $f(x) = a^x, o < a < 1, R$ is decreasing in R

Prove that the following functions are increasing in given intervals.

3. $f(x) = \log \sin x, \quad x \in (0, \pi/2)$
4. $f(x) = x^{100} + \sin x + 1, \quad x \in (0, \pi/2)$
5. $f(x) = (x-1)e^x + 1, \quad x > 0$
6. $f(x) = x^3 - 6x^2 + 12x - 1, \quad x \in R$

Prove that the following functions are decreasing in given intervals

7. $f(x) = \tan^{-1} x - x, \quad x \in R$
8. $f(x) = \sin^4 x + \cos^4 x, \quad x \in (0, \pi/4)$
9. $f(x) = 3/x + 5, \quad x \in R, x \neq 0$
10. $f(x) = x^2 - 2x + 3, \quad x < 1$

Find the intervals in which the following functions are increasing or decreasing

11. $f(x) = 2x^3 - 3x^2 - 36x + 7$
12. $f(x) = x^4 - 2x^2$
13. $f(x) = 9x^3 - 9x^2 + 12x + 5$
14. $f(x) = -2x^3 + 3x^2 + 12x + 5$
15. Find the least value of a function $f(x) = x^3 + 9x + 5$, when $f(x)$ is increasing in the interval $[1, 2]$
16. Prove that the function $f(x) = \tan^{-1}(\sin x + \cos x)$, is increasing function in the interval $(0, \pi/4)$

8.05 Tangents and normals

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

The slope of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$. So the equation of the tangent at (x_1, y_1) to the curve $y = f(x)$ is given by

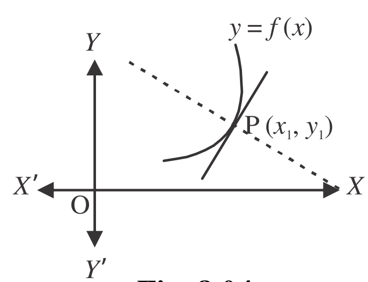

Fig. 8.04

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_1, y_1) is $-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$, if $f'(x_1) \neq 0$

Therefore, the equation of the normal to the curve $y = f(x)$ at (x_1, y_1) is given by

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}(x - x_1)$$

$$\Rightarrow \qquad (y - y_1)\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} + (x - x_1) = 0$$

Note: If a tangent line to the curve $y = f(x)$ makes an angle ψ with x-axis in the positive direction, then

$$\frac{dy}{dx} = \text{slope of the tangent} = \tan\psi$$

8.06 Particular cases

(i) If $\psi = 0$ means the tangent line is parallel to x-axis then $\frac{dy}{dx} = \tan 0 = 0$. In this case, the equation of the tangent at the point (x_1, y_1) is given by $y = y_1$

(ii) If $\psi = 90^\circ$, means the tangent line is perpendicular to the x-axis, i.e. parallel to the y-axis. In this case, the equation of the tangent at (x_1, y_1) is given by $x = x_1$

Illustrative Examples

Example 16. Find the equations of the tangent and normal to the curve $x^{2/3} + y^{2/3} = 2$ at (1, 1).

Solution : $\because$

$$x^{2/3} + y^{2/3} = 2$$

Differentiating with respect to x

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Slope of the tangent at (1, 1) is

$$\left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

So, the equation of the tangent at (1, 1) is

$$y - 1 = (-1)(x - 1)$$

$$\Rightarrow \qquad x + y - 2 = 0 \qquad (1)$$

So, the equation of the tangent at (1, 1) is

$$y - 1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}}(x - 1)$$

$$= -\frac{1}{(-1)}(x-1) = x-1 \qquad (2)$$

$$y - x = 0$$

$\Rightarrow$ (1) and (2) are required equations of tangent and normal.

Example 17. Find points on the curve $x^2 + y^2 - 2x - 3 = 0$ at whcih the tangents are

(i) Parallel to x-axis

(ii) Perpendicular to x-axis

(iii) Making equal angle with axes.

Solution : Equation of curve $x^2 + y^2 - 2x - 3 = 0 \qquad (1)$

Differentiating with respect to x

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1-x}{y}$$

(i) When tangent is parallel to x-axis, then

$$\psi = 0 \Rightarrow \frac{dy}{dx} = \tan 0 = 0$$

$$\Rightarrow \qquad \frac{1-x}{y} = 0 \Rightarrow 1 - x = 0$$

$$\Rightarrow \qquad x = 1$$

put $x = 1$ in (1)

$$y^2 - 4 = 0 \Rightarrow y = \pm 2$$

Hence required points are (1, 2) and (1, –2)

(ii) When tangent is perpendicualr to x-axis then

$$\psi = 90^\circ \Rightarrow \frac{dy}{dx} = \tan 90 = \infty$$

$$\Rightarrow \qquad \frac{1-x}{y} = \infty$$

$$\Rightarrow \qquad y = 0$$

Put $y = 0$ in (1)

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$\Rightarrow \qquad x = 3, -1$$

Hence required points are (3, 0) and (–1, 0) .

(iii) When tangents make equal angle with axes, then $\psi = \frac{\pi}{4}$

Hence slope of tangent $\frac{dy}{dx} = \tan\frac{\pi}{4} = 1$

$\Rightarrow$ $\frac{1-x}{y} = 1 \Rightarrow y = 1-x$ (2)

Put $y = 1-x$ in (1)

$$x^2 + (1-x)^2 - 2x - 3 = 0$$

$\Rightarrow$ $x^2 - 2x - 1 = 0$

$\Rightarrow$ $x = 1 \pm \sqrt{2}$

Put this value of x in (2)

$$y = \mp\sqrt{2}$$

Hence required points are $(1+\sqrt{2}, -\sqrt{2})$ and $(1-\sqrt{2}, \sqrt{2})$.

Example 18. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

Solution : Here $y = x^3 - 11x + 5$ (1)

$\Rightarrow$ $\frac{dy}{dx} = 3x^2 - 11$ (2)

Slope of tangent $y = x - 11$ is 1

From (2)

$$1 = 3x^2 - 11$$

$\Rightarrow$ $3x^2 = 12 \Rightarrow x = \pm 2$

Put $x = 2$ in equation (1)

$$y = 2^3 - 11(2) + 5 = -9$$

Put $x = -2$ in equation (1)

$$y = (-2)^3 - 11(-2) + 5 = 19$$

But point (–2, 19) does not lie on curve (1) hence the point at which the tangent is $y = x - 11$ is (–2, 9).

Example 19. Find the equation of all lines having slope zero that are tangents to the curve $y = \frac{1}{x^2 - 2x + 3}$

Solution : Here $y = \frac{1}{x^2 - 2x + 3}$ (1)

Differentiating with respect to x

$$\frac{dy}{dx} = -\frac{(2x-2)}{(x^2-2x+3)^2}$$

Here slope $= 0$

$\Rightarrow$ $$\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{-(2x-2)}{(x^2-2x+3)^2} = 0$$

$\Rightarrow$ $$2x-2=0$$

$\Rightarrow$ $$x=1$$

Put $x=1$ in (1)

$$y = \frac{1}{1^2-2(1)+3} = \frac{1}{2}$$

Hence at point $(1, 1/2)$ the slope of tangent $= 0$ and the equation of tangent is

$y - \frac{1}{2} = 0(x-1) \Rightarrow y = \frac{1}{2}$, which is required equation of tangent.

Example 20. Find the equation of normal for the curve $2x^2 - y^2 = 14$, which is parallel to the straight line $x+3y=6$.

Solution : Let a point $P(x_1, y_1)$ on $2x^2 - y^2 = 14$, where normal is parallel to $x+3y=6$

$\therefore$ $$2x_1^2 - y_1^2 = 14 \qquad (1)$$

$\because$ $$2x^2 - y^2 = 14$$

$\Rightarrow$ $$4x - 2y\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{4x}{2y} = \frac{2x}{y}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2x_1}{y_1}$$

$\because$ Normal at (x_1, y_1) is parallel to $x+3y=6$ hence slope of normal at (x_1, y_1) = slope of line $x+3y=6$

$\Rightarrow$ $$-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = -\frac{1}{3}$$

$$\Rightarrow \qquad \frac{y_1}{2x_1} = \frac{1}{3} \Rightarrow y_1 = \frac{2}{3}x_1$$

Put $y_1 = \frac{2}{3}x_1$, in (1)

$$2x_1^2 - \left(\frac{2}{3}x_1\right)^2 = 14$$

$$\Rightarrow \qquad \frac{14}{9}x_1^2 = 14 \Rightarrow x_1 = \pm 3$$

$$\text{at } x_1 = 3,\ y_1 = \frac{2}{3} \times 3 = 2$$

and $$\text{at } x_1 = -3,\ y_1 = \frac{2}{3}(-3) = -2$$

Hence at (3, 2) and (–3, –2) normal is parallel to $x + 3y = 6$. Hence the equation of normal at (3, 2) is

$$y - 2 = -1/3(x-3) \Rightarrow x + 3y = 9$$

Equation of normal at (–3, –2) is

$$y + 2 = -1/3(x+3) \Rightarrow x + 3y + 9 = 0.$$

Example 21. Find the equation of the tangent to curve $y = x^2 - 2x + 7$ which is

(i) parallel to the line $2x - y + 9 = 0$

(ii) perpendicular to the line $5y - 15x = 13$

Solution : Equation of curve is $y = x^2 - 2x + 7$ (1)

$$\Rightarrow \qquad \frac{dy}{dx} = 2x - 2 = 2(x-1) \qquad (2)$$

(i) Slope of the straight line $2x - y + 9 = 0$ or $y = 2x + 9$ is 2

$\because$ tangent is parallel to this line, hence

$$2(x-1) = 2$$

$$\Rightarrow \qquad x = 1$$

When $x = 1$, then from (1)

$$y = 1^2 - 2(1) + 7 = 6$$

Hence the equation of tangent at (1, 6) which is parallel to $2x - y + 9 = 0$ will be

$$y - 6 = 2(x-1)$$

$$\Rightarrow \qquad 2x - y + 4 = 0$$

(ii) Straight line $5y - 15x = 13$ or $5y = 15x + 13$

$\Rightarrow$ $y = 3x + 13/5$ Slope of line $= 3$

Slope of a line whcih is perpendicualr to $5y-15x=13$ is $-1/3$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{1}{3}$$

$$\Rightarrow \quad 2(x-1)=-1/3$$

$$\Rightarrow \quad 6x-6=-1$$

$$\Rightarrow \quad x=5/6$$

When $x=5/6$ then from (1)

$$y=\left(\frac{5}{6}\right)^2-2\left(\frac{5}{6}\right)+7 \quad =\frac{217}{36}$$

Hence the equation of tangent at $\left(\frac{5}{6},\frac{217}{36}\right)$ will be

$$y-\frac{217}{36}=-\frac{1}{3}\left(x-\frac{5}{6}\right)$$

$$\Rightarrow \quad \frac{36y-217}{36}=-\frac{1}{3}\left(\frac{6x-5}{6}\right)$$

$$\Rightarrow \quad 12x+36y-227=0$$

Which is the required equation of tangent.

Example 22. Prove that for every value of x, the straight line $\frac{x}{a}+\frac{y}{b}=2$, touches the curve $(x/a)^n+(y/b)^n=1$ at point (a, b).

Solution : Equation of curve $(x/a)^n+(y/b)^n=1$

Differentiating with respect to x

$$\frac{1}{a^n}nx^{n-1}+\frac{1}{b^n}ny^{n-1}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{b^n x^{n-1}}{a^n y^{n-1}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(a,b)}=-\frac{b^n\cdot a^{n-1}}{a^n b^{n-1}}=-\frac{b}{a}$$

Hence the equation of tangent at (a, b) is

$$y-b=-\frac{b}{a}(x-a)$$

$$\Rightarrow \quad ay-ab=-bx+ab$$

$$\Rightarrow \quad bx+ay=2ab$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}=2$$

Exericse 8.3

1. Find the slope of the tangent to the curve $y = x^3 - x$.

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.

3. Find the point at which the tangent to the curve $y = \sqrt{(4x-3)} - 1$ has its slope $2/3$.

4. Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

5. Find points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangent are

   (i) parallel to x-axis (ii) parallel to y-axis

6. Find the equation of tangent to the curve given by $x = a\sin^3 t,\ y = b\cos^3 t$ at a point where $t = \pi/2$.

7. Find the equation of normal to the curve $y = \sin^2 x$ at a point $\left(\frac{\pi}{3}, \frac{3}{4}\right)$.

8. Find the equations of the tangent and normal to the given curves at the indicated points:

   (a) $y = x^2 + 4x + 1$ at $x = 3$ (b) $y^2 = 4ax$ at $x = a$

   (c) $xy = a^2$, at $\left(at, \frac{a}{t}\right)$ (d) $y^2 = 4ax$, at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

   (e) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at $(a\sec\theta, b\tan\theta)$ (f) $y = 2x^2 - 3x - 1$, at $(1, -2)$

   (g) $x = at^2, y = 2at$, at $t = 1$ (h) $x = \theta + \sin\theta, y = 1 - \cos\theta$, at $\theta = \pi/2$

8.07 Approximation

In this section, we will use differential to approximate values of certain quantities.

Let $y = f(x)$ be the equation of given curve. Let Δx denote a small increment is x, whereas the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the followign (i) The differential of x, denoted by x is defined by $dx = \Delta x$. (ii) The differential by dy, is defined by $dy = f'(x)dx$ or $dy = \frac{dy}{dx}.\Delta x$

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

Illustrative Examples

Example 23. Use differential to approximate $\sqrt{26}$.

Solution : Let $y = \sqrt{x}$

Where $x = 25, \Delta x = 1$ and $x + \Delta x = 26$

$\because$ $$y = \sqrt{x} = x^{1/2} \qquad (1)$$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

$\because \qquad \Delta y = \dfrac{dy}{dx}.\Delta x = \dfrac{1}{2\sqrt{x}}\Delta x = \dfrac{1}{2\times 5}\times 1 = \dfrac{1}{10} = 0.1$

From (1) $\qquad y + \Delta y = (x+\Delta x)^{1/2}$

$\Rightarrow \qquad x^{1/2} + \Delta y = (x+\Delta x)^{1/2}$

Putting the value $\qquad (25)^{1/2} + 0.1 = (26)^{1/2}$

$\Rightarrow \qquad \sqrt{26} = 5 + 0.1 = 5.1.$

Example 24. Use diferential to approximate $(66)^{1/3}$

Solution : Let $\qquad y = x^{1/3} \qquad (1)$

Where $x = 64, \Delta x = 2$ and $x + \Delta x = 66$

$\because \qquad y = x^{1/3}$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{1}{3x^{2/3}}$

$\because \qquad \Delta y = \dfrac{dy}{dx}.\Delta x = \dfrac{1}{3x^{2/3}}.\Delta x = \dfrac{1}{3\times(64)^{2/3}}\times 2$

$$= \frac{1}{3\times(4)^2}\times 2 = \frac{1}{24}$$

Now From (1)

$$y + \Delta y = (x+\Delta x)^{1/3}$$

$\Rightarrow \qquad x^{1/3} + \dfrac{1}{24} = (66)^{1/3}$

$\Rightarrow \qquad (64)^{1/3} + \dfrac{1}{24} = (66)^{1/3}$

$\Rightarrow \qquad (4^3)^{1/3} + \dfrac{1}{24} = (66)^{1/3}$

$\Rightarrow \qquad 4 + 0.041 = (66)^{1/3}$

$\Rightarrow \qquad (66)^{1/3} = 4.041.$

Example 25. Use differential to approximate the following

(i) $\log_{10}(10.2)$ when $\log_{10} e = 0.4343$

(ii) $\log_e(4.04)$ when $\log_e 4 = 1.3863$

(iii) $\cos 61^\circ$ when $1^\circ = 0.01745$ Radian

Solution : (i) Let $y = \log_{10} x$ (1)

Where $x = 10, \Delta x = 0.2$

$\Rightarrow$ $x + \Delta x = 10.2$

$\because$ $y = \log_{10} x = \log_{10} e . \log_e x$

$\Rightarrow$ $\dfrac{dy}{dx} = (\log_{10}{}^{e})\dfrac{1}{x} = \dfrac{0.4343}{10}$

$\because$ $\Delta y = \dfrac{dy}{dx} . \Delta x = \dfrac{0.4343}{10} \times (0.2) = 0.008686$

From (1)

$$y + \Delta y = \log_{10}(x + \Delta x)$$

$\Rightarrow$ $\log_{10} x + \Delta y = \log_{10}(x + \Delta x)$

$\Rightarrow$ $\log_{10} 10 + 0.008686 = \log_{10}(10.2)$

$\Rightarrow$ $1 + 0.008686 = \log_{10}(10.2)$

$\Rightarrow$ $\log_{10}(10.2) = 1.008686$

(ii) Let $y = \log_e x$ (2)

Where $x = 4, \Delta x = 0.04$ and $x + \Delta x = 4.04$

$\because$ $y = \log_e x$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\because$ $\Delta y = \dfrac{dy}{dx} . \Delta x = \dfrac{\Delta x}{x} = \dfrac{0.04}{4} = 0.01$

From (2)

$$y + \Delta y = \log_e(x + \Delta x)$$

$\Rightarrow$ $\log_e x + \Delta y = \log_e(x + \Delta x)$

Putting values

$$\log_e 4 + 0.01 = \log_e(4.04)$$

$\Rightarrow$ $\log_e(4.04) = 1.3863 + 0.01$

$= 1.3963$

(iii) Let $y = \cos x$ (3)

When $x = 60^\circ, \Delta x = 1^\circ = 0.01745$ radian and $x + \Delta x = 61^\circ$

$\because$ $y = \cos x$

$\Rightarrow$ $\dfrac{dy}{dx} = -\sin x$

$\because \qquad \Delta y = \frac{dy}{dx}\Delta x = -\sin x.\Delta x$

$$= -\sin 60^\circ (0.01745)$$

$$= -0.1745 \times \frac{\sqrt{3}}{2} = -0.01511 \qquad (\because \sqrt{3} = 1.73205)$$

From (3)

$$y + \Delta y = \cos(x + \Delta x)$$

$\Rightarrow \qquad \cos x + \Delta y = \cos(x + \Delta x)$

$$\cos 60^\circ + (-0.01511) = \cos(61^\circ)$$

$\Rightarrow \qquad \cos 61^\circ = \frac{1}{2} - 0.01511$

$$= 0.48489.$$

Example 26. Prove that the approximation percentage error in calculating the volume of a sphere is almost three times the approximation percentage error in calculating the radius of sphere.

Solution : Let radius of sphere = r and volume = V

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$\because \qquad \Delta V = \frac{dV}{dr}.\Delta r$

$\Rightarrow \qquad \Delta V = 4\pi r^2 \Delta r$

$\Rightarrow \qquad \frac{\Delta V}{V} = \frac{4\pi r^2 \Delta r}{V} = \frac{4\pi r^2 \Delta r}{4/3\pi r^3} = 3\frac{\Delta r}{r}$

$\Rightarrow \qquad \frac{\Delta V}{V} \times 100 = 3\left(\frac{\Delta r}{r} \times 100\right)$

$\Rightarrow$ Percentage error in volume = 3 (percentage error in radius).

Example 27. Find the approximate vlaue of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$

Solution : Let $\qquad y = f(x) \qquad (1)$

Where $x = 5, \Delta x = 0.001$ and $x + \Delta x = 5.001$

From (1)

$$y + \Delta y = f(x + \Delta x)$$

$\Rightarrow \qquad f(x) + \frac{dy}{dx}.\Delta x = f(x + \Delta x) \qquad (2)$

$\because \qquad y = f(x) = x^3 - 7x^2 + 15$

$$\Rightarrow \qquad \frac{dy}{dx} = 3x^2 - 14x$$

Using in equation (2)

$$(x^3 - 7x^2 + 15) + (3x^2 - 14x).\Delta x = f(x + \Delta x)$$

Putting the value of x

$$(5)^3 - 7(5)^2 + 15 + \{3(5)^2 - 14(5)\} \times (0.001) = f(5.001)$$

$$\Rightarrow \qquad f(5.001) = 125 - 175 + 15 + (75 - 70)(0.001)$$

$$= -34.995$$

Example 28. Find the approximate change in the volume of a cube of side x metres caused by increasing the side by 1%.

Solution : Let volume of cube is V

$$\Delta x = x \text{ of } 1\% = \frac{x}{100}$$

$$\because \qquad V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$$

Hence change in volume of cube

$$dV = \frac{dV}{dx}\Delta x$$

$$= 3x^2 \times \frac{x}{100} = \frac{3}{100}x^3$$

$$= 0.03x^3 \text{ m}^3$$

Example 29. If the radius of a sphere is measured as 7 cm with an error of 0.02 cm, then find the approximate error in calculating its volume.

Solution : Radius of sphere = 7 cm

Error in measuring radius $\Delta r = 0.02$ cm

Let the volume of sphere be V

$$V = (4/3)\pi r^3$$

$$\Rightarrow \qquad \frac{dV}{dr} = 4\pi r^2$$

$$\because \qquad dV = \frac{dV}{dr}\Delta r = 4\pi r^2.\Delta r$$

$$= 4\pi(7)^2 \times .002 = 3.92\pi$$

Exercise 8.4

Using differentials, find the approximate value of each of the following.

1. $(0.009)^{1/3}$ 2. $(0.999)^{1/10}$ 3. $\sqrt{0.0037}$ 4. $\frac{1}{(2.002)^2}$

5. $(15)^{1/4}$ 6. $\sqrt{401}$ 7. $(3.968)^{3/2}$ 8. $(32.15)^{1/5}$

9. $\sqrt{0.6}$ 10. $\log_{10}(10.1)$, when $\log_{10} e = 0.4343$

11. $\log_e(10.02)$, when $\log_e 10 = 2.3026$

12. Find the approximate change in y when $y = x^2 + 4$ as x increases from 3 to 3.1.

13. Prove that the approximation percentage error in calculating the volume of a cubical box is almost three times the approximation percentage error in claculating the edge of cube.

14. If the radius of a sphere decreases from 10 cm to 9.8 cm, find the approximate error in calculating its volume.

8.08 Maxima and Minima

In this section, we will use the concept of derivatives to calcualte the maximum or minimum values of various functions.

Let us examine the graph of a function $y = f(x)$ in the interval $[a, b]$. Observe the ordiantes of points A, P, Q, R, S and B.

The function has maximum value in some neighbourhood of points P and R which are at the top of their respective hills (ordinates) where as the function has minimum value is some neighbourhood (interval) of each of the points Q and S. Point A has least ordinate and point B has maximum ordinate. Tangents drawn to the cuve at point P, Q, R and S are parallel to x-axis, i.e., their slope $\left(\frac{dy}{dx}\right)$ are zero. The P and R are called maximum points and points Q and S are called minimum points for the function. Maximum and minimum points of a function are also regarded as extreme points.

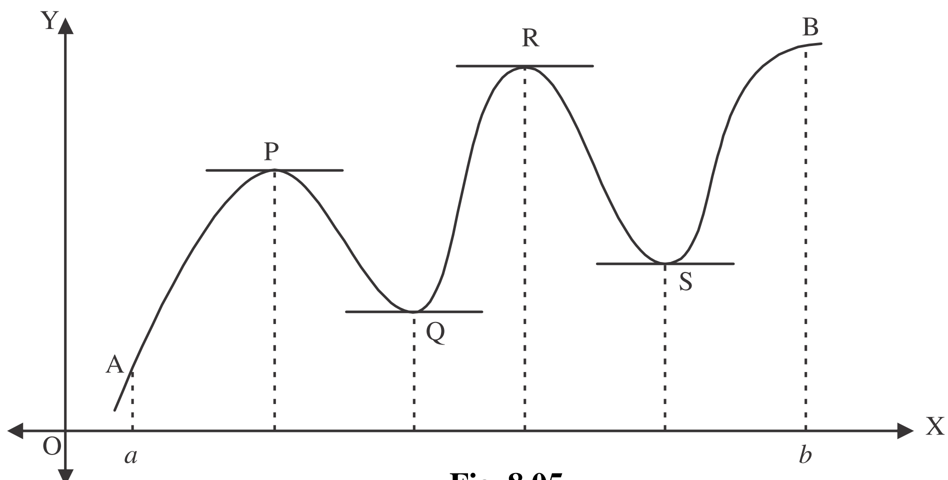

Fig. 8.05

8.09 Some Difinitions

(i) Relative maximum and minimum value

Let $f(x)$ be a real valued function and let c be an interior point he domain of f, then c is called a point in the domain of f, then c is called a point of relative maxima if there is $h > 0$ such that $f(x) \leq f(c), \quad \forall \; x \in (c - \hbar, c + \hbar)$, where h is very small. The value f(c) is called the relative maximum vlaue of f.

Similarly c is called a point of relative minimum if there is $h > 0$ such that $f(x) \geq f(c), \quad \forall \; x \in (c - \hbar, c + \hbar)$ the value f(c) is called the relative minimum value of f.

(ii) Absolute maximun and minimum value

Absolute maximum value: Any function $f(x)$ has its absolute maximum value at any point $x = a$ in its domain when.

$$f(x) \le f(c), \quad \forall \; x \in D$$

Absolute minimum value : Any function $f(x)$ has its absolute minimum value at any point $x = a$ in its domain when

$$f(x) \ge f(c), \quad \forall \; x \in D$$

Note: For a real valued function f(x) in a domain the maximum and minimum value of function may be more than one but absolute maximum and absolute minimum is only one.

8.10 Necessary condition for the extreme value of a function

Theorem : If $f(x)$ is a differentiable function then at $x = c$, necessary condition for the extreme value is $f'(c) = 0$

Note: For a function $f(x)$ at any point $x = c$, $f'(c) = 0$ is only necessary condition for maximum and minimum value of function, it is not sufficient conditiion.

For example if $f(x) = x^3$ then at $x = 0$, $f'(0) = 0$ but $f(0)$ is not extreme valeu of function because when $x > 0 \Rightarrow f(x) > f(0)$ and when $x < 0 \Rightarrow f(x) < f(0)$ and when $f(0)$ is neither minimum nor maximum.

Sufficient condition for the extreme value of a funciton

Theorem : (i) $f(x)$ will have its maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

(ii) $f(x)$ will have its minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$

Note: For a function $f(x)$ at any point $x = c$, $f'(c) = 0$, $f''(c) = 0$ but $f'''(c) \ne 0$ then this point is known as inflection point.

8.11 Properties of maxima and minima of a function

If $f(x)$ is continuous function and if its graph could be drawn then we may consider the following properties.

(i) There is at least one maxima or minima between two equal values of $f(x)$.

(ii) The maxima and minima of a function always occur alternatively.

(iii) If $f'(x)$ changes sign from positive to negative as x increases then $f(x)$ passes through maxima and when $f'(x)$ changes sign from negative to psotive then $f(x)$ passes through minima.

(iv) If $f'(x)$ does not change its sign then this point is called point of inflexion.

(v) At maxima and minima $f'(x) = 0$ then the line point is parallel to $x -$ axis.

8.12 Working method to find maxima and minima

1. First of all write the given function in the form of $y = f(x)$ and find $\frac{dy}{dx}$

2. Solve $\frac{dy}{dx} = 0$, let the solutions are $x = a_1, a_2, \ldots$

3. Find $\frac{d^2y}{dx^2}$ and find its value at $x = a_1, a_2, \ldots$

4. If $\frac{d^2y}{dx^2} < 0$ at $x = a_r$ $(r = 1, 2, \ldots)$ then $x = a_r$ function $f(x)$ will have maximum value.

5. If $\frac{d^2y}{dx^2} > 0$ at $x = a_r$ $(r = 1, 2, \ldots)$ then at $x = a_r$ function $f(x)$ will have minimum value. If $\frac{d^2y}{dx^2} = 0$ then we continue the process of differentiation.

6. If $\frac{d^2y}{dx^2} = 0$ $x = a_r$ $(r = 1, 2, \ldots)$ then find the values of $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$.......until $x = a_r$ becomes zero.

   (i) If non zero differential coefficient is of odd degree like $\frac{d^3y}{dx^3}, \frac{d^5y}{dx^5}$.... then at $x = a_r$. Function has neither maxima nor minima.

   (ii) If non zero differential coefficient is of even degree like $\frac{d^4y}{dx^4}, \frac{d^6y}{dx^6}$...., then repeat the same process as $\frac{d^2y}{dx^2} \neq 0$.

8.13 Stationary point

All points on which the rate of change of $f(x)$ with respect to x is zero i.e. $f'(x) = 0$, are called stationary points.

Note: Every extreme point is a stationary point but vice versa is not always true.

Illustrative Examples

Example 30. Find maximum and minimum value of following function (if exist)

(a) $y = (2x-1)^2 + 3$ (b) $y = 9x^2 + 12x + 2$

(c) $y = -(x-1)^2 + 10$ (d) $y = x^3 + 1$

Solution : (a) Minimum value of $(2x-1)^2$ is zero hence minimum value of $(2x-1)^2 + 3$ is 3. It is clear that there is not maximum value of function.

(b) $\because$
$$y = 9x^2 + 12x + 2$$
$$= (3x+2)^2 - 2$$

$\because$ Minimum value of $(3x+2)^2$ is zero. Hence minimum value of $(3x+2)^2 - 2$ is –2 which is at $3x + 2 = 0 \Rightarrow x = -\frac{2}{3}$. It is clear that no maximum value of $y = 9x^2 + 12x + 2$ is there.

(c) It is clear that the minimum value of $-(x-1)^2$ is zero. Hence the maximum value of function $y = -(x-1)^2 + 10$ is 10. There is no minimum value of function.

(d) $\therefore$ at $x \to \infty$, $y \to \infty$

and $x \to -\infty$, $y \to -\infty$

Hence given function has neither maximum nor minimum value.

Example 31. Find the minimum and maximum value of following functions:

(a) $x^5 - 5x^4 + 5x^3 - 2$ (b) $(x-2)^6(x-3)^5$ (c) $(x-1)^2 e^x$

Solution : (a) Let $y = x^5 - 5x^4 + 5x^3 - 2$

$\Rightarrow$ $\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$

and $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

For extreme point of function $\frac{dy}{dx} = 0$

$\Rightarrow$ $5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow$ $5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow$ $5x^2(x-1)(x-3) = 0$

$\Rightarrow$ $x = 0, 1, 3$

Now at $x = 0$, $\frac{d^2y}{dx^2} = 0$

So, $\frac{d^3y}{dx^3} = 60x^2 - 120x + 30$

at $x = 0$, $\frac{d^3y}{dx^3} = 30 \neq 0$

So, at $x = 0$, there is no extreme value of function.

at $x = 1$ $\frac{d^2y}{dx^2} = 20(1)^3 - 60(1)^2 + 30(1) = -10 < 0$

So at $x = 1$ function has maximum value and maximum value of function is

$= (1)^5 - 5(1)^4 + 5(1)^3 - 2 = -1$

Similarity at $x = 3$, $\frac{d^2y}{dx^2} = 20(3)^3 - 60(3)^2 + 30(3)$

$= 540 - 540 + 90 = 90 > 0$

So, at $x = 3$ function has minimum value and minimum value of fucntion is

$= (3)^5 - 5(3)^4 + 5(3)^3 - 2$

$= -29$

(b) Let $y=(x-2)^6(x-3)^5$

$\Rightarrow$ $\frac{dy}{dx}=6(x-2)^5(x-3)^5+(x-2)^6 5(x-3)^4$

$=(x-2)^5(x-3)^4\{6x-18+5x-10\}$

$=(x-2)^5(x-3)^4(11x-28)$

For maxima and minima $\frac{dy}{dx}=0$

$\Rightarrow$ $(x-2)^5(x-3)^4(11x-28)=0$

$\Rightarrow$ $x=2, 3, 28/11$

at $x=2$, $\frac{dy}{dx}$ changes its sign from positive to negative ($\because$ when $x<2$ then $\frac{dy}{dx}>0$ and $x>2$ then $\frac{dy}{dx}<0$)

So at $x=2$ functions has maximum value and maximum value = 0

at $\because x=3$, $\frac{dy}{dx}$ does not change its sign ($\because$ when $x<3$ then $\frac{dy}{dx}>0$ and $x>3$ then $\frac{dy}{dx}>0$)

at $x=3$ function has neither maxima nor minima

again at $x=\frac{28}{11}$, $\frac{dy}{dx}$ changes its sign from negative to positive ($\because$ when $x<\frac{28}{11}$ then $\frac{dy}{dx}<0$ and $x>\frac{28}{11}$ then $\frac{dy}{dx}>0$)

Hence at $x=\frac{28}{11}$ function has minimum value $=\left(\frac{28}{11}-2\right)^6\left(\frac{28}{11}-3\right)^5=-\frac{6^5.5^5}{11^{11}}$

(c) Let $y=(x-1)^2e^x$

$\Rightarrow$ $\frac{dy}{dx}=\{(x-1)^2+2(x-1)\}e^x$

and $\frac{d^2y}{dx^2}=\{(x-1)^2+4(x-1)+2\}e^x$

For extreme value $\frac{dy}{dx}=0$

$\Rightarrow$ $\{(x-1)^2+2(x-1)\}e^x=0$

$\Rightarrow$ $(x-1)^2+2(x-1)=0$ $\{\because e^x\neq 0\}$

$\Rightarrow \qquad x^2 - 1 = 0$

$\Rightarrow \qquad x = \pm 1$

Now at $x = 1$, $\qquad \frac{d^2y}{dx^2} = \{0 + 4(0) + 2\}e^1 = 2e > 0$

So at $x = 1$ function has minimum value and minimum value $= (1-1)^2 e^1 = 0$

again at $x = -1$ $\qquad \frac{d^2y}{dx^2} = \{(-1-1)^2 + 4(-1-1) + 2\}e^{-1}$

$$= \{4 - 8 + 2\}e^{-1} = \frac{-2}{e} < 0$$

So, at $x = -1$ function has maximum value and maximum value is $= (-1-1)^2 e^{-1} = \frac{4}{e}$.

Exercise 32. Find the maximum value of function $(1/x)^x$

Solution : Let $\qquad y = (1/x)^x$

$\Rightarrow \qquad \log y = x \log \frac{1}{x}$

$$= -x \log x = z \quad \text{Let}$$

Function y has maximum or minimum value if z has maximum or minimum value.

Now, $\qquad \frac{dz}{dx} = -x.\frac{1}{x} - 1.\log x = -(1 + \log x)$

ad $\qquad \frac{d^2z}{dx^2} = -\frac{1}{x}$

So, for maximum or minimum value

$$\frac{dz}{dx} = 0 \Rightarrow 1 + \log x = 0$$

$\Rightarrow \qquad \log x = -1$

$\Rightarrow \qquad x = e^{-1} = \frac{1}{e}$

at $x = 1/e$ $\qquad \frac{d^2z}{dx^2} = -\frac{1}{1/e} = -e < 0$

So at $x = 1/e$, y has maximum value and maximum value $= \left[\frac{1}{1/e}\right]^{1/e} = e^{1/e}$.

Example 33. Find the shortest distance of the point (0, a) from the parabola $x^2 = y$ where $a \in [0, 5]$.

Solution : Let a point (h, k) is on the parabola, let the distance between (0, a) and (h, k) is D, then

$$D = \sqrt{(h-o)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \qquad (1)$$

$\because$ point (h, k) is on parabola $x^2 = y$ hence $h^2 = k$ use this in (1)

$$D = \sqrt{k + (k-c)^2}$$

$\Rightarrow$ $$D(k) = \sqrt{k + (k-c)^2}$$

$\Rightarrow$ $$D'(k) = \frac{\{1 + 2(k-c)\}}{2\sqrt{k + (k-c)^2}} \qquad (2)$$

Now $$D'(k) = 0 \Rightarrow k = \frac{2c-1}{2}$$

when $k < \dfrac{2c-1}{2}$ then $2(k-c)+1 < 0$

$\Rightarrow$ $D'(k) < 0$ [from equation (2)]

and when $k > \dfrac{2c-1}{2}$ then $2(k-c)+1 < 0$

$\Rightarrow$ $D'(k) > 0$ [from equation (4)]

So at $k = \dfrac{2c-1}{2}$, D is minimum and the minimum distance

$$= \sqrt{\frac{2c-1}{2} + \left(\frac{2c-1}{2} - c\right)^2} = \frac{\sqrt{4c-1}}{2}.$$

Example 34. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(a) $f(x) = x^3, \quad x \in [-2, 2]$

(b) $f(x) = 4x - \dfrac{1}{2}x^2, \quad x \in \left[-2, 9/2\right]$

(c) $f(x) = (x-1)^2 + 3, \quad x \in [-3, 1]$

(d) $f(x) = \sin x + \cos x, \quad x \in [0, \pi]$

Solution : (a) Given $$f(x) = x^3,\ x \in [-2, 2]$$

$\Rightarrow$ $$f'(x) = 3x^2$$

$\therefore$ $$f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0 \Rightarrow f(0) = 0$$

Now $f(-2) = (-2)^3 = -8$; $f(0) = (0)^3 = 0$ and $f(2) = (2)^3 = 8$

The absolute maximum value of $f(x)$ is 8 which is obtained at $x = 2$ and absolute minimum value is -8 which is obtained at $x = -2$.

(b) Given $$f(x) = 4x - \frac{x^2}{2}$$

$\Rightarrow$ $$f'(x) = 4 - \frac{2x}{2} = 4 - x$$

for extreme value of $f(x)$ $\quad f'(x) = 0$

$\Rightarrow$ $$4 - x = 0$$

$\Rightarrow$ $$x = 4$$

Now, we find the value of function at points –2, 4 and 9 / 2.

$\because$ Given function is $f(x) = 4x - \frac{x^2}{2}$ So, $f(-2) = 4(-2) - \frac{(-2)^2}{2} = -10$; $f(4) = 4(4) - \frac{(4)^2}{2} = 8$

and $$f(9/2) = 4(9/2) - \frac{(9/2)^2}{2} = -9/4$$

Hence in the given interval [–2, 9/2] absolute maximum value = 8 and minimum value $= -10$

(c) Given function $$f(x) = (x-1)^2 + 3, x \in [-3, 1]$$

$\Rightarrow$ $$f'(x) = 2(x-1)$$

for extreme value of $f(x)$, $\quad f'(x) = 0$

$\Rightarrow$ $$2(x-1) = 0$$

$\Rightarrow$ $$x = 1$$

The values of $f(x)$ at $x = 1, -3, 0$, and 0 are

$f(1) = (1-1)^2 + 3 = 0 + 3 = 3$; $f(-3) = (-3-1)^2 + 3 = 16 + 3 = 19$ and $f(0) = (0-1)^2 + 3 = 1 + 3 = 4$

Hence in the given interval [–3, 1] absolute maximum value is 19 which is obtained at $x = -3$ and absolute minimum value is 3 which is at $x = 1$.

(d) Given function is $$f(x) = \sin x + \cos x, x \in [0, \pi]$$

$\Rightarrow$ $$f'(x) = \cos x - \sin x$$

For maxima and minima of $f(x)$ $\quad f'(x) = 0$

$\Rightarrow$ $$\cos x - \sin x = 0$$

$\Rightarrow$ $$\sin x = \cos x$$

$\Rightarrow$ $$\tan x = 1$$

$\Rightarrow$ $$x = \pi / 4$$

Now $$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

and $$f(\pi) = \sin \pi + \cos \pi = 0 + (-1) = -1$$

Hence maximum and minimum values of $f(x)$ are $\sqrt{2}$ and –1 respectively, for given interval [0, π]

Example 35. Find two positive numbers x and y such that

(a) $x + y = 60$ and xy^3 is maximum (b) $x + y = 16$ and $x^3 + y^3$ is minimum

Solution : (a) Let $p = xy^3$

Given $x + y = 60 \Rightarrow x = 60 - y$

$\therefore$ $p = (60 - y)y^3 = 60y^3 - y^4$

$\Rightarrow$ $\frac{dp}{dt} = 180y^2 - 4y^3$

and $\frac{d^2p}{dy^2} = 360y - 12y^2$

For extreme value of p, $\frac{dp}{dy} = 0$

$\Rightarrow$ $180y^2 - 4y^3 = 0$

$\Rightarrow$ $4y^2(45 - y) = 0$

$\Rightarrow$ $y = 45$ $\{\because y = 0 \text{ is not possible } y > 0\}$

Now $\left(\frac{d^2p}{dy^2}\right)_{y=45} = 360(45) - 12(45)^2 = -8100 < 0$

So at, $y = 45$, P has maximum value.

When $y = 45$ then $x = 60 - 45 = 15$

Hence numbers are $x = 15$ and $y = 45$.

(b) Let $p = x^3 + y^3$ (1)

Given $x + y = 16$

$\Rightarrow$ $y = 16 - x$ (2)

From equation (1)

$$p = x^3 + (16 - x)^3$$

$\Rightarrow$ $\frac{dp}{dx} = 3x^2 + 3(16 - x)^2(-1)$

$= 3x^2 - 3(256 - 32x + x^2)$

$= 3(32x - 256)$ (3)

Now $\frac{dp}{dx} = 0 \Rightarrow 3(32x - 256) = 0$

$\Rightarrow$ $x = \frac{256}{32} = 8$

From equation (3) $\frac{d^2p}{dx^2} = 96 > 0$

Hence at $x = 8$, p is minimum.
Hence required positive numbers and 8 and 8.

Exercise 8.5

1. Find maximum and minimum value of following functions:

(a) $2x^3 - 15x^2 + 36x + 10$ (b) $(x-1)(x-2)(x-3)$

(c) $\sin x + \cos 2x$ (d) $x^5 - 5x^4 + 5x^3 - 1$

2. Find the maximum and minimum value, if any:

(a) $-|x+1|+3$ (b) $|x+2|-1$ (c) $|\sin 4x + 3|$ (d) $\sin 2x + 5$

3. Find the maximum and minimum value of following function if any, in the given intervals.

(a) $2x^3 - 24x + 107$, $x \in [1, 3]$ (b) $3x^4 - 2x^3 - 6x^2 + 6x + 1$, $x \in [0, 2]$

(c) $x + \sin 2x$, $x \in [0, 2\pi]$ (d) $x^3 - 18x^2 + 96x$, $x \in [0, 9]$

4. Find extreme value of following functions

(a) $\sin x \cos 2x$ (b) $a \sec x + b \operatorname{cosec} x$, $o < a < b$

(c) $x^{1/x}$, $x > 0$ (d) $\frac{1}{x}.\log x$, $x \in (0, \infty)$

5. Prove that function $f(x) = \frac{x}{1 + x\tan x}$ has maximum value at $x = \cos x$.

6. Prove that $\sin^2 x(1 + \cos x)$ has maximum value at $\cos x = 1/3$

7. Prove that function $y = \sin^p \theta \cos^q \theta$ has maximum value at $\tan\theta = \sqrt{p/q}$

8.14 Applications of maxima and minima

With the help of following examples we shall use the application of derivatives in other branches as (i) Plane Geometry; (ii) Solid geometry; (iii) Mechanics; (iv) Commerce and Economics.

Illustrative Examples

Example 36. Show that of all the rectangles inscribed in a given fixed circle, the square has maximum area.

Solution : $PQRS$ is a rectangle, centre of circle is O and its radius is a

Let $PQ = 2x, QR = 2y$

In right ΔPQR

$$PQ^2 + QR^2 = PR^2$$

$$\Rightarrow \quad (2x)^2 + (2y)^2 = (2a)^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2$$

$$\Rightarrow \quad y = \sqrt{a^2 - x^2} \qquad (1)$$

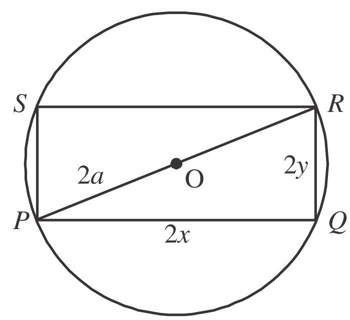

Fig. 8.06

Let area of rectangle $PQRS$ is A

$$A = (2x)\left(2\sqrt{a^2 - x^2}\right) = 4x\sqrt{a^2 - x^2}$$

$$\Rightarrow \qquad \frac{dA}{dx} = 4\left\{\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right\} = \frac{4(a^2 - 2x^2)}{\sqrt{a^2 - x^2}} \qquad (2)$$

For maximum or minimum value of A $\quad \frac{dA}{dx} = 0$

$$\Rightarrow \qquad \frac{4(a^2 - 2x^2)}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \qquad a^2 - 2x^2 = 0$$

$$\Rightarrow \qquad x = \frac{a}{\sqrt{2}}$$

From (2)

$$\frac{d^2A}{dx^2} = 4\left\{\frac{-4x}{\sqrt{a^2 - x^2}} - \frac{x(a^2 - 2x^2)}{(a^2 - x^2)^{3/2}}\right\}$$

at $\quad x = a/\sqrt{2}$, $\frac{d^2A}{dx^2} = -16 < 0$

So, at $\quad x = a/\sqrt{2}$, A is maximum.

Put $\quad x = a/\sqrt{2}$, in (1) $y = a/\sqrt{2}$

So $\quad x = y = a/\sqrt{2}$ hence area is maximum when $x = y$

$\Rightarrow 2x = 2y$ hence rectangle is a square.

Example 37. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Solution : Let slant height of cone = ℓ and the semi vertical angle of cone = θ

In right $\Delta OO'B$

$OO' = \ell\cos\theta = h$ (height of cone)

$O'B = \ell\sin\theta = r$ (radius of cone)

Volume of cone

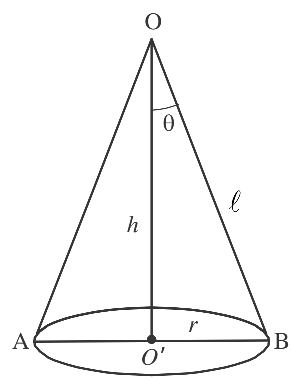

Fig. 8.07

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\ell^2\sin^2\theta.\ell\cos\theta$$

$$= \frac{1}{3}\pi\ell^3\sin^2\theta\cos\theta$$

$$\therefore \qquad \frac{dV}{d\theta} = \frac{1}{3}\pi\ell^3\{\sin^2\theta(-\sin\theta) + 2\sin\theta\cos\theta\cos\theta\}$$

$$= \frac{1}{3}\pi\ell^3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)$$

and $$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi\ell^3(2\cos\theta.\cos^2\theta - 4\sin\theta\cos\theta\sin\theta - 3\sin^2\theta\cos\theta)$$

$$= \frac{1}{3}\pi\ell^3(2\cos^3\theta - 7\sin^2\theta\cos\theta)$$

For maximum volume $\frac{dV}{d\theta} = 0$

$\Rightarrow$ $$\sin\theta(2\cos^2\theta - \sin^2\theta) = 0$$

$\Rightarrow$ $$\sin\theta\{2(1-\sin^2\theta) - \sin^2\theta\} = 0$$

$\Rightarrow$ $$\sin\theta\{2 - 3\sin^2\theta\} = 0$$

$\Rightarrow$ $$\sin\theta = 0, \sqrt{2/3}, -\sqrt{2/3}$$

Now $\sin\theta = \sqrt{2/3}$ or $\cos\theta = 1/\sqrt{3}$ then

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi\ell^3\left\{2\left(\frac{1}{\sqrt{3}}\right)^3 - 7\left(\sqrt{\frac{2}{3}}\right)^2.\frac{1}{\sqrt{3}}\right\}$$

$$= \frac{1}{3}\pi\ell^3\left\{\frac{2}{3\sqrt{3}} - \frac{14}{3\sqrt{3}}\right\} = -\frac{1}{3}\pi\ell^3\frac{12}{3\sqrt{3}} < 0$$

So, for maximum value $\sin\theta = \sqrt{2/3}$

Then $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{2/3}}{\sqrt{1/3}} = \sqrt{2}$$

$\therefore$ Semi vertical angle $\theta = \tan^{-1}(\sqrt{2})$.

Example 38. An open tank of fixed volume has square base. If inner surface is minimum then find the ratio of depth to length of the tank.

Solution : Let the depth and height of the tank are $\hbar$ and ℓ, then

Volume of tank $V = \ell^2\hbar$ (1)

Area of Inner surface of tank $S = \ell^2 + 4\ell\hbar$

$\Rightarrow$ $S = \ell^2 + 4\ell\left(\frac{V}{\ell^2}\right)$ [From (1)]

$\Rightarrow$ $S = \ell^2 + 4\frac{V}{\ell}$

$\Rightarrow \qquad \dfrac{dS}{d\ell} = 2\ell - \dfrac{4V}{\ell^2}$ and $\dfrac{d^2S}{d\ell^2} = 2 + \dfrac{2.4V}{\ell^3}$

For minimum surface area $\dfrac{dS}{d\ell} = 0$

$$\Rightarrow \qquad 2\ell - \frac{4V}{\ell^2} = 0$$

$$\Rightarrow \qquad \ell^3 = 2V$$

$$\Rightarrow \qquad \ell = (2V)^{1/3}$$

when $\ell = (2V)^{1/3}$ then $\dfrac{d^2S}{d\ell^2} = 2 + \dfrac{8V}{(2V)} > 0$

Hence inner surface is minimum.

From (1)

$$\hbar = \frac{V}{\ell^2} = \frac{1}{2}\frac{2V}{(2V)^{2/3}} = \frac{1}{2}(2V)^{1/3} = \frac{1}{2}.\ell$$

$$\Rightarrow \qquad \frac{\hbar}{\ell} = \frac{1}{2}$$

$\therefore$ Depth of tank : Length of tank $= 1:2$

Example 39. Manufacturer can sell x items at a price if rupees $\left(5 - \dfrac{x}{100}\right)$ each. The cost price of x items is rupees $\left(\dfrac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution : Let S be the selling price of x items and let c be the cost price of x items. Then, we have

$$S = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and $$C = \frac{x}{5} + 500$$

Let profit function be p then

$$p = S - C$$

$$= 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

$$= \frac{24}{5}x - \frac{x^2}{100} - 500$$

$$\Rightarrow \qquad \frac{dp}{dx} = \frac{24}{5} - \frac{x}{50} \text{ and } \frac{d^2 p}{dx^2} = -\frac{1}{50}$$

$$\therefore \qquad \frac{dp}{dx} = 0 \Rightarrow \frac{24}{5} - \frac{x}{50} = 0$$

$$\Rightarrow \qquad x = 240$$

and $$\left(\frac{d^2 p}{dx^2}\right)_{x=240} = -\frac{1}{50} < 0$$

Hence the manufacture can earn maximum profit if he sells 240 items.

Exercise 8.6

1. Prove that the maximum area of isosceles triangle, that can be inscribed in a circle, is an equilaterla triangle.
2. The sum of perimeter of a square and circumference of a circle is given. Prove that the sum of their areas will be minimum if the side of square is equal to the diameter of cirlce.
3. A cone is made from a sphere. Prove that the volumeof cone is maximum when height of cone is two third of diameter of sphere.
4. The expense for a steamer per hour is proportional tot he cube of its velocity. If velocity of stream is x km/h then prove that the maximum velocity of steamer per hour will be (2/3) x when the steamer runs against the direction of stream.
5. The sum of the length of the hypotenuse and any side of a right angled triangle is given. Prove that the area of the triangle is maximum when the angle between is then $\pi/3$.
6. A circle of radius a is inscribed in an equilateral triangle. Prove that the minimum perimeter of triangle is $6\sqrt{3}a$.
7. A normal is drawn to a point P on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that the maximum distance from centre of ellipse to normal is $a - b$.

Miscellaneous Exercise – 8

1. The radius of a cyclinder is r and height is h then find the rate of change in surface area of cylinder with respect to radius.
2. Find the values of x and y for function $y = x^2 + 21$, where the rate of change in y is thrice the rate of change in x.
3. Prove that exponential function e^x is an increasing function.
4. Prove that the function $f(x) = \log(\sin x)$, is increasing in $(0, \pi/2)$ and decreasing in $(\pi/2, \pi)$
5. If tangents OX and OY at a point on the curve $\sqrt{x} - \sqrt{y} = \sqrt{a}$ cut the axes at P and Q, then prove that $OP + OQ = a$, where O is the origin.
6. Find the equations of tangents to the curve $y = \cos(x + y)$, $x \in [-2\pi, 2\pi]$, which is parallel to line $x + 2y = 0$.
7. If the edge of a cube is measured with an error of 5%, then find the approximate error to calculate its volume.
8. A circle disc of radius 10 cm is being heated. Due to expansion, its radius increases 2%. Find the rate at which its area is increasing.
9. Prove that the volume of the largest cone inscribed in a sphere is 8/27 of the volume of sphere.

10. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}(1/3)$.

Important Points

1. If a function $f(x)$ is differentiable then at any point $x = c$ for extreme point / value it is necessary that $f'(c) = 0$
2. Function $f(x)$ will have maximum value at a point c if $f'(c) = 0$ and $f''(c) < 0$
3. Function $f(x)$ will have minimum value at a point c if $f'(c) = 0$ and $f''(c) > 0$

ANSWERS

Exercise 8.1

1. $6\pi\ \text{cm}^2/\text{s},\ 8\pi\ \text{cm}^2/\text{s}$ 2. $(1, 5/3), (-1, 1/3)$ 3. $-3/10$ radian / s

4. $900\ \text{cm}^3/\text{s}$ 5. $1/\pi$ cm / s 6. $\frac{27}{8}\pi(2x+1)^2$

7. 30.02 (approx) 8. $35.2\ \text{cm}^3/\text{sec.},\ 20\pi\ \text{cm}^3/\text{s}$ 9. $\frac{1}{48}\pi$ 10. 126

Exercise 8.2

11. increasing in $(-\infty, -2) \cup (3, \infty)$ and decreasing in $(-2, 3)$
12. increasing in $(-1, 0) \cup (1, \infty)$ and decreasing in $(-\infty, -1) \cup (0, 1)$
13. increasing in $(-\infty, -1) \cup (0, 1)$ and decreasign in $(1, 2)$
14. increasing in $(-1, 2)$ and decreasing in $(-\infty, -1) \cup (2, \infty)$
15. -2

Exercise 8.3

1. 11 2. $-1/64$ 3. $(3, 2)$ 4. $y - 2x + 2 = 0,\ y - 2x + 10 = 0$

5. (i) $(0, 5)$ and $(0, -5)$; (ii) $(2, 0)$ and $(-2, 0)$ 6. $y = 0$ 7. $24x + 12\sqrt{3}y = 8\pi + 9\sqrt{3}$

8.	**tangent**	**Normal**
(a)	$10x - y - 8 = 0$	$x + 10y - 223 = 0$
(b)	$y - x - a = 0$	$y + x - 3a = 0$
(c)	$x + yt^2 = 2at$	$xt^3 - yt = at^4 - a$
(d)	$y - mx = \frac{a}{m}$	$my + x = 2a + \frac{a}{m^2}$
(e)	$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$	$ax\cos\theta + by\cot\theta = a^2 + b^2$
(f)	$x - y - 3 = 0$	$x + y + 1 = 0$
(g)	$x - y + a = 0$	$x + y - 3a = 0$
(h)	$2x - 2y - \pi = 0$	$2x + 2y - \pi - 4 = 0$

Exercise 8.4

1. 0.2083	2. 0.9999	3. 0.0608	4. 0.2495	5. 1.968	6. 20.025	7. 7.904
8. 2.00187	9. 0.8	10. 1.004343	11. 2.3046	12. 0-6	14. 80π cm^3	

Exercise 8.5

1. (a) maximum at $x = 2$ and minimum at $x = 3$

(b) maximum at $x = \frac{6-\sqrt{3}}{3}$ and minimum at $x = \frac{6+\sqrt{3}}{3}$

(c) maximum at $x = \sin^{-1} 1/4, \pi - \sin^{-1} 1/4$ and minimum at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

(d) maximum at $x = 1$ and minimum at $x = 3$

2. (a) maximum value = 3, minimum value = does not exist
   (b) maximum value = does nto exist, minimum value = -1
   (c) maximum value =4, minimum value = 2
   (d) maximum value = 6, minimum vlaue = 4
3. (a) maximum value = 160, at $x = 4$
   minimum value = 75, at $x = 2$
   (b) maximum value = 21, at $x = 0$
   minimum value = 1, at $x = 0$
   (c) maximum value = 2p, at $x = 2\pi$
   minimum value = 0, at $x = 0$
   (d) maximum value = 160, at $x = 4$
   minimum = 0, at $x = 0$
4.

	Maximum value	Minimum value
(a)	$= 1, \frac{2}{3\sqrt{6}}$,	$= -1, \frac{-2}{3\sqrt{6}}$
(b)	$= (a^{2/3} + b^{2/3})^{3/2}$	$= -(a^{2/3} + b^{2/3})^{3/2}$
(c)	$= e^{1/e}$	
(d)	$= 1/e$	

Miscellaneous Exercise – 8

1. $4\pi r + 2\pi h$ 2. $x = \pm 1, \ y = 22, \ 2$ 6. $2x + 4y + 3\pi = 0$ and $2x + 4y - \pi = 0$

7. 15% 8. 4π cm^2